2 INTRODUCTION TO TENSOR CALCULUS

In the first part we have considered one-dimensional models to describe material behavior. The application of such models, however, is rather limited, since in the general case, both the stress and the strain state are three-dimensional. For the generalization of such concepts to higher dimensions than one, an appropriate mathematical instrumentarium is needed. In mechanics, the appropriate objects for the mathematical description are vectors and tensors. For readers not familiar with such objects, we will next give a brief introduction into tensor algebra and analysis.

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2.1 Vector and Tensor Algebra

2.1.1 Summation Convention

Let us consider the following sum

$$s = a_1 b_1 + a_2 b_2 + \ldots + a_N b_N = \sum_{i=1}^{N} a_i b_i \, .$$

We could also have written

$$s = \sum_{k=1}^{N} a_k b_k \, ,$$

i.e., the choice of the summation index (i or k) does not influence the result and is therefore called a **dummy index**. Since we often need sums, we introduce the following abbreviation:

> **Summation convention**
> *If an index appears twice in a product term, one has to sum over this index from 1 to N.*

Or: *one has to sum over dummy indices.*

The number N results from the context. In what follows, it is usually 3, equal to the dimension of the geometrical space.

Consequently, we can write for the previous example

$$s = a_i b_i = a_k b_k \quad \text{etc.}$$

We will need the component representation of a vector $\mathbf{v}$ with respect to some vector basis $\{\mathbf{g}_i\}$

$$\mathbf{v} = v^1 \mathbf{g}_1 + v^2 \mathbf{g}_2 + v^3 \mathbf{g}_3 = v^i \mathbf{g}_i \, .$$

If a_{ij} are the elements of a 3 × 3–matrix, the sum of the diagonal elements

$$a_{ii} = a_{11} + a_{22} + a_{33}$$

is called the **trace** of the matrix. The product of such matrices with elements a_{ij} and b_{ij} is

$$a_{ij}\, b_{jk} \;=\; a_{il}\, b_{lk} \;=\; a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + a_{i3}\, b_{3k}\,.$$

Note that only j and l are dummy indices, while i and k are not. i and k are called **free indices**, taking arbitrary values between 1 and N. Free indices stand on both sides of the equation only once in each term.

The trace of the resulting matrix $a_{ik}\, b_{ki}$ is the double sum over both i and k, and the order of the summation does not matter

$$\sum_{i=1}^{N}\sum_{k=1}^{N} a_{ik}\, b_{ki} \;=\; \sum_{k=1}^{N}\sum_{i=1}^{N} a_{ik}\, b_{ki} \;=\; a_{ik}\, b_{ki}\,.$$

It is always important to distinguish between dummy and free indices. We will show this for another example. One can present the three equations

$$\begin{aligned} y_1 &= a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 \\ y_2 &= a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 \\ y_3 &= a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 \end{aligned}$$

more briefly as

$$y_i \;=\; a_{ik}\, x_k$$

or equivalently as

$$y_p \;=\; a_{pm}\, x_m\,.$$

Here k and m are dummy, while i and p are free indices.

Very often a situation occurs where some term equals 1 if two indices coincide, or 0 otherwise, like in the following example

$$\begin{array}{lll} a_1 b_1 = 1 & a_1 b_2 = 0 & a_1 b_3 = 0 \\ a_2 b_1 = 0 & a_2 b_2 = 1 & a_2 b_3 = 0 \\ a_3 b_1 = 0 & a_3 b_2 = 0 & a_3 b_3 = 1\,. \end{array}$$

In order to abbreviate this notation, one introduces the **KRONECKER**[23] **symbol** δ_{ij} as the components of the unity matrix

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

i.e.

(2.1.1) $$\delta_{ij} \;=\; \begin{Bmatrix} 1 & \text{for} & i = j \\ 0 & \text{for} & i \neq j \end{Bmatrix} = \delta_{ji} \;=\; \delta^{j}{}_{i} \;=\; \delta_{i}{}^{j}.$$

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[23] Leopold Kronecker (1828-1891)

It shall not matter here whether the indices are notated in the upper or in the lower position. For the above 9 equations we can briefly write

$$a_i\, b_j \;=\; \delta_{ij}\,.$$

In reverse, one can use the extensions

$$a_{ik}\, x_k \;=\; a_{il}\, \delta_{lk}\, x_k$$

and

$$a_i\, b_i \;=\; a_i\, b_k\, \delta_{ik}$$

as well as

$$a_{ik}\, x_k \;-\lambda\, x_i \;=\; a_{ik}\;\; x_k \;-\; \lambda\, \delta_{ik}\, x_k \;=\; (a_{ik} - \lambda\, \delta_{ik})\, x_k\,.$$

In addition, we will introduce in three dimensions the **permutation symbol** or **LEVI-CIVITA**[24] **symbol** ε_{ijk} with three indices as

(2.1.2)
$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{if } ijk \text{ is no permutation von } 1,2,3 \end{cases}$$

such as

$$\varepsilon_{231} \;=\; +\,1 \qquad\qquad \varepsilon_{132} \;=\; -\,1 \qquad\qquad \varepsilon_{122} \;=\; 0$$

Consequently

$$\varepsilon_{ijk} \;=\; \varepsilon_{kij} \;=\; -\,\varepsilon_{ikj} \;=\; -\,\varepsilon_{kji}$$

etc.

2.1.2 Vectors

Definition. A **vector space** or **linear space** is a set $\mathscr{V}$ (of "vectors"), in which addition and multiplication with scalars or real numbers are defined in accordance with the following rules

$\mathbf{a} + \mathbf{b} \;=\; \mathbf{b} + \mathbf{a}$	(commutative)
$(\mathbf{a} + \mathbf{b}) + \mathbf{c} \;=\; \mathbf{a} + (\mathbf{b} + \mathbf{c})$	(associative)
$\mathbf{a} + \mathbf{o} \;=\; \mathbf{a}$	(zero vector)
$\mathbf{a} + (-\mathbf{a}) \;=\; \mathbf{o}$	(negative element)
$(\alpha\,\beta)\,\mathbf{a} \;=\; \alpha\,(\beta\,\mathbf{a})$	(associative)

[24] Tullio Levi-Civita (1873-1941)

$1\,\mathbf{a} = \mathbf{a}$	(unit element)
$\alpha(\mathbf{a}+\mathbf{b}) = \alpha\,\mathbf{a} + \alpha\,\mathbf{b}$	(distributive)
$(\alpha+\beta)\,\mathbf{a} = \alpha\,\mathbf{a} + \beta\,\mathbf{a}$	(distributive)
$\forall\, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathscr{V},\ \forall \alpha, \beta \in \mathscr{R}$.	

A maximal system of linear independent vectors $\{\mathbf{g}_i\} := \{\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_N\}$ forms a **vector basis** in $\mathscr{V}$. With respect to such a basis one can represent each vector as a linear combination

$$\mathbf{a} = a^i\,\mathbf{g}_i\,.$$

The scalars a^i are the **components** of the vector with respect to this basis. The following rules hold for the addition of two vectors

(2.1.3) $$\mathbf{a} + \mathbf{b} = a^i\,\mathbf{g}_i + b^j\,\mathbf{g}_j = (a^i + b^i)\,\mathbf{g}_i$$

and the scalar multiplication

(2.1.4) $$\alpha\,\mathbf{a} = \alpha\,(a^i\,\mathbf{g}_i) = (\alpha\,a^i)\,\mathbf{g}_i\,.$$

Some vector spaces have in addition an **inner product** or a **scalar product**

$$\cdot\ :\ \mathscr{V}\times\mathscr{V}\to\mathscr{R} \quad\Big|\quad (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a}\cdot\mathbf{b}$$

with the following rules

$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$	(commutative)
$(\alpha\,\mathbf{a})\cdot\mathbf{b} = \alpha\,(\mathbf{a}\cdot\mathbf{b})$	(associative)
$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = (\mathbf{a}\cdot\mathbf{c}) + (\mathbf{b}\cdot\mathbf{c})$	(distributive)
$\mathbf{a}\cdot\mathbf{a} > 0$ for $\mathbf{a} \neq \mathbf{o}$	(positive–definite)
$\forall\, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathscr{V},\ \forall \alpha \in \mathscr{R}$.	

With such a scalar product we obtain

(2.1.5) $$\mathbf{a}\cdot\mathbf{b} = (a^i\,\mathbf{g}_i)\cdot(b^k\,\mathbf{g}_k) = a^i\,b^k\,\mathbf{g}_i\cdot\mathbf{g}_k\,.$$

A scalar product induces a **norm** $|\mathbf{a}| := \sqrt{\mathbf{a}\cdot\mathbf{a}}$ (length) of a vector. One can also define an angle φ between two vectors as a solution of the equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\varphi\,.$$

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are mutually *orthogonal* we have

$$\mathbf{a}\cdot\mathbf{b} = 0\,.$$

Definition. Two vector bases $\{\mathbf{g}_i\}$ and $\{\mathbf{g}^k\}$ are called **dual** if

(2.1.6) $$\mathbf{g}_i\cdot\mathbf{g}^k = \delta_i^k$$

Consequently, all base vectors of $\{\mathbf{g}_i\}$ are orthogonal to those of $\{\mathbf{g}^k\}$ for $k \neq i$. For a given basis $\{\mathbf{g}_i\}$, (2.1.6) is an inhomogeneous system of linear equations for the components of the dual basis $\{\mathbf{g}^k\}$, whose coefficient determinant cannot be singular, so that it always possesses a unique solution.

Theorem. *For every vector basis* $\{\mathbf{g}_i\}$ *a unique dual basis* $\{\mathbf{g}^j\}$ *exists.*

The use of dual bases is always possible, and in many cases rather convenient. For example, for the inner product we have

$$\mathbf{a} \cdot \mathbf{b} = (a^i\, \mathbf{g}_i) \cdot (b_k\, \mathbf{g}^k) = a^i\, b_k\, \mathbf{g}_i \cdot \mathbf{g}^k$$

(2.1.7) $$= a^i\, b_k\, \delta_i^k = a^i\, b_i = a^k\, b_k\,.$$

Here we posed the dummy indices in counterpositions in order to indicate to which basis they are referred. We also have

$$\mathbf{a} \cdot \mathbf{b} = (a_i\, \mathbf{g}^i) \cdot (b^k\, \mathbf{g}_k) = a_i\, b^i\,,$$

but

$$\mathbf{a} \cdot \mathbf{b} = (a_i\, \mathbf{g}^i) \cdot (b_k\, \mathbf{g}^k) = a_i\, b_k\, (\mathbf{g}^i \cdot \mathbf{g}^k)\,.$$

If a basis $\{\mathbf{e}_i\}$ coincides with its dual $\{\mathbf{e}^j\}$

$$\mathbf{e}_i \equiv \mathbf{e}^i \qquad \text{for} \quad i = 1, 2, 3,$$

there is no need to distinguish between upper and lower indices anymore.

Definition. A vector basis $\{\mathbf{e}_i\}$ is called an **orthonormal basis** (**ONB**) if

(2.1.8) $$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik}\,.$$

In each vector space with inner product there are infinitely many vector bases or ONBs. If one uses simultaneously more than one basis, it becomes necessary to indicate to which basis the components are referred, like

$$\mathbf{a} = a_i\, \mathbf{e}_i = \overline{a}_i\, \overline{\mathbf{e}}_i$$

with

$$a_i = \mathbf{a} \cdot \mathbf{e}_i \qquad \text{and} \qquad \overline{a}_i = \mathbf{a} \cdot \overline{\mathbf{e}}_i\,.$$

With respect to an ONB we have simply

(2.1.9) $$\mathbf{e}_k \cdot \mathbf{e}_i = \delta_{ki}$$

(2.1.10) $$\mathbf{a} \cdot \mathbf{b} = a_i\, b_i$$

(2.1.11) $$|\mathbf{a}| = \sqrt{a_i\, a_i} = \sqrt{a_1^2 + a_2^2 + a_3^2}\,.$$

In three dimensions one can further introduce the **vector product** or **cross-product** with respect to a positively-oriented ONB with the aid of the permutation symbol by

(2.1.12) $$\mathbf{a} \times \mathbf{b} = a_i\, \mathbf{e}_i \times b_j\, \mathbf{e}_j = a_i\, b_j\, \varepsilon_{ijk}\, \mathbf{e}_k$$

and the **triple product** between three vectors as

(2.1.13) $$[\mathbf{a}\,, \mathbf{b}\,, \mathbf{c}] = [a_i\, \mathbf{e}_i\,, b_k\, \mathbf{e}_k\,, c_l\, \mathbf{e}_l] = a_i\, b_k\, c_l\, \varepsilon_{ikl}\,.$$

2.1.3 Dyads and Tensors

Definition. A mapping between vectors

$$\mathbf{F} : \mathscr{V} \rightarrow \mathscr{V} \qquad \mid \quad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x})$$

is called **linear** if

(2.1.14) $$\mathbf{F}(\mathbf{x}_1 + \alpha\, \mathbf{x}_2) = \mathbf{F}(\mathbf{x}_1) + \alpha\, \mathbf{F}(\mathbf{x}_2)$$

holds for all vectors $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathscr{V}$ and all scalars $\alpha \in \mathscr{R}$.

Problem 1. Linearity
Check the real function

$$y = f(x) := m\, x + n$$

for linearity. m and n are real constants.

Solution

The above-introduced definition of linearity can be written for a real function f as

(P1.1) $$f(\alpha\, a + b) = \alpha f(a) + f(b) \qquad \forall\, \alpha\,, a\,, b \in \mathscr{R}$$

The condition P1.1 gives

$$\alpha\, n = 0\,.$$

In general, this holds only for $n \equiv 0$. The concept of *linearity* used in algebra is stronger than the one used in real analysis or calculus. In the latter context such functions are sometimes called *affine* or *quasilinear*, whereas the present linearity is called *proportionality* in the context of calculus.

Let $\mathbf{a}$ and $\mathbf{b}$ be arbitrarily chosen, but fixed vectors. With their aid, one can construct a special linear mapping that assigns to each vector $\mathbf{x}$ another vector $\mathbf{y}$ after

$$\mathbf{y} = \mathbf{a}\,(\mathbf{b} \cdot \mathbf{x}) = (\mathbf{x} \cdot \mathbf{b})\,\mathbf{a} = (\mathbf{b} \cdot \mathbf{x})\,\mathbf{a}\,.$$

Thus, the resulting vector $\mathbf{y}$ is always parallel to $\mathbf{a}$.

Definition. The **dyadic product (tensor product)** between two vectors $\mathbf{a}$ and $\mathbf{b}$ or the **simple** or **collinear dyad** $\mathbf{a} \otimes \mathbf{b}$ is the mapping

$$\mathbf{a} \otimes \mathbf{b} : \mathscr{V} \to \mathscr{V} \quad \Big| \quad \mathbf{x} \mapsto \mathbf{a}\,(\mathbf{b} \cdot \mathbf{x})$$

so that

$$\mathbf{a} \otimes \mathbf{b}\,(\mathbf{x}) \; := \; \mathbf{a}\,(\mathbf{b} \cdot \mathbf{x}) \tag{2.1.15}$$

This mapping is linear since

$$\begin{aligned}\mathbf{a} \otimes \mathbf{b}\,(\mathbf{x}_1 + \alpha\,\mathbf{x}_2) &= \mathbf{a}\,[\mathbf{b} \cdot (\mathbf{x}_1 + \alpha\,\mathbf{x}_2)]\\ &= \mathbf{a}\,[\mathbf{b} \cdot \mathbf{x}_1 + \alpha\,\mathbf{b} \cdot \mathbf{x}_2]\\ &= \mathbf{a} \otimes \mathbf{b}\,(\mathbf{x}_1) + \alpha\,[\mathbf{a} \otimes \mathbf{b}\,(\mathbf{x}_2)]\end{aligned}$$

holds for all vectors $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathscr{V}$ and all scalars $\alpha \in \mathscr{R}$.

The dyadic product is in general *not* commutative since $\mathbf{a} \otimes \mathbf{b} \neq \mathbf{b} \otimes \mathbf{a}$ as long as both vectors do not happen to be collinear (parallel).

One can define the *sum of two dyads* as that particular linear mapping that gives for all vectors $\mathbf{x}$

$$\begin{aligned}(\mathbf{a} \otimes \mathbf{b} + \mathbf{c} \otimes \mathbf{d})\,(\mathbf{x}) \; &:= \; \mathbf{a} \otimes \mathbf{b}\,(\mathbf{x}) + \mathbf{c} \otimes \mathbf{d}\,(\mathbf{x})\\ &= \mathbf{a}\,(\mathbf{b} \cdot \mathbf{x}) + \mathbf{c}\,(\mathbf{d} \cdot \mathbf{x})\end{aligned} \tag{2.1.16}$$

and a *multiplication of a dyad* $\mathbf{a} \otimes \mathbf{b}$ *with a scalar* $\alpha \in \mathscr{R}$ as

$$(\alpha\,\mathbf{a} \otimes \mathbf{b})\,(\mathbf{x}) \; := \; \alpha\,[(\mathbf{a} \otimes \mathbf{b})\,(\mathbf{x})] = \alpha\,\mathbf{a}\,(\mathbf{b} \cdot \mathbf{x})\,. \tag{2.1.17}$$

These operations fulfil the axioms of the addition and multiplication with a scalar of vectorspaces. Moreover, the following rules hold for all vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathscr{V}$ and all scalars $\alpha \in \mathscr{R}$

$$(\mathbf{a} + \mathbf{b}) \otimes \mathbf{c} \; = \; \mathbf{a} \otimes \mathbf{c} + \mathbf{b} \otimes \mathbf{c} \tag{2.1.18}$$

$$\mathbf{a} \otimes (\mathbf{b} + \mathbf{c}) \; = \; \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{c} \tag{2.1.19}$$

$$\alpha\,(\mathbf{a} \otimes \mathbf{b}) \; = \; (\alpha\,\mathbf{a}) \otimes \mathbf{b} \; = \; \mathbf{a} \otimes (\alpha\,\mathbf{b}) \tag{2.1.20}$$

Accordingly, we can drop the brackets in the last line. The dyadic product is thus linear in both involved vectors. It follows that

$$(\mathbf{a} + \alpha\,\mathbf{b}) \otimes (\mathbf{c} + \beta\,\mathbf{d}) \; = \; \mathbf{a} \otimes \mathbf{c} + \beta\,\mathbf{a} \otimes \mathbf{d} + \alpha\,\mathbf{b} \otimes \mathbf{c} + \alpha\,\beta\,\mathbf{b} \otimes \mathbf{d}\,. \tag{2.1.21}$$

If we represent $\mathbf{a}$ and $\mathbf{b}$ with respect to a basis $\{\mathbf{g}_i\}$, we obtain

$$\begin{aligned}\mathbf{a} \otimes \mathbf{b} &= (a^i\,\mathbf{g}_i) \otimes (b^k\,\mathbf{g}_k)\\ &= a^i\,b^k\,\mathbf{g}_i \otimes \mathbf{g}_k\end{aligned} \tag{2.1.22}$$

and

$$(\mathbf{a} + \alpha\,\mathbf{b}) \otimes \mathbf{c} \; = \; (a^i + \alpha\,b^i)\,c^k\,\mathbf{g}_i \otimes \mathbf{g}_k \tag{2.1.23}$$

and

(2.1.24) $\mathbf{a} \otimes (\mathbf{b} + \alpha\,\mathbf{c}) \;=\; a^i\,(b^k + \alpha\, c^k)\,\mathbf{g}_i \otimes \mathbf{g}_k\,.$

For an ONB, these expressions do not become shorter. Only if we are evaluating scalar products, is the use of an ONB or the use of dual bases recommended. Because of the linearity we have with respect to an ONB $\{\mathbf{e}_i\}$

$$\begin{aligned}\mathbf{a} \otimes \mathbf{b}\,(\mathbf{x}) \;&=\; [(a_i\,\mathbf{e}_i) \otimes (b_k\,\mathbf{e}_k)]\,(x_l\,\mathbf{e}_l)\\ &=\; x_l\,a_i\,b_k\,(\mathbf{e}_i \otimes \mathbf{e}_k)\,(\mathbf{e}_l)\\ &=\; a_i\,b_k\,x_l\,\mathbf{e}_i\,(\mathbf{e}_k \cdot \mathbf{e}_l) \qquad (2.1.25)\\ &=\; a_i\,b_k\,x_l\,\mathbf{e}_i\,\delta_{kl}\\ &=\; a_i\,b_k\,x_k\,\mathbf{e}_i\,.\end{aligned}$$

Since the simple dyad $\mathbf{a} \otimes \mathbf{b}$ maps all vectors $\mathbf{x}$ in the direction of $\mathbf{a}$, it is a special linear mapping, called a **collinear** dyad.

The sum of two dyads $(\mathbf{a} \otimes \mathbf{b}) + (\mathbf{c} \otimes \mathbf{d})$ maps all vectors $\mathbf{x}$ into a linear combination of $\mathbf{a}$ and $\mathbf{c}$, i.e., into a plane spanned by $\mathbf{a}$ and $\mathbf{c}$. Therefore, one calls such a mapping a **planar** dyad.

Definition. The general linear mapping of a vector into a vector is called a **tensor** or a (**complete**) **dyad**.

If $\mathbf{T}$ is such a tensor, we have

(2.1.26) $\mathbf{y} \;=\; \mathbf{T}(\mathbf{x}) \;=\; \mathbf{T}(x_i\,\mathbf{e}_i) \;=\; x_i\,\mathbf{T}(\mathbf{e}_i)\,.$

A tensor is therefore completely determined if its action on every base vector is given. Since the brackets are not needed, we will no longer use them in what follows and instead use a dot which stands for the scalar product in the definition of the dyad

(2.1.27) $\mathbf{y} \;=\; \mathbf{T} \cdot \mathbf{x}\,.$

Since $\mathbf{T} \cdot \mathbf{e}_i$ is a vector, we can represent it with respect to an ONB $\{\mathbf{e}_k\}$ as

$$\mathbf{T} \cdot \mathbf{e}_i \;=\; T_{ki}\,\mathbf{e}_k$$

with the components

$$T_{ki} \;:=\; \mathbf{e}_k \cdot (\mathbf{T} \cdot \mathbf{e}_i)\,.$$

In this expression the brackets are not needed since there is no danger of confusion. Thus

$$\begin{aligned}\mathbf{T} \cdot \mathbf{x} \;&=\; \mathbf{T} \cdot (x_i\,\mathbf{e}_i) \;=\; x_i\,\mathbf{T} \cdot \mathbf{e}_i \;=\; x_i\,T_{ki}\,\mathbf{e}_k\\ &=\; x_i\,T_{km}\,\mathbf{e}_k \otimes \mathbf{e}_m \cdot \mathbf{e}_i\\ &=\; (T_{km}\,\mathbf{e}_k \otimes \mathbf{e}_m) \cdot \mathbf{x}\,.\end{aligned}$$

By comparison one obtains

Theorem. *Each tensor* $\mathbf{T}$ *can be uniquely represented with respect to an ONB* $\{\mathbf{e}_i\}$ *as*

(2.1.28) $$\mathbf{T} = T_{ki}\,\mathbf{e}_k \otimes \mathbf{e}_i$$

with the nine (base) dyads

$$\begin{matrix} \mathbf{e}_1 \otimes \mathbf{e}_1 & \mathbf{e}_1 \otimes \mathbf{e}_2 & \mathbf{e}_1 \otimes \mathbf{e}_3 \\ \mathbf{e}_2 \otimes \mathbf{e}_1 & \mathbf{e}_2 \otimes \mathbf{e}_2 & \mathbf{e}_2 \otimes \mathbf{e}_3 \\ \mathbf{e}_3 \otimes \mathbf{e}_1 & \mathbf{e}_3 \otimes \mathbf{e}_2 & \mathbf{e}_3 \otimes \mathbf{e}_3 \end{matrix}$$

and nine tensor components

(2.1.29) $$T_{ki} := \mathbf{e}_k \cdot \mathbf{T} \cdot \mathbf{e}_i \qquad \text{for } i, k = 1, 2, 3 .$$

$\{\mathbf{e}_k \otimes \mathbf{e}_i\}$ forms a **tensor basis**. The nine **tensor components** with respect to this basis can be assembled in the **matrix of components**

$$[T_{ij}] := \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} .$$

As a special case, a simple dyad gives

$$\mathbf{a} \otimes \mathbf{b} = a_i\, b_k\, \mathbf{e}_i \otimes \mathbf{e}_k$$

and therefore for the matrix of components

$$\begin{bmatrix} a_1\, b_1 & a_1\, b_2 & a_1\, b_3 \\ a_2\, b_1 & a_2\, b_2 & a_2\, b_3 \\ a_3\, b_1 & a_3\, b_2 & a_3\, b_3 \end{bmatrix} .$$

The dyadic product between two vectors can be performed by a matrix product in the FALK´s scheme

	b_1	b_2	b_3
a_1	$a_1 b_1$	$a_1 b_2$	$a_1 b_3$
a_2	$a_2 b_1$	$a_2 b_2$	$a_2 b_3$
a_3	$a_3 b_1$	$a_3 b_2$	$a_3 b_3$

This is the matrix product between the column vectors of $\mathbf{a}$ and the row vectors of $\mathbf{b}$.

Note that the components of a tensor depend on the choice of the basis, the same as vector components. In fact, if $\{\mathbf{e}_i\}$ and $\{\underline{\mathbf{e}}_i\}$ are two bases, then

$$\mathbf{T} = T_{ik}\,\mathbf{e}_i \otimes \mathbf{e}_k = \underline{T}_{ik}\,\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_k$$

with

$$T_{ik} = \mathbf{e}_i \cdot \mathbf{T} \cdot \mathbf{e}_k$$

$$\underline{T}_{ik} = \underline{\mathbf{e}}_i \cdot \mathbf{T} \cdot \underline{\mathbf{e}}_k ,$$

so that T_{ik} and $\underline{T}_{ik}$ are in general not equal. However, the tensor $\mathbf{T}$ itself is independent of the basis with respect to which it is represented.

If one wants to determine the value $\mathbf{y}$ of a vector $\mathbf{x}$ under the mapping of a tensor, one represents the tensor as before and also $\mathbf{x} = x_n\,\mathbf{e}_n$ and obtains

$$\begin{aligned} \mathbf{y} &= \mathbf{T}\cdot\mathbf{x} \\ = y_k\,\mathbf{e}_k &= (T_{ki}\,\mathbf{e}_k\otimes\mathbf{e}_i)\cdot(x_n\,\mathbf{e}_n) \\ &= T_{ki}\,x_n\,\mathbf{e}_k\,(\mathbf{e}_i\cdot\mathbf{e}_n) \\ &= T_{ki}\,x_n\,\mathbf{e}_k\,\delta_{in} \\ &= T_{ki}\,x_i\,\mathbf{e}_k \end{aligned} \tag{2.1.30}$$

which gives the equation for the components

$$y_k = T_{ki}\,x_i \qquad \text{for } k = 1, 2, 3.$$

In matrix form this is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

Accordingly, one can reduce the tensor operation to matrix operations after choosing a basis. This holds also for the sum of two tensors

$$\mathbf{S} = S_{ki}\,\mathbf{e}_k\otimes\mathbf{e}_i$$

$$\mathbf{T} = T_{ki}\,\mathbf{e}_k\otimes\mathbf{e}_i$$

as

$$\begin{aligned} \mathbf{S}+\mathbf{T} &= S_{ki}\,\mathbf{e}_k\otimes\mathbf{e}_i + T_{ki}\,\mathbf{e}_k\otimes\mathbf{e}_i \\ &= (S_{ki}+T_{ki})\,\mathbf{e}_k\otimes\mathbf{e}_i \end{aligned} \tag{2.1.31}$$

and the multiplication of a tensor $\mathbf{T}$ with a scalar α

$$\alpha\,\mathbf{T} = \alpha\,(T_{ki}\,\mathbf{e}_k\otimes\mathbf{e}_i) = (\alpha\,T_{ki})\,\mathbf{e}_k\otimes\mathbf{e}_i$$

which are defined in analogy to the same products between dyads after (2.1.16) and (2.1.17).

If $\mathbf{S}$ and $\mathbf{T}$ are two tensors and $\mathbf{x}$ a vector, then $\mathbf{S}\cdot\mathbf{x}$ is a vector, upon which we can apply $\mathbf{T}$

$$\mathbf{T}\cdot(\mathbf{S}\cdot\mathbf{x})\,.$$

Since the **composition** of linear mappings is again linear, $\mathbf{T}\cdot\mathbf{S}$ stands for another tensor after

$$\mathbf{T}\cdot(\mathbf{S}\cdot\mathbf{x}) := (\mathbf{T}\cdot\mathbf{S})\cdot\mathbf{x}\,.$$

Its components can be obtained by the following calculation

$$\mathbf{T}\cdot(\mathbf{S}\cdot\mathbf{x}) = (T_{pi}\,\mathbf{e}_p\otimes\mathbf{e}_i)\cdot[(S_{kl}\,\mathbf{e}_k\otimes\mathbf{e}_l)\cdot(x_m\,\mathbf{e}_m)]$$

$$= (T_{pi}\,\mathbf{e}_p \otimes \mathbf{e}_i) \cdot [(S_{kl}\, x_m\, \mathbf{e}_k)\, (\mathbf{e}_l \cdot \mathbf{e}_m)]$$
$$= (T_{pi}\,\mathbf{e}_p \otimes \mathbf{e}_i) \cdot [S_{kl}\, x_m\, \mathbf{e}_k\, \delta_{lm}]$$
$$= (T_{pi}\,\mathbf{e}_p \otimes \mathbf{e}_i) \cdot [S_{km}\, x_m\, \mathbf{e}_k]$$
$$= T_{pi}\, S_{km}\, x_m\, \mathbf{e}_p\, (\mathbf{e}_i \cdot \mathbf{e}_k)$$
$$= T_{pi}\, S_{km}\, x_m\, \mathbf{e}_p\, \delta_{ik}$$
$$= T_{pi}\, S_{im}\, x_m\, \mathbf{e}_p$$
$$= T_{pi}\, S_{iq}\, \delta_{qm}\, x_m\, \mathbf{e}_p$$
$$= (T_{pi}\, S_{iq}\, \mathbf{e}_p \otimes \mathbf{e}_q) \cdot (x_m\, \mathbf{e}_m)\ .$$

Here all brackets are unnecessary and are only used to emphasise the connections. By comparison we obtain the representation of the composed tensor

(2.1.32) $$\mathbf{T} \cdot \mathbf{S} = (T_{pi}\,\mathbf{e}_p \otimes \mathbf{e}_i) \cdot (S_{kl}\,\mathbf{e}_k \otimes \mathbf{e}_l) = T_{pi}\, S_{kl}\, \delta_{ik}\, \mathbf{e}_p \otimes \mathbf{e}_l$$
$$= T_{pi}\, S_{il}\, \mathbf{e}_p \otimes \mathbf{e}_l$$

i.e., one contracts the neighbouring base vectors by a scalar product, and the remaining ones by a dyadic product. Its component matrix is the result of a matrix product between the two component matrices. This operation is called a **simple contraction**.

It is important to note that this product between two tensors is in general *not* commutative, i.e., $\mathbf{T} \cdot \mathbf{S}$ does not equal $\mathbf{S} \cdot \mathbf{T}$. It is associative

(2.1.33) $$(\mathbf{T} \cdot \mathbf{S}) \cdot \mathbf{R} = \mathbf{T} \cdot (\mathbf{S} \cdot \mathbf{R})$$

so that we can drop the brackets. And it is linear in both factors

$$(\mathbf{T} + \alpha\,\mathbf{S}) \cdot \mathbf{R} = \mathbf{T} \cdot \mathbf{R} + \alpha\,\mathbf{S} \cdot \mathbf{R}$$
$$\mathbf{R} \cdot (\mathbf{T} + \alpha\,\mathbf{S}) = \mathbf{R} \cdot \mathbf{T} + \alpha\,\mathbf{R} \cdot \mathbf{S}\ .$$

The particular tensor that maps every vector into itself is the **unit** or **identity tensor**. With respect to any ONB $\{\mathbf{e}_i\}$ this tensor has the unity matrix representing the coefficients after (2.1.1)

(2.1.34) $$\mathbf{I} = \delta_{ik}\, \mathbf{e}_i \otimes \mathbf{e}_k = \mathbf{e}_i \otimes \mathbf{e}_i$$

since we have for every vector $\mathbf{x} = x_m\, \mathbf{e}_m$

$$\mathbf{I} \cdot \mathbf{x} = (\delta_{ik}\, \mathbf{e}_i \otimes \mathbf{e}_k) \cdot (x_m\, \mathbf{e}_m)$$
$$= \delta_{ik}\, x_m\, \mathbf{e}_i\, (\mathbf{e}_k \cdot \mathbf{e}_m)$$
$$= x_m\, \mathbf{e}_k\, \delta_{km} = x_m\, \mathbf{e}_m = \mathbf{x}\ .$$

If $\mathbf{T} = T_{ki}\, \mathbf{e}_k \otimes \mathbf{e}_i$ is an arbitrary tensor and $\mathbf{I}$ the identity tensor, then we have

$$\mathbf{T} \cdot \mathbf{I} = (T_{ki}\, \mathbf{e}_k \otimes \mathbf{e}_i) \cdot (\delta_{lm}\, \mathbf{e}_l \otimes \mathbf{e}_m)$$
$$= (T_{ki}\, \mathbf{e}_k \otimes \mathbf{e}_i) \cdot (\mathbf{e}_l \otimes \mathbf{e}_l)$$
$$= T_{ki}\, \mathbf{e}_k \otimes \mathbf{e}_l\, (\mathbf{e}_i \cdot \mathbf{e}_l)$$

$$= T_{ki}\, \mathbf{e}_k \otimes \mathbf{e}_l\, \delta_{il}$$

$$= T_{kl}\, \mathbf{e}_k \otimes \mathbf{e}_l$$

$$= \mathbf{T}$$

the same as the other way round

(2.1.35) $$\mathbf{I} \cdot \mathbf{T} = \mathbf{T}$$

for all tensors $\mathbf{T}$.

Scalar multiples $\alpha\, \mathbf{I}$ of $\mathbf{I}$ are called **spherical tensors** (or isotropic tensors). They multiply each vector

(2.1.36) $$\alpha\, \mathbf{I}\, \mathbf{v} = \alpha\, \mathbf{v}$$

keeping its direction constant. In particular, $\mathbf{0} := 0\, \mathbf{I}$ is the **zero tensor**, which maps every vector into the zero vector. Its coefficients are all zero with respect to whatever basis. For all tensors $\mathbf{T}$ we obtain

(2.1.37) $$\mathbf{0} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{0} = \mathbf{0}\,.$$

2.1.4 The Inverse of a Tensor

One may ask the question of whether a tensor possesses an inversion, i.e., if there exists for a tensor $\mathbf{T}$ an **inverse tensor** $\mathbf{T}^{-1}$ such that

$$\mathbf{T}^{-1} \cdot (\mathbf{T} \cdot \mathbf{x}) = \mathbf{x}$$

holds for all vectors $\mathbf{x}$. If such a mapping exists, then it must also be linear (and therefore also a tensor, notated as $\mathbf{T}^{-1}$). This is equivalent to

(2.1.38) $$\mathbf{T}^{-1} \cdot \mathbf{T} = \mathbf{I}\,.$$

In components with respect to an ONB this gives

$$(T^{-1}{}_{kl}\, \mathbf{e}_k \otimes \mathbf{e}_l) \cdot (T_{mn}\, \mathbf{e}_m \otimes \mathbf{e}_n) = \delta_{kn}\, \mathbf{e}_k \otimes \mathbf{e}_n$$

$$= T^{-1}{}_{kl}\, \mathbf{e}_l \cdot \mathbf{e}_m\, T_{mn}\, \mathbf{e}_k \otimes \mathbf{e}_n$$

$$= T^{-1}{}_{km}\, T_{mn}\, \mathbf{e}_k \otimes \mathbf{e}_n$$

(2.1.39) $\Rightarrow$ $$T^{-1}{}_{km}\, T_{mn} = \delta_{kn}\,.$$

Accordingly, the matrix of components $[T^{-1}{}_{ik}]$ of the inverse tensor $\mathbf{T}^{-1}$ is the inverse matrix of $[T_{mn}]$ in the sense of matrix algebra if for both tensors the same ONB is used.

We know from matrices that only the *non-singular* matrices are invertible. These are characterized by the property that their determinant is non-zero

$$det[T_{in}] \neq 0\,.$$

A tensor is in fact **invertible** if and only if the determinant of its matrix of components is non-zero with respect to an arbitrary basis (and hence for all bases).

Otherwise it is called **singular**. We will later define the determinant of a tensor, which will lead to an invertibility rule which is independent of the used basis.

A collinear or complanar dyad is always singular.

A spherical tensor $\alpha\,\mathbf{I}$ is invertible if and only if $\alpha \neq 0$. Its inverse is then $\alpha^{-1}\,\mathbf{I}$.

If $\mathbf{S}$ and $\mathbf{T}$ are both invertible tensors, then the composition $\mathbf{S} \cdot \mathbf{T}$ is also invertible, and vice versa, and we obtain

(2.1.40) $$(\mathbf{S} \cdot \mathbf{T})^{-1} = \mathbf{T}^{-1} \cdot \mathbf{S}^{-1}$$

since

$$(\mathbf{S} \cdot \mathbf{T})^{-1} \cdot (\mathbf{S} \cdot \mathbf{T}) = \mathbf{T}^{-1} \cdot \mathbf{S}^{-1} \cdot \mathbf{S} \cdot \mathbf{T} = \mathbf{T}^{-1} \cdot \mathbf{T} = \mathbf{I} .$$

2.1.5 The Transpose of a Tensor

For every tensor $\mathbf{T}$ the **transposed tensor** $\mathbf{T}^T$ is defined through the **bilinear form**

(2.1.41) $$\mathbf{a} \cdot (\mathbf{T}^T \cdot \mathbf{b}) = \mathbf{b} \cdot (\mathbf{T} \cdot \mathbf{a})$$

for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. It is sufficient to postulate this for two arbitrary base vectors

$$\mathbf{e}_k \cdot (\mathbf{T} \cdot \mathbf{e}_i) = \mathbf{e}_i \cdot (\mathbf{T}^T \cdot \mathbf{e}_k)$$

or

(2.1.42) $$T_{ki} = (T^T)_{ik} \qquad i, k = 1, 2, 3$$

which means that with respect to an ONB the matrix of the components of the transposed tensor equals the transposed matrix of the original tensor

$$\mathbf{T} = T_{ik}\,\mathbf{e}_i \otimes \mathbf{e}_k \quad \Leftrightarrow \quad \mathbf{T}^T = T_{ik}\,\mathbf{e}_k \otimes \mathbf{e}_i$$

$$= T_{ki}\,\mathbf{e}_k \otimes \mathbf{e}_i \qquad\qquad = T_{ki}\,\mathbf{e}_i \otimes \mathbf{e}_k .$$

The following rules hold for all tensors $\mathbf{T}$ and $\mathbf{S}$, all vectors $\mathbf{a}$ and $\mathbf{b}$, and all scalars α:

(2.1.43)	$(\mathbf{a} \otimes \mathbf{b})^T = (\mathbf{b} \otimes \mathbf{a})$
(2.1.44)	$(\mathbf{T} \cdot \mathbf{S})^T = \mathbf{S}^T \cdot \mathbf{T}^T$
(2.1.45)	$(\mathbf{T} + \mathbf{S})^T = \mathbf{T}^T + \mathbf{S}^T$
(2.1.46)	$(\alpha\,\mathbf{T})^T = \alpha\,(\mathbf{T}^T)$
(2.1.47)	$(\mathbf{T}^T)^T = \mathbf{T}$
(2.1.48)	$(\mathbf{T}^T)^{-1} = (\mathbf{T}^{-1})^T =: \mathbf{T}^{-T}$ for invertible tensors $\mathbf{T}$
	$\mathbf{I}^T = \mathbf{I}$
	$\mathbf{0}^T = \mathbf{0}$

One can also define a **left-product** between a vector and tensor

$$\mathbf{v} \cdot \mathbf{T} \ := \ \mathbf{T}^T \cdot \mathbf{v}$$

(2.1.49)
$$= (T_{km}\, \mathbf{e}_m \otimes \mathbf{e}_k) \cdot (v_i\, \mathbf{e}_i) \ = \ T_{km}\, v_i\, \mathbf{e}_m\, (\mathbf{e}_k \cdot \mathbf{e}_i) \ = \ T_{km}\, v_i\, \mathbf{e}_m\, \delta_{ki}$$
$$= v_i\, T_{im}\, \mathbf{e}_m \ = \ (v_i\, \mathbf{e}_i) \cdot (T_{mk}\, \mathbf{e}_m \otimes \mathbf{e}_k) \ = \ v_i\, T_{mk}\, (\mathbf{e}_i \cdot \mathbf{e}_m)\, \mathbf{e}_k$$
$$= v_i\, T_{mk}\, \delta_{im}\, \mathbf{e}_k \ = \ v_i\, T_{ik}\, \mathbf{e}_k \,.$$

For all vectors $\mathbf{v}$ and $\mathbf{w}$ we have then

(2.1.50) $$\mathbf{v} \cdot (\mathbf{T} \cdot \mathbf{w}) \ = \ (\mathbf{v} \cdot \mathbf{T}) \cdot \mathbf{w}$$

so that the brackets are not needed. Since we can express the left-product through the usual product from the right, it does not show any new properties.

If a tensor equals its transposed tensor we call it **symmetric**. As an example, every spherical tensor is symmetric. The definition of the symmetry coincides with the symmetry of the matrix of components with respect to an ONB.

If a tensor equals its negative transposed tensor

(2.1.51) $$\mathbf{T} \ = \ -\,\mathbf{T}^T$$

it is called **anti(sym)metric** or **skew**. Consequently, its components with respect to an ONB obey

(2.1.52) $$T_{ik} \ = \ -\,T_{ki}$$

and in particular (no summation)

$$T_{ii} \ = \ -\,T_{ii} \ = \ 0 \,.$$

While a symmetric tensor possesses six independent components

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}$$

a skew tensor has only three

$$\begin{bmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & 0 \end{bmatrix} .$$

So a skew tensor has the same DOFs as the underlying vector space. This gives rise to the supposition that its effect on some arbitrary vector $\mathbf{x}$ can also be obtained by an appropriate operation of some vector with $\mathbf{x}$. And in fact we find for every skew tensor $\mathbf{T}$ a unique **axial vector** $\mathbf{t}^A$ such that for all vectors $\mathbf{x}$

(2.1.53) $$\mathbf{T} \cdot \mathbf{x} \ = \ \mathbf{t}^A \times \mathbf{x}$$

holds. For determining $\mathbf{t}^A$ we choose an ONB $\{\mathbf{e}_i\}$ and obtain

$$(T_{ik}\, \mathbf{e}_i \otimes \mathbf{e}_k) \cdot (x_l\, \mathbf{e}_l) \ = \ t^A{}_m\, \mathbf{e}_m \times x_l\, \mathbf{e}_l$$

$$= T_{il}\, x_l\, \mathbf{e}_i \qquad\qquad = t^A{}_m\, x_l\, \varepsilon_{mli}\, \mathbf{e}_i$$

so that

(2.1.54) $$T_{il} = t^A{}_m\, \varepsilon_{mli}$$

or

$$T_{12} = t^A{}_1\, \varepsilon_{121} + t^A{}_2\, \varepsilon_{221} + t^A{}_3\, \varepsilon_{321} = -\, t^A{}_3$$

$$T_{23} = t^A{}_1\, \varepsilon_{132} + t^A{}_2\, \varepsilon_{232} + t^A{}_3\, \varepsilon_{332} = -\, t^A{}_1$$

$$T_{31} = t^A{}_1\, \varepsilon_{113} + t^A{}_2\, \varepsilon_{213} + t^A{}_3\, \varepsilon_{313} = -\, t^A{}_2\,.$$

Therefore

$$\mathbf{t}^A = -\, T_{23}\, \mathbf{e}_1 - T_{31}\, \mathbf{e}_2 - T_{12}\, \mathbf{e}_3$$

$$= +\, T_{32}\, \mathbf{e}_1 + T_{13}\, \mathbf{e}_2 + T_{21}\, \mathbf{e}_3$$

and

(2.1.55) $$t^A{}_m = \tfrac{1}{2}\, T_{il}\, \varepsilon_{mli}\,.$$

We obtain the following rules which can be easily proven.

- Any linear combination of (anti)symmetric tensors is again (anti)symmetric.

The composition of (anti)symmetric tensors, however, may lose this property.

- The inverse of symmetric invertible tensors is also symmetric.
- Skew tensors are singular.

One can uniquely decompose any tensor $\mathbf{T}$ into its symmetric part

(2.1.56) $$sym(\mathbf{T}) := \tfrac{1}{2}\, (\mathbf{T} + \mathbf{T}^T)$$

and its skew part

(2.1.57) $$skw\,(\mathbf{T}) := \tfrac{1}{2}\, (\mathbf{T} - \mathbf{T}^T)$$

so that

$$\mathbf{T} = sym(\mathbf{T}) + skw\,(\mathbf{T})\,.$$

In particular, we obtain for collinear dyads the symmetric part

$$sym(\mathbf{a} \otimes \mathbf{b}) = \tfrac{1}{2}\, (\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})$$

and the skew part

$$skw(\mathbf{a} \otimes \mathbf{b}) = \tfrac{1}{2}\, (\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a})$$

with the corresponding axial vector

$$\mathbf{t}^A = \tfrac{1}{2}\, \mathbf{b} \times \mathbf{a}\,.$$

2.1.6 Square Forms and Tensor Surfaces

If $\mathbf{x}$ is a vector and $\mathbf{T}$ a tensor, then $\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x}$ is a scalar. Such an expression is called a **square form**. Obviously, in such a form only the symmetric part of $\mathbf{T}$ is relevant.

A tensor $\mathbf{T}$ is called

- **positive definite** if $\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} > 0$
- **positive semidefinite** if $\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} \geq 0$
- **negative definite** if $\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} < 0$
- **negative semidefinite** if $\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} \leq 0$

holds for all vectors $\mathbf{x} \neq \mathbf{o}$, and **indefinite** otherwise.

If some tensor $\mathbf{T}$ is positive (semi)definite, then $-\mathbf{T}$ is negative (semi)definite. An example for a positive definite tensor is the identity tensor or a spherical tensor $\alpha\, \mathbf{I}$ with some positive scalar α.

A geometrical characterization of tensors can be obtained by their **tensor surfaces** which are defined in the following way: We consider all vectors $\mathbf{x}$ that fulfil the quadratic equation

$$\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} = 1 .$$

If we interpret the solution vectors as position vectors, then they describe a two-dimensional subset of the three-dimensional space, which is a surface. For a spherical tensor $\mathbf{T}$, this surface forms a sphere. If $\mathbf{T}$ is positive definite, then this surface is an ellipsoid. Other tensor surfaces will be considered after the treatment of eigenvalue problems.

2.1.7 Cross-Product between Vectors and Tensors

For some applications one needs the cross-product between a vector (from the left) and a tensor (from the right)

$$\mathbf{v} \times \mathbf{T}$$

which is defined by its action on an arbitrary vector $\mathbf{w}$ as

(2.1.58) $$(\mathbf{v} \times \mathbf{T}) \cdot \mathbf{w} := \mathbf{v} \times (\mathbf{T} \cdot \mathbf{w})$$

so that the brackets are not needed. With respect to an ONB this gives

$$\begin{aligned} \mathbf{v} \times \mathbf{T} \cdot \mathbf{w} &= v_i\, \mathbf{e}_i \times (T_{kl}\, \mathbf{e}_k \otimes \mathbf{e}_l) \cdot (w_m\, \mathbf{e}_m) \\ &= v_i\, \mathbf{e}_i \times T_{km}\, w_m\, \mathbf{e}_k \end{aligned}$$

$$= v_i\, T_{km}\, w_m\, \mathbf{e}_i \times \mathbf{e}_k$$

$$= v_i\, T_{km}\, w_m\, \varepsilon_{ikp}\, \mathbf{e}_p$$

$$= v_i\, T_{kl}\, (\mathbf{e}_i \times \mathbf{e}_k) \otimes \mathbf{e}_l \cdot (w_m\, \mathbf{e}_m)\,.$$

Accordingly

$$\begin{aligned} \mathbf{v} \times \mathbf{T} &= (v_i\, \mathbf{e}_i) \times (T_{kl}\, \mathbf{e}_k \otimes \mathbf{e}_l) \\ &= (v_i\, \mathbf{e}_i \times T_{kl}\, \mathbf{e}_k) \otimes \mathbf{e}_l \\ &= v_i\, T_{kl}\, (\mathbf{e}_i \times \mathbf{e}_k) \otimes \mathbf{e}_l \end{aligned}$$

or for linear dyads

$$\mathbf{v} \times (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{v} \times \mathbf{a}) \otimes \mathbf{b}\,.$$

The resulting tensor is linear in the three involved vectors. The brackets are not needed.

Its transpose is

$$\begin{aligned} (\mathbf{v} \times \mathbf{a} \otimes \mathbf{b})^T &= \mathbf{b} \otimes (\mathbf{v} \times \mathbf{a}) \\ &= -\,\mathbf{b} \otimes (\mathbf{a} \times \mathbf{v}) \\ &=: -\,(\mathbf{b} \otimes \mathbf{a}) \times \mathbf{v} \\ &= -\,(\mathbf{a} \otimes \mathbf{b})^T \times \mathbf{v} \end{aligned}$$

or, in general, for all tensors $\mathbf{T}$

(2.1.59) $$\mathbf{v} \times \mathbf{T} := -\,(\mathbf{T}^T \times \mathbf{v})^T.$$

In this way we have introduced the cross-product between a dyad (from the left) and a vector (from the right) as

$$\begin{aligned} (\mathbf{a} \otimes \mathbf{b}) \times \mathbf{v} &:= \mathbf{a} \otimes (\mathbf{b} \times \mathbf{v}) = -\,\mathbf{a} \otimes (\mathbf{v} \times \mathbf{b}) \\ &= -\,(\mathbf{v} \times \mathbf{b} \otimes \mathbf{a})^T = -\,[\mathbf{v} \times (\mathbf{a} \otimes \mathbf{b})^T]^T \end{aligned}$$

and, more generally, between a tensor (from the left) and a vector (from the right)

(2.1.60) $$\mathbf{T} \times \mathbf{v} := -\,(\mathbf{v} \times \mathbf{T}^T)^T$$

which is again linear in all factors. For the components with respect to an ONB we obtain

$$\begin{aligned} \mathbf{T} \times \mathbf{v} &= (T_{ik}\, \mathbf{e}_i \otimes \mathbf{e}_k) \times (v_m\, \mathbf{e}_m) \\ &= T_{ik}\, v_m\, \mathbf{e}_i \otimes (\mathbf{e}_k \times \mathbf{e}_m) \\ &= T_{ik}\, v_m\, \varepsilon_{kmp}\, \mathbf{e}_i \otimes \mathbf{e}_p \end{aligned}$$

where all brackets are again unnecessary.

The following rules hold for all scalars α, vectors $\mathbf{v}$, $\mathbf{w}$ and tensors $\mathbf{S}$, $\mathbf{T}$.

(2.1.61)	$(\mathbf{v} \times \mathbf{T}) \cdot \mathbf{w} = \mathbf{v} \times (\mathbf{T} \cdot \mathbf{w}) =: \mathbf{v} \times \mathbf{T} \cdot \mathbf{w}$
(2.1.62)	$\mathbf{v} \cdot (\mathbf{T} \times \mathbf{w}) = (\mathbf{v} \cdot \mathbf{T}) \times \mathbf{w} =: \mathbf{v} \cdot \mathbf{T} \times \mathbf{w}$
(2.1.63)	$(\mathbf{T} + \mathbf{S}) \times \mathbf{v} = \mathbf{T} \times \mathbf{v} + \mathbf{S} \times \mathbf{v}$
(2.1.64)	$\mathbf{v} \times (\mathbf{T} + \mathbf{S}) = \mathbf{v} \times \mathbf{T} + \mathbf{v} \times \mathbf{S}$
(2.1.65)	$\alpha(\mathbf{T} \times \mathbf{v}) = (\alpha \mathbf{T}) \times \mathbf{v} = \mathbf{T} \times (\alpha \mathbf{v}) =: \alpha \mathbf{T} \times \mathbf{v}$
(2.1.66)	$\alpha(\mathbf{v} \times \mathbf{T}) = (\alpha \mathbf{v}) \times \mathbf{T} = \mathbf{v} \times (\alpha \mathbf{T}) =: \alpha \mathbf{v} \times \mathbf{T}$
(2.1.67)	$\mathbf{T} \times (\mathbf{v} + \mathbf{w}) = \mathbf{T} \times \mathbf{v} + \mathbf{T} \times \mathbf{w}$
(2.1.68)	$(\mathbf{v} + \mathbf{w}) \times \mathbf{T} = \mathbf{v} \times \mathbf{T} + \mathbf{w} \times \mathbf{T}$

As a consequence of the rules for the triple product, we have additionally

(2.1.69) $$\mathbf{T} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{T} \times \mathbf{a}) \cdot \mathbf{b}$$

or for the product from the left

(2.1.70) $$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{T} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{T}) .$$

Here again all brackets are unnecessary since the operations only make sense in the given order.

If we choose in particular for $\mathbf{T}$ the identity $\mathbf{I}$, then we obtain for arbitrary vectors $\mathbf{w}$

$$(\mathbf{v} \times \mathbf{I}) \cdot \mathbf{w} = \mathbf{v} \times (\mathbf{I} \cdot \mathbf{w}) = \mathbf{v} \times \mathbf{w}$$
$$= \mathbf{I} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{I} \cdot \mathbf{v}) \times \mathbf{w} = (\mathbf{I} \times \mathbf{v}) \cdot \mathbf{w}$$

and therefore

(2.1.71) $$\mathbf{v} \times \mathbf{I} = \mathbf{I} \times \mathbf{v} .$$

On the other hand, we obtain with (2.1.60)

(2.1.72) $$\mathbf{I} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{I})^T = -(\mathbf{I} \times \mathbf{v})^T$$

so that $\mathbf{I} \times \mathbf{v} = \mathbf{v} \times \mathbf{I}$ must be antisymmetric. By comparison we conclude that $\mathbf{v}$ is the axial vector of the skew tensor $\mathbf{v} \times \mathbf{I}$.

We obtain with respect to an ONB

$$\mathbf{I} \times \mathbf{v} = \mathbf{e}_i \otimes \mathbf{e}_i \times v_j\, \mathbf{e}_j$$
$$= v_j\, \varepsilon_{ijk}\, \mathbf{e}_i \otimes \mathbf{e}_k$$

(2.1.73) $$= \mathbf{v} \times \mathbf{I} = v_j\, \mathbf{e}_j \times \mathbf{e}_i \otimes \mathbf{e}_i$$
$$= v_j\, \varepsilon_{jik}\, \mathbf{e}_k \otimes \mathbf{e}_i .$$

If this tensor is applied to a vector $\mathbf{w}$ we obtain

$$\mathbf{v} \times \mathbf{I} \cdot \mathbf{w} = v_j\, \varepsilon_{jik}\, \mathbf{e}_k (\mathbf{e}_i \cdot w_m\, \mathbf{e}_m)$$
$$= v_j\, w_i\, \varepsilon_{jik}\, \mathbf{e}_k .$$

2.1.8 Orthogonal Tensors

are those tensors which are compatible with the inner product such that

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{Q} \cdot \mathbf{a}) \cdot (\mathbf{Q} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \mathbf{b}$$

holds for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. This leads to

$$\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{I} \qquad \Leftrightarrow \qquad Q_{mi}\, Q_{mj} = \delta_{ij}$$

with respect to some ONB, or

(2.1.74) $$\mathbf{Q}^{-1} = \mathbf{Q}^T.$$

So for orthogonal tensors, the inverse equals the transpose. All orthogonal tensors are therefore invertible. We also have

$$\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{I} \qquad \Leftrightarrow \qquad Q_{im}\, Q_{jm} = \delta_{ij}$$

with respect to some ONB.

Accordingly, the transpose/ inverse of some orthogonal tensors is again orthogonal. If we represent an orthogonal tensor with respect to some ONB by its matrix of components, then this is an orthogonal matrix

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}.$$

Such orthogonal matrices have the property that both the row vectors and the column vectors are normalized and mutually orthogonal.

Orthogonal tensors describe rotations and reflections of vectors. If $\{\mathbf{e}_j\}$ is an ONB, then $\{\mathbf{Q} \cdot \mathbf{e}_i\}$ is also an ONB for every orthogonal tensor $\mathbf{Q}$. Occasionally $\mathbf{Q}$ may change the orientation of the basis.

Examples for orthogonal tensors are

- $\pm\, \mathbf{e}_1 \otimes \mathbf{e}_1 \pm \mathbf{e}_2 \otimes \mathbf{e}_2 \pm \mathbf{e}_3 \otimes \mathbf{e}_3$
- $\pm\, \mathbf{e}_1 \otimes \mathbf{e}_2 \pm \mathbf{e}_2 \otimes \mathbf{e}_3 \pm \mathbf{e}_3 \otimes \mathbf{e}_1$
- $\pm\, \mathbf{e}_1 \otimes \mathbf{e}_1 \pm \mathbf{e}_2 \otimes \mathbf{e}_3 \pm \mathbf{e}_3 \otimes \mathbf{e}_2$

for any ONB $\{\mathbf{e}_j\}$.

If $\mathbf{v}$ is a vector and $\mathbf{Q}$ an orthogonal tensor, then $\mathbf{Q} \cdot \mathbf{v}$ is the rotated and occasionally reflected vector. With respect to some ONB $\{\mathbf{e}_j\}$ we obtain

$$\mathbf{Q} \cdot \mathbf{v} = \mathbf{Q} \cdot v^i\, \mathbf{e}_i = v^i\, \mathbf{Q} \cdot \mathbf{e}_i$$

so that the mapped vector has the same components with respect to the ONB $\{\mathbf{Q} \cdot \mathbf{e}_i\}$ as the original one with respect to $\{\mathbf{e}_j\}$.

The same can be done with a tensor $\mathbf{T}$. If we rotate the tensorbasis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ into $\{(\mathbf{Q} \cdot \mathbf{e}_i) \otimes (\mathbf{Q} \cdot \mathbf{e}_j)\} = \{(\mathbf{Q} \cdot \mathbf{e}_i) \otimes (\mathbf{e}_j \cdot \mathbf{Q}^T)\}$, then we obtain the rotated tensor

(2.1.75) $$\mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}^T = \mathbf{Q} \cdot T_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{Q}^T = T_{ij}\, (\mathbf{Q} \cdot \mathbf{e}_i) \otimes (\mathbf{Q} \cdot \mathbf{e}_j)$$

again with the same components with respect to the ONB $\{(\mathbf{Q} \cdot \mathbf{e}_i) \otimes (\mathbf{Q} \cdot \mathbf{e}_j)\}$ as the original tensor with respect to an ONB $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$.

For representing an orthogonal tensor $\mathbf{Q}$ we choose a particular ONB, the $\mathbf{e}_1$ –direction of which coincides with the rotational axis of $\mathbf{Q}$, so that the matrix of components of $\mathbf{Q}$ is

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & +\sin\varphi & \cos\varphi \end{bmatrix}$$

with some angle φ. For $+1$ the tensor describes a pure rotation. In this case the tensor is called a **versor**[25] or **proper-orthogonal**, while for -1 an additional reflection at the $\mathbf{e}_2$–$\mathbf{e}_3$–plane takes place. One sees easily that also in this case the rows and columns of the matrix are mutually orthogonal.

For the rotation we obtain the following representation

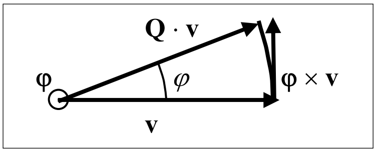
$$\mathbf{Q} = \mathbf{e}_1 \otimes \mathbf{e}_1 + (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \cos\varphi + (\mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3) \sin\varphi$$

(2.1.76) $$= \cos\varphi\, \mathbf{I} + (1 - \cos\varphi)\, \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{I} \times \mathbf{e}_1 \sin\varphi\,.$$

For *small rotations* one linearises this expression in φ

$$\sin\varphi \approx \varphi$$
$$\cos\varphi \approx 1$$

and obtains the more simple representation



(2.1.77) $$\mathbf{Q} \approx \mathbf{I} + \mathbf{I} \times \boldsymbol{\varphi} = \mathbf{I} + \boldsymbol{\varphi} \times \mathbf{I}$$

with the skew tensor $\mathbf{I} \times \boldsymbol{\varphi}$, the axial vector of which is $\boldsymbol{\varphi} = \varphi\, \mathbf{e}_1$. Then

$$\mathbf{Q} \cdot \mathbf{v} \approx \mathbf{v} + \boldsymbol{\varphi} \times \mathbf{v}\,.$$

Note that this linearised tensor is not orthogonal anymore.

[25] from lat. *vertere* = to turn

2.1.9 Transformations under Change of Basis

We introduced vectors and tensors without a basis. All operations and properties of tensors can also be written without referring to a basis (**direct** or **symbolic notation**).

On the other hand, we could see that after choosing a basis, all operations and properties of tensors could be related to analogous ones on the matrices of the components. This representation of tensor operations is general because this is always possible, but also special because one could have chosen any other basis as well.

Therefore the question arises of how the components of an arbitrary vector $\mathbf{v}$ or tensor $\mathbf{T}$ transform under changes of the basis. We will exclusively consider ONB, as we also did before. So letting $\{\mathbf{e}_i\}$ and $\{\underline{\mathbf{e}}_i\}$ be such ONBs, we obtain the representations

$$\mathbf{v} = v_i \, \mathbf{e}_i = \underline{v}_i \, \underline{\mathbf{e}}_i$$

$$\mathbf{T} = T_{ik} \, \mathbf{e}_i \otimes \mathbf{e}_k = \underline{T}_{ik} \, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_k \, .$$

It is always possible to transform one ONB into another ONB by rotations and occasionally reflections. This can be described by an orthogonal tensor $\mathbf{Q}$. If we take

$$\mathbf{Q} = \underline{\mathbf{e}}_i \otimes \mathbf{e}_i$$

then this tensor maps

(2.1.78) $$\mathbf{Q} \cdot \mathbf{e}_k = \underline{\mathbf{e}}_k$$

and vice versa

$$\mathbf{Q}^T \cdot \underline{\mathbf{e}}_k = \mathbf{e}_k \, .$$

This representation of $\mathbf{Q}$ is, however, trivial and not helpful for our purpose to derive the transformations of the components. For this purpose, we choose another representation

$$\mathbf{Q} = Q_{rs} \, \mathbf{e}_r \otimes \mathbf{e}_s$$

with

$$Q_{rs} = \mathbf{e}_r \cdot \mathbf{Q} \cdot \mathbf{e}_s = \mathbf{e}_r \cdot (\underline{\mathbf{e}}_i \otimes \mathbf{e}_i) \cdot \mathbf{e}_s$$

$$= (\mathbf{e}_r \cdot \underline{\mathbf{e}}_i)(\mathbf{e}_i \cdot \mathbf{e}_s) = \mathbf{e}_r \cdot \underline{\mathbf{e}}_s = cos \angle(\mathbf{e}_r \, , \underline{\mathbf{e}}_s) \, .$$

The component Q_{rs} is the directional cosine, i.e., the cosine of the angle between the base vectors $\mathbf{e}_r$ and $\underline{\mathbf{e}}_s$. With this representation of $\mathbf{Q}$ we obtain

$$\mathbf{v} = v_r \, \mathbf{e}_r = \underline{v}_i \, \underline{\mathbf{e}}_i = \underline{v}_i \, \mathbf{Q} \cdot \mathbf{e}_i$$

$$= \underline{v}_i \, (Q_{rs} \, \mathbf{e}_r \otimes \mathbf{e}_s) \cdot \mathbf{e}_i = \underline{v}_i \, Q_{rs} \, \mathbf{e}_r \, \delta_{si}$$

$$= \underline{v}_i \, Q_{ri} \, \mathbf{e}_r$$

and by comparing the components

(2.1.79) $$v_r = Q_{ri}\, \underline{v}_i \,.$$

In the reverse direction one would obtain

$$\underline{v}_i = v_r\, Q_{ri} \,.$$

Analogously one obtains for the components of a tensor

$$\begin{aligned} \mathbf{T} &= T_{ik}\, \mathbf{e}_i \otimes \mathbf{e}_k = \underline{T}_{lm}\, \underline{\mathbf{e}}_l \otimes \underline{\mathbf{e}}_m \\ &= \underline{T}_{lm}\, (\mathbf{Q} \cdot \mathbf{e}_l) \otimes (\mathbf{Q} \cdot \mathbf{e}_m) \\ &= Q_{il}\, Q_{km}\, \underline{T}_{lm}\, \mathbf{e}_i \otimes \mathbf{e}_k \end{aligned}$$

the tranformations

(2.1.80) $$T_{ik} = Q_{il}\, \underline{T}_{lm}\, Q_{km} \qquad \text{and} \qquad \underline{T}_{lm} = Q_{il}\, T_{ik}\, Q_{km}$$

or, if written in matrix form,

$$[T_{ik}] = [Q_{il}]\, [\underline{T}_{lm}]\, [Q_{km}]^T \qquad \text{and} \qquad [\underline{T}_{lm}] = [Q_{il}]^T\, [T_{ik}]\, [Q_{km}] \,.$$

2.1.10 Eigenvalues and Eigenvectors

In mechanics we are often confronted with the following problem: Find for a given tensor $\mathbf{T}$ vectors that $\mathbf{T}$ maps into their own direction, i.e.,

(2.1.81) $$\mathbf{T} \cdot \mathbf{a} = \lambda\, \mathbf{a}$$

for some real λ. Such a vector is called the **eigenvector** of $\mathbf{T}$ and λ the corresponding **eigenvalue** of $\mathbf{T}$.

First of all one states that the eigenvalue equation is trivially fulfilled for the zero vector for arbitrary λ. We will therefore only look for non-zero eigenvectors $\mathbf{a} \neq \mathbf{o}$.

If we take the α–multiple of some eigenvectors $\mathbf{a}$, then

$$\mathbf{T} \cdot (\alpha\, \mathbf{a}) = \alpha\, (\mathbf{T} \cdot \mathbf{a}) = \alpha\, \lambda\, \mathbf{a} = \lambda\, (\alpha\, \mathbf{a}) \,.$$

So every scalar multiple of some eigenvector is also an eigenvector with the same corresponding eigenvalue. This gives rise to a normalisation

(2.1.82) $$|\,\mathbf{a}\,| = 1 = \mathbf{a} \cdot \mathbf{a} = a_i\, a_i$$

where the sense of the direction of $\mathbf{a}$ still remains arbitrary. It would be more reasonable to talk about eigen*directions* instead of eigen*vectors*.

We can reformulate the above eigenvalue equation as

(2.1.83) $$(\mathbf{T} - \lambda\, \mathbf{I}) \cdot \mathbf{a} = \mathbf{o}$$

or in component form with respect to some ONB $\{\mathbf{e}_i\}$ as

(2.1.84) $(T_{ik} - \lambda\, \delta_{ik})\, a_k = 0$ for $i = 1, 2, 3.$

This gives the equations

$$(T_{11} - \lambda)\, a_1 + T_{12}\, a_2 + T_{13}\, a_3 = 0$$
$$T_{21}\, a_1 + (T_{22} - \lambda)\, a_2 + T_{23}\, a_3 = 0$$
$$T_{31}\, a_1 + T_{32}\, a_2 + (T_{33} - \lambda)\, a_3 = 0\,.$$

These are the well-known eigenvalue equations from matrix algebra. It is a system of three linear and homogeneous equations for the components a_i of the eigenvectors **a**, which allows for a non-zero solution if and only if the determinant of the matrix of coefficients is zero

(2.1.85) $$det\,[T_{ik} - \lambda\, \delta_{ik}] = det \begin{bmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{bmatrix} = 0\,.$$

This leads to a cubic polynomial in λ

$$(T_{11} - \lambda)\,[(T_{22} - \lambda)(T_{33} - \lambda) - T_{32}\, T_{23}]$$
$$- T_{12}\,[T_{21}(T_{33} - \lambda) - T_{31}\, T_{23}]$$
$$+ T_{13}\,[T_{21}\, T_{32} - T_{31}(T_{22} - \lambda)] = 0$$

called the **characteristic equation** of the matrix $[T_{in}]$. It can always be brought into the form

(2.1.86) $$\lambda^3 - I_{\mathbf{T}}\, \lambda^2 + II_{\mathbf{T}}\, \lambda - III_{\mathbf{T}} = 0$$

with the three scalar coefficients $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$, called the **principal invariants** of **T**. Since the eigenvalue equations (2.1.83) do not depend on a basis, this should also hold for the characteristic polynomial and, therefore, for the principal invariants.

The first of these is the **trace** of the tensor

(2.1.87) $$I_{\mathbf{T}} = tr\,\mathbf{T} = T_{ii} = T_{11} + T_{22} + T_{33}\,.$$

The second one is the sum of the minors of the main diagonal

$$II_{\mathbf{T}} = det\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} + det\begin{bmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{bmatrix} + det\begin{bmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{bmatrix}$$

(2.1.88) $$= T_{11}\, T_{22} - T_{12}\, T_{21} + T_{22}\, T_{33} - T_{23}\, T_{32} + T_{11}\, T_{33} - T_{13}\, T_{31}$$
$$= \tfrac{1}{2}\,(T_{ii}\, T_{kk} - T_{ik}\, T_{ki})$$
$$= \tfrac{1}{2}\,\{tr^2(\mathbf{T}) - tr(\mathbf{T}^2)\}\,.$$

The third is the **determinant** of the tensor

$$III_{\mathbf{T}} = det(\mathbf{T}) = det\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

(2.1.89)
$$= \varepsilon_{ikm}\, T_{1i}\, T_{2k}\, T_{3m}$$
$$= T_{11}\, T_{22}\, T_{33} - T_{11}\, T_{23}\, T_{32} - T_{12}\, T_{21}\, T_{33}$$
$$+ T_{12}\, T_{23}\, T_{31} + T_{13}\, T_{21}\, T_{32} - T_{13}\, T_{22}\, T_{31}$$
$$= {}^{1}\!/_{6}\, tr^{3}(\mathbf{T}) - {}^{1}\!/_{2}\, tr(\mathbf{T})\, tr(\mathbf{T}^{2}) + {}^{1}\!/_{3}\, tr(\mathbf{T}^{3}) \,.$$

All of these representations with components hold only if referred to some ONB.

If we rotate the tensor by some versor, then the characteristic polynomial remains the same

$$det(\mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^{T} - \lambda\, \mathbf{I}) = det[\mathbf{Q} \cdot (\mathbf{A} - \lambda\, \mathbf{I}) \cdot \mathbf{Q}^{T}] = det(\mathbf{A} - \lambda\, \mathbf{I})$$

and therefore the invariants and the eigenvalues remain the same. Only the eigenvectors rotate.

For the principal invariants, the following rules hold for all scalars α, vectors $\mathbf{a}$ and $\mathbf{b}$, tensors $\mathbf{A}$ and $\mathbf{B}$, and orthogonal tensors $\mathbf{Q}$.

(2.1.90)	$tr(\mathbf{A} + \alpha\, \mathbf{B}) = tr(\mathbf{A}) + \alpha\, tr(\mathbf{B})$	linearity
(2.1.91)	$tr\, \mathbf{A} = tr(\mathbf{A}^{T})$	
(2.1.92)	$tr(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$	
(2.1.93)	$tr(\alpha\, \mathbf{I}) = 3\alpha$	in three dimensions
(2.1.94)	$tr(\mathbf{Q}\, \mathbf{A}\, \mathbf{Q}^{T}) = tr\, \mathbf{A}$	

(2.1.95)	$det(\mathbf{A}) = det(\mathbf{A}^{T})$	
(2.1.96)	$det(\alpha\, \mathbf{A}) = \alpha^{3}\, det(\mathbf{A})$	in three dimensions
(2.1.97)	$det(\mathbf{A} \cdot \mathbf{B}) = det(\mathbf{A})\, det(\mathbf{B})$	determinant rule
(2.1.98)	$det^{-1}(\mathbf{A}) = det(\mathbf{A}^{-1})$	for invertible tensors
(2.1.99)	$det\, \mathbf{I} = 1$	
(2.1.100)	$det\, \mathbf{Q} = det(\mathbf{Q}^{-1}) = \pm 1$ for orthogonal tensors	
(2.1.101)	$det(\mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^{T}) = det\, \mathbf{A}$	
for orthogonal tensors $\mathbf{Q}$ and arbitrary tensors $\mathbf{A}$		

A traceless tensor is called a **deviator**. One can uniquely decompose every tensor into its deviatoric and its spherical part. The deviatoric part is

(2.1.102) $$\mathbf{T}' := \mathbf{T} - {}^{1}\!/_{3}\, tr(\mathbf{T})\, \mathbf{I} \,.$$

After the theorem of VIETA[26] one can represent every polynomial by its roots (nulls)

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$
$$= \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\,\lambda^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\,\lambda - \lambda_1\lambda_2\lambda_3 \,.$$

By comparison with the characteristic polynomial, we obtain a representation for the invariants by the eigenvalues λ_i

(2.1.103)
$$I_{\mathbf{T}} = \lambda_1 + \lambda_2 + \lambda_3$$
$$II_{\mathbf{T}} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$
$$III_{\mathbf{T}} = \lambda_1\lambda_2\lambda_3 \,.$$

Theorem. *A tensor has*
- *either three real eigenvalues*
- *or one real and two conjugate complex ones.*

Proof. From the behaviour of such cubic polynomials for very small and very large values of λ and its continuity we conclude that at least one real root λ_1 must exist. Let $\mathbf{e}_1$ be the corresponding eigenvector. Then there exists an ONB $\{\mathbf{e}_i\}$ with respect to which the tensor has the following components

$$\begin{bmatrix} \lambda_1 & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & T_{32} & T_{33} \end{bmatrix} .$$

The trace of a tensor is real

$$I_{\mathbf{T}} = \lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + T_{22} + T_{33}$$

and therefore also the sum of the other eigenvalues

$$\lambda_2 + \lambda_3 = T_{22} + T_{33} \,.$$

We make for them a complex ansatz with the imaginary unity i

$$\lambda_2 = \alpha_2 + i\,\beta_2$$
$$\lambda_3 = \alpha_3 + i\,\beta_3 \,.$$

Its sum is only real if $\beta_2 = -\beta_3$. The determinant of the tensor is also real

$$III_{\mathbf{T}} = \lambda_1\lambda_2\lambda_3 = \lambda_1(T_{22}T_{33} - T_{32}T_{23})$$
$$\Rightarrow \qquad \lambda_2\lambda_3 = T_{22}T_{33} - T_{32}T_{23}$$

so that the product of the two other eigenvalues is also real

$$\lambda_2\lambda_3 = (\alpha_2 + i\,\beta_2)(\alpha_3 - i\,\beta_2) = \alpha_2\alpha_3 + \beta_2^2 + i\,(\alpha_3 - \alpha_2)\,\beta_2$$

and thus $(\alpha_3 - \alpha_2)\,\beta_2 = 0$. For $\beta_2 = 0$ all eigenvalues are real. If $\alpha_3 - \alpha_2 = 0$, then the two remaining eigenvalues are conjugate complex; *q. e. d.*

26 Francois Viète (1540-1603)

Let $\mathbf{T}$ be a tensor with real eigenvalues λ_1 , λ_2 , λ_3 and corresponding eigenvectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. First, we consider the case in which two different eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ have the same eigenvalue λ

$$\mathbf{T} \cdot \mathbf{a}_1 = \lambda\, \mathbf{a}_1$$

$$\mathbf{T} \cdot \mathbf{a}_2 = \lambda\, \mathbf{a}_2 \qquad\qquad \mathbf{a}_1 \neq \mathbf{a}_2$$

$$\Rightarrow \quad \mathbf{T} \cdot \mathbf{a}_1 + \mathbf{T} \cdot \mathbf{a}_2 = \lambda\, \mathbf{a}_1 + \lambda\, \mathbf{a}_2$$

$$= \mathbf{T} \cdot (\mathbf{a}_1 + \mathbf{a}_2) = \lambda\, (\mathbf{a}_1 + \mathbf{a}_2) \, .$$

So the sum of two eigenvectors corresponding to the same eigenvalue is also an eigenvector for the same eigenvalue. The same holds for all linear combinations of the two eigenvectors. The plane spanned by such eigenvectors is an **eigenspace** of the tensor. If the same holds for three eigenvectors which are linear independent, then the entire vector space is an eigenspace.

Examples

- For the *zero tensor* $\mathbf{0}$ every vector is an eigenvector with the (triple) eigenvalue 0 since

$$\mathbf{0} \cdot \mathbf{a} = \mathbf{o} = 0\, \mathbf{a} \, .$$

- For the *identity tensor* $\mathbf{I}$ every vector is an eigenvector with the (triple) eigenvalue 1 since

$$\mathbf{I} \cdot \mathbf{a} = \mathbf{a} = 1\, \mathbf{a} \, .$$

- For a *spherical tensor* $\alpha\, \mathbf{I}$ every vector is an eigenvector with the (triple) eigenvalue α .

Since for spherical tensors all directions are eigendirections, we conclude

Theorem. *If* $\mathbf{T}$ *is a tensor and* $\alpha\, \mathbf{I}$ *a spherical tensor, then* $\mathbf{T}$ *and* $\mathbf{T} + \alpha\, \mathbf{I}$ *have the same eigendirections.*

Accordingly, the eigenvectors of a tensor depend only on its deviatoric part.

- For an *orthogonal tensor* (2.1.76) the axial vector $\mathbf{e}_1$ is an eigenvector with eigenvalue ± 1. The positive sign holds for pure rotations (versors), the negative one for additional reflections. The other two eigenvalues are conjugate complex.
- For *dyads* $\mathbf{a} \otimes \mathbf{b}$, $\mathbf{a}$ is an eigenvector corresponding to the eigenvalue $\mathbf{a} \cdot \mathbf{b}$, and every vector perpendicular to $\mathbf{b}$ is an eigenvector for the double eigenvalue 0 . If $\mathbf{a}$ is perpendicular to $\mathbf{b}$, there is a triple eigenvalue 0 .

2.1.11 Spectral Forms of Symmetric Tensors

In mechanics, the eigenvalue problem is mainly posed for symmetric tensors, for which the following important theorem holds.

Theorem. *Symmetric tensors have three (not necessary different) real eigenvalues.*

Proof. By the relation of the previous proof we obtain the equation

$$\begin{aligned}(\lambda_2 - \lambda_3)^2 &= \lambda_2^2 + \lambda_3^2 - 2\,\lambda_2\,\lambda_3 \\ &= (\lambda_2 + \lambda_3)^2 - 4\,\lambda_2\,\lambda_3 \\ &= (T_{22} + T_{33})^2 - 4\,(T_{22}\,T_{33} - T_{23}\,T_{32}) \\ &= (T_{22} - T_{33})^2 + 4\,T_{23}^2\,.\end{aligned}$$

Both terms on the right -hand side are non-negative, and so

$$0 \leq (\lambda_2 - \lambda_3)^2 = (i\,\beta_2 + i\,\beta_2)^2 = 4\,i^2\,\beta_2^{\,2} = -4\,\beta_2^2.$$

This is only possible if $\beta_2 = 0$, so that all eigenvalues must be real; *q. e. d.*

If we consider the case of two eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ with *different* eigenvalues $\lambda_1 \neq \lambda_2$, then

$$\mathbf{T} \cdot \mathbf{a}_1 = \lambda_1\,\mathbf{a}_1 \qquad \text{and} \qquad \mathbf{T} \cdot \mathbf{a}_2 = \lambda_2\,\mathbf{a}_2$$

$$\Rightarrow \qquad \mathbf{a}_2 \cdot \mathbf{T} \cdot \mathbf{a}_1 = \lambda_1\,\mathbf{a}_2 \cdot \mathbf{a}_1$$

$$\mathbf{a}_1 \cdot \mathbf{T} \cdot \mathbf{a}_2 = \lambda_2\,\mathbf{a}_1 \cdot \mathbf{a}_2$$

and because of the assumed symmetry of $\mathbf{T}$ we obtain for the difference

$$\mathbf{a}_1 \cdot \mathbf{T} \cdot \mathbf{a}_2 - \mathbf{a}_2 \cdot \mathbf{T} \cdot \mathbf{a}_1 = 0 = (\lambda_2 - \lambda_1)\,(\mathbf{a}_1 \cdot \mathbf{a}_2)$$

i.e., $\mathbf{a}_1$ is perpendicular to $\mathbf{a}_2$. Thus, we have shown the following

Theorem. *Eigenvectors of symmetric tensors with different eigenvalues are mutually orthogonal.*

We consider now the three possible cases.

1st case: three different eigenvalues.

In this case, the eigenvectors form an ONB called the **eigenbasis** of the tensor, which is unique (up to changes of sign). The eigenspaces are three one-dimensional mutually orthogonal vector spaces. With respect to the eigenbasis, the matrix of the components has **diagonal form** or **spectral form**

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

with the three eigenvalues λ_1 , λ_2 , λ_3 .

2nd case: two of three eigenvalues are equal.

The eigenspace corresponding the equal eigenvalues is two-dimensional. It is a plane perpendicular to the eigenvector of the third eigenvalue. All vectors in this plane are eigenvectors. A diagonal form is also possible, which is (after an appropriate ordering)

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

3rd case: all (three) eigenvalues are equal.

In this case, all directions are eigendirections, and the eigenspace coincides with the underlying vector space. The matrix of the components has diagonal form with respect to each ONB

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} .$$

The tensor is the spherical tensor $\lambda\,\mathbf{I}$.

Theorem. *A tensor is symmetric if and only if there exists an ONB* $\{\mathbf{e}^{\mathrm{p}}_i\}$ *with respect to which the tensor has a* ***spectral form***

(2.1.104) $$\mathbf{T} = \sum_{i=1}^{3} \lambda_i \, \mathbf{e}^{\mathrm{p}}_i \otimes \mathbf{e}^{\mathrm{p}}_i$$

with λ_i *: real eigenvalues,*

$\mathbf{e}^{\mathrm{p}}_i$ *: normed eigenvectors.*

Theorem. *A symmetric tensor is positive definite if and only if all eigenvalues are positive. It is positive semidefinite, if and only if all eigenvalues are non-negative.*

We obtain the following classification of the symmetric tensors if we order the eigenvalues after their value, i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3$. The columns 3 - 5 in the table contain the intersections with different planes.

sign of eigenvals.	square form	$\mathbf{e}^{\mathrm{p}}_1$ - $\mathbf{e}^{\mathrm{p}}_2$ –plane	$\mathbf{e}^{\mathrm{p}}_2$ - $\mathbf{e}^{\mathrm{p}}_3$ –plane	$\mathbf{e}^{\mathrm{p}}_1$ - $\mathbf{e}^{\mathrm{p}}_3$ –plane	tensor surface
+ + +	positive def.	ellipse	ellipse	ellipse	ellipsoid
+ + *0*	pos.-semidef.	ellipse	straight lines	straight lines	ellipt. cylinder
+ *0 0*	pos.-semidef.	straight lines	-	straight lines	parall. planes
+ + –	indef.	ellipse	hyperbola	hyperbola	1fld hyperboloid
+ – –	indef.	hyperbola	-	hyperbola	2fld hyperboloid

Problem 2. Products between Vectors

Two vectors $\mathbf{v}_1 = 0.6\,\mathbf{e}_1 + 0.8\,\mathbf{e}_3$ and $\mathbf{v}_2 = -\,0.8\,\mathbf{e}_1 + 0.6\,\mathbf{e}_3$ are given with respect to an ONB $\{\mathbf{e}_i\}$. Find a third vector $\mathbf{v}_3$ such that $\{\mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3\}$ forms an ONB. Represent the tensor $\mathbf{A} := \mathbf{v}_1 \otimes \mathbf{v}_3 + \mathbf{v}_3 \otimes \mathbf{v}_1$ with respect to $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ and calculate the product $\mathbf{A} \cdot \mathbf{v}_2$.

<u>Solution</u>

An ONB is defined by the equations $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$. We first make sure that $\mathbf{v}_1$ and $\mathbf{v}_2$ fulfil this. In fact,

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\,0.48\,\mathbf{e}_1 \cdot \mathbf{e}_1 + 0.36\,\mathbf{e}_1 \cdot \mathbf{e}_3 - 0.64\,\mathbf{e}_3 \cdot \mathbf{e}_1 + 0.48\,\mathbf{e}_3 \cdot \mathbf{e}_3 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 0.36\,\mathbf{e}_1 \cdot \mathbf{e}_1 + 0.48\,\mathbf{e}_1 \cdot \mathbf{e}_3 + 0.48\,\mathbf{e}_3 \cdot \mathbf{e}_1 + 0.64\,\mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

$$\mathbf{v}_2 \cdot \mathbf{v}_2 = 0.64\,\mathbf{e}_1 \cdot \mathbf{e}_1 - 0.48\,\mathbf{e}_1 \cdot \mathbf{e} - 0.48\,\mathbf{e}_3 \cdot \mathbf{e}_1 + 0.36\,\mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

The mixed products $\mathbf{e}_1 \cdot \mathbf{e}_3$ and $\mathbf{e}_3 \cdot \mathbf{e}_1$ are all zero for an ONB $\{\mathbf{e}_i\}$, while $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1$ and $\mathbf{e}_3 \cdot \mathbf{e}_3 = 1$. For the determination of $\mathbf{v}_3$ we use the equations $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$, $\mathbf{v}_3 \cdot \mathbf{v}_3 = 1$. We make the ansatz

$$\mathbf{v}_3 = v^3{}_1\,\mathbf{e}_1 + v^3{}_2\,\mathbf{e}_2 + v^3{}_3\,\mathbf{e}_3$$

and obtain a system of equations for the components $v^3{}_1$, $v^3{}_2$, and $v^3{}_3$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = 0.6\,v^3{}_1 + 0.8\,v^3{}_3 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\,0.8\,v^3{}_1 + 0.6\,v^3{}_3 = 0$$

$$\mathbf{v}_3 \cdot \mathbf{v}_3 = (v^3{}_1)^2 + (v^3{}_2)^2 + (v^3{}_3)^2 = 1$$

The first two equations are only fulfilled if $v^3{}_1 = v^3{}_3 = 0$, so that $v^3{}_2$ can only be

(P2.1) $$v^3{}_2 = \pm 1 .$$

We want to determine $\mathbf{v}_3$ such that $\{\mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3\}$ is a positively oriented system. Thus, we postulate $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$. One could also have used this equation for the calculation of $\mathbf{v}_3$

$$\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2 = (0.6\,\mathbf{e}_1 + 0.8\,\mathbf{e}_3) \times (-\,0.8\,\mathbf{e}_1 + 0.6\,\mathbf{e}_3)$$

$$= -\,0.48\,\mathbf{e}_1 \times \mathbf{e}_1 + 0.36\,\mathbf{e}_1 \times \mathbf{e}_3 - 0.64\,\mathbf{e}_3 \times \mathbf{e}_1 + 0.48\,\mathbf{e}_3 \times \mathbf{e}_3$$

$$= -\,0.48\,\varepsilon_{11k}\,\mathbf{e}_k + 0.36\,\varepsilon_{13k}\,\mathbf{e}_k - 0.64\,\varepsilon_{31k}\,\mathbf{e}_k + 0.48\,\varepsilon_{33k}\,\mathbf{e}_k$$

The last equation has for $k \equiv 2$ only non-zero solutions. With $\varepsilon_{132} = -1$ and $\varepsilon_{312} = 1$ ($\varepsilon_{11k} = 0$ and $\varepsilon_{33k} = 0$ for all k) we obtain $\mathbf{v}_3 = -\,\mathbf{e}_2$. So we have to choose after P2.1 -1 for a positively oriented system or $+1$ for a negatively oriented one. We continue with $\mathbf{v}_3 \equiv -\mathbf{e}_2$. $\mathbf{A}$ is then

$$\begin{aligned}\mathbf{A} &= \mathbf{v}_1 \otimes \mathbf{v}_3 + \mathbf{v}_3 \otimes \mathbf{v}_1 \\ &= -(0.6\,\mathbf{e}_1 + 0.8\,\mathbf{e}_3) \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes (0.6\,\mathbf{e}_1 + 0.8\,\mathbf{e}_3) \\ &= -0.6\,\mathbf{e}_1 \otimes \mathbf{e}_2 - 0.8\,\mathbf{e}_3 \otimes \mathbf{e}_2 - 0.6\,\mathbf{e}_2 \otimes \mathbf{e}_1 - 0.8\,\mathbf{e}_2 \otimes \mathbf{e}_3 \\ &= A_{12}\,\mathbf{e}_1 \otimes \mathbf{e}_2 + A_{32}\,\mathbf{e}_3 \otimes \mathbf{e}_2 + A_{21}\,\mathbf{e}_2 \otimes \mathbf{e}_1 + A_{32}\,\mathbf{e}_2 \otimes \mathbf{e}_3 .\end{aligned}$$

We can put the components of $\mathbf{A}$ with respect to this basis in a matrix

$$[A^{\mathbf{ee}}{}_{ij}] = \begin{bmatrix} 0 & -0.6 & 0 \\ -0.6 & 0 & -0.8 \\ 0 & -0.8 & 0 \end{bmatrix} .$$

We now compute

$$\mathbf{b} := \mathbf{A} \cdot \mathbf{v}_2$$

$$\text{(P2.2)} \qquad = (-0.6\,\mathbf{e}_1 \otimes \mathbf{e}_2 - 0.8\,\mathbf{e}_3 \otimes \mathbf{e}_2 - 0.6\,\mathbf{e}_2 \otimes \mathbf{e}_1 - 0.8\,\mathbf{e}_2 \otimes \mathbf{e}_3) \cdot (-0.8\,\mathbf{e}_1 + 0.6\,\mathbf{e}_3) .$$

Since we multiply $\mathbf{v}_2$ from the right side to $\mathbf{A}$, the right vector of the base dyad must be contracted with $\mathbf{v}_2$. The left vector of the base dyads remains. Since $\{\mathbf{e}_i\}$ is an ONB, only two terms remain

$$\mathbf{b} = (-0.6)(-0.8)\,\mathbf{e}_2 + (-0.8)(0.6)\,\mathbf{e}_2 = \mathbf{o} .$$

With respect to the basis $\{\mathbf{v}_i\}$ we obtain the component matrix for $\mathbf{A}$

$$[A^{\mathbf{vv}}{}_{ij}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

The calculation of $\mathbf{b}$ with respect to the basis $\{\mathbf{v}_i\}$ is rather simple (compare with P2.2)

$$\mathbf{b} = \mathbf{A} \cdot \mathbf{v}_2 = (\mathbf{v}_1 \otimes \mathbf{v}_3 + \mathbf{v}_3 \otimes \mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{o}$$

because of the orthogonality of the basis $\{\mathbf{v}_i\}$. Since both $\{\mathbf{e}_i\}$ and $\{\mathbf{v}_i\}$ are ONBs, the matrices of the components with respect to both bases must have the same eigenvalues and principal invariants: $I_{\mathbf{A}} = 0$, $II_{\mathbf{A}} = -1$, $III_{\mathbf{A}} = 0$ after (2.1.87-89).

Problem 3. Direct Notation and Index Notation
Determine $\mathbf{v} \times \mathbf{T} \times \mathbf{v}$ between a vector $\mathbf{v}$ and a tensor $\mathbf{T}$ in components with respect to an ONB $\{\mathbf{e}_i\}$. Formulate the result as compact as possible and bring it - as far as possible - in a direct notation.

Solution

For the components with respect to an ONB we obtain

$$\mathbf{v} \times \mathbf{T} \times \mathbf{v} = v_i \, \mathbf{e}_i \times T_{jk} \, \mathbf{e}_j \otimes \mathbf{e}_k \times v_l \, \mathbf{e}_l$$

$$= v_i \, T_{jk} \, v_l \, (\mathbf{e}_i \times \mathbf{e}_j) \otimes (\mathbf{e}_k \times \mathbf{e}_l) \; .$$

Since the cross-product between vectors gives a vector, we obtain as a result dyads or second-order tensors. We use the permutation symbol of (2.1.2) to evaluate the cross-products

(P3.1) $$\mathbf{v} \times \mathbf{T} \times \mathbf{v} = \varepsilon_{ijm} \, \varepsilon_{kln} \, v_i \, T_{jk} \, v_l \, \mathbf{e}_m \otimes \mathbf{e}_n \, .$$

If we want to determine the component belonging to the base dyad $\mathbf{e}_1 \otimes \mathbf{e}_2$ we set $m \equiv 1$ and $n \equiv 2$ and obtain the sum $\varepsilon_{ij1} \, \varepsilon_{kl2} \, v_i \, T_{jk} \, v_l$. We have to sum over 4 dummy indices, so that we obtain $3^4 = 81$ terms. Most of the terms, however, are zero since only six index combinations in ε_{ijm} out of 27 are non-zero. For the compactification, the rule

$$\varepsilon_{ijm} \, \varepsilon_{kln} = det \begin{bmatrix} \delta_{ik} & \delta_{jk} & \delta_{mk} \\ \delta_{il} & \delta_{jl} & \delta_{ml} \\ \delta_{in} & \delta_{jn} & \delta_{mn} \end{bmatrix}$$

helps. One assembles a matrix with elements δ_{ij} such that the indices coincide with the three indices of the permutation symbol row-wise and column-wise. The determinant of this matrix equals the product of the two permutations symbols. Calculation of the determinant after the SARRUS[27] rule gives

$$\varepsilon_{ijm} \, \varepsilon_{kln} = \delta_{ik} \, \delta_{jl} \, \delta_{mn} + \delta_{jk} \, \delta_{ml} \, \delta_{in} + \delta_{mk} \, \delta_{il} \, \delta_{jn} - \delta_{mk} \, \delta_{jl} \, \delta_{in}$$

$$- \, \delta_{ik} \, \delta_{ml} \, \delta_{jn} - \delta_{jk} \, \delta_{il} \, \delta_{mn} \, .$$

We insert this into P3.1

$$\mathbf{v} \times \mathbf{T} \times \mathbf{v} = (\delta_{ik} \, \delta_{jl} \, \delta_{mn} + \delta_{jk} \, \delta_{ml} \, \delta_{in} + \delta_{mk} \, \delta_{il} \, \delta_{jn}$$

$$- \, \delta_{mk} \, \delta_{jl} \, \delta_{in} - \delta_{ik} \, \delta_{ml} \, \delta_{jn} - \delta_{jk} \, \delta_{il} \, \delta_{mn}) \, v_i \, T_{jk} \, v_l \, \mathbf{e}_m \otimes \mathbf{e}_n \; .$$

[27] Pierre Frédéric Sarrus (1798-1861)

We can now make reductions after rule (2.1.1) for the KRONECKER symbols. One can rename the dummy index in one KRONECKER symbol and thus eliminate the other KRONECKER symbol. We find

$$\mathbf{v} \times \mathbf{T} \times \mathbf{v} = v_j T_{ji} v_i \, \mathbf{e}_m \otimes \mathbf{e}_m + v_i T_{jj} v_l \, \mathbf{e}_l \otimes \mathbf{e}_i$$

$$+ v_i T_{jk} v_i \, \mathbf{e}_k \otimes \mathbf{e}_j - v_i T_{jk} v_j \, \mathbf{e}_k \otimes \mathbf{e}_i$$

$$- v_i T_{ji} v_l \, \mathbf{e}_l \otimes \mathbf{e}_j - v_i T_{jj} v_i \, \mathbf{e}_m \otimes \mathbf{e}_m .$$

In each term only two dummy indices appear in the base dyads. We can transform the result back into a direct notation

$$\mathbf{v} \times \mathbf{T} \times \mathbf{v} = (\mathbf{v} \cdot \mathbf{T} \cdot \mathbf{v}) \, \mathbf{I} + tr(\mathbf{T}) \, \mathbf{v} \otimes \mathbf{v}$$

$$+ (\mathbf{v} \cdot \mathbf{v}) \, \mathbf{T}^T - (\mathbf{v} \cdot \mathbf{T}) \otimes \mathbf{v} - \mathbf{v} \otimes (\mathbf{T} \cdot \mathbf{v}) - tr(\mathbf{T}) \, (\mathbf{v} \cdot \mathbf{v}) \, \mathbf{I}$$

with

$$v_j T_{ji} v_i = \mathbf{v} \cdot \mathbf{T} \cdot \mathbf{v} \qquad \mathbf{I} = \mathbf{e}_m \otimes \mathbf{e}_m \qquad T_{jj} = tr \, \mathbf{T}$$

$$v_i v_l \, \mathbf{e}_l \otimes \mathbf{e}_i = \mathbf{v} \otimes \mathbf{v} \qquad v_i v_i = \mathbf{v} \cdot \mathbf{v} \qquad T_{jk} \mathbf{e}_k \otimes \mathbf{e}_j = \mathbf{T}^T.$$

Problem 4. Orthogonal Tensors

Given the two ONBs $\{\mathbf{e}_i\}$ and $\{\mathbf{v}_i\}$ from Problem 2, find the tensor $\mathbf{Q}$ that describes the transformation of $\{\mathbf{e}_i\}$ into $\{\mathbf{v}_i\}$. Determine the components of $\mathbf{Q}$ with respect to the bases $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$, $\{\mathbf{v}_i \otimes \mathbf{v}_j\}$, $\{\mathbf{v}_i \otimes \mathbf{e}_j\}$, and $\{\mathbf{e}_i \otimes \mathbf{v}_j\}$. Note the difference between change of basis and transformation of a vector.

Solution

It is obvious that

(P4.1) $$\mathbf{Q} = \mathbf{v}_i \otimes \mathbf{e}_i = \mathbf{v}_1 \otimes \mathbf{e}_1 + \mathbf{v}_2 \otimes \mathbf{e}_2 + \mathbf{v}_3 \otimes \mathbf{e}_3$$

performs the desired transformation. The matrix of the components with respect to $\{\mathbf{v}_i \otimes \mathbf{e}_j\}$ is then

$$[Q^{\mathrm{ve}}{}_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For the representation with respect to the basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ we insert the vectors $\mathbf{v}_i$ from Problem 2

$$\mathbf{Q} = (0.6 \, \mathbf{e}_1 + 0.8 \, \mathbf{e}_3) \otimes \mathbf{e}_1 + (-0.8 \, \mathbf{e}_1 + 0.6 \, \mathbf{e}_3) \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3$$

and obtain the components

$$[Q^{ee}{}_{ij}] = \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0 & 0 & -1 \\ 0.8 & 0.6 & 0 \end{bmatrix}.$$

The components with respect to $\{\mathbf{v}_i \otimes \mathbf{v}_j\}$ can be achieved by scalar products of the tensor by vectors $\mathbf{v}_i$ from the left and $\mathbf{v}_j$ from the right

$$Q^{vv}{}_{ij} = \mathbf{v}_i \cdot \mathbf{Q} \cdot \mathbf{v}_j$$

$$= (\mathbf{v}_k \cdot \mathbf{v}_i)\,(\mathbf{e}_k \cdot \mathbf{v}_j) = \delta_{ik}\,\mathbf{e}_k \cdot \mathbf{v}_j = \mathbf{e}_i \cdot \mathbf{v}_j\,.$$

With $\mathbf{v}_1 = 0.6\,\mathbf{e}_1 + 0.8\,\mathbf{e}_3$, $\mathbf{v}_2 = -\,0.8\,\mathbf{e}_1 + 0.6\,\mathbf{e}_3$ and $\mathbf{v}_3 = -\,\mathbf{e}_2$ we obtain

$$[Q^{vv}{}_{ij}] = \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0 & 0 & -1 \\ 0.8 & 0.6 & 0 \end{bmatrix} = [Q^{ee}{}_{ij}]\,.$$

Thus, $Q^{ee}{}_{ij} = Q^{vv}{}_{ij}$ holds. In the same way we find

$$Q^{ev}{}_{ij} = \mathbf{Q} \cdot\cdot\, \mathbf{e}_i \otimes \mathbf{v}_j = Q^{ee}{}_{kl}\,\mathbf{e}_k \otimes \mathbf{e}_l \cdot\cdot\, \mathbf{e}_i \otimes \mathbf{v}_j$$

$$= Q^{ee}{}_{kl}\,\delta_{ik}\,\mathbf{e}_l \cdot \mathbf{v}_j = Q^{ee}{}_{il}\,\mathbf{e}_l \cdot \mathbf{v}_j\,.$$

With the previous result $\mathbf{e}_i \cdot \mathbf{v}_j = Q^{ee}{}_{ij}$ we obtain

$$[Q^{ev}{}_{ij}] = [Q^{ee}{}_{il}]\,[Q^{ee}{}_{lj}] = \begin{bmatrix} 0.36 & -0.48 & 0.8 \\ -0.8 & -0.6 & 0 \\ 0.48 & -0.64 & -0.6 \end{bmatrix}.$$

Since $\mathbf{Q}$ maps a right-hand-system into another one, it is orientation-preserving and therefore a proper-orthogonal tensor or a versor. So its determinant must be $+1$, which can easily be proven. Such tensors are pure rotations

$$\mathbf{v}_i = \mathbf{Q} \cdot \mathbf{e}_i \qquad \Leftrightarrow \qquad \mathbf{e}_i = \mathbf{Q}^T \cdot \mathbf{v}_i\,.$$

One should note the difference between this vector transformation and a change of basis. In the latter case, one changes both the basis and the components simultaneously, such that the vector itself remains the same, while only its representation changes

$$\mathbf{b} = b^{e}{}_i\,\mathbf{e}_i = b^{e}{}_i\,\mathbf{Q}^T \cdot \mathbf{v}_i = b^{e}{}_i\,[Q^{vv}{}_{jk}]^T\,\mathbf{v}_j \otimes \mathbf{v}_k \cdot \mathbf{v}_i$$

$$= b^{\mathbf{e}}{}_i \, Q^{\mathrm{vv}}{}_{kj} \, \delta_{ik} \, \mathbf{v}_j \;=\; b^{\mathbf{e}}{}_i \, Q^{\mathrm{vv}}{}_{ij} \, \mathbf{v}_j \;=\; b^{\mathbf{v}}{}_i \, \mathbf{v}_j \,.$$

In contrast to this, a rotation of a vector would lead to a change of the components with respect to the same basis

$$\mathbf{b}^* \;=\; b^{\mathbf{e}*}{}_i \, \mathbf{e}_i \;=\; \mathbf{Q} \cdot \mathbf{b} \;=\; Q^{\mathrm{ee}}{}_{ij} \, b^{\mathbf{e}}{}_k \, \mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{e}_k$$

$$= Q^{\mathrm{ee}}{}_{ij} \, b^{\mathbf{e}}{}_k \, \delta_{jk} \, \mathbf{e}_i \;=\; Q^{\mathrm{ee}}{}_{ij} \, b^{\mathbf{e}}{}_j \, \mathbf{e}_i \,.$$

Because of $Q^{\mathrm{ee}}{}_{ij} = Q^{\mathrm{vv}}{}_{ij}$ we have for the change of the basis

$$\mathbf{b} \;=\; b^{\mathbf{e}}{}_i \, \mathbf{e}_i \;=\; b^{\mathbf{e}}{}_i \, Q^{\mathrm{ee}}{}_{ij} \, (\mathbf{Q} \cdot \mathbf{e}_j) \,,$$

while the rotation of the vector leads to

$$\mathbf{b}^* \;=\; \mathbf{Q} \cdot \mathbf{b} \;=\; b_i^{\mathbf{e}*} \, \mathbf{e}_i \;=\; Q_{ij}^{\mathrm{ee}} \, b_j^{\mathbf{e}} \, \mathbf{e}_i \,.$$

Problem 5. Eigenvalues and Invariants

Determine the eigenvalues and eigenvectors and the principal invariants of a tensor, which is given with respect to an ONB $\{\mathbf{e}_i\}$ by

$$\mathbf{A} \;=\; 10 \, (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + 5 \, (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + 20 \, \mathbf{e}_3 \otimes \mathbf{e}_3 \,.$$

Represent $\mathbf{A}$ with respect to its eigenbasis $\{\mathbf{v}^i\}$. Find the tensor $\mathbf{Q}$ for the change of the basis from $\{\mathbf{e}_i\}$ to $\{\mathbf{v}^i\}$. Calculate the inverse of $\mathbf{A}$ with respect to $\{\mathbf{e}_i\}$ and to $\{\mathbf{v}^i\}$.

<u>Solution</u>

The matrix of the components of $\mathbf{A}$ with respect to $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ is

$$[A_{ij}] = \begin{bmatrix} 10 & 5 & 0 \\ 5 & 10 & 0 \\ 0 & 0 & 20 \end{bmatrix} .$$

With (2.1.87) to (2.1.89) we can calculate the principal invariants as

$$I_{\mathbf{A}} \;=\; A_{11} + A_{22} + A_{33} \;= 40$$

$$II_{\mathbf{A}} \;=\; A_{11} A_{22} - A_{12} \; A_{21} + A_{22} A_{33} - A_{23} A_{32} + A_{11} A_{33} - A_{13} \; A_{31}$$

$$= \; 100 - 25 + 200 + 200 \;= 475$$

$$III_{\mathbf{A}} \;=\; A_{11} \, A_{22} \, A_{33} - A_{11} \, A_{23} \, A_{32} - A_{12} \, A_{21} \, A_{33} \, +$$

$$A_{12} \, A_{23} \, A_{31} + A_{13} \, A_{21} \, A_{32} \; - A_{13} \, A_{22} \, A_{31}$$

$$= 2000 - 500 \;= 1500$$

The characteristic polynomial (2.1.86) is then

(P5.1) $$\lambda^3 - 40\,\lambda^2 + 475\,\lambda - 1500 = 0$$

which can be used to determine the eigenvalues. Since $A_{13} = A_{23} = A_{31} = A_{32} = 0$ we see that $\mathbf{A} \cdot \mathbf{e}_3 = 20\,\mathbf{e}_3$. So $\lambda = 20$ is the eigenvalue of the eigenvector $\mathbf{e}_3$. Polynomial division of P5.1 by $\lambda - 20$ gives

$$(\lambda^3 - 40\lambda^2 + 475\lambda - 1500) / (\lambda - 20) = \lambda^2 - 20\lambda + 75$$

$$\underline{\lambda^3 - 20\lambda^2}$$

$$-20\lambda^2 + 475\lambda$$

$$\underline{-20\lambda^2 + 400\lambda}$$

$$75\lambda - 1500$$

$$\underline{75\lambda - 1500}$$

$$0$$

The solutions of the resulting quadratic equation are

$$\lambda_{1,2} = 10 \pm 5\,.$$

So we have the following eigenvalues: $\lambda_1 = 5$, $\lambda_2 = 15$, $\lambda_3 = 20$, where the order is arbitrary. The third eigenvector is already known: $\mathbf{v}^3 = \mathbf{e}_3$. We compute $\mathbf{v}^1$ and $\mathbf{v}^2$ using (2.1.83), which results in the component form $(A_{ij} - \lambda\,\delta_{ij})\,v_j$. For λ_1 we obtain

$$\begin{bmatrix} 10-5 & 5 & 0 \\ 5 & 10-5 & 0 \\ 0 & 0 & 20-5 \end{bmatrix} \begin{bmatrix} v_1^{\,1} \\ v_2^{\,1} \\ v_3^{\,1} \end{bmatrix} = \begin{bmatrix} 5 & 5 & 0 \\ 5 & 5 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} v_1^{\,1} \\ v_2^{\,1} \\ v_3^{\,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

One can easily see that $v_3^{\,1} = 0$, and that the first two equations for the components lead to $v_1^{\,1} = -\,v_2^{\,1}$. Norming $|\mathbf{v}^1|$ determines $v_1^{\,1}$ and $v_2^{\,1}$ only up to their sign, so that $|v_1^{\,1}| = |v_2^{\,1}| = 1/\sqrt{2}$. For λ_2 we obtain by the same procedure $v_1^{\,2} = v_2^{\,2}$, $|v_1^{\,2}| = |v_2^{\,2}| = 1/\sqrt{2}$, $v_1^{\,3} = 0$. The spectral form of $\mathbf{A}$ is after (2.1.104)

$$\mathbf{A} = \lambda_1\,\mathbf{v}^1 \otimes \mathbf{v}^1 + \lambda_2\,\mathbf{v}^2 \otimes \mathbf{v}^2 + \lambda_3\,\mathbf{v}^3 \otimes \mathbf{v}^3.$$

The sense of direction of the $\mathbf{v}^i$ does not matter, since these vectors appear in all base dyads twice. $\mathbf{A}$ has the following components with respect to $\{\mathbf{v}^i \otimes \mathbf{v}^j\}$

$$[A^{vv}{}_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 20 \end{bmatrix}.$$

A change of the basis from $\{\mathbf{e}_i\}$ to $\{\mathbf{v}^i\}$ is performed by $\mathbf{Q} = \mathbf{v}^i \otimes \mathbf{e}_i$. For this purpose, we have to fix the senses of $\mathbf{v}^1$ and $\mathbf{v}^2$: $\mathbf{v}^1 = 1/\sqrt{2}\ \mathbf{e}_1 - 1/\sqrt{2}\ \mathbf{e}_2$, $\mathbf{v}^2 = 1/\sqrt{2}\ \mathbf{e}_1 + 1/\sqrt{2}\ \mathbf{e}_2$, $\mathbf{v}^3 = \mathbf{e}_3$. Inserting them gives us the components of $\mathbf{Q}$ with respect to $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$

$$[Q^{ee}{}_{ij}] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The inverse of $\mathbf{A}$ can be most easily determined in the spectral form

$$\mathbf{A}^{-1} = \lambda_1^{-1}\ \mathbf{v}^1 \otimes \mathbf{v}^1 + \lambda_2^{-1}\ \mathbf{v}^2 \otimes \mathbf{v}^2 + \lambda_3^{-1}\ \mathbf{v}^3 \otimes \mathbf{v}^3$$

or as a matrix

$$[A^{vv}{}_{ij}]^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/15 & 0 \\ 0 & 0 & 1/20 \end{bmatrix}.$$

The representation with respect to $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ can be most easily calculated by a change of the basis

$$\mathbf{A}^{-1} = [A^{vv}{}_{ij}]^{-1}\ \mathbf{v}^i \otimes \mathbf{v}^j = [A^{vv}{}_{ij}]^{-1}\ (\mathbf{Q} \cdot \mathbf{e}_i) \otimes (\mathbf{Q} \cdot \mathbf{e}_j)$$
$$= [A^{vv}{}_{ij}]^{-1}\ Q^{ee}{}_{lk}\ Q^{ee}{}_{mn}\ (\mathbf{e}_l \otimes \mathbf{e}_k \cdot \mathbf{e}_i) \otimes (\mathbf{e}_m \otimes \mathbf{e}_n \cdot \mathbf{e}_j)$$
$$= [A^{vv}{}_{ij}]^{-1} Q^{ee}{}_{li}\ Q^{ee}{}_{mj}\ \mathbf{e}_l \otimes \mathbf{e}_m .$$

A comparison of the components with the representation $\mathbf{A}^{-1} = [A^{ee}{}_{lm}]^{-1}\ \mathbf{e}_l \otimes \mathbf{e}_m$ shows that one can determine the components $[A^{ee}{}_{lm}]^{-1}$ by the matrix multiplication

$$[A^{ee}{}_{lm}]^{-1} = Q^{ee}{}_{li}\ [A^{vv}{}_{ij}]^{-1}\ Q^{ee}{}_{jm}{}^{T}$$

as

$$[A^{ee}{}_{lm}]^{-1} = \begin{bmatrix} 4/30 & -2/30 & 0 \\ -2/30 & 4/30 & 0 \\ 0 & 0 & 1/20 \end{bmatrix}.$$

Problem 6. Spectral Form
The well-known FIBONACCI[28] series is defined by

$$a_n = a_{n-1} + a_{n-2} \qquad a_1 = 1,\ a_2 = 1.$$

This can be written in matrix form as

$$\begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{n-2} \\ a_{n-1} \end{bmatrix}.$$

Find an explicit formula for a_n using the spectral form.

Solution

With the given initial values we can write

$$\text{(P6.1)} \qquad \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [M^{n-2}] \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We obtain

$$a_n = [M^{n-2}]_{21} + [M^{n-2}]_{22}.$$

By the spectral form one can determine the $n-2$th power of the matrix of coefficients M. The eigenvalues and normed eigenvectors are then

$$\lambda_1 = \tfrac{1}{2}(1+\sqrt{5}) \qquad \mathbf{v} = \left[-\frac{1-\sqrt{5}}{\sqrt{10-\sqrt{20}}}, \sqrt{\frac{5+\sqrt{5}}{10}} \right]$$

$$\lambda_2 = \tfrac{1}{2}(1-\sqrt{5}) \qquad \mathbf{w} = \left[-\frac{1+\sqrt{5}}{\sqrt{10+\sqrt{20}}}, \sqrt{\frac{2}{5+\sqrt{5}}} \right].$$

The matrices of the coefficients $[M_{ij}]^{n-2}$ can be obtained from the spectral form by applying the exponents to the eigenvalues using the scheme for matrix multiplications

$$M_{ij}^{n-2} = \lambda_1^{n-2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{matrix} [v_1 \quad v_2] \\ \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{matrix} + \lambda_2^{n-2} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \begin{matrix} [w_1 \quad w_2] \\ \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{matrix}.$$

The calculation is somewhat laborious. The result is

$$a_n = (\lambda_1^n - \lambda_2^n) / \sqrt{5}.$$

[28] Fibonacci (Leonardo of Pisa) around 1170-1250

2.1.12 Time-Dependent Vectors and Tensors

and their derivatives will be needed in dynamical problems. So let t be the time or a time-like parameter. Let $\alpha(t)$ be a scalar function of t, $\mathbf{v}(t)$ a time-dependent vector, and $\mathbf{T}(t)$ a time-dependent tensor. The time-derivatives are defined by the limits

$$\alpha(t)^\bullet = \frac{d\alpha}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\alpha(t+\Delta t) - \alpha(t)]$$

$$\mathbf{v}(t)^\bullet = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{v}(t+\Delta t) - \mathbf{v}(t)]$$

$$\mathbf{T}(t)^\bullet = \frac{d\mathbf{T}}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{T}(t+\Delta t) - \mathbf{T}(t)]$$

assuming that they exist.

The following rules hold for all scalars α, vectors $\mathbf{a}$ and $\mathbf{b}$, and tensors $\mathbf{T}$ and $\mathbf{S}$, all being differentiable functions of time. Then

$$(\mathbf{a} + \mathbf{b})^\bullet = \mathbf{a}^\bullet + \mathbf{b}^\bullet$$

$$(\mathbf{a} \cdot \mathbf{b})^\bullet = \mathbf{a}^\bullet \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}^\bullet$$

$$(\mathbf{a} \times \mathbf{b})^\bullet = \mathbf{a}^\bullet \times \mathbf{b} + \mathbf{a} \times \mathbf{b}^\bullet$$

$$(\mathbf{T} + \mathbf{S})^\bullet = \mathbf{T}^\bullet + \mathbf{S}^\bullet$$

$$(\mathbf{T} \cdot \mathbf{S})^\bullet = \mathbf{T}^\bullet \cdot \mathbf{S} + \mathbf{T} \cdot \mathbf{S}^\bullet$$

$$(\alpha\,\mathbf{T})^\bullet = \alpha^\bullet\,\mathbf{T} + \alpha\,\mathbf{T}^\bullet$$

$$(\mathbf{T} \cdot \mathbf{a})^\bullet = \mathbf{T}^\bullet \cdot \mathbf{a} + \mathbf{T} \cdot \mathbf{a}^\bullet$$

$$(\mathbf{a} \cdot \mathbf{T})^\bullet = \mathbf{a}^\bullet \cdot \mathbf{T} + \mathbf{a} \cdot \mathbf{T}^\bullet$$

$$(\mathbf{T} \times \mathbf{a})^\bullet = \mathbf{T}^\bullet \times \mathbf{a} + \mathbf{T} \times \mathbf{a}^\bullet$$

$$(\mathbf{a} \times \mathbf{T})^\bullet = \mathbf{a}^\bullet \times \mathbf{T} + \mathbf{a} \times \mathbf{T}^\bullet$$

$$(\mathbf{T}^T)^\bullet = (\mathbf{T}^\bullet)^T$$

If we represent a time-dependent vector with respect of a time-independent ONB

$$\mathbf{v} = v^i(t)\,\mathbf{e}_i\,,$$

then by $\mathbf{e}_j^\bullet = \mathbf{o}$ we get

$$\mathbf{v}^\bullet = v^{i\bullet}\,\mathbf{e}_i$$

so that the time-derivative of a vector is reduced to the time-derivatives of its scalar components. Analogously we obtain for a tensor

$$\mathbf{T} = T^{ij}(t)\,\mathbf{e}_i \otimes \mathbf{e}_j$$

the time-derivative

$$\mathbf{T}^\bullet \;=\; T^{ij\,\bullet}\, \mathbf{e}_i \otimes \mathbf{e}_j \, .$$

If $\mathbf{T}$ is invertible at all times, we get

(2.1.105) $$(\mathbf{T}^{-1})^\bullet \;=\; -\mathbf{T}^{-1} \cdot \mathbf{T}^\bullet \cdot \mathbf{T}^{-1}$$

since

$$(\mathbf{T}^{-1} \cdot \mathbf{T})^\bullet \;=\; \mathbf{0} \;=\; (\mathbf{T}^{-1})^\bullet \cdot \mathbf{T} + \mathbf{T}^{-1} \cdot \mathbf{T}^\bullet .$$

We now consider as a special case a time-dependent orthogonal tensor $\mathbf{Q}(t)$. By $\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{I}$ we see that

(2.1.106) $$(\mathbf{Q} \cdot \mathbf{Q}^T)^\bullet \;=\; \mathbf{0} \;=\; \mathbf{Q}^\bullet \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \mathbf{Q}^{T\,\bullet}$$
$$=\; \mathbf{Q}^\bullet \cdot \mathbf{Q}^T + (\mathbf{Q}^\bullet \cdot \mathbf{Q}^T)^T .$$

Thus, $\mathbf{Q}^\bullet \cdot \mathbf{Q}^T$ is skew. If $\mathbf{Q}(t)$ rotates a fixed ONB $\{\mathbf{e}_i\}$ into some time-dependent ONB $\{\underline{\mathbf{e}}_i(t)\}$, this allows the representation

$$\mathbf{Q} \;=\; \underline{\mathbf{e}}_i(t) \otimes \mathbf{e}_i \quad \Leftrightarrow \quad \mathbf{Q}^T = \mathbf{e}_i \otimes \underline{\mathbf{e}}_i(t)$$

and

$$\mathbf{Q}^\bullet \;=\; \underline{\mathbf{e}}_i(t)^\bullet \otimes \mathbf{e}_i$$

and

$$\mathbf{Q}^\bullet \cdot \mathbf{Q}^{-1} \;=\; \mathbf{Q}^\bullet \cdot \mathbf{Q}^T$$
$$=\; \underline{\mathbf{e}}_i(t)^\bullet \otimes \mathbf{e}_i \cdot \mathbf{e}_j \otimes \underline{\mathbf{e}}_j(t) \;=\; \underline{\mathbf{e}}_i(t)^\bullet \otimes \underline{\mathbf{e}}_i \, .$$

2.1.13 Rigid Body Dynamics

As a demonstration for the application of tensors, we will next consider the laws of motion, and specify them for rigid bodies. It will be shown that a direct tensor notation enables us to give these laws a very clear and compact form.

Let us consider a (deformable) body $\mathscr{B}$ which moves in the space being subjected to forces and torques. Let O be a fixed reference point in space and $\mathbf{r}_O$ the position vector of some other point with respect to O . The **centre of mass** of the body $\mathscr{B}$ is defined by its position vector

(2.1.107) $$\mathbf{r}_M \;:=\; \frac{1}{m} \int_{\mathscr{V}} \mathbf{r}_O\, \rho\, dV \;=\; \frac{1}{m} \int_{\mathscr{V}} \mathbf{r}_O\, dm$$

with

m mass of $\mathscr{B}$

ρ mass density in $\mathscr{B}$

$\mathscr{V}$ the current region of space occupied by $\mathscr{B}$

If we decompose the position vector of an arbitrary point of the body into

$$\mathbf{r}_O = \mathbf{r}_M + \mathbf{x} \,,$$

we obtain by the definition of the centre of mass the useful formula

(2.1.108) $$\int_{\mathscr{V}} \mathbf{x}\, dm = \mathbf{o}$$

as well as

$$\int_{\mathscr{V}} \mathbf{x}^{\bullet}\, dm = \mathbf{o} \,, \quad \int_{\mathscr{V}} \mathbf{x}^{\bullet\bullet}\, dm = \mathbf{o}$$

etc.

The **linear momentum** of the body is defined as the time-dependent vector

(2.1.109) $$\mathbf{p} := \int_{\mathscr{V}} \mathbf{r}_O^{\bullet}\, dm = \int_{\mathscr{V}} (\mathbf{r}_M^{\bullet} + \mathbf{x}^{\bullet})\, dm = \mathbf{r}_M^{\bullet}\, m \,,$$

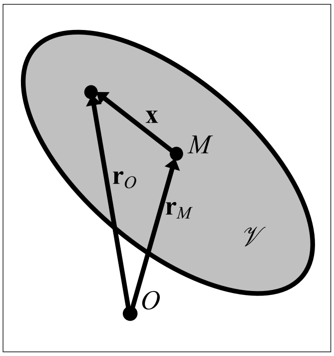
and the **angular momentum** or **moment of momentum** with respect to O is the vector

(2.1.110) $$\mathbf{d}_O := \int_{\mathscr{V}} \mathbf{r}_O \times \mathbf{r}_O^{\bullet}\, dm \,.$$

The **balance of linear momentum** (NEWTON 1687) is in general

(2.1.111) $$\mathbf{p}^{\bullet} = \frac{d}{dt} \int_{\mathscr{V}} \mathbf{r}_O^{\bullet}\, dm = \int_{\mathscr{V}} \mathbf{r}_O^{\bullet\bullet}\, dm = \mathbf{r}_M^{\bullet\bullet}\, m$$

$$= \mathbf{f} = \text{resulting force acting on } \mathscr{B}$$



and the **balance of angular momentum** (EULER[29] 1775) with respect to O

[29] Leonhard Euler (1707-1783)

(2.1.112) $$\mathbf{d}_O^{\bullet} = \frac{d}{dt} \int_{\mathscr{V}} \mathbf{r}_O \times \mathbf{r}_O^{\bullet}\, dm = \int_{\mathscr{V}} \mathbf{r}_O \times \mathbf{r}_O^{\bullet\bullet}\, dm$$

$$= \mathbf{m}_O = \text{resulting torque with respect to } O \text{ acting on the body.}$$

We now use the above decomposition of the position vector $\mathbf{r}_O = \mathbf{r}_M + \mathbf{x}$

$$\mathbf{d}_O^{\bullet} = \int_{\mathscr{V}} \mathbf{r}_O \times \mathbf{r}_O^{\bullet\bullet}\, dm = \int_{\mathscr{V}} (\mathbf{r}_M + \mathbf{x}) \times (\mathbf{r}_M + \mathbf{x})^{\bullet\bullet}\, dm$$

$$= \int_{\mathscr{V}} \mathbf{r}_M \times \mathbf{r}_M^{\bullet\bullet}\, dm + \int_{\mathscr{V}} \mathbf{r}_M \times \mathbf{x}^{\bullet\bullet}\, dm$$

$$+ \int_{\mathscr{V}} \mathbf{x} \times \mathbf{r}_M^{\bullet\bullet}\, dm + \int_{\mathscr{V}} \mathbf{x} \times \mathbf{x}^{\bullet\bullet}\, dm$$

$$= \mathbf{r}_M \times \mathbf{r}_M^{\bullet\bullet}\, m + \mathbf{r}_M \times \int_{\mathscr{V}} \mathbf{x}^{\bullet\bullet}\, dm$$

$$+ \int_{\mathscr{V}} \mathbf{x}\, dm \times \mathbf{r}_M^{\bullet\bullet} + \int_{\mathscr{V}} \mathbf{x} \times \mathbf{x}^{\bullet\bullet}\, dm$$

and by (2.1.108)

$$\mathbf{d}_O^{\bullet} = \mathbf{r}_M \times \mathbf{r}_M^{\bullet\bullet}\, m + \int_{\mathscr{V}} \mathbf{x} \times \mathbf{x}^{\bullet\bullet}\, dm$$

$$= \mathbf{m}_O\,.$$

By using VARIGNON′s principle, we can also refer the torques to M

$$\mathbf{m}_O = \mathbf{r}_M \times \mathbf{f} + \mathbf{m}_M\,.$$

If we multiply the balance of linear momentum (2.1.111) by $\mathbf{r}_M$

$$\mathbf{r}_M \times \mathbf{r}_M^{\bullet\bullet}\, m = \mathbf{r}_M \times \mathbf{f}$$

then the difference of the result and the previous equations gives the balance of angular momentum with respect to the centre of mass

(2.1.113) $$\mathbf{d}_M^{\bullet} = \mathbf{m}_M \qquad \text{with } \mathbf{d}_M := \int_{\mathscr{V}} \mathbf{x} \times \mathbf{x}^{\bullet}\, dm$$

i.e., in the same form as for a fixed point (2.1.112). In contrast, if we transform the balance of angular momentum to some other moving point, it has to be enlarged by additional terms.

If we want to further reduce the balance of angular momentum, we restrict our consideration to **rigid bodies**. The displacement of any point of a rigid body with position vector $\mathbf{r}_O = \mathbf{r}_M + \mathbf{x}$ can be decomposed into the displacement of the centre of mass

$$\mathbf{u}_M(t) = \mathbf{r}_M(t) - \mathbf{r}_M(0)$$

and a rotation of $\mathbf{x}$ around M, which can be performed by a versor $\mathbf{Q}(t)$

$$\mathbf{x}(t) = \mathbf{Q}(t) \cdot \mathbf{x}(0)\,.$$

Its time-derivative is

$$\mathbf{x}^\bullet(t) \;=\; \mathbf{Q}^\bullet(t)\cdot\mathbf{x}(0) \;=\; \mathbf{Q}^\bullet(t)\cdot\mathbf{Q}^T(t)\cdot\mathbf{x}(t)$$

$$=\; \boldsymbol{\omega}(t)\times\mathbf{x}(t)$$

with $\boldsymbol{\omega}(t)$ being the axial vector of the skew tensor $\mathbf{Q}^\bullet \cdot \mathbf{Q}^T$ after (2.1.106) called the **angular velocity**. The total velocity of an arbitrary point of the body is thus

(2.1.114)	$\mathbf{v}(t) \;=\; \mathbf{r}_O^\bullet \;=\; \mathbf{r}_M(t)^\bullet + \mathbf{x}(t)^\bullet \;=\; \mathbf{v}_M(t) + \boldsymbol{\omega}(t)\times\mathbf{x}(t)$

This is **EULER´s velocity formula** of rigid body kinematics (wherein one could also refer to any other reference point of the body instead of the centre of mass). Consequently, each vector $\mathbf{x}$ being fixed to the body rotates with the same angular velocity $\mathbf{x}^\bullet \;=\; \boldsymbol{\omega}\times\mathbf{x}$.

The angular momentum with respect to M is

$$\mathbf{d}_M(t) \;=\; \int_{\mathcal{V}} \mathbf{x}\times\mathbf{x}^\bullet\; dm$$

$$=\; \int_{\mathcal{V}} \mathbf{x}\times(\boldsymbol{\omega}\times\mathbf{x})\; dm$$

and after the formula for double cross-products

$$=\; \int_{\mathcal{V}} [\boldsymbol{\omega}\,(\mathbf{x}\cdot\mathbf{x}) - \mathbf{x}\,(\mathbf{x}\cdot\boldsymbol{\omega})]\; dm$$

$$=\; [\int_{\mathcal{V}} (\mathbf{x}\cdot\mathbf{x}\,\mathbf{I} - \mathbf{x}\otimes\mathbf{x})\; dm]\cdot\boldsymbol{\omega}$$

(2.1.115)	$\mathbf{d}_M \;=\; \mathbf{J}_M\cdot\boldsymbol{\omega}$

with the **tensor of inertia** with respect to M

$$\mathbf{J}_M \;:=\; \int_{\mathcal{V}} (\mathbf{x}^2\,\mathbf{I} - \mathbf{x}\otimes\mathbf{x})\; dm\;.$$

Its components with respect to some ONB are the **moments of inertia**

$$J_{ik} \;=\; \mathbf{e}_i\cdot\mathbf{J}_M\cdot\mathbf{e}_k \;=\; \int_{\mathcal{V}} [(x_1^2 + x_2^2 + x_3^2)\,\delta_{ik} \;-\; x_i\,x_k]\; dm$$

i.e.

$$J_{11} \;=\; \int_{\mathcal{V}} (x_2^2 + x_3^2)\; dm \qquad J_{12} \;=\; J_{21} \;=\; -\int_{\mathcal{V}} x_1\,x_2\; dm$$

$$J_{22} \;=\; \int_{\mathcal{V}} (x_3^2 + x_1^2)\; dm \qquad J_{23} \;=\; J_{32} \;=\; -\int_{\mathcal{V}} x_2\,x_3\; dm$$

$$J_{33} = \int_{\mathcal{V}} (x_1^2 + x_2^2)\, dm \qquad J_{31} = J_{13} = -\int_{\mathcal{V}} x_3\, x_1\, dm$$

The tensor of inertia is symmetric for all bodies. Accordingly, there exists an eigenbasis $\{\mathbf{e}^{\mathrm{p}}{}_i\}$ which gives the spectral form

$$\mathbf{J}_M = \sum_{i=1}^{3} J^{\mathrm{p}}{}_i\, \mathbf{e}^{\mathrm{p}}{}_i \otimes \mathbf{e}^{\mathrm{p}}{}_i\,.$$

These **principal axes of inertia** indicated by the vectors $\mathbf{e}^{\mathrm{p}}{}_i$ are for all bodies and all motions fixed to the body, but not fixed in space. The principal moments of inertia $J^{\mathrm{p}}{}_i$ are time-independent and positive, so that $\mathbf{J}_M$ is positive definite.

For evaluating the balance of angular momentum, we will need the time derivative of the tensor of inertia, which is by use of EULER´s velocity formula (2.1.114)

$$\mathbf{J}_M^{\bullet} = \sum_{i=1}^{3} J^{\mathrm{p}}{}_i\, [\mathbf{e}^{\mathrm{p}\bullet}_i \otimes \mathbf{e}^{\mathrm{p}}{}_i + \mathbf{e}^{\mathrm{p}}{}_i \otimes \mathbf{e}^{\mathrm{p}\bullet}_i\,]$$

(2.1.116)
$$= \sum_{i=1}^{3} J^{\mathrm{p}}{}_i\, [(\boldsymbol{\omega} \times \mathbf{e}^{\mathrm{p}}{}_i) \otimes \mathbf{e}^{\mathrm{p}}{}_i + \mathbf{e}^{\mathrm{p}}{}_i \otimes (\boldsymbol{\omega} \times \mathbf{e}^{\mathrm{p}}{}_i)]$$

$$= \sum_{i=1}^{3} J^{\mathrm{p}}{}_i\, [\boldsymbol{\omega} \times (\mathbf{e}^{\mathrm{p}}{}_i \otimes \mathbf{e}^{\mathrm{p}}{}_i) - (\mathbf{e}^{\mathrm{p}}{}_i \otimes \mathbf{e}^{\mathrm{p}}{}_i) \times \boldsymbol{\omega}]$$

$$= \boldsymbol{\omega} \times \mathbf{J}_M - \mathbf{J}_M \times \boldsymbol{\omega}\,.$$

Accordingly, the rate of the angular momentum equals

$$\mathbf{d}_M^{\bullet} = \mathbf{J}_M^{\bullet} \cdot \boldsymbol{\omega} + \mathbf{J}_M \cdot \boldsymbol{\omega}^{\bullet}$$

$$= (\boldsymbol{\omega} \times \mathbf{J}_M - \mathbf{J}_M \times \boldsymbol{\omega}) \cdot \boldsymbol{\omega} + \mathbf{J}_M \cdot \boldsymbol{\omega}^{\bullet}.$$

The term in the middle is zero after the rules of the triple product. Therefore, the balance of angular momentum becomes with respect to the centre of mass of a rigid body

(2.1.117)
$$\mathbf{m}_M = \mathbf{J}_M \cdot \boldsymbol{\omega}^{\bullet} + \boldsymbol{\omega} \times \mathbf{J}_M \cdot \boldsymbol{\omega}$$

If we also represent the angular velocity with respect to the eigenbasis $\{\mathbf{e}^{\mathrm{p}}{}_i\}$

$$\boldsymbol{\omega}(t) = \omega^{\mathrm{p}}{}_i(t)\, \mathbf{e}^{\mathrm{p}}{}_i(t)$$

then its time-derivative is

$$\boldsymbol{\omega}(t)^{\bullet} = \omega^{\mathrm{p}\bullet}_i\, \mathbf{e}^{\mathrm{p}}{}_i + \omega^{\mathrm{p}}{}_i\, \mathbf{e}^{\mathrm{p}\bullet}_i$$

$$= \omega^{\mathrm{p}\bullet}_i\, \mathbf{e}^{\mathrm{p}}{}_i + \omega^{\mathrm{p}}{}_i\, (\boldsymbol{\omega} \times \mathbf{e}^{\mathrm{p}}{}_i)$$

$$= \omega^{\mathrm{p}\bullet}_i\, \mathbf{e}^{\mathrm{p}}{}_i + \boldsymbol{\omega} \times \boldsymbol{\omega}$$

$$= \omega^{\mathrm{p}\bullet}_i\, \mathbf{e}^{\mathrm{p}}{}_i\,.$$

This holds analogously for each ONB $\{\mathbf{e}_i\}$ that is fixed to the body. The component form of the balance of angular momentum with respect to such a basis is

$$m_{Mi} = J_{il}\,\omega_l^{\bullet} + \omega_l\, J_{kp}\, \omega_p\, \varepsilon_{lki}$$

and in particular with respect to the eigenbasis

$$m_M{}^p{}_1 = J^p{}_1\, \omega^p{}_1{}^{\bullet} + \omega^p{}_2\, \omega^p{}_3\, (J^p{}_3 - J^p{}_2)$$

(2.1.118)
$$m_M{}^p{}_2 = J^p{}_2\, \omega^p{}_2{}^{\bullet} + \omega^p{}_1\, \omega^p{}_3\, (J^p{}_1 - J^p{}_3)$$

$$m_M{}^p{}_3 = J^p{}_3\, \omega^p{}_3{}^{\bullet} + \omega^p{}_2\, \omega^p{}_1\, (J^p{}_2 - J^p{}_1)$$

These are **EULER´s equations for gyroscopes** (1758) with respect to the centre of mass and to the principal axes of inertia.

The **kinetic energy** of the rigid body is

$$K = \tfrac{1}{2} \int_{\mathscr{V}} \mathbf{r}_O^{\bullet 2}\, dm$$

$$= \tfrac{1}{2} \int_{\mathscr{V}} (\mathbf{r}_M^{\bullet} + \mathbf{x}^{\bullet})^2\, dm$$

$$= \tfrac{1}{2} \int_{\mathscr{V}} (\mathbf{r}_M^{\bullet} + \boldsymbol{\omega} \times \mathbf{x})^2\, dm$$

$$= \tfrac{1}{2}\, \mathbf{r}_M^{\bullet 2}\, m + \tfrac{1}{2} \int_{\mathscr{V}} (\boldsymbol{\omega} \times \mathbf{x})^2\, dm + \mathbf{r}_M^{\bullet} \cdot \boldsymbol{\omega} \times \int_{\mathscr{V}} \mathbf{x}\; dm$$

(2.1.119)
$$= \tfrac{1}{2}\, \mathbf{r}_M^{\bullet 2}\, m + \tfrac{1}{2} \int_{\mathscr{V}} (\boldsymbol{\omega} \times \mathbf{x})^2\, dm$$

$$= K_{trans} + K_{rot}$$

i.e., a sum of translatoric and rotatoric energy. The latter can be reformulated as

$$K_{rot} = \tfrac{1}{2} \int_{\mathscr{V}} (\boldsymbol{\omega} \times \mathbf{x}) \cdot (\boldsymbol{\omega} \times \mathbf{x})\, dm$$

$$= \tfrac{1}{2} \int_{\mathscr{V}} \boldsymbol{\omega} \cdot [\mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x})]\, dm$$

$$= \boldsymbol{\omega} \cdot \tfrac{1}{2} \int_{\mathscr{V}} [(\mathbf{x} \cdot \mathbf{x})\, \boldsymbol{\omega} - (\mathbf{x} \cdot \boldsymbol{\omega})\, \mathbf{x}]\, dm$$

(2.1.120)
$$= \boldsymbol{\omega} \cdot \tfrac{1}{2} \int_{\mathscr{V}} [\mathbf{x}^2\, \mathbf{I} - (\mathbf{x} \otimes \mathbf{x})]\, dm \cdot \boldsymbol{\omega}$$

$$= \tfrac{1}{2}\, \boldsymbol{\omega} \cdot \mathbf{J}_M \cdot \boldsymbol{\omega}$$

$$= \tfrac{1}{2}\, \boldsymbol{\omega} \cdot \mathbf{d}_M$$

$$= \tfrac{1}{2}\,(J_{11}\,\omega_1^{\,2} + J_{22}\,\omega_2^{\,2} + J_{33}\,\omega_3^{\,2})$$
$$+ (\omega_1\; J_{12}\,\omega_2 + \omega_2\; J_{23}\,\omega_3 + \omega_3\,J_{31}\,\omega_1)\;.$$

This is a positive definite square form in the angular velocity, i.e., the rotatoric energy is positive for all $\boldsymbol{\omega} \neq \mathbf{o}$. With respect to the principal axes of inertia the last bracket vanishes.

If we now consider the particular case of a *load-free gyroscope* ($\mathbf{m}_M \equiv \mathbf{o}$, $\mathbf{f} \equiv \mathbf{o}$). In this case we have

- conservation of linear momentum (2.1.121)

$$\mathbf{p} \;=\; \mathbf{v}_M\, m \;=\; constant \qquad \Rightarrow \qquad \mathbf{v}_M = \; constant$$

- conservation of angular momentum (2.1.122)

$$\mathbf{d}_M \;=\; \mathbf{J}_M \cdot \boldsymbol{\omega} \;=\; constant$$

- conservation of energy (2.1.123)

$$K \;=\; \tfrac{1}{2}\,\mathbf{v}_M^{\,2}\; m + \tfrac{1}{2}\,\boldsymbol{\omega} \cdot \mathbf{J}_M \cdot \boldsymbol{\omega} \;=\; constant$$

We will later show that the conservation of energy is a result of conservation of linear momentum and angular momentum and thus does not give another independent balance law.

Using (2.1.123) we conclude with (2.1.121)

$$\boldsymbol{\omega} \cdot \mathbf{J}_M \cdot \boldsymbol{\omega} \;=\; constant \qquad \text{(energy ellipsoid)}$$

and with (2.1.122)

$$\mathbf{d}_M^{\,2} \;=\; \boldsymbol{\omega} \cdot \mathbf{J}_M^{\,2} \cdot \boldsymbol{\omega} \;=\; constant \qquad \text{(angular momentum ellipsoid)}$$

The tensor surfaces of $\mathbf{J}_M$ and $\mathbf{J}_M^{\,2}$ are two ellipsoids, both fixed to the body and with the same axes, namely the principal axes of inertia of the gyroscope. The geometric interpretation is that $\boldsymbol{\omega}$ is located on both ellipsoids and, therefore, on the intersection of them. $\boldsymbol{\omega}$ describes the *polecone*. The normal $\mathbf{n}$ of the energy ellipsoid lies in the direction of

$$\mathbf{J}_M \cdot \boldsymbol{\omega} \;=\; \mathbf{d}_M \;=\; constant$$

and is fixed in space after (2.1.122). The energy ellipsoid is rolling on a tangential plane fixed in space (POINSOT[30]′s rolling motion).

On the other hand we can conclude from (2.1.122)

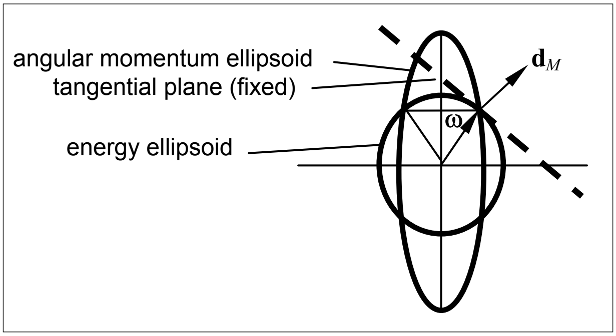
$$\mathbf{d}_M^{\,2} \;=\; \mathbf{d}_M \cdot \mathbf{I} \cdot \mathbf{d}_M \;=\; constant \qquad \text{(sphere)}$$

and from (2.1.123)

$$\boldsymbol{\omega} \cdot \mathbf{J}_M \cdot \boldsymbol{\omega} \;=\; \mathbf{d}_M \cdot \mathbf{J}_M^{-1} \cdot \mathbf{d}_M \;=\; constant$$

(McCULLAGH ellipsoid)

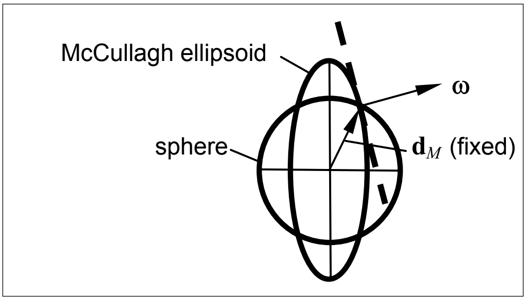
[30] Louis Poinsot (1777-1859)



The first describes a sphere fixed in space and with respect to the body. The latter is the McCULLAGH[31] ellipsoid fixed with respect to the body. The vector of angular momentum lies in the intersection of the body-fixed ellipsoid and a sphere and describes the *polehode*. The normal to $\mathbf{J}_M^{-1}$ is in the direction of

$$\mathbf{J}_M^{-1} \cdot \mathbf{d}_M = \omega$$

and swings in space.



2.1.14 Bending of Bars

We consider the three-dimensional bending problem of originally straight bars. We make the following assumptions.

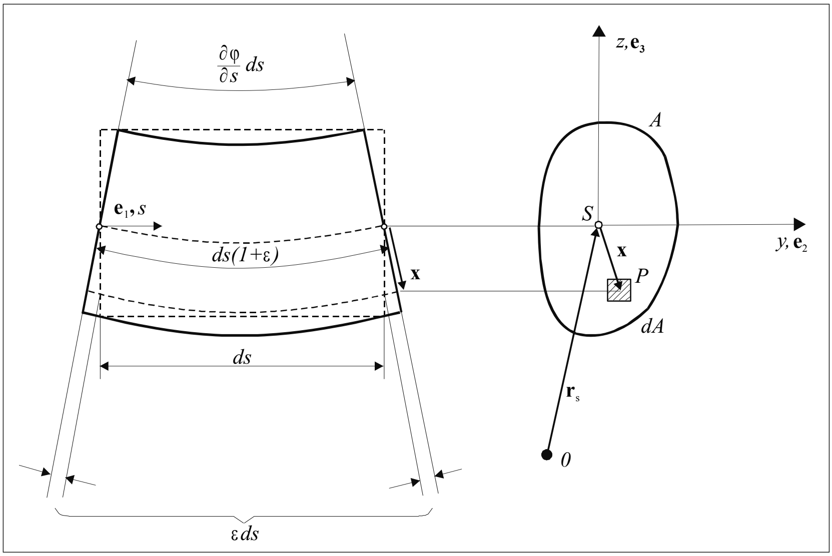
- BERNOULLI´s hypothesis: plain cross-sections remain plain and perpendicular to the central axis of the bar
- small displacements, small rotations, and small deformations

[31] James McCullagh (1809-1847)

- linear-elastic material: $\sigma = E\,\varepsilon$ (HOOKE´s law)
- no axial loads.

Let

s	be the coordinate of the arc length of the undeformed bar
$'$	the derivative with respect to s
$\boldsymbol{\varphi}(s)$	the (infinitesimal) rotation vector of the cross-section in s
$\mathscr{A}$	the cross-section
$A(s)$	the area of the cross-section
$\mathbf{x}$	the position vector of the neutral fibre to an arbitrary point of the cross-section
$\mathbf{t}$	the normed tangent vector to the axis of the bar ($\lvert \mathbf{t} \rvert = 1$)



We consider some cross-sections at a constant s. As we have already seen before, one can represent an infinitesimal rotation by an axial vector $\boldsymbol{\varphi}(s)$. For the rotation of the cross-section plane, this vector is contained in this plane. The displacement resulting from bending is

$$\mathbf{u} = \boldsymbol{\varphi}(s) \times \mathbf{x} + \mathbf{u}_o$$

with some constant part $\mathbf{u}_o$. The derivative of the tangential part of the displacement $\mathbf{u} \cdot \mathbf{t}$ is the stretch in axial direction

$$\varepsilon_t = \frac{d}{ds}(\mathbf{u} \cdot \mathbf{t}) = \mathbf{u}' \cdot \mathbf{t} + \mathbf{u} \cdot \mathbf{t}'$$

$$= \boldsymbol{\varphi}(s)' \times \mathbf{x} \cdot \mathbf{t} + \boldsymbol{\varphi}(s) \times \mathbf{x}' \cdot \mathbf{t} + \boldsymbol{\varphi}(s) \times \mathbf{x} \cdot \mathbf{t}'$$

(2.1.124) $$= \boldsymbol{\kappa} \times \mathbf{x} \cdot \mathbf{t} \qquad \text{with} \quad \boldsymbol{\kappa} := \frac{d}{ds}\boldsymbol{\varphi}(s)$$

since $\mathbf{x}$ does not depend on s, $\mathbf{t}'$ lies in the cross-section plane, while $\boldsymbol{\varphi}(s) \times \mathbf{x}$ is perpendicular to it. The traction vector in some point of the cross-section indicated by $\mathbf{x}$ is after HOOKE´s law

$$\boldsymbol{\sigma} = \sigma_t\, \mathbf{t} = E\, \varepsilon_t\, \mathbf{t} = E\, \{(\boldsymbol{\kappa} \times \mathbf{x}) \cdot \mathbf{t}\}\, \mathbf{t}\,.$$

Since we assumed that no resulting normal force acts on the cross-section, we obtain

$$\int_{\mathscr{A}} \sigma_t\, dA = E \int_{\mathscr{A}} (\boldsymbol{\kappa} \times \mathbf{x}) \cdot \mathbf{t}\, dA = E\,(\mathbf{t} \times \boldsymbol{\kappa}) \cdot \int_{\mathscr{A}} \mathbf{x}\, dA = 0$$

so that the neutral fibre must go through the centroid of the cross-section. The resulting torque with respect to the centroid is

$$\mathbf{m} = \int_{\mathscr{A}} \mathbf{x} \times \boldsymbol{\sigma}\, dA = E \int_{\mathscr{A}} \{\boldsymbol{\kappa} \cdot (\mathbf{x} \times \mathbf{t})\}\, \mathbf{x} \times \mathbf{t}\, dA$$

$$= E\,(\int_{\mathscr{A}} (\mathbf{x} \times \mathbf{t}) \otimes (\mathbf{x} \times \mathbf{t})\, dA) \cdot \boldsymbol{\kappa}$$

(2.1.125) $$\mathbf{m} = E\, \mathbf{J}_A \cdot \boldsymbol{\kappa}$$

with the **tensor of inertia of area**

$$\mathbf{J}_A := \int_{\mathscr{A}} (\mathbf{x} \times \mathbf{t}) \otimes (\mathbf{x} \times \mathbf{t})\, dA$$

obviously being symmetric. We choose an ONB $\{\mathbf{e}_i\}$ with $\mathbf{e}_1 \equiv \mathbf{t}$ so that

$$\mathbf{x}(y, z) = y\, \mathbf{e}_2 + z\, \mathbf{e}_3$$

and

$$J_{ik} = \mathbf{e}_i \cdot \mathbf{J}_A \cdot \mathbf{e}_k$$

$$= \int_{\mathscr{A}} \mathbf{e}_i \cdot \{(y\, \mathbf{e}_2 + z\, \mathbf{e}_3) \times \mathbf{e}_1\}\, \{(y\, \mathbf{e}_2 + z\, \mathbf{e}_3) \times \mathbf{e}_1\} \cdot \mathbf{e}_k\, dA$$

$$= \int_{\mathscr{A}} (y\, \varepsilon_{i21} + z\, \varepsilon_{i31})\, (y\, \varepsilon_{21k} + z\, \varepsilon_{31k})\, dA$$

and also

$$J_{yy} = \int_{\mathscr{A}} z^2\, dA \qquad J_{zz} = \int_{\mathscr{A}} y^2\, dA \qquad J_{yz} = J_{zy} = -\int_{\mathscr{A}} y\, z\, dA$$

while all other components with respect to this basis are zero

$$[J_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & J_{yy} & J_{yz} \\ 0 & J_{yz} & J_{zz} \end{bmatrix}.$$

$\mathbf{e}_1$ is eigenvector of $\mathbf{J}_F$ with eigenvalue 0. The two non-trivial eigenvalues $J^p{}_1$ and $J^p{}_2$ are the roots of the characteristic equation of $\mathbf{J}_F$

$$III_{\mathbf{J}} - J^p{}_i\, II_{\mathbf{J}} + (J^p{}_i)^2\, I_{\mathbf{J}} - (J^p{}_i)^3 = 0$$
$$= 0 - J^p{}_i\,(J_{yy}\,J_{zz} - J_{yz}{}^2) + (J^p{}_i)^2\,(J_{yy} + J_{zz}) - (J^p{}_i)^3$$

with the solutions

$$J^p{}_{2,3} = ½\,(J_{yy} + J_{zz}) \pm \sqrt{\tfrac{1}{4}(J_{yy} + J_{zz})^2 - J_{yy}\,J_{zz} + J_{yz}^2}$$

$$= ½\,(J_{yy} + J_{zz}) \pm \sqrt{\tfrac{1}{4}(J_{yy} - J_{zz})^2 + J_{yz}^2}$$

called the **principal moments of inertia**.

For the transformation of an arbitrary basis $\{\mathbf{e}_i\}$ into the eigenbasis $\{\mathbf{e}^p{}_i\}$ we use the transformations of the components

$$J^p{}_{ik} = Q_{il}\,J_{lm}\,Q_{km}$$

with

$$[Q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

and the angle of rotation α. The matrix has spectral form with respect to the principal axes of inertia

$$[J^p{}_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & J^p{}_2 & 0 \\ 0 & 0 & J^p{}_3 \end{bmatrix}.$$

Thus

$$J^p{}_{23} = 0 = \sin\alpha\cos\alpha\,(J_{22} - J_{33}) + (\cos^2\alpha - \sin^2\alpha)\,J_{23}.$$

With the trigonometric relations

$$2\sin\alpha\,\cos\alpha = \sin(2\alpha)$$
$$2\cos^2\alpha = 1 + \cos(2\alpha)$$
$$2\sin^2\alpha = 1 - \cos(2\alpha)$$

we conclude

$$\tfrac{1}{2}\, sin(2\alpha)\,(J_{22} - J_{33}) + cos(2\alpha)\; J_{23} \;=\; 0$$

or

$$tan(2\alpha) \;=\; \frac{2J_{23}}{J_{33} - J_{22}}$$

as an equation to determine α and the eigenbasis

$$\mathbf{e}^{p}{}_{1} \;=\; \mathbf{e}_{1}$$

$$\mathbf{e}^{p}{}_{2} \;=\; cos\,\alpha\,\mathbf{e}_{2} + sin\,\alpha\,\mathbf{e}_{3}$$

$$\mathbf{e}^{p}{}_{3} \;=\; -\,sin\,\alpha\,\mathbf{e}_{2} + cos\,\alpha\,\mathbf{e}_{3}$$

2.1.15 Higher-Order Tensors

The scalar product of two vectors (which we will consider from now on as 1st-order tensors) gives the real number

$$\mathbf{v} \cdot \mathbf{x}\,.$$

Up to now, we have only introduced the twofold or dyadic tensor product between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ by its action on some vector $\mathbf{x}$

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \cdot \mathbf{x} \;:=\; \mathbf{v}_1\,(\mathbf{v}_2 \cdot \mathbf{x}) \;=\; (\mathbf{v}_2 \cdot \mathbf{x})\,\mathbf{v}_1$$

(simple contraction). Analogously one defines the three-fold tensor product between three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ by its action on some vector $\mathbf{x}$, the result of which is the 2nd-order tensor

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \cdot \mathbf{x} \;:=\; \mathbf{v}_1 \otimes \mathbf{v}_2\,(\mathbf{v}_3 \cdot \mathbf{x})$$

(simple contraction). The three-fold tensor product is called a **triad** or **3rd-order tensor**.

One can continue this way up to the introduction of a ***K*-fold tensor product** of K vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ... , $\mathbf{v}_K$ by its action on some vector $\mathbf{x}$, the result of which is the $K{-}1$-fold tensor product

$$\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_K \cdot \mathbf{x} \;:=\; \mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_{K-1}\,(\mathbf{v}_K \cdot \mathbf{x})$$

(simple contraction).

We had already introduced the simple contraction of two dyads as a composition of two linear mappings

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \cdot \mathbf{x}_1 \otimes \mathbf{x}_2 \;=\; (\mathbf{v}_2 \cdot \mathbf{x}_1)\,\mathbf{v}_1 \otimes \mathbf{x}_2\,.$$

A *multiple contraction* can be achieved if one contracts more couples of adjacent vectors by scalar products. The order in which these contractions are performed matters, and it can be defined in different ways. An example is the double contraction of two dyads, the result of which is a scalar

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \cdot\cdot\, \mathbf{x}_1 \otimes \mathbf{x}_2 \;:=\; (\mathbf{v}_1 \cdot \mathbf{x}_1)\,(\mathbf{v}_2 \cdot \mathbf{x}_2)\,.$$

This rule can be generalised in the following way.

Definition. The P**-fold contraction** of a K-fold tensor product with an M-fold tensor product for $K \geq P \leq M$ is the (K+M–$2P$)-fold tensor product

(2.1.126)
$$\begin{aligned}&(\mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_K) \cdot\ldots\cdot (\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_M)\\ &= \mathbf{v}_1 \otimes \ldots \otimes \mathbf{v}_{K-P} \otimes \mathbf{x}_{P+1} \otimes \ldots \otimes \mathbf{x}_M\\ &\quad (\mathbf{v}_{K-P+1} \cdot \mathbf{x}_1)\,(\mathbf{v}_{K-P+2} \cdot \mathbf{x}_2) \ldots (\mathbf{v}_K \cdot \mathbf{x}_P)\,.\end{aligned}$$

wherein " ·...·" stands for P scalar products.

Examples

$K\equiv 1,\ M\equiv 1,\ P\equiv 1$	$\mathbf{v} \cdot \mathbf{x}$	a scalar
$K\equiv 2,\ M\equiv 1,\ P\equiv 1$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \cdot \mathbf{x} = \mathbf{v}_1 (\mathbf{v}_2 \cdot \mathbf{x})$	a vector
$K\equiv 3,\ M\equiv 1,\ P\equiv 1$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \cdot \mathbf{x} = \mathbf{v}_1 \otimes \mathbf{v}_2\, (\mathbf{v}_3 \cdot \mathbf{x})$	a dyad
$K\equiv 1,\ M\equiv 2,\ P\equiv 1$	$\mathbf{v} \cdot \mathbf{x}_1 \otimes \mathbf{x}_2 = (\mathbf{v} \cdot \mathbf{x}_1)\, \mathbf{x}_2$	a vector
$K\equiv 2,\ M\equiv 2,\ P\equiv 1$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \cdot \mathbf{x}_1 \otimes \mathbf{x}_2 = (\mathbf{v}_2 \cdot \mathbf{x}_1)\, \mathbf{v}_1 \otimes \mathbf{x}_2$	a dyad
$K\equiv 2,\ M\equiv 2,\ P\equiv 2$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \cdot\cdot\, \mathbf{x}_1 \otimes \mathbf{x}_2 = (\mathbf{v}_1 \cdot \mathbf{x}_1)\,(\mathbf{v}_2 \cdot \mathbf{x}_2)$	a scalar
$K\equiv 3,\ M\equiv 1,\ P\equiv 1$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \cdot \mathbf{x}_1 = \mathbf{v}_1 \otimes \mathbf{v}_2\, (\mathbf{v}_3 \cdot \mathbf{x}_1)$	a dyad
$K\equiv 3,\ M\equiv 2,\ P\equiv 1$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \cdot \mathbf{x}_1 \otimes \mathbf{x}_2 = \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{x}_2\, (\mathbf{v}_3 \cdot \mathbf{x}_1)$	a triad
$K\equiv 3,\ M\equiv 2,\ P\equiv 2$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \cdot\cdot\, \mathbf{x}_1 \otimes \mathbf{x}_2 = \mathbf{v}_1 (\mathbf{v}_2 \cdot \mathbf{x}_1)\,(\mathbf{v}_3 \cdot \mathbf{x}_2)$	a vector
$K\equiv 2,\ M\equiv 3,\ P\equiv 1$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \cdot \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3 = \mathbf{v}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3\, (\mathbf{v}_2 \cdot \mathbf{x}_1)$	a triad
$K\equiv 1,\ M\equiv 3,\ P\equiv 1$	$\mathbf{v}_1 \cdot \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3 = (\mathbf{v}_1 \cdot \mathbf{x}_1)\, \mathbf{x}_2 \otimes \mathbf{x}_3$	a dyad
$K\equiv 3,\ M\equiv 3,\ P\equiv 3$	$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \cdot\cdot\cdot\, \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3 = (\mathbf{v}_1 \cdot \mathbf{x}_1)\,(\mathbf{v}_2 \cdot \mathbf{x}_2)\,(\mathbf{v}_3 \cdot \mathbf{x}_3)$	a scalar

etc.

Linear combinations of tensor products of equal order are achieved in analogy to those of dyads.

Tensors of *K*-th-order with $K \geq 0$ can be generated by K-fold tensor products between base vectors of some ONB

$$\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \ldots \otimes \mathbf{e}_{i_K}$$

as linear combinations

(2.1.127)
$$\mathbf{C} \;=\; C_{i_1 i_2 \ldots i_K}\; \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \ldots \otimes \mathbf{e}_{i_K}$$

with 3^K components in three-dimensions. For $K \equiv 1$ this is a vector, for $K \equiv 0$ it is defined as a scalar, i.e., a real.

Notations

tensors of 0th-order:	**scalars** with $3^0 = 1$ component
tensors of 1st-order:	**vectors** with $3^1 = 3$ components
tensor of 2nd-order:	**dyads** with $3^2 = 9$ components
tensors of 3rd-order:	**triads** with $3^3 = 27$ components
tensors of 4th-order:	**tetrads** with $3^4 = 81$ components.

If $\{\underline{\mathbf{e}}_i\}$ is another ONB, then we can represent a K-th-order tensor $\mathbf{C}$ as

$$\mathbf{C} = \underline{C}_{ij\ldots kl}\ \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \ldots \otimes \underline{\mathbf{e}}_k \otimes \underline{\mathbf{e}}_l \,.$$

The transformations of the components under change of ONBs results from the orthogonal mappings

$$\mathbf{Q} := \underline{\mathbf{e}}_i \otimes \mathbf{e}_i = Q_{rs}\, \mathbf{e}_r \otimes \mathbf{e}_s$$

$$\Rightarrow \qquad \underline{\mathbf{e}}_i = \mathbf{Q} \cdot \mathbf{e}_i = (Q_{rs}\, \mathbf{e}_r \otimes \mathbf{e}_s) \cdot \mathbf{e}_i = Q_{ri}\, \mathbf{e}_r$$

generalising those of vectors (2.1.79) and of dyads (2.1.80) as

(2.1.128) $$C_{ij\ldots kl} = Q_{im}\, Q_{jn} \ldots Q_{kr}\, Q_{ls}\ \underline{C}_{mn\ldots rs} \,.$$

A tensor of K-th-order can be applied to some vector by a simple contraction as

$$\begin{aligned}
\mathbf{C} \cdot \mathbf{x} &= (C_{ij\ldots kl}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_k \otimes \mathbf{e}_l) \cdot (x_m\, \mathbf{e}_m) \\
&= C_{ij\ldots kl}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_k\, (\mathbf{e}_l \cdot \mathbf{e}_m)\, x_m \\
&= C_{ij\ldots kl}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_k\, \delta_{lm}\, x_m \\
&= C_{ij\ldots kl}\, x_l\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_k
\end{aligned}$$

giving a (K–1)-th-order tensor. For $K \equiv 2$ this operation coincides with the linear mapping between vectors.

Similarly, one can define a simple contraction between a K-th-order tensor and a dyad as a generalisation of the composition of two 2nd-order tensors

$$\begin{aligned}
\mathbf{C} \cdot \mathbf{T} &= (C_{i\ldots jkl}\, \mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) \cdot (T_{mp}\, \mathbf{e}_m \otimes \mathbf{e}_p) \\
&= C_{i\ldots jkl}\, \mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_p\, (\mathbf{e}_l \cdot \mathbf{e}_m)\, T_{mp} \\
&= C_{i\ldots jkl}\, \mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_p\ \ \delta_{lm}\, T_{mp} \\
&= C_{i\ldots jkl}\, T_{lp}\, \mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_p
\end{aligned}$$

and obtains a K-th-order tensor. Thus, the two adjacent base vectors are contracted in a scalar product.

One can also introduce a double contraction between a K-th-order tensors and a dyad as

$$\begin{aligned}
\mathbf{C} \cdot\cdot\, \mathbf{T} &= (C_{i\ldots jkl}\, \mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) \cdot\cdot\, (T_{mp}\, \mathbf{e}_m \otimes \mathbf{e}_p) \\
&= C_{i\ldots jkl}\, \mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_j\, (\mathbf{e}_k \cdot \mathbf{e}_m)\, (\mathbf{e}_l \cdot \mathbf{e}_p)\, T_{mp}
\end{aligned}$$

$$= C_{i...jkl}\, \mathbf{e}_i \otimes ... \otimes \mathbf{e}_j\ \delta_{km}\ \delta_{lp}\ T_{mp}$$

$$= C_{i...jkl}\, T_{kl}\, \mathbf{e}_i \otimes ... \otimes \mathbf{e}_j$$

giving a $(K\text{–}2)$th-order tensor.

In analogy to (2.1.126) one can continue like this until the ***P*-fold contraction** of a ***K*th-order tensor** with an ***M*th-order tensor** (K, $M \geq P$), resulting in a **(*K*+*M*–2*P*)th-order tensor**.

Theorem. *A linear mapping of an M-th-order tensor into an L-th-order tensor can be uniquely represented by an (L+M)-th-order tensor through an M-fold contraction.*

This motivates the use of higher-order tensors.

An interesting particular case is with $L \equiv 0$ and M arbitrary, an M-fold contraction of two M-th-order tensors. The result is a 0-th-order tensor, i.e., a scalar

$$\mathbf{C} \cdot ... \cdot \mathbf{D} = (C_{ij...k}\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_k)$$

$$\cdot ... \cdot (D_{mp...q}\ \mathbf{e}_m \otimes \mathbf{e}_p \otimes ... \otimes \mathbf{e}_q)$$

(2.1.129) $$= C_{ij...k}\ D_{mp...q}\ (\mathbf{e}_i \cdot \mathbf{e}_m)\ (\mathbf{e}_j \cdot \mathbf{e}_p) ... (\mathbf{e}_k \cdot \mathbf{e}_q)$$

$$= C_{ij...k}\ D_{mp...q}\ \delta_{im}\ \delta_{jp} ... \delta_{kq}$$

$$= C_{ij...k}\ D_{ij...k}\,.$$

One can interpret this operation as a **scalar product** in the space of L-th-order tensors, since the according axioms of a scalar product are fulfilled. This scalar product induces a **norm**

(2.1.130) $$|\mathbf{C}| := \surd(\mathbf{C} \cdot ... \cdot \mathbf{C})$$

as well as lengths of and angles between tensors of arbitrary order in analogy to those of vectors.

For $L \equiv 2$ this gives

$$\mathbf{S} \cdot\cdot\, \mathbf{T} = (S_{il}\, \mathbf{e}_i \otimes \mathbf{e}_l) \cdot\cdot\, (T_{mp}\, \mathbf{e}_m \otimes \mathbf{e}_p)$$

$$= S_{il}\, (\mathbf{e}_i \cdot \mathbf{e}_m)\, (\mathbf{e}_l \cdot \mathbf{e}_p)\, T_{mp}$$

$$= S_{il}\, \delta_{im}\, \delta_{lp}\, T_{mp}$$

$$= S_{il}\, T_{il}$$

which can be expressed by the trace as

(2.1.131) $$\mathbf{S} \cdot\cdot\, \mathbf{T} = tr(\mathbf{S} \cdot \mathbf{T}^T) = tr(\mathbf{S}^T \cdot \mathbf{T}) = tr(\mathbf{T}^T \cdot \mathbf{S}) = tr(\mathbf{T} \cdot \mathbf{S}^T)\,.$$

In the literature, this scalar product between tensors is sometimes called a *double* scalar product (because of the *double* contraction). It commutes and is linear in both factors. In contrast to this, one can show that $tr(\mathbf{S} \cdot \mathbf{T})$ does *not* define a scalar product, since it is not positive definite for $\mathbf{S} \equiv \mathbf{T}$.

In particular, we obtain for $\mathbf{T} \equiv \mathbf{I}$ a representation for the trace of a tensor $\mathbf{S}$

(2.1.132) $\mathbf{S} \cdot\cdot\, \mathbf{I} \;=\; tr(\mathbf{S} \cdot \mathbf{I}) \;=\; tr(\mathbf{S})\,.$

For simple dyads we get

$$(\mathbf{a} \otimes \mathbf{b}) \cdot\cdot\, (\mathbf{c} \otimes \mathbf{d}) \;=\; (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d})\,.$$

With respect to this scalar product, the symmetric and the skew tensors are mutually orthogonal. In fact, if $\mathbf{T}$ is an arbitrary dyad, $\mathbf{S}$ symmetric and $\mathbf{A}$ skew, then

(2.1.133) $\mathbf{S} \cdot\cdot\, \mathbf{A} \;=\; 0$

and therefore

$$\mathbf{S} \cdot\cdot\, \mathbf{T} \;=\; \mathbf{S} \cdot\cdot\, \tfrac{1}{2}\,(\mathbf{T} + \mathbf{T}^T) \;=\; \mathbf{S} \cdot\cdot\, sym(\mathbf{T})$$

$$\mathbf{A} \cdot\cdot\, \mathbf{T} \;=\; \mathbf{A} \cdot\cdot\, \tfrac{1}{2}\,(\mathbf{T} - \mathbf{T}^T) \;=\; \mathbf{A} \cdot\cdot\, skw(\mathbf{T})\,.$$

Using this scalar product, we are able to introduce a tensorial ONB.

Definition. A tensorbasis $\{\mathbf{f}_i \otimes \mathbf{g}_j\}$ is called an **orthonormal basis** (**ONB**) if

$$\mathbf{f}_i \otimes \mathbf{g}_j \cdot\cdot\, \mathbf{f}_k \otimes \mathbf{g}_l \;=\; \delta_{ik}\, \delta_{jl} \qquad i, j, k, l = 1, 2, 3.$$

The following examples are given.

Let $\{\mathbf{e}_i\}$ be a vectorial ONB. Then

- $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ is an ONB in the 9-dimensional space of 2nd-order tensors,
- $\{\, \mathbf{e}_i \otimes \mathbf{e}_i$ and $\sqrt{2}\;\; sym(\mathbf{e}_i \otimes \mathbf{e}_j)\,,\, i < j,\;\; i,\; j = 1, 2, 3\}$ is an ONB in the 6-dimensional space of symmetric tensors,
- $\{\, \sqrt{2}\;\; skw(\mathbf{e}_i \otimes \mathbf{e}_j)\,,\, i < j\}$ is an ONB in the 3-dimensional space of skew tensors.

Only tensors of even-order ($2K$) can be interpreted as linear mappings between tensors of the same order (K). Their invertibility can be defined in complete analogy to 2nd-order tensors. Among the $2K$-th-order tensors is a distinguished element, namely the ***2K*-th-order identity**

$$\mathbf{I} \;=\; \delta_{i_1\, i_{K+1}}\; \delta_{i_2\, i_{K+2}} \ldots\; d_{i_K\, i_{2K}}\;\; \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \ldots \otimes \mathbf{e}_{i_{2K}}$$

$$=\; \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \ldots \otimes \mathbf{e}_{i_K} \otimes \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_K}\,.$$

Generalising the symmetry definition of dyads to $2K$-th-order (even) tensors by K-fold contractions from both sides with arbitrary K-th-order tensors $\mathbf{S}$ and $\mathbf{T}$ gives

$$\mathbf{S} \cdot \ldots \cdot \mathbf{C} \cdot \ldots \cdot \mathbf{T} \;=\; \mathbf{T} \cdot \ldots \cdot \mathbf{C} \cdot \ldots \cdot \mathbf{S}\,.$$

This is equivalent to the conditions for the components with respect to some ONB

$$C_{i_1\; i_2\; \cdots\; i_K\;\; i_{K+1}\; \cdots\; i_{2K}} \;=\; C_{i_{K+1}\; i_{K+2}\;\; i_{2K}\;\; i_1\; \cdots\; i_K}$$

for all indices. *Symmetry* does therefore not include the invariance under arbitrary interchanges of indices, but only under interchanges of the first index group and the second.

For even order (*2K*) tensors one can define **eigenvalue problems** in analogy to Chapter 2.1.10. The eigendirections are directions in the space of *K*-th-order tensors. Instead of eigenvectors, we are now looking for *K*-th-order **eigentensors**. The resulting characteristic equation again contains principal invariants as its coefficients. Many properties of the eigenvalue problem of dyads can also be applied analogously to such even-order tensors. In particular, if such tensor is symmetric, a spectral form can always be achieved with respect to its eigenbasis.

For odd-order tensors, however, an eigenvalue problem in this form can not be defined.

2.1.16 Tetrads

Since we deal in mechanics mainly with dyads or 2nd-order tensors, we will occasionally need 4th-order tensors or tetrads being used as linear mappings between dyads

$$\begin{aligned} \boldsymbol{C} \cdot\cdot \mathbf{T} &= (C_{ijkl}\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) \cdot\cdot (T_{mp}\, \mathbf{e}_m \otimes \mathbf{e}_p) \\ &= C_{ijkl}\ \mathbf{e}_i \otimes \mathbf{e}_j\, (\mathbf{e}_k \cdot \mathbf{e}_m)\, (\mathbf{e}_l \cdot \mathbf{e}_p)\, T_{mp} \\ &= C_{ijkl}\ \mathbf{e}_i \otimes \mathbf{e}_j\, \delta_{km}\, \delta_{lp}\, T_{mp} \\ &= C_{ijkl}\, T_{kl}\, \mathbf{e}_i \otimes \mathbf{e}_j\,. \end{aligned} \tag{2.1.134}$$

In the linear theory of elasticity, tetrads are frequently used for linear dependences of the stress tensor and the deformation tensor, as we will see later. The composition of such tensors $\boldsymbol{C}$ and $\boldsymbol{D}$ maps a 2nd-order tensor $\mathbf{T}$ into

$$(\boldsymbol{C} \cdot\cdot \boldsymbol{D}) \cdot\cdot \mathbf{T} \ := \ \boldsymbol{C} \cdot\cdot (\boldsymbol{D} \cdot\cdot \mathbf{T}) \ = \ \boldsymbol{C} \cdot\cdot \boldsymbol{D} \cdot\cdot \mathbf{T}\,.$$

So the brackets are not needed.

The 4th-order **zero tensor** $\boldsymbol{O}$ maps all dyads $\mathbf{T}$ into the second-order zero

$$\boldsymbol{O} \cdot\cdot \mathbf{T} \ = \ \mathbf{0}\,.$$

All of its components are zero with respect to all bases.

The 4th-order **identity tensor** $\boldsymbol{I}$ maps any dyad into itself

$$\boldsymbol{I} \cdot\cdot \mathbf{T} \ = \ \mathbf{T}\,.$$

One obtains its component representation with respect to some ONB as

$$\boldsymbol{I} \ = \ \delta_{ik}\, \delta_{jl}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \ = \ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j\,. \tag{2.1.135}$$

The *eigenvalue problem* of a tetrad consists of finding some 2nd-order **eigentensors** $\mathbf{A}$ for eigenvalues λ that fulfil the eigenvalue equation

(2.1.136) $$\boldsymbol{C} \cdot\cdot\, \mathbf{A} = \lambda\, \mathbf{A} .$$

Here we expect 9 (not necessarily different) eigenvalues and 9 corresponding eigentensors, being either real or complex.

The inverse $\boldsymbol{C}^{-1}$ of an invertible tetrad $\boldsymbol{C}$ gives

$$\boldsymbol{C}^{-1} \cdot\cdot\, \boldsymbol{C} \cdot\cdot\, \mathbf{T} = \mathbf{T}$$

for all dyads $\mathbf{T}$. This is equivalent to

(2.1.137) $$\boldsymbol{C}^{-1} \cdot\cdot\, \boldsymbol{C} = \boldsymbol{I} = \boldsymbol{C} \cdot\cdot\, \boldsymbol{C}^{-1}.$$

In general, a tetrad has $3^4 = 81$ independent components C_{ijkl} , $i, j, k, l = 1, 2, 3$. To represent them in a matrix, it is necessary to enlarge the concept of a matrix to a *hypermatrix*, i.e., a matrix whose components are also matrices. This can be achieved in the following form

$$\begin{bmatrix} \begin{bmatrix} C_{1111} & C_{1112} & C_{1113} \\ C_{1121} & C_{1122} & C_{1123} \\ C_{1131} & C_{1132} & C_{1133} \end{bmatrix} & \begin{bmatrix} C_{1211} & C_{1212} & C_{1213} \\ C_{1221} & C_{1222} & C_{1223} \\ C_{1231} & C_{1232} & C_{1233} \end{bmatrix} & \begin{bmatrix} C_{1311} & C_{1312} & C_{1313} \\ C_{1321} & C_{1322} & C_{1323} \\ C_{1331} & C_{1332} & C_{1333} \end{bmatrix} \\ \begin{bmatrix} C_{2111} & C_{2112} & C_{2113} \\ C_{2121} & C_{2122} & C_{2123} \\ C_{2131} & C_{2132} & C_{2133} \end{bmatrix} & \begin{bmatrix} C_{2211} & C_{2212} & C_{2213} \\ C_{2221} & C_{2222} & C_{2223} \\ C_{2231} & C_{2232} & C_{2233} \end{bmatrix} & \begin{bmatrix} C_{2311} & C_{2312} & C_{2313} \\ C_{2321} & C_{2322} & C_{2323} \\ C_{2331} & C_{2332} & C_{2333} \end{bmatrix} \\ \begin{bmatrix} C_{3111} & C_{3112} & C_{3113} \\ C_{3121} & C_{3122} & C_{3123} \\ C_{3131} & C_{3132} & C_{3133} \end{bmatrix} & \begin{bmatrix} C_{3211} & C_{3212} & C_{3213} \\ C_{3221} & C_{3222} & C_{3223} \\ C_{3231} & C_{3232} & C_{3233} \end{bmatrix} & \begin{bmatrix} C_{3311} & C_{3312} & C_{3313} \\ C_{3321} & C_{3322} & C_{3323} \\ C_{3331} & C_{3332} & C_{3333} \end{bmatrix} \end{bmatrix}$$

The number of 81 independent components can eventually be drastically reduced by symmetries. While the components of a dyad have only two indices, with respect to which a symmetry may exist, we have four indices in the case of tetrads and, therefore, have to distinguish different symmetries. This will be done next.

The **transposition** of a tetrad is introduced in analogy to that of a dyad (2.1.41) by the relation

(2.1.138) $$\mathbf{S} \cdot\cdot\, \boldsymbol{C}^{T} \cdot\cdot\, \mathbf{T} = \mathbf{T} \cdot\cdot\, \boldsymbol{C} \cdot\cdot\, \mathbf{S}$$

which shall be valid for all dyads $\mathbf{S}$ and $\mathbf{T}$. A tetrad possesses the (main) **symmetry** if it coincides with its transpose. For the components with respect to some ONB this is equivalent to

$$C_{ijkl} = C_{klij} .$$

These are 36 independent conditions. Consequently, a symmetric tetrad has $n^2/2 + n/2 = 45$ independent components with $n \equiv 9$ in our case.

Independently from the (main) symmetry, we define two more types of symmetries for tetrads, namely the **left subsymmetry** if

(2.1.139) $$\mathbf{T} \cdot\cdot \boldsymbol{C} = \mathbf{T}^T \cdot\cdot \boldsymbol{C}$$

holds for all tensors $\mathbf{T}$, and the **right subsymmetry** if

(2.1.140) $$\boldsymbol{C} \cdot\cdot \mathbf{T} = \boldsymbol{C} \cdot\cdot \mathbf{T}^T$$

holds. For the components with respect to some ONB this is equivalent to the conditions

$$C_{ijkl} = C_{jikl} \qquad \text{(left subsymmetry)}$$

$$C_{ijkl} = C_{ijlk} \qquad \text{(right subsymmetry).}$$

These two subsymmetries are particularly helpful if a tetrad $\boldsymbol{C}$ is used to map symmetric dyads into symmetric ones. For such applications the following theorems hold.

- A tetrad $\boldsymbol{C}$ has the left subsymmetry if and only if $\boldsymbol{C} \cdot\cdot \mathbf{T}$ is symmetric for arbitrary dyads $\mathbf{T}$.
- A tetrad $\boldsymbol{C}$ has the right subsymmetry if and only if $\boldsymbol{C} \cdot\cdot \mathbf{T}^A = \mathbf{0}$ holds for all skew dyads $\mathbf{T}^A$. Therefore, all skew tensors are eigentensors of $\boldsymbol{C}$ with a triple eigenvalue 0.
- If a tetrad possesses a subsymmetry, it is singular (non-invertible).

In the linear theory of elasticity, one considers linear functions between the (symmetric) deformation tensor and the (symmetric) stress tensor. For such linear functions between symmetric dyads we can assume both subsymmetries without influencing the relevant part of the function. In doing so, we can reduce the number of independent components of the tetrad to $6 \times 6 = 36$. In this case it is convenient to represent symmetric dyads as members of a 6-dimensional linear space

(2.1.141)
$$\begin{aligned}
\mathbf{T} &= T_{11}\ \mathbf{e}_1 \otimes \mathbf{e}_1 + T_{22}\ \mathbf{e}_2 \otimes \mathbf{e}_2 + T_{33}\ \mathbf{e}_3 \otimes \mathbf{e}_3 \\
&\quad + T_{23}\,(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) \\
&\quad + T_{31}\,(\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3) \\
&\quad + T_{12}\,(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \\
&= T_{11}\,\mathbf{E}_{V1} + T_{22}\,\mathbf{E}_{V2} + T_{33}\,\mathbf{E}_{V3} \\
&\quad + \sqrt{2}\,T_{23}\,\mathbf{E}_{V4} + \sqrt{2}\,T_{31}\,\mathbf{E}_{V5} + \sqrt{2}\,T_{12}\,\mathbf{E}_{V6} \\
&= T_{V\alpha}\,\mathbf{E}_{V\alpha} \qquad \text{(sum over } \alpha \text{ from } 1 \text{ to } 6\text{)}
\end{aligned}$$

with the matrix of components

$$\{T_{V1}, T_{V2}, T_{V3}, T_{V4}, T_{V5}, T_{V6}\} := \{T_{11}, T_{22}, T_{33}, \sqrt{2}\,T_{23}, \sqrt{2}\,T_{31}, \sqrt{2}\,T_{12}\}$$

with respect to the symmetric tensor basis

$$\mathbf{E}_{V1} := \mathbf{e}_1 \otimes \mathbf{e}_1$$

$$\mathbf{E}_{V2} := \mathbf{e}_2 \otimes \mathbf{e}_2$$

$$\mathbf{E}_{V3} := \mathbf{e}_3 \otimes \mathbf{e}_3$$

$$\mathbf{E}_{V4} := 1/\sqrt{2}\,(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2)$$

$$\mathbf{E}_{V5} := 1/\sqrt{2}\,(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1)$$

$$\mathbf{E}_{V6} := 1/\sqrt{2}\,(\mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2)$$

where the square roots $\sqrt{2}$ are normalisation factors of the basis $\{\mathbf{E}_{V\alpha}\}$

$$\mathbf{E}_{V\alpha} \cdot\cdot\, \mathbf{E}_{V\beta} = \delta_{\alpha\beta}$$

such as

$$\mathbf{E}_{V6} \cdot\cdot\, \mathbf{E}_{V6} = 1/\sqrt{2}\,(\mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2) \cdot\cdot\; 1/\sqrt{2}\,(\mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2) = 1\,.$$

This turns $\{\mathbf{E}_{V\alpha}\}$ into an ONB in the 6-dimensional space of symmetric dyads. It leads to a **VOIGT**[32] **representation** (1882) of a tetrad with the two subsymmetries

(2.1.142) $$\boldsymbol{C} = C_{V\alpha\beta}\, \mathbf{E}_{V\alpha} \otimes \mathbf{E}_{V\beta}$$

with summation over Greek indices from *1* to *6* . The components of the tetrad can now be given as a 6 × 6 matrix

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1113} & \sqrt{2}C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2213} & \sqrt{2}C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & \sqrt{2}C_{3323} & \sqrt{2}C_{3313} & \sqrt{2}C_{3312} \\ \sqrt{2}C_{2311} & \sqrt{2}C_{2322} & \sqrt{2}C_{2333} & 2C_{2323} & 2C_{2313} & 2C_{2312} \\ \sqrt{2}C_{1311} & \sqrt{2}C_{1322} & \sqrt{2}C_{1333} & 2C_{1323} & 2C_{1313} & 2C_{1312} \\ \sqrt{2}C_{1211} & \sqrt{2}C_{1222} & \sqrt{2}C_{1233} & 2C_{1223} & 2C_{1213} & 2C_{1212} \end{bmatrix}$$

We shall mention that in the literature the normalisation is sometimes not applied which leads to a slightly different representation (without the *2* and the $\sqrt{2}$).

Obviously, the tetrad with the two subsymmetries possesses also the (main) symmetry if and only if the VOIGT 6×6 matrix is symmetric. In this case only 21 independent components remain.

The 4th-order identity tensor possesses the (main) symmetry, but no subsymmetry since this would lead to a loss of invertibility. The identity tensor is invertible of course.

The identity tetrad must be distinguished from that particular tetrad $\boldsymbol{I}^S$ which possesses both subsymmetries and maps every dyad into its symmetric part. We call it a **symmetriser**

$$\boldsymbol{I}^S = \tfrac{1}{4}\, \delta_{ik}\, \delta_{jl}\, (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i) \otimes (\mathbf{e}_k \otimes \mathbf{e}_l + \mathbf{e}_l \otimes \mathbf{e}_k)$$

[32] Woldemar Voigt (1850-1919)

$$= \tfrac{1}{4}\,(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i)$$

(2.1.143) $$= \tfrac{1}{2}\,(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i) \otimes (\mathbf{e}_i \otimes \mathbf{e}_j)$$

$$= \tfrac{1}{2}\,(\mathbf{e}_i \otimes \mathbf{e}_j) \otimes (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i)$$

with the VOIGT representation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

This tensor is used as the identity on the symmetric dyads.

In analogy to 2nd-order tensors we define the following properties of tetrads.

Definition. A (not necessarily symmetric) tetrad $\mathcal{C}$ with the property

$\mathbf{A} \cdot\cdot\, \mathcal{C} \cdot\cdot\, \mathbf{A} > 0 \quad \forall\, \mathbf{A} \neq \mathbf{0}$ is called **positive definite**

$\mathbf{A} \cdot\cdot\, \mathcal{C} \cdot\cdot\, \mathbf{A} \geq 0 \quad \forall\, \mathbf{A}$ is called **positive semidefinite**

$\mathbf{A} \cdot\cdot\, \mathcal{C} \cdot\cdot\, \mathbf{A} < 0 \quad \forall\, \mathbf{A} \neq \mathbf{0}$ is called **negative definite**

$\mathbf{A} \cdot\cdot\, \mathcal{C} \cdot\cdot\, \mathbf{A} \leq 0 \quad \forall\, \mathbf{A}$ is called **negative semidefinite**.

If nothing of the above holds, the tetrad is called **indefinite**. Evidently, only the symmetric part of the tetrad enters into these definitions. For a symmetric $\mathcal{C}$ the positive definiteness is equivalent to the positivity of all eigenvalues.

Such classifications can be made for all tensors of even-order.

Problem 7. Multiple Contraction

Between tensors of higher-order, a multiple contraction was introduced. Simplify the following expression for two vectors $\mathbf{a}$ and $\mathbf{b}$

(P7.1) $$\mathbf{v} = \boldsymbol{\varepsilon} \cdot\cdot\, \mathbf{a} \otimes \mathbf{b}$$

as far as possible, with $\boldsymbol{\varepsilon} = \varepsilon_{ijk}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$.

Solution

$\boldsymbol{\varepsilon}$ is a triad, while $\mathbf{a} \otimes \mathbf{b}$ is a simple dyad. The double contraction reduces both tensors by the order two, so that the result is a 1st-order tensor or a vector $\mathbf{v}$. By using indices, we obtain

$$\mathbf{v} = \varepsilon_{ijk}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \cdot\cdot\, a_m\, b_n\, \mathbf{e}_m \otimes \mathbf{e}_n$$

$$= \varepsilon_{ijk}\, a_m\, b_n\, \delta_{jm}\delta_{kn}\, \mathbf{e}_i \;=\; \varepsilon_{ijk}\, a_j\, b_k\, \mathbf{e}_i \;=\; \varepsilon_{jki}\, a_j\, b_k\, \mathbf{e}_i$$

$$= \mathbf{a} \times \mathbf{b}\ .$$

$\mathbf{v}$ is perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. The tensor $\mathbf{a} \otimes \mathbf{b}$ is a simple dyad with $II_{\mathbf{a} \otimes \mathbf{b}} = III_{\mathbf{a} \otimes \mathbf{b}} = 0$.

2.2 Vector and Tensor Analysis

2.2.1 The Directional Differential

Let f be a real-valued differentiable function of a real variable. Its linear approximation at x is the differential

$$df(x, dx) = \frac{df}{dx} dx = f(x)' \, dx$$

with the derivative

$$f(x)' = \frac{df}{dx} := \lim_{\Delta x \to 0} \frac{1}{\Delta x} [f(x + \Delta x) - f(x)] .$$

This concept shall now be generalised to tensor functions.

Let $\mathbf{r}$ be some position vector in the EUCLIDean space and $\phi(\mathbf{r})$ a real field (scalar field), i.e., a function that assigns to each position vector a scalar. Examples: the temperature field, or the field of mass density or energy density. The **differential of ϕ at $\mathbf{r}$ in the direction of $\mathbf{dr}$** is defined as the limit

(2.2.1)
$$d\phi(\mathbf{r}, \mathbf{dr}) := \lim_{h \to 0} \frac{1}{h} [\phi(\mathbf{r} + h\, \mathbf{dr}) - \phi(\mathbf{r})]$$
$$= \frac{d}{dh} \phi(\mathbf{r} + h\, \mathbf{dr}) \Big|_{h=0}$$

If the function $\phi(\mathbf{r})$ is sufficiently smooth, then the differential is linear in $\mathbf{dr}$, and therefore there exists a vector field $grad\, \phi(\mathbf{r})$, called the **gradient** of $\phi(\mathbf{r})$, such that

(2.2.2)
$$d\phi(\mathbf{r}, \mathbf{dr}) = grad\, \phi(\mathbf{r}) \cdot \mathbf{dr} .$$

Other notations for the gradient are

$$grad\, \phi(\mathbf{r}) = \frac{d\phi}{d\mathbf{r}} = \phi(\mathbf{r})'.$$

With respect to a fixed ONB $\{\mathbf{e}_i\}$ we have the component representations

$$\mathbf{r} = x_j \, \mathbf{e}_j \qquad \text{and} \qquad \mathbf{dr} = dx_i \, \mathbf{e}_i$$

and for the differential of ϕ at $\mathbf{r}$ in the direction $\mathbf{dr} \equiv \mathbf{e}_i$

(2.2.3)
$$d\phi(\mathbf{r}, \mathbf{e}_i) = \lim_{h \to 0} \frac{1}{h} [\phi(\mathbf{r} + h\, \mathbf{e}_i) - \phi(\mathbf{r})] = grad\, \phi(\mathbf{r}) \cdot \mathbf{e}_i$$

which corresponds to the i-th component of the gradient. If, e.g., $i \equiv 1$, then $\mathbf{r} + h\, \mathbf{e}_1$ has the components $\{x_1 + h, x_2, x_3\}$, and the limit is the partial derivative of $\phi(x_i\,\mathbf{e}_i) = \phi(x_1, x_2, x_3)$ with respect to x_1, which is often denoted by "$,_1$"

$$\frac{\partial\phi(x_1,x_2,x_3)}{\partial x_1} := d\phi(\mathbf{r}, \mathbf{e}_1) = grad\ \phi(\mathbf{r}) \cdot \mathbf{e}_1 = \phi,_1 .$$

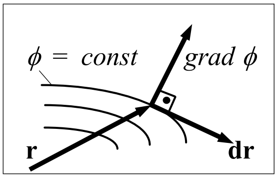
With this we obtain the component representation of the gradient with respect to some ONB

(2.2.4) $$grad\ \phi(\mathbf{r}) = \frac{\partial\phi(x_1,x_2,x_3)}{\partial x_i}\,\mathbf{e}_i = \phi,_i\,\mathbf{e}_i$$

We consider an iso-surface of ϕ, i.e., a surface in the EUCLIDean space on which ϕ is constant. If $\mathbf{dr}$ is tangential to this surface, then

$$d\phi(\mathbf{r}, \mathbf{dr}) = grad\ \phi(\mathbf{r}) \cdot \mathbf{dr} = 0 .$$

Consequently, $grad\ \phi$ is orthogonal or normal to the tangential plane of the iso-surface in $\mathbf{r}$ and points in the direction in which ϕ has the largest increase.



As an example we consider the **temperature field**

$$\phi(\mathbf{r}) \equiv \theta(\mathbf{r}) .$$

The **temperature gradient**

(2.2.5) $$\mathbf{g}(\mathbf{r}) := grad\ \theta(\mathbf{r}) = \frac{\partial\theta(x_1,x_2,x_3)}{\partial x_i}\,\mathbf{e}_i = \theta,_i\,\mathbf{e}_i$$

points in the direction of the largest temperature increase. If there is a linear relation between $\mathbf{g}$ and the **heat flux vector** $\mathbf{q}$ (*FOURIER's*[33] *law of heat conduction*), then it can be represented by the heat conduction tensor $\mathbf{K}$ as

(2.2.6) $$\mathbf{q} = -\mathbf{K} \cdot \mathbf{g} .$$

After the *CASIMIR-ONSAGER*[34] *reciprocal relations*, $\mathbf{K}$ is symmetric and therefore allows for a spectral representation

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[33] Jean Baptiste Joseph de Fourier (1768-1830)
[34] Hendrik Brugt Gerhard Casimir (1909-2000), Lars Onsager (1903-1976)

$$\mathbf{K} = \sum_{i=1}^{3} \kappa_i \; \mathbf{e}^{\mathrm{p}}{}_i \otimes \mathbf{e}^{\mathrm{p}}{}_i$$

with real eigenvalues κ_i. According to the experimental result that the heat flux is always directed from the hot to the cold, the heat conduction tensor is positive semidefinite (we will later see that this is a consequence of the second law of thermodynamics), and thus

$$\kappa_i \geq 0 \quad \text{for } i = 1, 2, 3 .$$

If the heat conduction ability is equal in all directions (isotropic heat conduction), then

$$\kappa_1 = \kappa_2 = \kappa_3 =: \kappa$$

and

$$\mathbf{K} = \sum_{i=1}^{3} \kappa \, \mathbf{e}_i \otimes \mathbf{e}_i = \kappa \mathbf{I}$$

is a spherical tensor. In this case, the *isotropic FOURIER's law of heat conduction* is reduced to

(2.2.7) $$\mathbf{q} = -\kappa \mathbf{g}$$

with the (non-negative) coefficient of heat conduction κ. Its components with respect to some ONB are

(2.2.8) $$q_i = -\kappa \frac{\partial\theta(x_1, x_2, x_3)}{\partial x_i} = -\kappa\, \theta(x_1, x_2, x_3)_{,i} .$$

Let us next consider a *vector field* $\mathbf{v}(\mathbf{r})$. Examples are the displacement field, the velocity field, the force field, and the heat flux field. We determine the **differential** of $\mathbf{v}$ at $\mathbf{r}$ in the direction of $\mathbf{dr}$ analogously to (2.2.1) as

(2.2.9) $$\mathbf{dv}(\mathbf{r}, \mathbf{dr}) := \lim_{h \to 0} \frac{1}{h} [\mathbf{v}(\mathbf{r} + h\, \mathbf{dr}) - \mathbf{v}(\mathbf{r})] .$$

This expression is again linear in $\mathbf{dr}$ if the field $\mathbf{v}(\mathbf{r})$ is sufficiently smooth. Thus, there exists a tensor field $grad\, \mathbf{v}(\mathbf{r})$, the **gradient** of $\mathbf{v}$, which gives

(2.2.10) $$\mathbf{dv}(\mathbf{r}, \mathbf{dr}) = \frac{\mathbf{dv}(\mathbf{r})}{\mathbf{dr}} \cdot \mathbf{dr} = grad\, \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr} .$$

After choosing some ONB $\{\mathbf{e}_i\}$, the differential in the direction of $\mathbf{e}_i$ is

$$\mathbf{dv}(\mathbf{r}, \mathbf{e}_i) := \lim_{h \to 0} \frac{1}{h} [\mathbf{v}(\mathbf{r} + h\, \mathbf{e}_i) - \mathbf{v}(\mathbf{r})]$$

$$= \frac{\partial \mathbf{v}(x_1, x_2, x_3)}{\partial x_i} = \mathbf{v}(x_1, x_2, x_3)_{,i}$$

$$= grad\, \mathbf{v}(\mathbf{r}) \cdot \mathbf{e}_i$$

containing the partial derivatives $\mathbf{v}(x_1, x_2, x_3)_{,i}$. If we represent the vector field by its components with respect to a *fixed* ONB $\{\mathbf{e}_i\}$, which are scalar fields,

$$\mathbf{v}(\mathbf{r}) = v_i(x_1, x_2, x_3)\, \mathbf{e}_i$$

then we can express the partial derivatives of the vector field by those of its scalar components as

$$\mathbf{v}(x_1, x_2, x_3)_{,i} = [v_j(x_1, x_2, x_3)\, \mathbf{e}_j]_{,i} = v_j(x_1, x_2, x_3)_{,i}\, \mathbf{e}_j$$

so that

$$grad\, \mathbf{v} \cdot \mathbf{e}_i = \mathbf{v}(x_1, x_2, x_3)_{,i} = v_j(x_1, x_2, x_3)_{,i}\, \mathbf{e}_j$$

$$= [v_j(x_1, x_2, x_3)_{,k}\, \mathbf{e}_j \otimes \mathbf{e}_k] \cdot \mathbf{e}_i .$$

Accordingly, the gradient of the vector field $\mathbf{v}$ is the tensor field

(2.2.11) $$grad\, \mathbf{v} = v_j(x_1, x_2, x_3)_{,k}\, \mathbf{e}_j \otimes \mathbf{e}_k$$

The matrix of its components is

$$\begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} .$$

The transposed field of the gradient is notated as $grad^T \mathbf{v}$. The symmetric part of the gradient will often be needed in kinematics. It is sometimes called the **deformator** of the vector field

(2.2.12) $$def\, \mathbf{v} := ½\, (grad\, \mathbf{v} + grad^T \mathbf{v}) = ½\, (v_{j,k} + v_{k,j})\, \mathbf{e}_j \otimes \mathbf{e}_k .$$

Its component matrix is with respect to some ONB

$$\begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) & \frac{\partial v_3}{\partial x_3} \end{bmatrix} .$$

The trace of the gradient of a vector field is called the **divergence** (-field)

(2.2.13) $$div\,\mathbf{v} \;:=\; tr(grad\,\mathbf{v}) \;=\; v_{j\,,\,j} \;=\; v_{1\,,\,1} + v_{2\,,\,2} + v_{3\,,\,3} \;=\; \frac{\partial \mathbf{v}}{\partial x_i} \cdot \mathbf{e}_i\,.$$

Thus, the divergence of a vector field is a scalar field.

The **curl** of a vector field is defined by the axial vector $\mathbf{t}^A$ of the skew part of the gradient

$$\mathbf{W} \;:=\; ½\,(grad\,\mathbf{v} - grad^{\,T}\mathbf{v})$$

as

(2.2.14) $$curl\,\mathbf{v} \;:=\; 2\,\mathbf{t}^A\,.$$

The component matrix of $\mathbf{W}$ is then

$$\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) & 0 \end{bmatrix}.$$

The component representation of the curl is

(2.2.15) $$curl\,\mathbf{v} \;=\; (v_{3\,,\,2} - v_{2\,,\,3})\,\mathbf{e}_1 + (v_{1\,,\,3} - v_{3\,,\,1})\,\mathbf{e}_2 + (v_{2\,,\,1} - v_{1\,,\,2})\,\mathbf{e}_3$$

$$=\; v_{i\,,\,k}\,\varepsilon_{kil}\,\mathbf{e}_l\,.$$

Similarly one can define the *divergence of a tensor field* $\mathbf{T}$ as that particular vector field which acts on an arbitrary constant vector $\mathbf{a}$ as

$$\mathbf{a} \cdot div\,\mathbf{T} \;=\; (div\,\mathbf{T}) \cdot \mathbf{a} \;:=\; div\,(\mathbf{a} \cdot \mathbf{T}) \;=\; div\,(\mathbf{T}^T \cdot \mathbf{a})\,.$$

If we set $\mathbf{a} \equiv \mathbf{e}_i$, we obtain the i-th component of $div\,\mathbf{T}$ with (2.2.13)

$$\mathbf{e}_i \cdot div\,\mathbf{T} \;=\; div(\mathbf{e}_i \cdot \mathbf{T}) \;=\; div(T_{im}\;\mathbf{e}_m) \;=\; T_{im\,,\,k}\,\mathbf{e}_m \cdot \mathbf{e}_k \;=\; T_{im\,,\,m}$$

and therefore

(2.2.16) $$div\,\mathbf{T} \;=\; T_{im\,,\,m}\,\mathbf{e}_i\,,$$

a vector field.

Similarly, one defines the *curl of a tensor field* $\mathbf{T}$ by its effect on a constant vector $\mathbf{a}$

(2.2.17) $$(curl\,\mathbf{T}) \cdot \mathbf{a} \;:=\; curl(\mathbf{a} \cdot \mathbf{T}) \;=\; curl(\mathbf{T}^T \cdot \mathbf{a})\,.$$

If we set $\mathbf{a} \equiv \mathbf{e}_i$, then with (2.2.15)

$$(curl\,\mathbf{T}) \cdot \mathbf{e}_i \;=\; curl(\mathbf{e}_i \cdot \mathbf{T}\,) \;=\; curl(T_{ij}\;\mathbf{e}_j) \;=\; T_{ij\,,\,k}\,\varepsilon_{kjl}\,\mathbf{e}_l$$

$$=\; T_{mj\,,\,k}\,\varepsilon_{lkj}\,\mathbf{e}_l \otimes \mathbf{e}_m \cdot \mathbf{e}_i$$

and therefore

(2.2.18) $$curl\,\mathbf{T} \;=\; T_{mj\,,\,k}\,\varepsilon_{lkj}\,\mathbf{e}_l \otimes \mathbf{e}_m$$

which is also a tensor field.

We will later need the **tensor of incompatibility** of a symmetric tensor field $\mathbf{T}$ as

(2.2.19) $$inc\,\mathbf{T} \;:=\; -\,curl\;curl\,\mathbf{T} \;=\; \varepsilon_{ikj}\,\varepsilon_{lmn}\,T_{jm\,,\,kn}\;\mathbf{e}_i \otimes \mathbf{e}_l\,.$$

This gives again a 2nd-order tensor field.

One should note that the definitions of *grad*, *div*, and *curl* are not unique in the literature.

We obtain the general rules for these operations:

- the gradient operation increases the order of a field by *1*
- the divergence operation decreases it by *1*
- the curl leaves it equal.

The chain rule also holds for gradients, divergence, and curl operations.

2.2.2 The Nabla Operator

In the literature, the calculus in linear spaces is often formalised by the **nabla**[35] **operator**, which has a double function as a differential operator and as a (co-) vector. This needs some explanation. Nabla is introduced as

(2.2.20) $$\nabla \;:=\; \frac{\partial}{\partial x_i}\,\mathbf{e}_i$$

with respect to some fixed ONB $\{\mathbf{e}_i\}$ and with tensor fields of arbitrary order algebraically connected, i.e., by a scalar product or a simple contraction, a tensor product, a cross-product, or another product.

The rules for the application are then:

1.) Apply the differential operator $\dfrac{\partial}{\partial x_i}$ to all of these fields (if necessary by use of the product rule).

2.) Connect $\mathbf{e}_i$ with the result according to the given algebraic product.

It is important to note that the order of these two steps is in general not interchangeable.

As an example we choose the *gradient of a scalar field* after (2.2.4):

[35] from greek $\nu\alpha\beta\lambda\alpha$ = harp

(2.2.21) $$grad\ \phi(\mathbf{r}) \;=\; \frac{\partial\phi(x_1, x_2, x_3)}{\partial x_i}\,\mathbf{e}_i \;=\; \phi_{,\,i}\,\mathbf{e}_i \;=\; \nabla\phi \;=\; \phi\nabla.$$

In this case, the algebraic product between ϕ and ∇ is a scalar multiplication of a scalar with a (co)vector, where the scalar can be written before or after the vector.

For a *gradient of a vector field* the order is no longer arbitrary, since we have to use the tensor product in this case after (2.2.11), which does not commute,

(2.2.22) $$grad\,\mathbf{v} \;=\; \frac{\partial v_j\,(x_1, x_2, x_3)}{\partial x_k}\,\mathbf{e}_j \otimes \mathbf{e}_k \;=\; v_j(x_1, x_2, x_3)\,\mathbf{e}_j \otimes \frac{\partial}{\partial x_k}\,\mathbf{e}_k$$

$$=\; \mathbf{v} \otimes \nabla.$$

For the divergence (2.2.13) and the curl (2.2.15) of a vector field we can write

(2.2.23) $$div\,\mathbf{v} \;=\; tr(\mathbf{v} \otimes \nabla) \;=\; \mathbf{v} \cdot \nabla \;=\; \nabla \cdot \mathbf{v}$$

(2.2.24) $$curl\,\mathbf{v} \;=\; \nabla \times \mathbf{v} \;=\; -\,\mathbf{v} \times \nabla$$

and for the divergence of a tensor field (2.2.16)

(2.2.25) $$div\,\mathbf{T} \;=\; \mathbf{T} \cdot \nabla \;=\; \nabla \cdot \mathbf{T}^T$$

and for the curl of a tensor field (2.2.17)

(2.2.26) $$curl\,\mathbf{T} \;=\; \nabla \times \mathbf{T}^T \;=\; -\,(\mathbf{T} \times \nabla)^T.$$

The tensor of incompatibility (2.2.19) is in nabla notation

(2.2.27) $$inc\,\mathbf{T} \;:=\; -\,curl\,curl\,\mathbf{T} \;=\; \nabla \times \mathbf{T} \times \nabla$$

from which we see the identity

(2.2.28) $$div\,inc\,\mathbf{T} \;=\; \nabla \times \mathbf{T} \times \nabla \cdot \nabla \;=\; \mathbf{o}\,.$$

The deformator (2.2.12) is

(2.2.29) $$def\,\mathbf{v} \;:=\; \tfrac{1}{2}\,(grad\,\mathbf{v} + grad^{\,T}\mathbf{v}) \;=\; \tfrac{1}{2}\,(\mathbf{v} \otimes \nabla + \nabla \otimes \mathbf{v})\,.$$

If the differential operation of nabla does not apply to the whole term, but only to a part of it, then we can indicate this by brackets or by a superimposed arrow. As an example, we consider the gradient of two differentiable scalar fields ψ and φ

$$\nabla(\phi\,\psi) \;=\; (\nabla\phi)\,\psi + (\nabla\psi)\,\phi \;=\; \nabla\overset{\downarrow}{\phi}\;\psi + \nabla\phi\,\overset{\downarrow}{\psi}$$

after (2.2.31).

For all differentiable scalar fields $\psi(\mathbf{r})$, $\varphi(\mathbf{r})$, vector fields $\mathbf{u}(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$, and tensor fields $\mathbf{T}(\mathbf{r})$, $\mathbf{S}(\mathbf{r})$ the following rules hold.

$$grad(\psi + \varphi) = \nabla(\phi + \psi) = \nabla\phi + \nabla\psi = grad\;\psi + grad\;\varphi \tag{2.2.30}$$

$$grad(\psi\,\varphi) = \varphi\,grad\;\psi + \psi\,grad\;\varphi \tag{2.2.31}$$

$$\begin{aligned} grad(\mathbf{u} + \mathbf{v}) &= (\mathbf{u} + \mathbf{v}) \otimes \nabla = \mathbf{u} \otimes \nabla + \mathbf{v} \otimes \nabla \\ &= grad\,\mathbf{u} + grad\,\mathbf{v} \end{aligned} \tag{2.2.32}$$

$$div(\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \cdot \nabla = \mathbf{u} \cdot \nabla + \mathbf{v} \cdot \nabla = div\,\mathbf{u} + div\,\mathbf{v} \tag{2.2.33}$$

$$curl(\mathbf{u} + \mathbf{v}) = \nabla \times (\mathbf{u} + \mathbf{v}) = \nabla \times \mathbf{u} + \nabla \times \mathbf{v} = curl\,\mathbf{u} + curl\,\mathbf{v} \tag{2.2.34}$$

$$\begin{aligned} grad(\varphi\,\mathbf{v}) &= (\varphi\,\mathbf{v}) \otimes \nabla = \varphi\,(\mathbf{v} \otimes \nabla) + \mathbf{v} \otimes (\varphi\nabla) \\ &= \varphi\,grad\,\mathbf{v} + \mathbf{v} \otimes grad\;\varphi \end{aligned} \tag{2.2.35}$$

$$\begin{aligned} div(\varphi\,\mathbf{v}) &= (\varphi\,\mathbf{v}) \cdot \nabla = \varphi\,(\mathbf{v} \cdot \nabla) + \mathbf{v} \cdot (\varphi\nabla) \\ &= \varphi\,div\,\mathbf{v} + \mathbf{v} \cdot grad\;\varphi \end{aligned} \tag{2.2.36}$$

$$\begin{aligned} grad(\mathbf{u} \cdot \mathbf{v}) &= (\mathbf{u} \cdot \mathbf{v})\,\nabla = (\nabla \otimes \mathbf{u}) \cdot \mathbf{v} + (\nabla \otimes \mathbf{v}) \cdot \mathbf{u} \\ &= \mathbf{v} \cdot (\mathbf{u} \otimes \nabla) + \mathbf{u} \cdot (\mathbf{v} \otimes \nabla) = \mathbf{v} \cdot grad\,\mathbf{u} + \mathbf{u} \cdot grad\,\mathbf{v} \\ &= grad^{T}(\mathbf{u}) \cdot \mathbf{v} + grad^{T}(\mathbf{v}) \cdot \mathbf{u} \end{aligned} \tag{2.2.37}$$

$$\begin{aligned} div(\mathbf{u} \otimes \mathbf{v}) &= (\mathbf{u} \otimes \mathbf{v}) \cdot \nabla = \mathbf{u}\,(\mathbf{v} \cdot \nabla) + (\mathbf{u} \otimes \nabla) \cdot \mathbf{v} \\ &= \mathbf{u}\;\,div\,\mathbf{v} + grad(\mathbf{u}) \cdot \mathbf{v} \end{aligned} \tag{2.2.38}$$

$$\begin{aligned} div(\mathbf{T} \cdot \mathbf{v}) &= \nabla \cdot (\mathbf{T} \cdot \mathbf{v}) = (\mathbf{T} \cdot \mathbf{v}) \cdot \nabla \\ &= \mathbf{T} \cdot\cdot (\nabla \otimes \mathbf{v}) + (\nabla \cdot \mathbf{T}) \cdot \mathbf{v} \\ &= tr(\mathbf{T} \cdot grad\,\mathbf{v}) + div(\mathbf{T}^{T}) \cdot \mathbf{v} = \mathbf{T}^{T} \cdot\cdot\, grad\,\mathbf{v} + div(\mathbf{T}^{T}) \cdot \mathbf{v} \end{aligned} \tag{2.2.39}$$

$$\begin{aligned} div(\mathbf{v} \cdot \mathbf{T}) &= (\mathbf{v} \cdot \mathbf{T}) \cdot \nabla = \nabla \cdot (\mathbf{v} \cdot \mathbf{T}) \\ &= \mathbf{T} \cdot\cdot (\mathbf{v} \otimes \nabla) + \mathbf{v} \cdot (\mathbf{T} \cdot \nabla) \\ &= tr(\mathbf{T}^{T} \cdot grad\,\mathbf{v}) + div(\mathbf{T}) \cdot \mathbf{v} = \mathbf{T} \cdot\cdot\, grad\,\mathbf{v} + div(\mathbf{T}) \cdot \mathbf{v} \end{aligned} \tag{2.2.40}$$

$$\begin{aligned} div(\varphi\,\mathbf{T}) &= (\varphi\,\mathbf{T}) \cdot \nabla = \varphi\,(\mathbf{T} \cdot \nabla) + \mathbf{T} \cdot (\varphi\nabla) \\ &= \varphi\,div\,\mathbf{T} + \mathbf{T} \cdot grad\;\varphi \end{aligned} \tag{2.2.41}$$

$$div(\mathbf{T} + \mathbf{S}) = (\mathbf{T} + \mathbf{S}) \cdot \nabla = \mathbf{T} \cdot \nabla + \mathbf{S} \cdot \nabla = div\,\mathbf{T} + div\,\mathbf{S} \tag{2.2.42}$$

With the nabla notation one sees the following identities for all differentiable scalar fields $\varphi(\mathbf{r})$, vector fields $\mathbf{v}(\mathbf{r})$, and tensor fields $\mathbf{T}(\mathbf{r})$.

(2.2.43) $$curl\, grad\, \varphi = \nabla \times (\nabla\, \varphi) = \mathbf{o}$$

(2.2.44) $$curl\, grad^T\, \mathbf{v} = \nabla \times (\mathbf{v} \otimes \nabla) = (\nabla \times \mathbf{v}) \otimes \nabla = grad\, curl\, \mathbf{v}$$

(2.2.45) $$curl\, grad\, \mathbf{v} = \nabla \times (\nabla \otimes \mathbf{v}) = \mathbf{0}$$

(2.2.46) $$div\, curl\, \mathbf{v} = \nabla \cdot (\nabla \times \mathbf{v}) = (\nabla \times \nabla) \cdot \mathbf{v} = 0$$

(2.2.47) $$curl\, curl\, \mathbf{v} = \nabla \times (\nabla \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\, \nabla - (\mathbf{v} \otimes \nabla) \cdot \nabla$$
$$= grad\, div\, \mathbf{v} - div\, grad\, \mathbf{v}$$

(2.2.48) $$div\, grad^T\, \mathbf{v} = (\nabla \otimes \mathbf{v}) \cdot \nabla = (\nabla \cdot \mathbf{v})\, \nabla = grad\, div\, \mathbf{v}$$

(2.2.49) $$div\, grad\, grad\, \varphi = [(\nabla \varphi) \otimes \nabla] \cdot \nabla = [(\nabla \varphi) \cdot \nabla]\, \nabla$$
$$= grad\, div\, grad\, \varphi$$

(2.2.50) $$div\, grad\, grad\, \mathbf{v} = [(\mathbf{v} \otimes \nabla) \otimes \nabla] \cdot \nabla = [(\mathbf{v} \otimes \nabla) \cdot \nabla] \otimes \nabla$$
$$= grad\, div\, grad\, \mathbf{v}$$

(2.2.51) $$div\, div\, grad\, \mathbf{v} = \nabla \cdot [(\mathbf{v} \otimes \nabla) \cdot \nabla] = [(\mathbf{v} \cdot \nabla)\, \nabla] \cdot \nabla$$
$$= div\, grad\, div\, \mathbf{v}$$

(2.2.52) $$div\, grad\, curl\, \mathbf{v} = [(\nabla \times \mathbf{v}) \otimes \nabla] \cdot \nabla = \nabla \times [(\mathbf{v} \otimes \nabla) \cdot \nabla]$$
$$= curl\, div\, grad\, \mathbf{v}$$

(2.2.53) $$div\, (curl\, \mathbf{T})^T = -\mathbf{T} \times \nabla \cdot \nabla = \mathbf{o}$$

The **LAPLACE**[36] **operator** is defined as

(2.2.54) $$\Delta := \nabla \cdot \nabla\,,$$

which equals the operations *div grad*. Examples: Let φ be a scalar field, then

(2.2.55) $$\Delta\, \varphi = \varphi \nabla \cdot \nabla = div\, grad\, \varphi = \frac{\partial}{\partial x_i} \mathbf{e}_i \cdot \frac{\partial}{\partial x_k} \mathbf{e}_k\, \varphi = \phi_{,ii}$$

is a scalar field. Let $\mathbf{v}$ be a vector field, then

(2.2.56) $$\Delta\, \mathbf{v} = \mathbf{v}\, \Delta = \mathbf{v}\, (\nabla \cdot \nabla) = \mathbf{v} \otimes \nabla \cdot \nabla = div\, grad\, \mathbf{v}$$

is also a vector field.

[36] Pierre Simon Laplace (1749-1827)

2.2.3 Cylindrical Coordinates

If one uses coordinates to describe the geometrical space, the simplest coordinate system is a Cartesian one. However, if the problem under consideration comprises a rotational symmetry, which is quite often the case, it is not recommended to use Cartesian coordinates, but instead cylindrical ones. We will therefore derive the most important representations in cylindrical coordinates.

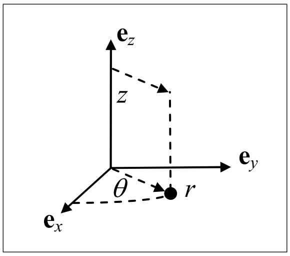
Here, a point is described by three real numbers r , θ , z , where r is the distance of the z–axis, z the height above the base plane, and θ the angle of the projection into this plane, which can take values between 0^o and 360^o. Note that for both endpoints the angle is not unique. Therefore we use the open interval and leave out points with $\theta \equiv 0$ or with $r \equiv 0$.

Cylindrical coordinates are rectangular and curved. If $\{x , y , z\}$ are Cartesian coordinates with associated ONB $\{\mathbf{e}_x , \mathbf{e}_y , \mathbf{e}_z\}$, then we obtain the transformations

$$x = r \cos \theta \qquad y = r \sin \theta$$

$$r^2 = x^2 + y^2 \qquad \theta = \arctan \frac{y}{x}$$

for $r \neq 0$, while z is identical in both systems.



In a point with coordinates $\{r , \theta , z\}$ we can introduce an associated local ONB by

$$\mathbf{e}_r(\theta) := \cos \theta \ \mathbf{e}_x + \sin \theta \ \mathbf{e}_y$$

$$\mathbf{e}_\theta(\theta) := -\sin \theta \ \mathbf{e}_x + \cos \theta \ \mathbf{e}_y$$

$$\mathbf{e}_z$$

This gives for the differentials

$$d\mathbf{e}_r = \frac{d\mathbf{e}_r}{d\theta} \, d\theta = d\theta \ \mathbf{e}_\theta$$

$$d\mathbf{e}_\theta = \frac{d\mathbf{e}_\theta}{d\theta} \, d\theta = -d\theta \ \mathbf{e}_r$$

The position vector is

$$\mathbf{r} = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$$
$$= r\,\mathbf{e}_r(\theta) + z\,\mathbf{e}_z$$

and its differential

$$d\mathbf{r} = dx\,\mathbf{e}_x + dy\,\mathbf{e}_y + dz\,\mathbf{e}_z$$
$$= dr\,\mathbf{e}_r(\theta) + r\,d\mathbf{e}_r(\theta) + dz\,\mathbf{e}_z$$
$$= dr\,\mathbf{e}_r(\theta) + r\,d\theta\,\mathbf{e}_\theta(\theta) + dz\,\mathbf{e}_z$$

The *gradient of a scalar field* $\phi(r,\theta,z)$ can be obtained by the differential

$$d\phi(r,\theta,z) = grad\,\phi\cdot d\mathbf{r}$$
$$= \phi,_r\,dr + \phi,_\theta\,d\theta + \phi,_z\,dz$$
$$= grad\,\phi\cdot[dr\,\mathbf{e}_r(\theta) + r\,d\theta\,\mathbf{e}_\theta(\theta) + dz\,\mathbf{e}_z]$$
$$= \phi,_r\,\mathbf{e}_r\cdot dr\,\mathbf{e}_r + \phi,_\theta\,\frac{1}{r}\,\mathbf{e}_\theta\cdot r\,d\theta\,\mathbf{e}_\theta + \phi,_z\,\mathbf{e}_z\cdot dz\,\mathbf{e}_z$$

and by comparison

(2.2.57) $$grad\,\phi = \phi,_r\,\mathbf{e}_r + \phi,_\theta\,\frac{1}{r}\,\mathbf{e}_\theta + \phi,_z\,\mathbf{e}_z\,.$$

The *gradient of a vector field* $\mathbf{v}(r,\theta,z)$ is obtained by its differential

$$d\mathbf{v}(\mathbf{r},\mathbf{dr}) = \mathbf{v},_r\,dr + \mathbf{v},_\theta\,d\theta + \mathbf{v},_z\,dz$$
$$= grad\,\mathbf{v}\cdot d\mathbf{r}$$

with the partial derivatives

$$\mathbf{v},_r = (v^i\,\mathbf{e}_i),_r = v^r,_r\,\mathbf{e}_r + v^\theta,_r\,\mathbf{e}_\theta + v^z,_r\,\mathbf{e}_z$$
$$\mathbf{v},_\theta = (v^i\,\mathbf{e}_i),_\theta = v^r,_\theta\,\mathbf{e}_r + v^r\,\mathbf{e}_r,_\theta + v^\theta,_\theta\,\mathbf{e}_\theta + v^\theta\,\mathbf{e}_\theta,_\theta + v^z,_\theta\,\mathbf{e}_z$$
$$= v^r,_\theta\mathbf{e}_r + v^r\,\mathbf{e}_\theta + v^\theta,_\theta\,\mathbf{e}_\theta - v^\theta\,\mathbf{e}_r + v^z,_\theta\,\mathbf{e}_z$$
$$\mathbf{v},_z = (v^i\,\mathbf{e}_i),_z = v^r,_z\,\mathbf{e}_r + v^\theta,_z\,\mathbf{e}_\theta + v^z,_z\,\mathbf{e}_z$$

By inserting them and comparison we obtain the matrix of components of *grad* $\mathbf{v}$ as

(2.2.58) $$\begin{bmatrix} v^r,_r & \frac{1}{r}\left(v^r,_\theta - v^\theta\right) & v^r,_z \\ v^\theta,_r & \frac{1}{r}\left(v^\theta,_\theta + v^r\right) & v^\theta,_z \\ v^z,_r & \frac{1}{r}v^z,_\theta & v^z,_z \end{bmatrix}.$$

The divergence is its trace

(2.2.59) $$div\,\mathbf{v} = v^r,_r + \frac{1}{r}\,(v^\theta,_\theta + v^r) + v^z,_z$$

and the curl is

(2.2.60) $$curl\,\mathbf{v} = (\frac{1}{r}\, v^z{}_{,\theta} - v^\theta{}_{,z})\ \mathbf{e}_r + (v^r{}_{,z} - v^z{}_{,r})\ \mathbf{e}_\theta + [v^\theta{}_{,r} + \frac{1}{r}\,(v^\theta - v^r{}_{,\theta})]\ \mathbf{e}_z\,.$$

The *divergence of a tensor field* is

(2.2.61) $$div\,\mathbf{T} = (T^{rr}{}_{,r} + \frac{1}{r}\,T^{r\theta}{}_{,\theta} + \frac{T^{rr} - T^{\theta\theta}}{r} + T^{rz}{}_{,z})\ \mathbf{e}_r + (T^{\theta r}{}_{,r} + \frac{1}{r}\,T^{\theta\theta}{}_{,\theta} + \frac{T^{r\theta} + T^{\theta r}}{r} + T^{\theta z}{}_{,z})\ \mathbf{e}_\theta + (T^{zr}{}_{,r} + \frac{1}{r}\,T^{z\theta}{}_{,\theta} + \frac{T^{zr}}{r} + T^{zz}{}_{,z})\ \mathbf{e}_z\,.$$

Nabla has the following representation in cylindrical coordinates

(2.2.62) $$\nabla = \frac{\partial}{\partial r}\ \mathbf{e}_r + \frac{\partial}{r\partial\theta}\ \mathbf{e}_\theta + \frac{\partial}{\partial z}\ \mathbf{e}_z$$

and the LAPLACE operator

(2.2.63) $$\Delta := \nabla\cdot\nabla = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2} + \frac{\partial^2}{\partial z^2}\,.$$

Problem 8. Cylindrical Coordinates I

The sketched windmill with only one rotor blade rotates with a constant angular velocity ω. The mass of the blade is m, and we approximate it by a homogeneous bar with moment of inertia J with respect to the rotor axis. We consider the load-free case. Determine the moment at the foundation A caused by the imbalance of the blade.

<u>Solution</u>

First we cut the windmill free. At the foundation point we introduce the torque $\mathbf{m}_A(t)$. This is the only torque acting on the mill. The angular momentum with respect to A is

$$\mathbf{d}(t) = m\,\mathbf{r}_{AM}(t) \times \mathbf{r}_{AM}(t)^\bullet + J\,\omega(t)\,\mathbf{e}_z\,.$$

The first term represents the motion of the centre of mass of the blade (after STEINER´s[37] theorem). The second term comes from the rotation of the blade with respect to its centre of mass. The moment of inertia of a

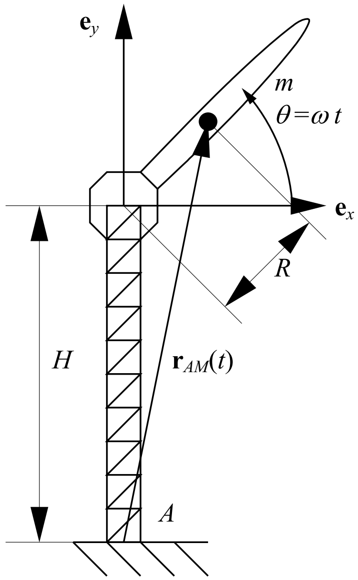
[37] Jakob Steiner (1796-1863)

homogeneous bar with respect to some axis perpendicular to its own axis through the centre of mass is

$$J = m\,l^2/12 = m\,(2R)^2/12 = m\,R^2/3\,.$$

The axis of rotation points in the $\mathbf{e}_z$–direction. After the balance of angular momentum, the resulting moments equal the rate of moment of momentum

$$\mathbf{m}_A(t) = \mathbf{d}(t)^\bullet = [m\,\mathbf{r}_{AM}(t) \times \mathbf{r}_{AM}(t)^\bullet]^\bullet + J\omega(t)^\bullet\,\mathbf{e}_z\,.$$



If the angular velocity is constant, then $\omega(t)^\bullet = 0$ and only

$$\mathbf{m}_A(t) = m\,\mathbf{r}_{AM}(t)^\bullet \times \mathbf{r}_{AM}(t)^\bullet + m\,\mathbf{r}_{AM}(t) \times \mathbf{r}_{AM}(t)^{\bullet\bullet}$$

(P8.1) $$= m\,\mathbf{r}_{AM}(t) \times \mathbf{r}_{AM}(t)^{\bullet\bullet}$$

remains. The position vector is assigned with respect to a basis in the hub of the mill. Its distance from the foundation is most easily represented by a fixed ONB, while for the blade a cylindrical system is preferable

$$\mathbf{r}_{AM}(t) = H\,\mathbf{e}_y + R\,\mathbf{e}_r(\theta)\,.$$

The angle θ results from the initial condition $\theta(0) = 0$ as $\theta = \omega\,t$. We determine the acceleration $\mathbf{r}_{AM}^{\bullet\bullet}(t)$ by the chain rule using

$$\frac{d\mathbf{e}_r(\theta)}{d\theta} = \mathbf{e}_\theta(\theta) \qquad\qquad \frac{d\mathbf{e}_\theta(\theta)}{d\theta} = -\,\mathbf{e}_r(\theta)$$

so that (Chapter 2.2.3)

$$\mathbf{r}_{AM}(t)^\bullet = [H\,\mathbf{e}_y + R\,\mathbf{e}_r(\theta)]^\bullet = R\,\omega\,\mathbf{e}_\theta(\theta)$$

$$\mathbf{r}_{AM}(t)^{\bullet\bullet} = -R\,\omega^2\,\mathbf{e}_r(\theta)\,.$$

Now we use P8.1

$$\mathbf{m}_A(t) = m\,\mathbf{r}_{AM}(t) \times \mathbf{r}_{AM}(t)^{\bullet\bullet}$$

$$= -R\,\omega^2 m\,[H\,\mathbf{e}_y + R\,\mathbf{e}_r(\theta)] \times \mathbf{e}_r(\theta)$$

$$= -R\,\omega^2 m\,H\,\mathbf{e}_y \times \mathbf{e}_r(\theta)\,,$$

and with $\mathbf{e}_r = \cos\theta\;\mathbf{e}_x + \sin\theta\;\mathbf{e}_y$

$$\mathbf{m}_A(t) = R\,\omega^2 m\,H\cos\theta\,\mathbf{e}_z\,.$$

This must be supported by the foundation.

Problem 9. Cylindrical Coordinates II

Calculate the angular momentum with respect to the axis of rotation of a homogenous circular plate rotating with angular velocity ω around $\mathbf{e}_z$ with a thickness D, radius R, and mass density ρ.

<u>Solution</u>

We represent the angular momentum after (2.1.110) as a volume integral in cylindrical coordinates

$$\mathbf{d} = \int_0^D \int_0^R \int_0^{2\pi} \mathbf{r} \times \mathbf{r}^{\bullet}\,\rho\,r\,d\theta\,dr\,dz\,.$$

The position vector is $\mathbf{r} = r\,\mathbf{e}_r(\omega\,t)$. We apply the chain rule and

$$\frac{d\mathbf{e}_r(\theta)}{d\theta} = \mathbf{e}_\theta(\theta)$$

to obtain

$$\mathbf{r}^{\bullet} = r\,\omega\,\mathbf{e}_\theta(\omega\,t)\,.$$

Then the angular momentum is

$$\mathbf{d} = \omega D\rho \int_0^R \int_0^{2\pi} r^3\,\mathbf{e}_r \times \mathbf{e}_\theta\,d\theta\,dr\,.$$

With $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z$, independent of r and θ, we find

$$\mathbf{d} = \tfrac{1}{2}\,\pi\,\omega D\rho\;R^4\,\mathbf{e}_z\,.$$

2.2.4 Integral Transformations

are needed to transform volume integrals into surface integrals, or vice versa.

Divergence Theorem (GAUSS–OSTROGRADSKI[38]) *Let $\mathscr{V}$ be a three-dimensional regular volumetric region in the EUCLIDean space, and $\mathscr{A}$ its surface with outer normal* $\mathbf{n}$. *Let further* $\mathbf{U}$ *be a tensor field of arbitrary order, and* $\oplus$ *an arbitrary product between* $\mathbf{U}$ *and* $\mathbf{n}$. *Then*

(2.2.64) $$\int_{\mathscr{A}} \mathbf{U} \oplus \mathbf{n}\, dA = \int_{\mathscr{V}} \mathbf{U} \oplus \nabla\, dV$$

The following special cases will be important for us.

- $\mathbf{U}$ is a scalar field ϕ and $\oplus$ the scalar multiplication of a scalar and vector. Then

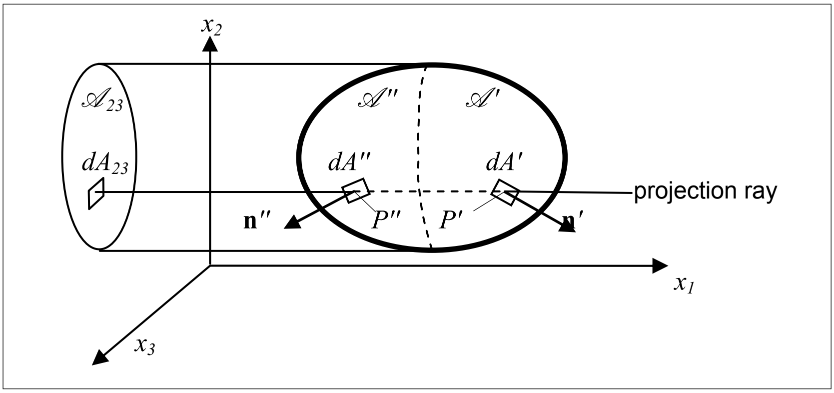
(2.2.65) $$\int_{\mathscr{A}} \phi\, \mathbf{n}\, dA = \int_{\mathscr{V}} \phi \nabla\, dV = \int_{\mathscr{V}} grad\, \phi\, dV.$$

- $\mathbf{U}$ is a vector field $\mathbf{u}$ and $\oplus$ the scalar product between vectors. Then

(2.2.66) $$\int_{\mathscr{A}} \mathbf{u} \cdot \mathbf{n}\, dA = \int_{\mathscr{V}} \mathbf{u} \cdot \nabla\, dV = \int_{\mathscr{V}} div\, \mathbf{u}\, dV.$$

We give a sketch of the proof of this form, from which the other forms can be easily derived. We write down the component form with respect to some ONB

$$\int_{\mathscr{V}} div\, \mathbf{u}\, dV = \int_{\mathscr{V}} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) dx_1\, dx_2\, dx_3\,.$$



[38] Carl Friedrich Gauß (1777-1855), Mikhail Vasilevich Ostrogradski (1801-1862)

We first assume that the region $\mathscr{V}$ is convex. If we project this region into the x_2–x_3–coordinate plane, we obtain the area $\mathscr{A}_{23}$. We next decompose the surface of $\mathscr{V}$ into its positive part $\mathscr{A}'$ with respect to x_1 and its negative part $\mathscr{A}''$. A projection ray parallel to the x_1–axis penetrates the surface $\mathscr{A}$ of $\mathscr{V}$ twice at the points P' and P''. In these points we notate the elements of area as dA' and dA'' and the normal vectors as $\mathbf{n}'$ and $\mathbf{n}''$, respectively. The projections of these area elements into the x_2–x_3–plane shall be $dA_{23} = dx_2\, dx_3$. For them we obtain the relation

$$dA_{23} = dA' \, cos(\mathbf{n}', \mathbf{e}_1) = dA'' \, cos(\mathbf{n}'', -\mathbf{e}_1)$$

wherein the cosine of the angle spanned by $\mathbf{n}'$ and $\mathbf{e}_1$ can be calculated by the scalar product

$$cos(\mathbf{n}', \mathbf{e}_1) = \mathbf{n}' \cdot \mathbf{e}_1$$

and analogously

$$cos(\mathbf{n}'', -\mathbf{e}_1) = -\mathbf{n}'' \cdot \mathbf{e}_1$$

so that

$$dA_{23} = \mathbf{n}' \cdot \mathbf{e}_1 \, dA' = -\mathbf{n}'' \cdot \mathbf{e}_1 \, dA''.$$

The first term of the integral is by partial integration

$$\int_{\mathscr{V}} \frac{\partial u_1}{\partial x_1} dx_1\, dx_2\, dx_3 = \int_{\mathscr{A}_{23}} \int_{P''}^{P'} \frac{\partial u_1}{\partial x_1} dx_1\, dA_{23}$$

$$= \int_{\mathscr{A}_{23}} [u_1(P') - u_1(P'')] \, dA_{23}$$

$$= \int_{\mathscr{A}'} u_1 \, \mathbf{n}' \cdot \mathbf{e}_1 \, dA' - \int_{\mathscr{A}''} u_1 (-\mathbf{n}'' \cdot \mathbf{e}_1) \, dA''$$

$$= \int_{\mathscr{A}} u_1 \, \mathbf{e}_1 \cdot \mathbf{n} \, dA \, .$$

For the other two terms of the integral we obtain analogously

$$\int_{\mathscr{V}} \frac{\partial u_2}{\partial x_2} dx_1\, dx_2\, dx_3 = \int_{\mathscr{A}} u_2 \, \mathbf{e}_2 \cdot \mathbf{n} \, dA$$

and

$$\int_{\mathscr{V}} \frac{\partial u_3}{\partial x_3} dx_1\, dx_2\, dx_3 = \int_{\mathscr{A}} u_3 \, \mathbf{e}_3 \cdot \mathbf{n} \, dA$$

and for the sum of the three

$$\int_{\mathscr{V}} div \, \mathbf{u} \, dV = \int_{\mathscr{A}} \mathbf{u} \cdot \mathbf{n} \, dA \, .$$

If the region is not convex, the projection ray penetrates the surface possibly more than twice. Then we can also project these parts of $\mathscr{A}$ into the $\mathbf{e}_2$– $\mathbf{e}_3$–plane. Finally we will obtain again the same formula. If the region contains internal voids, then one must also integrate over the internal surfaces.

For $\mathbf{u} \equiv \mathbf{v} \cdot \mathbf{T}$ with a vector field $\mathbf{v}$ and a tensor field $\mathbf{T}$ we obtain by the divergence theorem with (2.2.40)

$$\int_{\mathscr{A}} \mathbf{v} \cdot \mathbf{T} \cdot \mathbf{n} \, dA = \int_{\mathscr{V}} div\,(\mathbf{v} \cdot \mathbf{T}) \, dV$$

(2.2.67)
$$= \int_{\mathscr{V}} [\mathbf{T} \cdot\cdot (\mathbf{v} \otimes \nabla) + \mathbf{v} \cdot (\mathbf{T} \cdot \nabla)] \, dV$$

$$= \int_{\mathscr{V}} (\mathbf{T} \cdot\cdot \, grad\, \mathbf{v} + \mathbf{v} \cdot div\, \mathbf{T}) \, dV.$$

- $\mathbf{U}$ is a vector field $\mathbf{u}$ and $\oplus$ the cross-product between vectors. Then

$$\int_{\mathscr{A}} \mathbf{u} \times \mathbf{n} \, dA = \int_{\mathscr{V}} \mathbf{u} \times \nabla \, dV = -\int_{\mathscr{V}} curl\, \mathbf{u} \, dV$$

or

(2.2.68)
$$\int_{\mathscr{A}} \mathbf{n} \times \mathbf{u} \, dA = \int_{\mathscr{V}} curl\, \mathbf{u} \, dV$$

hold.

- $\mathbf{U}$ is a vector field $\mathbf{u}$ and $\oplus$ the tensorial product between vectors. Then

(2.2.69)
$$\int_{\mathscr{A}} \mathbf{u} \otimes \mathbf{n} \, dA = \int_{\mathscr{V}} \mathbf{u} \otimes \nabla \, dV = \int_{\mathscr{V}} grad\, \mathbf{u} \, dV$$

holds.

- $\mathbf{U}$ is a tensor field of arbitrary order and $\oplus$ a simple contraction. Then

(2.2.70)
$$\int_{\mathscr{A}} \mathbf{U} \cdot \mathbf{n} \, dA = \int_{\mathscr{V}} \mathbf{U} \cdot \nabla \, dV = \int_{\mathscr{V}} div\, \mathbf{U} \, dV.$$

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