Formal Concept Analysis from the Standpoint of Possibility Theory

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Abstract. Formal concept analysis (FCA) and possibility theory (PoTh) have been developed independently. They address different concerns in information processing: while FCA exploits relations linking objects and properties, and has applications in data mining and clustering, PoTh deals with the modeling of (graded) epistemic uncertainty. However, making a formal parallel between FCA and PoTh is fruitful. The four set-functions at work in PoTh have meaningful counterparts in FCA; this leads to consider operators neglected in FCA, and thus new fixed point equations. One of these pairs of equations, paralleling the one defining formal concepts in FCA, defines independent sub-contexts of objects and properties that have nothing in common. The similarity of the structures underlying FCA and PoTh is still more striking, using a cube of opposition (a device extending the traditional square of opposition in logic). Beyond the parallel between FCA and PoTh, this invited contribution, which largely relies on several past publications by the authors, also addresses issues pertaining to the possible meanings, degree of satisfaction vs. degree of certainty, of graded object-property links, which calls for distinct manners of handling the degrees. Other lines of interest for further research are briefly mentioned.

1 Introduction

Formal concept analysis (FCA) and possibility theory (PoTh) are two theoretical frameworks that are addressing different concerns in the processing of information. Namely FCA builds concepts from a relation linking objects to the properties they satisfy, which has applications in data mining, clustering and related fields, while PoTh deals with the modeling of (graded) epistemic uncertainty. This difference of focus explains why the two settings have been developed completely independently for a very long time. However, it is possible to build a formal analogy between FCA and PoTh. Both theories heavily rely on the comparison of sets, in terms of containment or overlap. The four set-functions at work in PoTh actually determine all possible relative positions of two sets. Then the FCA operator defining the set of objects sharing a set of properties, which is at the basis of the definition of formal concepts, appears to be the counterpart of the set function expressing strong (or guaranteed) possibility in PoTh. Then, it

J. Baixeries et al. (Eds.): ICFCA 2015, LNAI 9113, pp. 21–38, 2015.
DOI: 10.1007/978-3-319-19545-2_2

suggests that the three other set functions existing in PoTh should also make sense in FCA, which leads to consider their FCA counterparts and new fixed point equations in terms of the new operators. One of these pairs of equations, paralleling the one defining formal concepts, define independent sub-contexts of objects and properties that have nothing in common.

The parallel of FCA with PoTh can still be made more striking using a cube of opposition (a device extending the traditional square of opposition existing in logic, and exhibiting a structure at work in many theories aiming at representing some aspects of the handling of information).

In this survey we shall indicate various issues pertaining to FCA that could be worth studying in the future. For instance, the object-property links in formal contexts of FCA may be a matter of degree. These degrees may refer to very different notions, such as the degree of satisfaction of a gradual property, the degree of certainty that an object has, or not, a property, or still the typicality of an object with respect to a set of properties. These different intended semantics call for distinct manners of handling the degrees, as advocated in the presentation.

Lastly, other examples of lines of interest for further research, such as the extension of the parallel of FCA with PoTh to conceptual pattern structures, or the applications to the fusion of conflicting pieces of information, to the clustering of sets of objects on the basis of approximate concepts, or to the building of conceptual analogical proportions, are briefly mentioned.

2 Possibility Theory and Formal Concept Analysis - A Parallel

Formal concept analysis [5,30,43] associates objects with the set of their properties, through a formal context which is a binary relation R on the Cartesian product of the set of objects $\mathcal{O}$ and the set of properties $\mathcal{P}$. Thus, knowing only that an object x has some property y, the set $R^t(y) = \{x \in \mathcal{O} | (x, y) \in R\}$ is the set of the *possible* objects corresponding to the elementary piece of knowledge "the object has property y"(in the context R). This suggests a possibilistic reading of formal concept analysis and leads to considering the formal counterpart to possibility theory set-functions in this framework. After introducing some notations, we first provide a short refresher on possibility theory [18,21,47].

2.1 Describing Objects

An object, or item, is denoted by x, or x_i in case we consider several ones at the same time. A subset of objects is denoted by a capital letter X, and we write $X = \{x_1, \ldots, x_i, \ldots, x_m\}$. A set of objects associated with their respective sets of properties defines a formal context $R \subseteq \mathcal{O} \times \mathcal{P}$ [30]. An object x is associated with its description, denoted $\partial(x)$. In the following, we only consider simple descriptions, expressible in terms of a subset Y of properties y_j, namely, $Y = \{y_1, \ldots, y_j, \ldots, y_n\}$. In such a case, we write $\partial(x) = Y$.

Besides, a useful kind of structured description is in terms of attributes. Let a, and $A = \{a_1, \ldots, a_k, \ldots, a_r\}$, respectively denote an attribute, and a set of attributes. The value of attribute a for x is denoted $a(x) = u$, where u belongs to the attribute domain U_a. In this case, we shall write $\partial(x) = (a_1(x), \ldots, a_k(x), \ldots, a_r(x)) = (u_1, \ldots, u_k, \ldots, u_r)$. This corresponds to a completely informed situation where all the considered attribute values are known for x. When it is not the case, the precise value $a_k(x)$ will be replaced by the possibility distribution $\pi_{a_k(x)}$. Such a possibility distribution [47] is a mapping from U_{a_k} to $[0, 1]$, or more generally any linearly ordered scale. Then $\pi_{a_k(x)}(u) \in [0, 1]$ estimates to what extent it is possible that the value of a_k for x is u. 0 means impossibility; several distinct values may be fully possible (i.e. at degree 1). The characteristic function of an ordinary subset is a particular case of a possibility distribution. Precise information corresponds to the characteristic function of singletons. An elementary property y can be viewed as a subset of a single attribute domain, i.e. $y \subseteq U$. Note that while a set of properties Y is *conjunctive* (in the sense that an object possesses *all* properties in Y), each property y corresponds to a subset of some attribute domain U that is *disjunctive* [23]: it is a set of mutually exclusive values, since object x having property y possesses a single attribute value $a(x) = u$ in U.

Taking inspiration from the existence of four set functions in possibility theory [20], new operators have been suggested in the setting of formal concept analysis [16]. These set functions are now recalled, emphasizing the symmetrical roles played by the object x and the attribute value u, a point of view unusual in possibility theory, but echoing the symmetrical role played by objects and properties in formal concept analysis. See [20,21] for more complete introductions and surveys on possibility theory.

2.2 Possibility Theory

Let $\pi_{a(x)}(u)$ denote the possibility that object x has value $u \in U$ (for attribute a). For simplicity, we only consider the single-attribute case here. We assume that π_a is bi-normalized: $\forall x\ \exists u\ \pi_{a(x)}(u) = 1$ and $\forall u\ \exists x\ \pi_{a(x)}(u) = 1$. This means that for any object x, there is some fully possible value for attribute a, and that for any value u there is an object x that takes this value. Let X be a set of objects, and $y \subseteq U$ be a property. Then, one can define

(i) the possibility measures [47], denoted by Π:

$$\Pi(X) = \max_{x \in X} \pi_{a(x)}(u) \text{ and } \Pi(y) = \max_{u \in y} \pi_{a(x)}(u).$$

$\Pi(X)$ estimates to what extent it is possible that there is an object in X having value u, while $\Pi(y)$ is the possibility that object x has property y. Π is an indicator of non-empty intersection of the fuzzy set induced by the possibility distribution with an ordinary subset. They are measures of "*weak*, or potential *possibility*". Clearly, Π is max-decomposable with respect to set union.

(ii) the dual measures of necessity N (or "strong or actual necessity") [17]:

$$N(X) = \min_{x \notin X} 1 - \pi_{a(x)}(u) \text{ and } N(y) = \min_{u \notin y} 1 - \pi_{a(x)}(u)$$

$N(X)$ estimates to what extent it is certain (necessarily true) that an object has value u is in X, while $N(y)$ is the certainty that object x has property y. Note that $N(y) = 1 - \Pi(\overline{y})$ where $\overline{y} = U \setminus y$. N may be viewed as a degree of inclusion of the fuzzy set induced by the possibility distribution into an ordinary subset. N is min-decomposable with respect to set intersection.

(iii) the measures of "strong (or actual, or guaranteed) possibility" [19]

$$\Delta(X) = \min_{x \in X} \pi_{a(x)}(u) \text{ and } \Delta(y) = \min_{u \in y} \pi_{a(x)}(u)$$

$\Delta(X)$ estimates to what extent it is possible that *all* objects in X have value u, while $\Delta(y)$ estimates the possibility that object x takes any value in y. Δ may be viewed as a degree of inclusion of an ordinary subset into the fuzzy set induced by the possibility distribution. Δ is min-decomposable with respect to set union.

(iv) the dual measures of "weak (or potential) necessity or certainty" [19]

$$\nabla(X) = 1 - \min_{x \notin X} \pi_{a(x)}(u) \text{ and } \nabla(y) = 1 - \min_{u \notin y} \pi_{a(x)}(u)$$

$\nabla(X)$ estimates to what extent there exists at least one object outside X that has a low degree of possibility of having value u, while $\nabla(y)$ measures to what extent x has a low possibility value outside y. Note that $\nabla(y) = 1 - \Delta(\overline{y})$. ∇ is an indicator of non-full coverage of the considered universe by the fuzzy set induced by the possibility distribution together with an ordinary subset. ∇ is max-decomposable with respect to set intersection.

2.3 Formal Context Setting

The classical setting of formal concept analysis defined from a formal context relies on a single operator that associates a subset of objects with the set of properties shared by them (and the dual operator). In [16], this framework has been enlarged with the introduction of three other operators. We now recall the four operators which are counterparts to the possibility theory set functions in the setting of a formal context.

Namely, let R be the formal context. Then $R(x) = \{y \in \mathcal{P} | (x, y) \in R\}$ is the set of properties of object x, and $R^t(y) = \{x \in \mathcal{O} | (x, y) \in R\}$ is the set of objects having properties y. Then, four remarkable sets can be associated with a subset X of objects (the notations have been chosen here in order to emphasize the parallel with possibility theory):

– the set $R^{\Pi}(X)$ of properties that are possessed by *at least one* object in X:

$$R^{\Pi}(X) = \{y \in \mathcal{P} | R^t(y) \cap X \neq \emptyset\} = \bigcup_{x \in X} R(x).$$

Clearly, we have $R^{\Pi}(X_1 \cup X_2) = R^{\Pi}(X_1) \cup R^{\Pi}(X_2)$.

– the set $R^N(X)$ of properties s. t. any object that satisfies *one* of them is necessarily in X:

$$R^N(X) = \{y \in \mathcal{P} | R^t(y) \subseteq X\} = \bigcap_{x \notin X} \overline{R(x)}.$$

In other words, having any property in $R^N(X)$ is a sufficient condition for belonging to X. Moreover, we have $R^N(X) = \overline{R^{\Pi}(\overline{X})} = \mathcal{P} \setminus R^{\Pi}(\overline{X})$, and $R^N(X_1 \cap X_2) = R^N(X_1) \cap R^N(X_2)$.

– the set $R^{\triangle}(X)$ of properties shared by *all* objects in X:

$$R^{\triangle}(X) = \{y \in \mathcal{P} | R^t(y) \supseteq X\} = \bigcap_{x \in X} R(x).$$

In other words, satisfying all properties in $R^{\triangle}(X)$ is a necessary condition for an object for belonging to X. $R^{\triangle}(X)$ is a partial conceptual characterization of objects in X: objects in X have all the properties of $R^{\triangle}(X)$ and may have some others (that are not shared by all objects in X). It is worth noticing that $\overline{R^{\Pi}(X)}$ provides a negative conceptual characterization of objects in X since it gathers all the properties that are never satisfied by any object in X. Moreover, we have $R^{\triangle}(X_1 \cup X_2) = R^{\triangle}(X_1) \cap R^{\triangle}(X_2)$. Besides, as can be seen, $R^N(X) \cap R^{\triangle}(X)$ is the set of properties possessed by all objects in X and only by them.

– the set $R^{\nabla}(X)$ of properties that are not satisfied by at least one object in $\overline{X}$.

$$R^{\nabla}(X) = \{y \in \mathcal{P} | R^t(y) \cup X \neq \mathcal{O}\} = \bigcup_{x \notin X} \overline{R(x)}.$$

Note that $R^{\nabla}(X) = \overline{R^{\triangle}(\overline{X})} = \mathcal{P} \setminus R^{\triangle}(\overline{X})$. In other words, in context R, for any property in $R^{\nabla}(X)$, there exists at least one object outside X that misses it. Moreover, we have $R^{\nabla}(X_1 \cap X_2) = R^{\nabla}(X_1) \cup R^{\nabla}(X_2)$.

Note that $R^{\Pi}(X)$ and $R^N(X)$ become larger when X increases, while $R^{\triangle}(X)$ and $R^{\nabla}(X)$ get smaller. The four subsets $R^{\Pi}(X)$, $R^N(X)$, $R^{\triangle}(X)$, and $R^{\nabla}(X)$ have been considered by different authors (with different notations) without any reference to possibility theory. Düntsch *et al.* [26,27] calls $R^{\triangle}$ a *sufficiency* operator, and its representation capabilities are studied in the theory of Boolean algebras. Taking inspiration as the previous authors from rough sets [40], Yao [45,46] also considers these four subsets. In both cases, the four operators were introduced. See also [33,41].

2.4 The Cube of Opposition in FCA

Before being able to present the structures of opposition relating the four operators introduced in the previous section, we need to start with a refresher on the Aristotelian square of opposition [39]. The traditional square involves four logically related statements exhibiting universal or existential quantifications: it has been noticed that a statement **A** of the form "every x is p"is negated by the statement **O** "some x is not p", while a statement like **E** "no x is p"is clearly in even stronger opposition to the first statement **A**. These three statements, together with the negation of the last one, namely **I** "some x is p", give birth to the Aristotelian square of opposition in terms of quantifiers **A**: $\forall x\ p(x)$, **E**: $\forall x\ \neg p(x)$, **I**: $\exists x\ p(x)$, **O**: $\exists x\ \neg p(x)$, pictured in Fig. 1. Such a square is usually denoted by the letters **A**, **I** (affirmative half) and **E**, **O** (negative half). The names of the vertices come from a traditional Latin reading: **A**ff**I**rmo, n**E**g**O**).

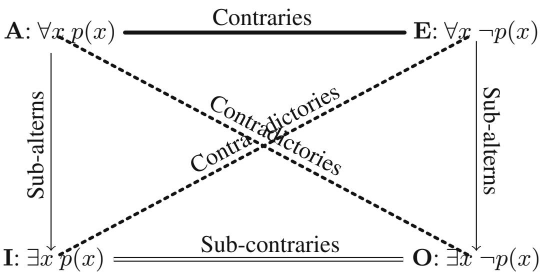

Fig. 1. Square of opposition

As can be seen, different relations hold between the vertices. Namely,

(a) **A** and **O** are the negation of each other, as well as **E** and **I**;
(b) **A** entails **I**, and **E** entails **O** (we assume that there are some x for avoiding existential import problems);
(c) **A** and **E** cannot be true together, but may be false together;
(d) **I** and **O** cannot be false together, but may be true together.

Recently, it has been noticed that such a square can be generated by a binary relation and a subset that can be composed together [12]. Indeed, let R be a binary relation on a Cartesian product $\mathcal{X} \times \mathcal{Y}$ (nothing forbids $\mathcal{Y} = \mathcal{X}$ in the construction we are going to describe). We assume $R \neq \emptyset$. Let R^t denote the transposed relation ($(y, x) \in R^t$ iff $(x, y) \in R$). Moreover, we assume that $\forall x,\ R(x) \neq \emptyset$, which means that the relation R is *serial*, namely $\forall x, \exists y$ such that $(x, y) \in R$; this is also referred to in the following as the $\mathcal{X}$*-normalization condition*. In the same way R^t is also supposed to be serial, i.e., $\forall y,\ R^t(y) \neq \emptyset$ ($\mathcal{Y}$-normalization). We further assume that the complementary relation $\overline{R}$ ($(x, y) \in \overline{R}$ iff $(x, y) \notin R$), and its transpose are also serial, i.e. $\forall x,\ R(x) \neq \mathcal{Y}$ and $\forall y,\ R^t(y) \neq \mathcal{X}$. These conditions enforce a non trivial relation between $\mathcal{X}$ and $\mathcal{Y}$. In the following, set complementations will be denoted by means of overbars.

Let S be a subset of $\mathcal{Y}$. We assume $S \neq \emptyset$ and $S \neq \mathcal{Y}$. The relation R and the subset S give birth to the following subset of X, namely the (left) image of S by R

$$R(S) = \{x \in X \mid \exists s \in S, (x,s) \in R\} = \{x \in X \mid S \cap R(x) \neq \emptyset\}.$$

Similarly, we consider $R(\overline{S})$, $\overline{R(S)}$, and $\overline{R(\overline{S})} = \{x \in X \mid \forall s \in \overline{S}, (x,s) \notin R\} = \{x \in X \mid R(x) \subseteq S\}$. The four subsets thus defined can be nicely organized into a square of opposition. See Fig. 2. Indeed, it can be checked that the set counterparts of the relations existing between the logical statements of the traditional square of oppositions still hold here. Namely, $\overline{R(\overline{S})}$ and $R(\overline{S})$ are complements of each other, as $\overline{R(S)}$ and $R(S)$; we have $\overline{R(\overline{S})} \subseteq R(S)$ and $\overline{R(S)} \subseteq R(\overline{S})$, thanks to $\mathcal{X}$-normalization condition; $\overline{R(\overline{S})} \cap \overline{R(S)} = \emptyset$; $R(S) \cup R(\overline{S}) = \mathcal{X}$.

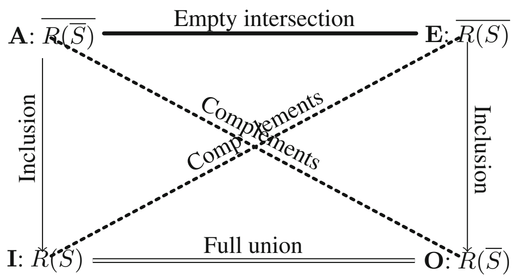

Fig. 2. Square of oppositions induced by a relation R and a subset S

Let us now consider the complementary relation $\overline{R}$. We further assume that $\overline{R} \neq \emptyset$ (i.e., $R \neq \mathcal{X} \times \mathcal{Y}$). Moreover we have also assumed the $\mathcal{X}$-normalization of $\overline{R}$, i.e. $\forall x, \exists y \; (x,y) \notin R$. In the same way as previously, we get four other subsets of $\mathcal{X}$ from $\overline{R}$. Namely, $\overline{R}(\overline{S}) = \{x \in \mathcal{X} \mid \exists s \in \overline{S}, (x,s) \notin R\} = \{x \in \mathcal{X} \mid S \cup R(x) \neq \mathcal{X}\}$; $\overline{R}(S)$; $\overline{\overline{R}(\overline{S})}$; $\overline{\overline{R}(S)} = \{x \in \mathcal{X} \mid \forall s \in S, (x,s) \in R\} = \{x \in \mathcal{X} \mid S \subseteq R(x)\}$. This generates a second square of opposition denoted by **aeoi**.

As can be seen, when R is a formal context (i.e., $\mathcal{X} = \mathcal{O}$, $\mathcal{Y} = \mathcal{P}$), we have $R^{\Pi}(S) = R(S)$, $R^{N}(S) = \overline{R(\overline{S})}$, $R^{\Delta}(S) = \overline{\overline{R}(S)}$, $R^{\nabla}(S) = \overline{R}(\overline{S})$. The eight subsets involving R and its complement can be organized into a cube of opposition as in Fig. 3. The four formal concept analysis operators correspond to the left side facet of the cube of oppositions. The full cube is then obtained by introducing their complements, giving birth to the right side facet. Since $\overline{R^{\Pi}(S)} = R^{N}(\overline{S})$, and $\overline{R^{\Delta}(S)} = R^{\nabla}(\overline{S})$, the classical square of oppositions **AEOI** is given by the four corners $R^{N}(S)$, $R^{N}(\overline{S})$, $R^{\Pi}(\overline{S})$, and $R^{\Pi}(S)$, and the second square **aeoi** on the back of the cube is given by $R^{\Delta}(S)$, $R^{\Delta}(\overline{S})$, $R^{\nabla}(\overline{S})$, and $R^{\nabla}(S)$.

Moreover, in the side facets, all edges are uni-directed, including the diagonal ones, and express inclusions. Indeed, as already established in [16], under the $\mathcal{X}$- and $\mathcal{Y}$–normalization hypotheses, the following inclusion relation holds:

$$R^{N}(S) \cup R^{\Delta}(S) \subseteq R^{\Pi}(S) \cap R^{\nabla}(S).$$

$R^N(S)$, $R^{\Delta}(S)$, $R^{\Pi}(S)$, and $R^{\nabla}(S)$ constitute, four distinct pieces of information [16], which are only (weakly) related by the above relation.

Lastly, it can be checked that we also have $R^{\Delta}(S) \cap R^N(\overline{S}) = \emptyset$ and $R^{\Delta}(\overline{S}) \cap R^N(S) = \emptyset$ on the one hand, and $R^{\nabla}(S) \cup R^{\Pi}(\overline{S}) = \mathcal{X}$ and $R^{\nabla}(\overline{S}) \cup R^{\Pi}(S) = \mathcal{X}$ on the other hand. These are the relations that holds on the top and on the bottom facets of the cube respectively.

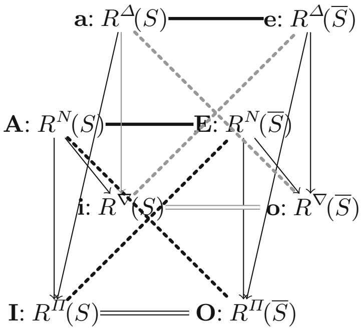

Fig. 3. Cube of opposition in formal concept analysis

The cube of oppositions not only underlie FCA (and PoTh) [25], but also is a setting of interest for building bridges with rough set theory [40] (see [12]), or even formal argumentation [1]!

3 Formal Context Decomposition

In FCA, a formal concept [30] is defined as a pair $(X, Y) \in \mathcal{O} \times \mathcal{P}$ such that

$$R^{\triangle}(X) = Y \text{ and } R^{t\triangle}(Y) = X,$$

where $R^{t\triangle}(Y) = \{x \in \mathcal{O} | R(x) \supseteq Y\} = \bigcap_{y \in Y} R^t(y)$ is the set X of objects having all properties in Y, and in this case Y is also the maximal set of properties shared by all objects in X. A formal concept (X, Y) is a maximal sub-rectangle in the formal context, i.e. is such that $X \times Y \subseteq R$. It can be checked that $R^{\triangledown}$ gives back the same Galois connection as the one defined from $R^{\triangle}$, while R^N (or R^{Π}) induces another connection, which is now described.

Consider the connection defined from R^N in a similar formal way as when defining formal concepts. It was proposed by Popescu [41] and studied in a general setting of residuated algebras, but not in the usual Boolean setting. Namely, let us consider pairs (X, Y) s.t. $R^N(X) = Y$ and $R^{tN}(Y) = X$. As suggested in [22], the pairs (X, Y) s.t. $R^N(X) = Y$ and $R^{tN}(Y) = X$ allow us to characterize independent sub-contexts (i.e. that have no common objects and no common properties). They are thus of interest for the decomposition of a formal context into smaller independent ones. This is expressed through the following property, proved in [13,24]:[1]

Proposition 1. *The following properties of pairs* (X, Y) *are equivalent*

1. $R^N(X) = Y$ *and* $R^{tN}(Y) = X$
2. $R^N(\overline{X}) = \overline{Y}$ *and* $R^{tN}(\overline{Y}) = \overline{X}$
3. $R^{\Pi}(X) = Y$ *and* $R^{t\Pi}(Y) = X$
4. $R \subseteq (X \times Y) \cup (\overline{X} \times \overline{Y})$

Proof. Let us first show that Property 1 implies Property 4. First it is clear that: $R^N(X) = Y \Leftrightarrow \bigcap_{x \in \overline{X}} \overline{R(x)} = Y \Leftrightarrow \bigcup_{x \in \overline{X}} R(x) = \overline{Y}$.

Denoting $X + Y = \overline{\overline{X} \times \overline{Y}}$, it implies $R \subseteq X + \overline{Y}$.

Likewise due to $R^{tN}(Y) = X$, $\quad R^t \subseteq Y + \overline{X}$ holds.

Finally: $R \subseteq (X + \overline{Y}) \cap (Y + \overline{X})$, which equivalently writes: $R \subseteq (X \times Y) \cup (\overline{X} \times \overline{Y})$.

Conversely assume Property 4. Then it is clear that $R^N(X) \subseteq Y$ and $R^{tN}(Y) \subseteq X$ hold since there is no property possessed by any object in X outside Y, and no object outside X that possesses a property outside Y. Suppose $R^N(X) \subset Y$, i.e. $\exists y^* \in Y$ such that property y^* is possessed by objects outside X. But then $R(x, y^*) = 1$ for some $x \in X, y \in \overline{Y}$. So Property 4 does not hold. Contradiction.

The invariance of Property 4 with respect to complementation proves that the choice of (X, Y) versus $(\overline{X}, \overline{Y})$ in Property 1 is immaterial. Hence the equivalence with Property 2. For Property 3, note that $R^N(X) = Y$ is equivalent to $R^{\Pi}(\overline{X}) = \overline{Y}$. □

Thus, (X, Y) and $(\overline{X}, \overline{Y})$ are two independent sub-contexts in R, in the sense that there is no object / property pair (x, y) of the context R either in $X \times \overline{Y}$ or in $\overline{X} \times Y$. The above proposition does not involve any minimality in the inclusion Property 4 of the above proposition. In particular, the pair $(\mathcal{O}, \mathcal{P})$ trivially satisfies it. However, this result leads to a decomposition of R into a disjoint union of *minimal* independent sub-contexts. Indeed, suppose two pairs (X_1, Y_1), (X_2, Y_2) satisfy Proposition 1. It implies that for instance, the pair $(X_1 \cap X_2, Y_1 \cap Y_2)$ satisfies it (it can be checked that $R^N(X_1 \cap X_2) = Y_1 \cap Y_2$), and likewise with any element of the partition refining both partitions $(X_1, \overline{X_1})$ and $(X_2, \overline{X_2})$. Due to point 4 of Proposition 1, it yields

$$R \subseteq ((X_1 \times Y_1) \cup (\overline{X_1} \times \overline{Y_1})) \cap ((X_2 \times Y_2) \cup (\overline{X_2} \times \overline{Y_2})),$$

where the intersection on the right-hand side comes down to the union of subcontexts $(X_1 \cap X_2) \times (Y_1 \cap Y_2)$, $(X_1 \cap \overline{X_2}) \times (Y_1 \cap \overline{Y_2})$, $(\overline{X_1} \cap X_2) \times (\overline{Y_1} \cap Y_2)$, $(\overline{X_1} \cap \overline{X_2}) \times (\overline{Y_1} \cap \overline{Y_2})$. The decomposition of R into minimal subcontexts is achieved by taking the following intersection

$$\bigcap_{(X,Y): R^N(X) = Y, R^{tN}(Y) = X} (X \times Y) \cup (\overline{X} \times \overline{Y}).$$

[1] We again provide the proof for the sake of self-containedness.

Example 1. The table below presents a formal context. Pairs $(\{6, 7, 8\}, \{c, d, e\})$, or $(\{5, 6, 7, 8\}, \{d, e\})$, or $(\{2, 3, 4\}, \{g, h\})$ are examples of formal concepts, while $(\{5, 6, 7, 8\}, \{a, b, c, d, e\})$, $(\{2, 3, 4\}, (\{f, g, h\})$, $(\{1\}, \{i\})$ are minimal subcontexts.

		objects							
p		1	2	3	4	5	6	7	8
r	a							×	
o	b					×	×		
p	c						×	×	×
e	d					×	×	×	×
r	e					×	×	×	×
t	f		×		×				
i	g		×	×	×				
e	h		×	×	×				
s	i	×							

Thus, through the notions of formal sub-contexts and of formal concepts, one sees two key aspects of granulation at work. Namely, on the one hand independent sub-contexts are separated, while *inside* each sub-context, formal concepts (X, Y) are identified where each object in X is associated with each property in Y. However, objects in the extension of a formal concept may not be fully similar since they may also possess properties outside the intension of the concept. They are only similar with respect to the properties associated to the formal concept.

Thus, the classical Galois connection founding formal concept analysis (associated with the actual possibility operator), and the other connection induced by the actual necessity operator, respectively embed two basic ideas associated with the idea of a cluster (see, e.g., [35]), namely

1. any pair of elements in a cluster should be closely related in some sense, and
2. any element of a cluster should be sufficiently separated from any element outside it.

Moreover, formal concept analysis is also useful for conceptual clustering, where clusters should be associated with labels, obtained in this case as a conjunction of the properties shared by the objects in the cluster [11].

Such an idea can be also stated in terms of graph clustering, taking advantage of an exact parallel between formal concept analysis and bipartite graph analysis [31], as viewing an (ideal) cluster as a group of vertices

1. either with no missing link inside the group,
2. or with no link with vertices outside the group.

These two complementary views are also clearly at the basis of cluster analysis for unipartite graphs [42].

In practice, it is important to introduce some tolerance in the evaluation of the similarity between the members of a cluster and in the separatedness of the clusters, leading to a more permissive and approximate view of granules or clusters; see, e.g., [32].

The other (mixed) connections $R^{\Theta}(S) = T$ and $R^{t\Lambda}(T) = S$ where $\Theta, \Lambda \in \{\Pi, N, \Delta, \nabla\}$ with $\Theta \neq \Lambda$ are also worth studying. They have still to be better understood and to be investigated systematically. See [15] for a preliminary discussion, and [7,9] for results in the graded case.

4 Graded Links Between a Property and an Object

Fuzzy extensions of FCA where R is a fuzzy relation in $L^{\mathcal{O}\times\mathcal{P}}$ with L often taken as the unit interval have been proposed early [6,10]. However, the development of a fuzzy formal concept analysis theory requires an appropriate algebra of fuzzy sets [6,8]. While many theoretical studies have been developed, the different gradual interpretations of a fuzzy formal context have not been much discussed. Following [14], this section highlights some basic issues regarding the fact that a "fuzzy" or graded extension of binary formal contexts may convey different semantics: graded satisfaction of properties vs. uncertainty.

4.1 Gradual Properties: Unipolar Vs. Bipolar Scale Interpretation

In this first interpretation, the values in the table (which are scalars in L) may be understood as providing a refinement of the cross marks. Namely, they represent to what extent an object has a property, while in the classical model, this relationship was not a matter of degree. It is important to remark that in this view, we do not refine the absence of a property for an object (the blank mark is always replaced by the bottom element 0 of L). This view will be referred to as the *positive unipolar interpretation*. In this interpretation, $R^t(y)$ (resp. $R(x)$) is considered as the support of the fuzzy set of objects (resp. properties) satisfying the property y (resp. the object x). One could also consider the opposite convention namely the *negative unipolar interpretation* where degrees would represent to which extent an object does not have a property and equivalently provide a refinement of the blank marks.

The most commonly used interpretations, through existing FCA proposals, are implicitly based on the positive unipolar interpretation that allows to map a formal context with quantitative attributes into a fuzzy formal context. In this spirit, conceptual scale theory [44] may be used to achieve a suitable (Boolean) representation by successive subsumptions.

Example 2. For instance, the formal context illustrated in Table 2 is obtained from Table 1 by a conceptual scaling of both many-valued attributes "Age"and "Salary". As can be seen, we have two sets of properties with obvious subsumption relations between them. Pairs $(\{Peter, Sophie, Mike, Joe\}, \{age \geq 20, salary \geq 1000\})$, $(\{Sophie, Mike\}, \{age \geq 20, age \geq 25, salary \geq 1000, salary \geq 1200\})$, or $(\{Mike\}, \{age \geq 20, age \geq 25, age \geq 30, salary \geq 1000, salary \geq 1200, salary \geq 1400\})$ are formal concepts.

Table 1. Many-valued relation

R_1	*Pierre*	*Sophie*	*Mike*	*Nahla*
Age	22	28	30	22
Salary	1100	1300	1500	1500

Table 2. Context subsumption

R_2	*Pierre*	*Sophie*	*Mike*	*Nahla*
age ≥ 20	×	×	×	×
age ≥ 25		×	×	
age ≥ 30			×	
salaire ≥ 1000	×	×	×	×
salaire ≥ 1200		×	×	×
salaire ≥ 1400			×	×

Table 3. Context summarization

R_3	*Pierre*	*Sophie*	*Mike*	*Nahla*
age 'young'	1	0.7	0.6	1
salary 'low'	1	0.8	0.6	0.6

Knowing the ages and the salaries, the formal context R_2 can be re-encoded in a more compact way, using two fuzzy sets 'young' and 'small' with decreasing membership functions, as illustrated in Table 3.

Observe also that R_3 offers a more precise representation of initial data than Table 2. The context in Table 3, event though more compact than Table 2 highlights the fact that *Mike*, and to a lesser extent *Sophie* are not very young and have a salary that is not really low. It constitutes in some sense the negative of the picture shown on Table 1. Note that the type of representation on Table 3 can be obtained even without providing interpretable fuzzy sets and thus, by normalizing in L the domain of attribute values. This approach is used in [36].

Another interpretation of the degrees, maybe more in the standard spirit of fuzzy logic would be to replace both the *cross* marks and the *blank* marks by values in the scale L ($L = [0, 1]$). Then L possesses a mid-point acting as a pivoting value between the situations where the object possesses the property to some extent and the converse situation where the object possesses the opposite property to some extent. Under this view, a fuzzy formal concept should be learnt together with its negation. This view corresponds to a *bipolar scale interpretation*.

4.2 Uncertainty

Neither the standard FCA approach nor its fuzzy extension are equipped for representing situations of partial or complete ignorance. To this end, in the

Boolean case, we need to introduce a proper representation of partial uncertainty including ignorance in the relational table of the formal context. One may think of introducing gradations of uncertainty by changing crosses and blanks in the table into probability degrees, or by possibility or necessity degrees. In the probabilistic case, one number shall assess the probability that a considered property holds for a given object (its complement to 1 corresponding to the probability it does not hold). However, this is assuming precise knowledge on the probability values, which is not really appropriate if we have to model the state of complete ignorance. It is why we investigate the use of the possibilistic setting in the following.

In the possibilistic setting, crosses may be replaced by positive degrees of necessity for expressing some certainty that an object satisfies a property. The blanks could be refined by possibility degrees less than 1, expressing that it is little possible that an object satisfies a property. However, this convention using a single number in the unit interval for each entry in the context may be misleading as when the number replaces a blank or a cross, the meaning of the number is not the same.

In the possibilistic setting, possibility and necessity functions are related by the duality relation $N(A) = 1 - \Pi(\overline{A})$, that holds for any event A, where $\overline{A}$ denotes the opposite event [18]. Then, for entries (x, y) in the table, we use a representation as *a pair* of necessity degrees $(\alpha, 1-\beta)$ where $\alpha = N((x, y) \in R)$ (resp. $1-\beta = N((x, y) \notin R)$) corresponds to the necessity (certainty) that object x has (resp. does not have) property y. Moreover, we should respect the property $\min(\alpha, 1-\beta) = 0$, since $\min(N(A), N(\overline{A})) = 0$ in agreement with complete ignorance, in which case nothing (i.e., neither A nor $\overline{A}$) is even somewhat certain. Pairs (1,0) and (0,1) correspond to completely informed situations where it is known that object x has, respectively does not have, property y. The pair (0,0) reflects total ignorance, whereas pairs $(\alpha, 1-\beta)$ s.t. $1 > \max(\alpha, 1-\beta) > 0$ correspond to partial ignorance.

An uncertain formal context is thus represented by

$$R^U = \{(\alpha(x, y), 1 - \beta(x, y)) \mid x \in \mathcal{O}, y \in \mathcal{P}\}$$

where $\alpha(x, y) \in [0, 1]$, $\beta(x, y) \in [0, 1]$. A relational database with fuzzily-known attribute values is *theoretically* equivalent to the fuzzy set of all ordinary databases corresponding to the different possible ways of completing the information consistently with the fuzzy restrictions on the attribute values. So, an uncertain formal context may be viewed as a weighted family of all standard formal contexts obtained by changing uncertain entries into sure ones. More precisely, one may consider all the completions of an uncertain formal context. This is done by substituting entries (x, y) that are uncertain, i.e., such that $1 > \max(\alpha(x, y), 1 - \beta(x, y))$ by a pair (1,0), or a pair (0,1). Replacing $(\alpha(x, y), 1 - \beta(x, y))$ by $(1, 0)$ is possible at degree $\beta(x, y)$, the possibility that x has property y. Similarly, replacing $(\alpha(x, y), 1 - \beta(x, y))$ by $(0, 1)$ is possible at degree $1 - \alpha(x, y)$, the possibility that x does *not* have the property y. In this way, one may determine to what extent a particular completion (a context C) is possible, by aggregating the possibility degrees associated with each completed entry (using min operator). Formally, one can write

$$\pi(C) = \min(\min_{(x,y):(x,y)\in C}\beta(x,y), \min_{(x,y):(x,y)\notin C}1-\alpha(x,y)).$$

Likewise the degree of possibility that (X, Y) is a formal concept of R^U is

$$\pi(X, Y) = \sup\{\pi(C) : C \text{ such that } (X, Y) \text{ is a formal concept of } C\}.$$

Useful completions are those where partial certainty becomes full certainty. Indeed, given an uncertain formal context and a threshold pair (u, v), let us replace all entries of the form $(\alpha, 0)$ such that $\alpha \geqslant u$ with (1, 0) and entries of the form $(0, 1-\beta)$ such that $1-\beta \geqslant v$ with (0, 1). All such replacements have possibility 1 according to the above formula. Remaining entries, which are more uncertain, can be systematically substituted either by (1,0), or by (0,1). Considering, the two extreme cases where *all* such entries are changed into (1,0) and the case when where *all* such entries are changed into (0,1) gives birth to upper and lower completions, respectively. In this way, two classical (Boolean) formal contexts, denoted $R^*_{(u,v)}$ and $R_{*(u,v)}$ are obtained as respective results of the two completions. They allow to determine, for a given threshold (u, v), maximal extensions (resp. minimal intensions) and minimal extensions (resp. maximal intensions) of uncertain formal concepts. It is clear that $R_{*(u,v)} \subseteq R^*_{(u,v)}$. Let us illustrate the idea with an example.

Example 3. Table 4 exhibits a formal context where some entries are pervaded with uncertainty. Let us examine the situation regarding formal concepts. Take $u = 0.7, v = 0.5$ for instance. In context $\mathcal{R}_{*(0.7,0.5)}$, examples of formal concepts are pairs $(\{6, 7, 8\}, \{c, d, e\})$, or $(\{5, 6, 7, 8\}, \{d, e\})$, or $(\{2, 3, 4\}, \{g, h\})$, although with $u = 0.9$, the last formal concept would reduce to $(\{2, 3\}, \{g, h\})$, i.e., the extent of the concept is smaller.

Now consider $R^*_{(0.7,0.5)}$, where the entries with low certainty levels (either in favor or against the existence of the link between x and y) are turned into

Table 4. Uncertain formal concepts

	1	2	3	4	5	6	7	8
a							×	
b					×	×		
c					(0.5,0)	×	×	×
d					×	×	×	×
e					×	×	×	×
f		(0, 0.8)		×	(0, 0.3)			
g		×	×	(0.8, 0)				
h		×	×	(0.8, 0)				
i	×							

positive links. Then, $(\{2,3,4\},\{g,h\})$ remains unchanged as a formal concept, while a larger concept now emerges, namely $(\{5,6,7,8\},\{c,d,e\})$. However, one may prefer to consider the results obtained from $R_{*(0.7,0.5)}$, where only the almost certain information is changed into positive links. In the example, if we move down u to 0.5, and use $R_{*(0.5,0.5)}$ we still validate the larger former concept $(\{5,6,7,8\},\{c,d,e\})$. This illustrates the fact that becoming less and less demanding on the level of certainty, may enable the fusion of close concepts (here $(\{6,7,8\},\{c,d,e\})$, and $(\{5,6,7,8\},\{d,e\})$, providing a more synthetic view of the formal context.

This small example is intended to illustrate several points. First of all, it should be clear that being uncertain about the existence of a link between an object and a property is not the same as being certain about a gradual link. Second, under uncertainty, there are formal concepts whose boundaries are not affected by uncertainty, while others are. Lastly, regarding certain enough pieces of information as fully certain may help simplifying the analysis of the formal context. Besides, the proposed setting may also handle inconsistent information by relaxing the constraint $\min(\alpha, 1-\beta) = 0$. This would amount to introducing paraconsistent links between objects and properties.

5 More Lines for Further Research

Let us briefly conclude this survey of works in FCA inspired by PoTh by mentioning other examples of lines of interest for further research:

- The parallel of FCA with PoTh leading to the introduction of new operators extends to conceptual pattern structures [28,29], where the description $\partial(x)$ of an object x, may, e.g., be a possibilistic knowledge base [2];
- Applications of FCA to the fusion of conflicting pieces of information issued from multiple sources using pattern structures for labeling sets of possible values in terms of sources supporting them [3];
- The clustering of sets of objects on the basis of approximate concepts [24,32], with labeling of the clusters [38];
- The building of conceptual analogical proportions [37] on the basis of the formal definition of analogical proportions in non-distributive lattices [34], conceptualization and analogical reasoning being two basic cognitive activities [4].

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http://www.springer.com/978-3-319-19544-5

Formal Concept Analysis
13th International Conference, ICFCA 2015, Nerja, Spain,
June 23-26, 2015, Proceedings
Baixeries, J.; Sacarea, C.; Ojeda-Aciego, M. (Eds.)
2015, XII, 319 p. 57 illus., Softcover
ISBN: 978-3-319-19544-5