1
Transformations of the Plane

1.1 Introduction

The two main areas of application which are considered in this textbook are computer graphics and computer-aided design (CAD). In computer graphics applications, geometric objects are defined in terms of a number of basic building blocks called *graphical primitives*. There are primitives which correspond to points, lines, curves, and surfaces. For example, a rectangle can be defined by its four sides. Each side is constructed from a line segment primitive by applying a number of geometric operations, called transformations, which position, orientate or scale the line primitive. Five types of transformation are particularly relevant in applications, namely, translations, scalings, reflections, rotations, and shears. These are introduced in Sections 1.2–1.6. Applications of transformations are considered in Section 1.8. In particular, Section 1.8.1 exemplifies, in more detail, how objects can be defined by applying transformations to graphical primitives by a process called *instancing*. Each primitive has a mathematical representation which can be expressed as a data or type structure for storage and manipulation by a computer. The mathematical representation of primitives is discussed in Chapters 5–9.

Given a fixed unit of length, and two perpendicular lines of reference called the x-axis and the y-axis, each point $\mathbf{P}$ of the plane is represented by an ordered pair of real numbers (x, y) such that the perpendicular distance of $\mathbf{P}$ from the y-axis is x units and the distance of $\mathbf{P}$ from the x-axis is y units. The ordered pair (x, y) is called the *Cartesian* or *affine coordinates* of $\mathbf{P}$, and the set of all ordered pairs of real numbers (x, y) is called the *Cartesian* or *affine plane* and denoted $\mathbb{R}^2$. The axes intersect in a point $\mathbf{O}$, with coordinates $(0, 0)$, called the *origin.* The point $\mathbf{P}$ with coordinates (x, y) will be denoted $\mathbf{P}(x, y)$. For the purposes of computation the point may also be represented by the row vector (x, y) or the row matrix $\begin{pmatrix} x & y \end{pmatrix}$.

For constants A, B, C (A and B not both zero) the set of points (x, y) satisfying the equation

$$Ax + By + C = 0$$

is a *line* which is said to be defined in *implicit form.* The line through a point (p_1, p_2) in the direction of the vector (v_1, v_2) can be defined *parametrically* by

$$(x(t), y(t)) = (p_1 + v_1 t, p_2 + v_2 t) \ .$$

Each value of the parameter t corresponds to a point on the line. For instance, evaluating $x(t)$ and $y(t)$ at $t = 0$ yields the point (p_1, p_2), and evaluating at $t = 1$ yields the point $(p_1 + v_1, p_2 + v_2)$. Any parametrically defined line can be expressed in implicit form by eliminating t from $x = p_1 + v_1 t$ and $y = p_2 + v_2 t$, to give

$$v_2 x - v_1 y + (p_2 v_1 - p_1 v_2) = 0 \ .$$

It also follows that the line with equation $Ax + By + C = 0$ has the direction of the vector $\pm(-B, A)$ and normal direction (the direction perpendicular to the line) $\pm(A, B)$.

The line through the two points $\mathbf{P}$ and $\mathbf{Q}$ is denoted $\overline{\mathbf{PQ}}$. The line *segment* $\mathbf{PQ}$ (with *endpoints* $\mathbf{P}$ and $\mathbf{Q}$) is the portion of the line $\overline{\mathbf{PQ}}$ between the points $\mathbf{P}$ and $\mathbf{Q}$.

Example 1.1

Consider the line passing through the point (a, b), and making an angle α with the x-axis. By elementary trigonometry, a point (x, y) on the line satisfies $\tan(\alpha) = (y - b)/\,(x - a)$. Hence the line is given in implicit form by $\tan(\alpha)x - y + b - \tan(\alpha)a = 0$.

Example 1.2

Consider two lines $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ with directions $\mathbf{v} = (-B_1, A_1)$ and $\mathbf{w} = (-B_2, A_2)$ respectively. Suppose θ is the angle between the lines. Then the vector identity $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$ and the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ give

$$\cos\theta = \frac{A_1 A_2 + B_1 B_2}{\left(A_1^2 + B_1^2\right)\left(A_2^2 + B_2^2\right)}, \quad \sin\theta = \frac{A_1 B_2 - B_1 A_2}{\left(A_1^2 + B_1^2\right)\left(A_2^2 + B_2^2\right)} \ .$$

Hence

$$\tan\theta = \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\,.$$

It follows that the two lines are parallel if and only if $\theta = 0$, that is, if and only if $A_1B_2 = A_2B_1$.

Exercises

1.1 Show that the angle α that the line $Ax + By + C = 0$ makes with the x-axis is given by $\tan(\alpha) = -A/B$.

1.2 Determine an implicit equation for the line $(2+3t, 5-4t)$. Determine the angle that the line makes with the x-axis.

1.3 Show that, for points $\mathbf{P}(a_1, a_2)$ and $\mathbf{Q}(b_1, b_2)$, the line $\overline{\mathbf{PQ}}$ can be expressed in the parametric form $(1-t)(p_1, p_2) + t(q_1, q_2)$, that is, $(x(t), y(t)) = (p_1 - tp_1 + tq_1, p_2 - tp_2 + tq_2)$ for $t \in \mathbb{R}$. Show also that the segment $\mathbf{PQ}$ is given by the same equation and $t \in [0, 1]$.

1.4 Show that $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are perpendicular if and only if $A_1A_2 + B_1B_2 = 0$.

Definition 1.1

A (linear) *transformation* of the plane is a mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane to itself of the form

$$L(x, y) = (ax + by + c, dx + ey + f)\,, \tag{1.1}$$

for some constant real numbers a, b, c, d, e, f. The point $\mathbf{P}' = L(\mathbf{P})$ is called the *image* of $\mathbf{P}$. If S is a subset of $\mathbb{R}^2$, then $L(S) = \{L(x, y) : (x, y) \in S\}$ is called the *image* of S.

Example 1.3

Let $L(x, y) = (2x + 3y + 4, 5x + 6y + 7)$. The images of the points $(4, 2)$, $(2, 1)$, and $(0, 0)$ are $L(4, 2) = (18, 39)$, $L(2, 1) = (11, 23)$, and $L(0, 0) = (4, 7)$.

Lemma 1.1

If $aB - bA$ and $dB - eA$ are not both zero, then the transformation L given by (1.1) maps the line $Ax + By + C = 0$ (A and B not both zero) to the line

$$(eA - dB)\,x + (aB - bA)\,y + ((bf - ce)\,A - (af - cd)\,B + (ae - bd)\,C) = 0\,. \tag{1.2}$$

If $aB - bA = 0$ and $eA - dB = 0$, then $ae - bd = 0$ and L maps every point on the line to the point $((cB - bC)/B, (fB - eC)/B)$.

Proof

Let L be the transformation given by (1.1). Consider the line $Ax + By + C = 0$, and suppose $B \neq 0$. (The case $B = 0$ is left as an exercise to the reader.) Then each point on the line has the form $\left(t, -\frac{A}{B}t - \frac{C}{B}\right)$. So $L\left(t, -\frac{A}{B}t - \frac{C}{B}\right) = (x, y)$ where

$$x = \frac{(aB - bA)t - bC + cB}{B} \text{ and } y = \frac{(dB - eA)t - eC + fB}{B}. \tag{1.3}$$

If $aB - bA \neq 0$ or $dB - eA \neq 0$, then t can be eliminated from equations (1.3) to give (1.2) and the first part of the lemma is proved.

Suppose $aB - bA = 0$ and $eA - dB = 0$. Since A and B are not both zero, it follows that $ae - bd = 0$. Every point on the line maps to the point $(X, Y) = ((cB - bC)/B, (fB - eC)/B)$. □

Definition 1.2

A transformation L given by (1.1) is said to be *singular* whenever

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd = 0, \tag{1.4}$$

and *non-singular* otherwise.

Exercises

1.5 The proof of Lemma 1.1 shows that whenever a linear transformation L given by (1.1) maps a line to a point, then $aB - bA = dB - eA = 0$. Hence $ae - bd = 0$, and L is singular. Show the converse, that if L is singular ($ae - bd = 0$), then there exists a line $Ax + By + C = 0$ which is mapped by L to a point.

1.6 Suppose L is a non-singular transformation. Show that the line segment with endpoints $\mathbf{P}(p_1, p_2)$ and $\mathbf{Q}(q_1, q_2)$ maps to the line segment with endpoints $L(\mathbf{P})$ and $L(\mathbf{Q})$.

Remark 1.1

Throughout the book the term *object* is used rather vaguely. A planar object is a subset of $\mathbb{R}^2$, and a spatial object is a subset of $\mathbb{R}^3$. In most applications an object has a geometrical structure such as that of being a 'point', a 'line', a 'curve', a 'collection of curves', or a 'region of points'.

1.2 Translations

A *translation* is a transformation which maps a point $\mathbf{P}(x, y)$ to a point $\mathbf{P}'(x', y')$ by adding a constant amount to each coordinate so that

$$x' = x + h, \quad y' = y + k ,$$

for some constants h and k. The translation has the effect of moving $\mathbf{P}$ in the direction of the x-axis by h units, and in the direction of the y-axis by k units. If $\mathbf{P}$ and $\mathbf{P}'$ are written as row vectors, then

$$(x', y') = (x, y) + (h, k) .$$

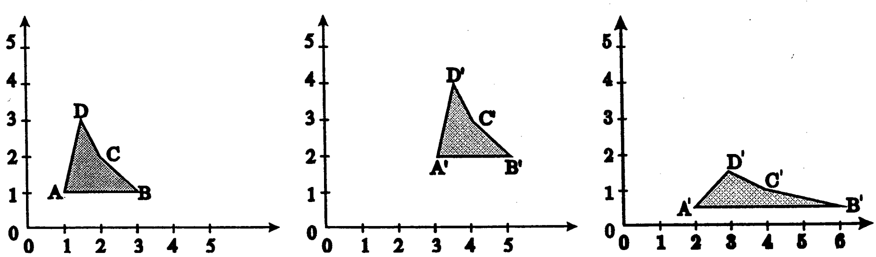
To translate an object it is necessary to add the vector (h, k) to every point of that object. The translation is denoted $\mathsf{T}(h, k)$. A translation can also be executed using matrix addition if (x, y) is represented as the row matrix $\begin{pmatrix} x & y \end{pmatrix}$.

Example 1.4

Consider a quadrilateral with vertices $\mathbf{A}(1, 1)$, $\mathbf{B}(3, 1)$, $\mathbf{C}(2, 2)$, and $\mathbf{D}(1.5, 3)$. Applying the translation $\mathsf{T}(2, 1)$, the images of the vertices are

$$\begin{aligned} \mathbf{A}' &= (1,1) + (2,1) = (3,2) \ , \\ \mathbf{B}' &= (3,1) + (2,1) = (5,2) \ , \\ \mathbf{C}' &= (2,2) + (2,1) = (4,3) \ , \text{ and} \\ \mathbf{D}' &= (1.5,3) + (2,1) = (3.5,4) \ . \end{aligned}$$

Fig. 1.1 shows (a) the original, and (b) the translated quadrilateral.



(a) Original quadrilateral (b) Effect of a translation (c) Effect of a scaling

Fig. 1.1.

Definition 1.3

The transformation which leaves all points of the plane unchanged is called the *identity transformation* and denoted I. The *inverse transformation* of L, denoted L^{-1}, is the transformation such that (i) L^{-1} maps every image point $L(\mathbf{P})$ back to its original position $\mathbf{P}$, and (ii) L maps every image point $L^{-1}(\mathbf{P})$ to $\mathbf{P}$. Inverse transformations will be discussed further in Section 2.5.1.

Example 1.5

Consider the translation $\mathsf{T}(h, k)$ which maps a point $\mathbf{P}(x, y)$ to $\mathbf{P}'(x+h, y+k)$. The transformation T^{-1} required to map $\mathbf{P}'$ back to $\mathbf{P}$ is the inverse translation $\mathsf{T}(-h, -k)$. For instance, applying $\mathsf{T}(-2, -1)$ to the point $\mathbf{A}'$ of Example 1.4 gives $(3, 2) + (-2, -1) = (1, 1)$, and hence maps $\mathbf{A}'$ back to $\mathbf{A}$. The reader can check that the same translation returns the other images to their original locations.

Exercise 1.7

(a) Apply the translation $\mathsf{T}(3, -2)$ to the quadrilateral of Example 1.4, and make a sketch of the transformed quadrilateral.

(b) Determine the inverse transformation of $\mathsf{T}(3, -2)$. Apply the inverse to the transformed quadrilateral to verify that the inverse returns the quadrilateral to its original position.

1.3 Scaling about the Origin

A *scaling about the origin* is a transformation which maps a point $\mathbf{P}(x, y)$ to a point $\mathbf{P}'(x', y')$ by multiplying the x and y coordinates by non-zero constant *scaling factors* s_x and s_y, respectively, to give

$$x' = s_x x \text{ and } y' = s_y y \ .$$

A scaling factor s is said to be an *enlargement* if $|s| > 1$, and a *contraction* if $|s| < 1$. A scaling transformation is said to be *uniform* whenever $s_x = s_y$. By representing a point (x, y) as a row matrix $\begin{pmatrix} x & y \end{pmatrix}$, the scaling transformation can be performed by a matrix multiplication

$$\mathbf{P}' = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} = \begin{pmatrix} s_x x & s_y y \end{pmatrix} \ .$$

The matrix

$$\mathsf{S}(s_x, s_y) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

is called the *scaling transformation matrix*.

Example 1.6

To apply the scaling transformation $\mathsf{S}(2, 0.5)$ to the quadrilateral of Example 1.4, the coordinates of the four vertices of the quadrilateral are represented by the rows of the 4×2 matrix

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 2 \\ 1.5 & 3 \end{pmatrix},$$

and multiplied by the scaling transformation matrix

$$\begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \\ \mathbf{C}' \\ \mathbf{D}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 2 \\ 1.5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 2 & 0.5 \\ 6 & 0.5 \\ 4 & 1 \\ 3.0 & 1.5 \end{pmatrix}.$$

The rows of the resulting matrix are the coordinates of the images of the vertices. The original quadrilateral and its scaled image are shown in Fig. 1.1(a) and (c). The quadrilateral is scaled by a factor 2 in the x-direction and by a factor 0.5 in the y-direction.

Remark 1.2

The quadrilateral of Example 1.6 has experienced a translation due to the fact that scaling transformations are performed 'about the origin **O**'. (Scalings about an arbitrary point are considered in Section 2.4.2.) The true effect of a scaling about the origin is to scale the position vectors $\overrightarrow{\mathbf{OP}}$ of each point **P** in the plane. For instance, in Example 1.6 vectors $\overrightarrow{\mathbf{OA}}$, $\overrightarrow{\mathbf{OB}}$, $\overrightarrow{\mathbf{OC}}$, and $\overrightarrow{\mathbf{OD}}$ have been scaled by the factors 2 and 0.5 in the x- and y-directions as shown in Fig. 1.2. Since the positions of all four points **A**, **B**, **C**, and **D** have changed, there is a combined effect of scaling and translating of the object. The origin is the only point unaffected by a scaling about the origin.

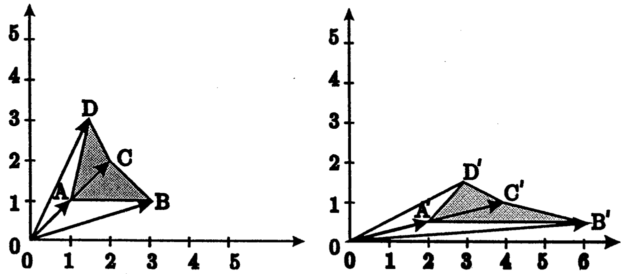


Fig. 1.2. Effect of scaling on position vectors

Exercises

1.8 Apply the scaling transformation $S(-1,1)$ to the quadrilateral of Example 1.4. Describe the effect of the transformation.

1.9 Show that the inverse transformation $S(s_x, s_y)^{-1}$ of a scaling $S(s_x, s_y)$ (with $s_x \neq 0$ and $s_y \neq 0$) is the scaling $S(1/s_x, 1/s_y)$.

1.4 Reflections

Two effects which are commonly used in CAD or computer drawing packages are the horizontal and vertical 'flip' or 'mirror' effects. Pictures which have undergone a horizontal or vertical flip are shown in Fig. 1.3(a). A flip of an object is obtained by applying a transformation known as a *reflection.* Consider a fixed line ℓ in the plane. The reflected image of a point $\mathbf{P}$, a distance d from ℓ, is determined as follows. If $d = 0$ then $\mathbf{P}$ is a point on ℓ and the image is $\mathbf{P}$. Otherwise, take the unique line ℓ_1 through $\mathbf{P}$ and perpendicular to ℓ. Then, as showed in Fig. 1.3, there are two distinct points on ℓ_1, $\mathbf{P}$ and $\mathbf{P}'$, which are a distance d away from ℓ. The point $\mathbf{P}'$ is the required image of $\mathbf{P}$.

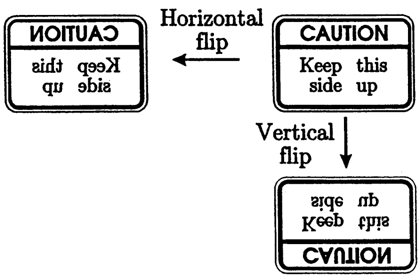
It is easily verified that the reflection R_x in the x-axis is the transformation $L(x,y) = (x,-y)$, and the reflection R_y in the y-axis is $L(x,y) = (-x,y)$. The reflection R_x can be computed by the matrix multiplication

$$\mathsf{R}_x \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} x & -y \end{pmatrix},$$

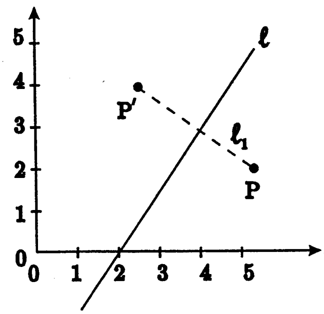
and R_y by

$$\mathsf{R}_y \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -x & y \end{pmatrix}.$$

The reflection R_y was encountered in Exercise 1.7. Reflections in arbitrary lines are discussed in Section 2.5.3.



(a) Horizontal and vertical flips



(b) Reflection in the line ℓ

Fig. 1.3.

Exercises

1.10 Apply the reflection R_x to the quadrilateral of Example 1.4.

1.11 Verify that $\mathsf{R}_x = \mathsf{S}(1,-1)$ and $\mathsf{R}_y = \mathsf{S}(-1,1)$.

1.12 Show that the inverse of R_x is R_x, that is, $\mathsf{R}_x^{-1} = \mathsf{R}_x$. Similarly, show that $\mathsf{R}_y^{-1} = \mathsf{R}_y$.

1.5 Rotation about the Origin

A *rotation* about the origin through an angle θ has the effect that a point $\mathbf{P}(x,y)$ is mapped to a point $\mathbf{P}'(x',y')$ so that the initial point $\mathbf{P}$ and its image point $\mathbf{P}'$ are the same distance from the origin, and the angle between lines $\overline{\mathbf{OP}}$ and $\overline{\mathbf{OP}'}$ is θ. There are two possible image points which satisfy these properties depending on whether the rotation is carried out in a clockwise or anticlockwise direction. It is the convention that a positive angle θ represents an *anticlockwise* direction so that a $\pi/2$ rotation about the origin maps points on the x-axis to points on the y-axis.

Referring to Fig. 1.4, let $\mathbf{P}'(x',y')$ denote the image of a point $\mathbf{P}(x,y)$ following a rotation about the origin through an angle θ (in an anticlockwise direction). Suppose the line $\overline{\mathbf{OP}}$ makes an angle ϕ with the x-axis, and that $\mathbf{P}$ is a distance r from the origin. Then $(x,y) = (r\cos\phi, r\sin\phi)$. $\mathbf{P}'$ makes an angle $\theta+\phi$ with the x-axis, and therefore $(x',y') = (r\cos(\theta+\phi), r\sin(\theta+\phi))$. The addition formulae for trigonometric functions yields

$$\begin{aligned} x' &= r\cos(\theta+\phi) = r\cos\theta\cos\phi - r\sin\theta\sin\phi = x\cos\theta - y\sin\theta\ , \text{ and} \\ y' &= r\sin(\theta+\phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi = x\sin\theta + y\cos\theta\ . \end{aligned}$$

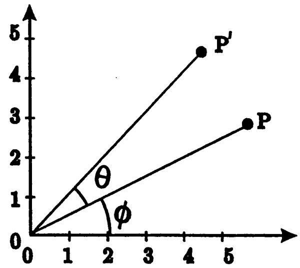


Fig. 1.4. Rotation of a point **P** about the origin

The coordinates (x', y') can be obtained from (x, y) by the matrix multiplication

$$\mathbf{P}' = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta \end{pmatrix} .$$

The matrix

$$\text{Rot}\,(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is called the *rotation matrix.*

Example 1.7

The rotation matrices of rotations about the origin through $\pi/2$, π, and $3\pi/2$ radians are

$$\text{Rot}\,(\pi/2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{Rot}\,(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \text{Rot}\,(3\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Example 1.8

Applying the rotation Rot $(\pi/2)$ to the quadrilateral of Example 1.4, gives the points

$$\begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \\ \mathbf{C}' \\ \mathbf{D}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 2 \\ 1.5 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 3 \\ -2 & 2 \\ -3 & 1.5 \end{pmatrix} .$$

The image of the quadrilateral is shown in Fig. 1.5.

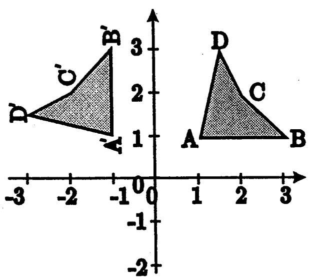


Fig. 1.5. Rotation of the quadrilateral about the origin through $\pi/2$

Exercises

1.13 Apply rotations about the origin through the angles $\pi/3$, $2\pi/3$, and $\pi/4$ to the triangle with vertices $\mathbf{P}(1,1)$, $\mathbf{Q}(3,1)$, and $\mathbf{R}(2,2)$. Sketch the resulting triangles.

1.14 Show that $\mathsf{Rot}(\theta)^{-1} = \mathsf{Rot}(-\theta)$.

1.15 Do the transformations $\mathsf{Rot}(\pi/2)$ and R_y have the same effect?

1.6 Shears

Given a fixed direction in the plane specified by a unit vector $\mathbf{v} = (v_1, v_2)$, consider the lines ℓ_d with direction $\mathbf{v}$ and a distance d from the origin as shown in Fig. 1.6. A *shear about the origin of factor r in the direction* $\mathbf{v}$ is defined to be the transformation which maps a point $\mathbf{P}$ on ℓ_d to the point $\mathbf{P}' = \mathbf{P} + rd\mathbf{v}$. Thus the points on ℓ_d are translated along ℓ_d (that is, in the direction of $\mathbf{v}$) through a distance of rd.

Example 1.9

To determine a shear in the direction of the x-axis with factor r, let $\mathbf{v} = (1,0)$. The line in the direction of $\mathbf{v}$ through an arbitrary point $\mathbf{P}(x_0, y_0)$ has equation $y = y_0$. The line is a distance y_0 from the origin. Thus $\mathbf{P}$ is mapped to $\mathbf{P}'(x_0 + ry_0, y_0)$ and hence

$$\begin{pmatrix} x' & y' \end{pmatrix} = \begin{pmatrix} x_0 + ry_0 & y_0 \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$

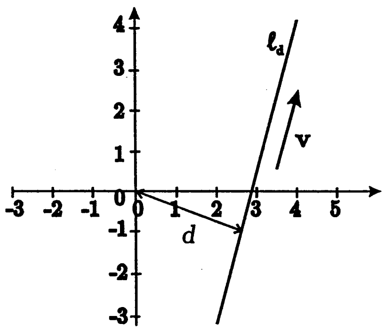


Fig. 1.6. Shear in the direction **v**

The general shear transformation matrix is determined as follows. The line through $\mathbf{P}(x_0, y_0)$ with direction $\mathbf{v} = (v_1, v_2)$ has equation

$$v_2 x - v_1 y + (v_1 y_0 - v_2 x_0) = 0 \, .$$

Since **v** is a unit vector, the distance from this line to the origin is

$$d = v_1 y_0 - v_2 x_0 \, .$$

There are two lines a given distance away from the origin with a specified direction, and the lines on either side of ℓ_0 (the line through the origin with direction **v**) are distinguished by the sign of $v_1 y_0 - v_2 x_0$. It follows that the shear transformation maps $\mathbf{P}(x_0, y_0)$ to

$$\mathbf{P}' = \mathbf{P} + rd\mathbf{v} = (x_0 + r(v_1 y_0 - v_2 x_0)v_1, y_0 + r(v_1 y_0 - v_2 x_0)v_2) \, .$$

Thus the shear has transformation matrix

$$\mathsf{Sh}(\mathbf{v}, r) = \begin{pmatrix} 1 - rv_1 v_2 & -rv_2^2 \\ rv_1^2 & 1 + rv_1 v_2 \end{pmatrix} .$$

In particular,

$$\mathsf{Sh}((1,0), r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

verifying the result of Example 1.9.

Example 1.10

The shear in the direction $\mathbf{v} = (2/\sqrt{5}, 1/\sqrt{5})$ with a factor $r = 1.5$ has transformation matrix

$$\mathsf{Sh}\left(\left(2/\sqrt{5}, 1/\sqrt{5}\right), 1.5\right) = \begin{pmatrix} 1 - 1.5\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) & -1.5\left(\frac{1}{\sqrt{5}}\right)^2 \\ 1.5\left(\frac{2}{\sqrt{5}}\right)^2 & 1 + 1.5\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0.4 & -0.3 \\ 1.2 & 1.6 \end{pmatrix}.$$

Applying the shear to the quadrilateral of Example 1.4,

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 2 \\ 1.5 & 3 \end{pmatrix} \begin{pmatrix} 0.4 & -0.3 \\ 1.2 & 1.6 \end{pmatrix} = \begin{pmatrix} 1.6 & 1.3 \\ 2.4 & 0.7 \\ 3.2 & 2.6 \\ 4.2 & 4.35 \end{pmatrix}.$$

The effect of the shear is shown in Fig. 1.7.

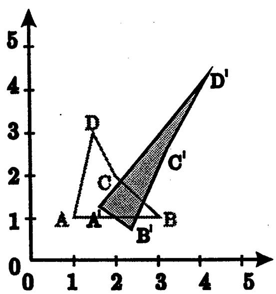


Fig. 1.7.

Exercise 1.16

Determine the transformation matrix for a shear with (a) direction $(3,-4)$ and factor $r = 4$, and (b) direction $(8,6)$ and factor $r = -1$.

1.7 Concatenation of Transformations

In many applications it is desirable to apply more than one transformation to an object. For instance, a translation and a rotation may be required to position and orientate an object. The process of following one transformation by another to form a new transformation with a combined effect is called *concatenation* or *composition* of transformations. The term *concatenation* is the most commonly used in computer graphics. All of the transformations described in the earlier sections can be concatenated to obtain new transformations.

Example 1.11

A rotation about the origin through an angle $\pi/3$ is obtained by applying the matrix Rot $(\pi/3)$

$$\begin{aligned}\begin{pmatrix} x' & y' \end{pmatrix} &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ -\sin(\pi/3) & \cos(\pi/3) \end{pmatrix} \\ &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} .\end{aligned}$$

Next apply a scaling by a factor of 6 in the x-direction and 2 in the y-direction

$$\begin{aligned}\begin{pmatrix} x'' & y'' \end{pmatrix} &= \begin{pmatrix} x' & y' \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} .\end{aligned}$$

Hence, the concatenated transformation has transformation matrix

$$\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & \sqrt{3} \\ -3\sqrt{3} & 1 \end{pmatrix} .$$

A problem is encountered whenever translations are concatenated with other types of transformation since it is necessary to combine a matrix (or vector) addition for the translation with a matrix multiplication for the other transformations. This is an awkward procedure remedied only by the introduction of homogeneous coordinates, as discussed in Chapter 2. Thus concatenation will not be discussed any further, and the approach of using 2×2 matrix multiplications will be abandoned. The homogeneous coordinate system offers the following advantages for the execution of transformations.

1. All transformations can be represented by matrices, and performed by matrix multiplication.
2. Concatenation of transformations is performed by matrix multiplication of the transformation matrices.
3. Inverse transformations are obtained by taking a matrix inverse.

The effort expended has not been in vain since the 'homogeneous' transformation matrices are closely related to those described in this chapter.

1.8 Applications

1.8.1 Instancing

A geometric object is created by defining the different parts which make up the object. For example, the front of a house in Fig. 1.8 consists of a number of rectangles, or rather scaled squares, which form the walls, windows, and door of the house. The square is an example of a *picture element.* For convenience, picture elements are defined in their own local coordinate system called the *modelling coordinate system*, and are constructed from *graphical primitives* which are the basic building blocks. Picture elements are defined once, but may be used many times in the construction of objects. The number and type of graphical primitives available depends on the computer graphics system.

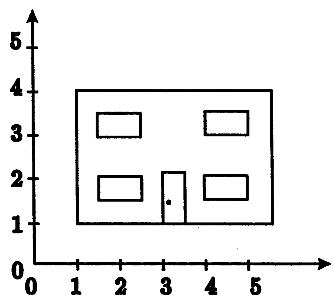


Fig. 1.8. Front of a house obtained from instances of **Square** and **Point**

For example, a square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ can be obtained using the graphical primitive for the line segment, denoted **Line**, which joins the points $(0,0)$ and $(1,0)$. One possible construction of the square is obtained in the following manner.

1. Draw **Line**. This produces the horizontal base of the square.
2. Apply a rotation about the origin through an angle $\pi/2$ to a copy of **Line**, and then apply a translation of 1 unit in the x-direction. This gives the right vertical edge of the square.
3. Apply a translation of 1 unit in the y-direction to a copy of **Line**. This gives the top of the square.
4. Apply a rotation about the origin through an angle $\pi/2$ to a copy of **Line**. This gives the left vertical edge of the square.

A transformed copy of a graphical primitive or picture element is called an *instance*. The square, denoted **Square**, is defined by four instances of **Line** as depicted in Fig. 1.9.

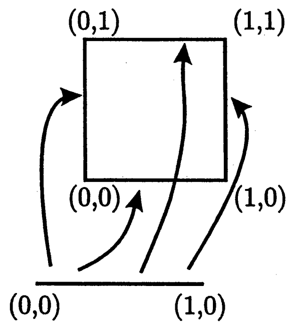


Fig. 1.9. Square obtained from four instances of **Line**

The completed 'real' object is defined in *world coordinates* by applying a *modelling coordinate transformation* to each picture element. The house of Figure 1.8 is defined by six instances of the picture element **Square,** and one instance of the primitive **Point** (for the door handle). In particular, the front door is obtained by applying a scaling of 0.5 unit in the x-direction, followed by a translation of 3 units in the x-direction and 1 unit in the y-direction.

In the above discussion, instancing has been described in words since without homogeneous coordinates the concatenation of transformations is awkward. In the proposed homogeneous coordinate system described in the next chapter each instance of a picture element or object can be represented by a single modelling transformation matrix.

Exercises

1.17 Each window and the outline of the house is obtained by instances of **Square.** Describe in words the sequence of transformations used for each instance.

1.18 Investigate the graphical primitives available in graphics systems such as PHIGS, GKS, and OpenGL. See for example [17] and the web page for the book.

1.8.2 Robotics

Consider a planar 2R robot manipulator arm (Fig. 1.10) consisting of two rigid links. The first link is attached to the base by a revolute joint J_1. A revolute joint permits the link to rotate about a point. The second link is attached to the first link by a second revolute joint J_2. The robot hand or end effector is attached to the second link. The position and orientation of the robot hand is controlled by turning the links about the two joints.

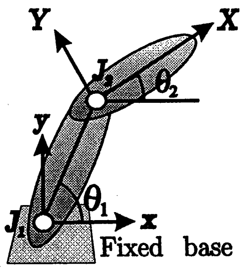


Fig. 1.10. 2R robot

Define an (x, y)-coordinate system with J_1 as the origin as shown in Fig. 1.10. The second link is given its own (X, Y)-coordinate system with J_2 as the origin. Suppose that the distance between J_1 and J_2 is d, that link 1 makes an angle θ_1 with the x-axis, and link 2 makes an angle θ_2 with the x-axis. The position and orientation of the second link is obtained by applying a rotation $\mathsf{Rot}(\theta_2)$ followed by a translation $\mathsf{T}(d\cos\theta_1, d\sin\theta_1)$. Given the (X, Y) coordinates of a point $\mathbf{P}$, the (x, y)-coordinates of $\mathbf{P}$ are obtained by the transformation

$$\begin{aligned}\begin{pmatrix} x & y \end{pmatrix} &= \begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} + \begin{pmatrix} d\cos\theta_1 & d\sin\theta_1 \end{pmatrix} \\ &= \begin{pmatrix} X\cos\theta_2 - Y\sin\theta_2 + d\cos\theta_1 & X\sin\theta_2 + Y\cos\theta_2 + d\sin\theta_1 \end{pmatrix}.\end{aligned}$$

The ultimate aim is to express such concatenations with one matrix multiplication with the assistance of homogeneous coordinates.

Exercises

1.19 Suppose an affine transformation $L(x, y) = (ax + by + c, dx + ey + f)$ is applied to a triangle $\mathcal{T}$ with vertices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and area $\mathcal{A}$. Show that the area of $L(\mathcal{T})$ is $(ad - bc) \cdot \mathcal{A}$.

1.20 Prove that a transformation maps the midpoint of a line segment to the midpoint of the image.

1.21 Write a computer program or use a computer package to implement the various types of transformation. Apply the program to the examples of the chapter.