Chapter 2
Generalities on Distributions

2.1 Definition

Let X be an open set in R^n, $n \in \mathrm{N}$ a fixed integer.

Definition 2.1 Every linear continuous map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is called a distribution or generalized function. In other words, a distribution is a linear map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ such that $u(\phi_n) \longrightarrow_{n\longrightarrow\infty} u(\phi)$ for every sequence $\{\phi_n\}_{n=1}^\infty$ in $\mathscr{C}_0^\infty(X)$ converging to $\phi \in \mathscr{C}_0^\infty(X)$ as $n \longrightarrow \infty$.

The space of distributions on X will be denoted by $\mathscr{D}'(X)$. We will write $u(\phi)$ or (u, ϕ) for the value of the functional (generalized function, distribution) $u \in \mathscr{D}'(X)$ on the element $\phi \in \mathscr{C}_0^\infty(X)$.

Example 2.1 Suppose $0 \in X$ and take the map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ defined as follows

$$u(\phi) = \phi(0) \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X).$$

Let $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X)$ and $\alpha_1, \alpha_2 \in \mathrm{C}$. As

$$u(\phi_1) = \phi_1(0), \quad u(\phi_2) = \phi_2(0),$$

$$u(\alpha_1\phi_1 + \alpha_2\phi_2) = (\alpha_1\phi_1 + \alpha_2\phi_2)(0) = \alpha_1\phi_1(0) + \alpha_2\phi_2(0) = \alpha_1 u(\phi_1) + \alpha_2 u(\phi_2),$$

$u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is linear. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ for which $\phi_n \longrightarrow_{n\longrightarrow\infty} \phi$ in $\mathscr{C}_0^\infty(X)$. Then there exists a compact set $K \subset X$ such that $\mathrm{supp}\phi_n \subset K$ for every $n \in \mathrm{N}$ and $D^\alpha\phi_n \longrightarrow D^\alpha\phi$ uniformly in X for every multi-index $\alpha \in \mathrm{N} \cup \{0\}$. In particular, $\phi_n(0) \longrightarrow_{n\longrightarrow\infty} \phi(0)$, and therefore $u(\phi_n) \longrightarrow_{n\longrightarrow\infty} u(\phi)$. Consequently the linear map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is continuous, in other words it is a distribution on X.

S.G. Georgiev, *Theory of Distributions*, DOI 10.1007/978-3-319-19527-8_2

Exercise 2.1 Let $0 \in X$. For each multi-index α prove that the map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$, defined by

$$u(\phi) = D^\alpha \phi(0) \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X),$$

is a distribution on $\mathscr{C}_0^\infty(X)$.

Exercise 2.2 Denote by δ_a or $\delta(x-a)$, $a \in \mathbb{C}^n$, Dirac's "delta" function at the point a:

$$\delta_a(\phi) = \phi(a) \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X).$$

Prove that δ_a is a distribution on $\mathscr{C}_0^\infty(X)$.

Exercise 2.3 Prove that the map $1 : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$, defined by

$$1(\phi) = \int_X \phi(x)dx \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X),$$

is a distribution on $\mathscr{C}_0^\infty(X)$.

Exercise 2.4 For $u \in L^p_{\text{loc}}(X)$, $p \geq 1$, we define $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ by

$$u(\phi) = \int_X u(x)\phi(x)dx.$$

Prove that u is a distribution on $\mathscr{C}_0^\infty(X)$.

Exercise 2.5 Let $P\frac{1}{x} : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ be the map defined by

$$P\frac{1}{x}(\phi) = P.V. \int_X \frac{\phi(x) - \phi(0)}{x} dx \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X).$$

Prove that $P\frac{1}{x} \in \mathscr{D}'(X)$.

Definition 2.2 The distributions $u, v \in \mathscr{D}'(X)$ are said to be equal if

$$u(\phi) = v(\phi)$$

for any $\phi \in \mathscr{C}_0^\infty(X)$.

Definition 2.3 The linear combination $\lambda u + \mu v$ of the distributions $u, v \in \mathscr{D}'(X)$ is the functional acting by the rule

$$(\lambda u + \mu v)(\phi) = \lambda u(\phi) + \mu v(\phi), \qquad \phi \in \mathscr{C}_0^\infty(X).$$

This makes the set $\mathscr{D}'(X)$ a vector space.

Definition 2.4 Let $u \in \mathscr{D}'(X)$. We define a distribution $\overline{u} \in \mathscr{D}'(X)$, called the complex conjugate of u, by

$$\overline{u}(\phi) = \overline{u(\overline{\phi})}, \qquad \phi \in \mathscr{C}_0^\infty(X).$$

The distributions

$$\mathrm{Re}(u) = \frac{u + \overline{u}}{2}, \qquad \mathrm{Im}(u) = \frac{u - \overline{u}}{2i}$$

are respectively called the real and imaginary parts of u. Equivalently,

$$u = \mathrm{Re}(u) + i\mathrm{Im}(u), \qquad \overline{u} = \mathrm{Re}(u) - i\mathrm{Im}(u).$$

If $\mathrm{Im}(u) = 0$, u is said to be a real distribution.

Exercise 2.6 Prove that the delta function is a real distribution.

Here are elementary properties of distributions. If $u_1, u_2 \in \mathscr{D}'(X)$, then

1. $u_1 \pm u_2 \in \mathscr{D}'(X)$,
2. $\alpha u_1 \in \mathscr{D}'(X) \quad \text{for} \quad \forall \alpha \in \mathrm{C}$.

These properties follow from the definition, so their proof is omitted.
For $u \in \mathscr{D}'(X)$ and $a \in \mathrm{C}^n$, $|a| \neq 0$, $b \in \mathrm{C}$, $b \neq 0$, we define following distributions

1. $u(\phi)(x + a) = u(\phi(x - a))(x) \quad \forall \phi \in \mathscr{C}_0^\infty(X)$,
2. $u(\phi)(bx) = \frac{1}{|b|^n} u\left(\phi\left(\frac{x}{b}\right)\right)(x) \quad \forall \phi \in \mathscr{C}_0^\infty(X)$.

Example 2.2 For $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ we have

$$\delta(\phi)(x + 1 - 2i) = \delta(\phi(x - 1 + 2i))(x) = \phi(-1 + 2i),$$
$$\delta(\phi)(2ix) = \frac{1}{2}\delta\left(\phi\left(\frac{x}{2i}\right)\right)(x) = \frac{1}{2}\phi(0).$$

Exercise 2.7 Compute

$$\delta(\phi)(2x + 3i)$$

for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$.

Answer $\frac{1}{2}\phi\left(-\frac{3i}{2}\right)$.

If u is a distribution on X, then for every compact subset K of X there exist constants C and k so that the inequality

$$|u(\phi)| \leq C \sum_{|\alpha|\leq k} \sup_K \left| D^\alpha \phi(x) \right| \tag{2.1}$$

holds for every $\phi \in \mathscr{C}_0^\infty(K)$. Actually, we suppose there exists a compact set K in X so that

$$|u(\phi_n)| > n \sum_{\alpha\in \mathrm{N}^n \cup \{0\}} \sup_K \left| D^\alpha \phi_n(x) \right| \tag{2.2}$$

holds for $\phi_n \in \mathscr{C}_0^\infty(K)$. We set

$$\psi_n(x) = \frac{\phi_n(x)}{n \sum_{\alpha\in \mathrm{N}^n \cup \{0\}} \sup_K \left| D^\alpha \phi_n(x) \right|}.$$

From (2.2) we obtain

$$|u(\psi_n)| \geq 1. \tag{2.3}$$

By the definition of $\psi_n(x)$ it follows that $\psi_n \longrightarrow_{n\longrightarrow\infty} 0$ in $\mathscr{C}_0^\infty(X)$. Since $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is continuous, we have

$$u(\psi_n) \longrightarrow_{n\longrightarrow\infty} 0,$$

which contradicts (2.3).
If $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is a linear map such that for every compact set K in X there exist constants $C > 0$ and $k \in \mathrm{N} \cup \{0\}$ for which (2.1) holds, then u is a distribution on X. To show this we will prove that $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is continuous at 0. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ with $\phi_n \longrightarrow_{n\longrightarrow\infty} 0$ in $\mathscr{C}_0^\infty(X)$. Then

$$\sup_K \left| D^\alpha \phi_n(x) \right| \longrightarrow_{n\longrightarrow\infty} 0$$

for every $|\alpha| \leq k$. Hence with (2.1) we conclude

$$u(\phi_n) \longrightarrow_{n\longrightarrow 0} 0.$$

Exercise 2.8 The function $H(x)$, $x \in \mathrm{R}^1$, defined by

$$H(x) = \begin{cases} 1 & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x < 0 \end{cases}$$

is called Heaviside function. We define

$$H(\phi) = \int_{\mathrm{R}^1} H(x)\phi(x)dx,$$

$\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Using inequality (2.1) prove that $H \in \mathscr{D}'(\mathrm{R}^1)$.

2.2 Order of a Distribution

Definition 2.5 If inequality (2.1) holds for some integer k independent of the compact set $K \subset X$, the distribution u is said to be of finite order. The smallest such k is called the order of the distribution u.

The space of distributions on X of finite order is denoted by $D'_F(X)$, and the space of distributions of order $\leq k$ is denoted by $D'^k(X)$. Then

$$D'_F(X) = \bigcup_k D'^k(X).$$

Example 2.3 Dirac's δ function is a distribution of order 0.

Exercise 2.9 Prove that $P\frac{1}{x}$ has order 1 on R^1.

Exercise 2.10 Prove that $P\frac{1}{x}$ is of order 0 on $\mathrm{R}^1 \backslash \{0\}$.

Let

$$\omega_\epsilon(a(x)) = \begin{cases} C_\epsilon e^{-\frac{\epsilon^2}{\epsilon^2 - |a(x)|^2}} & \text{when} \quad |a(x)| \leq \epsilon, \\ 0 \quad \text{when} \quad |a(x)| > \epsilon \end{cases}$$

for $a(x) \in \mathscr{C}^1(X)$ and C_ϵ a constant. It is easy to see that

$$\delta(a(x)) = \lim_{\epsilon \longrightarrow 0} \omega_\epsilon(a(x)).$$

If $a(x) \in \mathscr{C}^1(\mathrm{R}^1)$ has isolated simple zeros $x_1, x_2, \ldots$, then

$$\delta(a(x)) = \sum_k \frac{\delta(x - x_k)}{|a'(x_k)|}.$$

It is enough to prove the assertion on a neighbourhood of the simple zero x_k. Since x_k is an isolated simple zero of $a(x)$, there exists $\epsilon_k > 0$ such that $a(x) \neq 0$ for every

$x \in (x_k - \epsilon_k, x_k + \epsilon_k), x \neq x_k, a(x_k) = 0$. As

$$\Big(\delta(a(x)), \phi(x)\Big) = \int_{x_k-\epsilon_k}^{x_k+\epsilon_k} \delta(a(x))\phi(x)dx =$$

$$= \lim_{\epsilon \longrightarrow 0} \int_{x_k-\epsilon_k}^{x_k+\epsilon_k} \omega_\epsilon(a(x))\phi(x)dx \qquad (a(x) = y)$$

$$= \lim_{\epsilon \longrightarrow 0} \int_{a(x_k-\epsilon_k)}^{a(x_k+\epsilon_k)} \omega_\epsilon(y) \frac{\phi(a^{-1}(y))}{|a'(a^{-1}(y))|} dy$$

$$= \lim_{\epsilon \longrightarrow 0} \int_{a(x_k-\epsilon_k)}^{a(x_k+\epsilon_k)} \omega_\epsilon(y) \frac{\phi(a^{-1}(a(x)))}{|a'(a^{-1}(a(x)))|} dy$$

$$= \frac{\phi(x_k)}{|a'(x_k)|} = \Big(\frac{\delta(x-x_k)}{|a'(x_k)|}, \phi(x)\Big)$$

for $\phi \in \mathscr{C}_0^\infty(x_k - \epsilon_k, x_k + \epsilon_k)$, it follows that

$$\delta(a(x)) = \frac{\delta(x - x_k)}{|a'(x_k)|}$$

on a neighbourhood of the point x_k.

Example 2.4 Let us consider $\delta(\cos x)$. Here $a(x) = \cos x$ and its isolated zeros are $x_k = \frac{(2k+1)\pi}{2}$, $k \in \mathrm{Z}$. We notice that

$$|a'(x_k)| = 1 \qquad \text{for} \qquad k \in \mathrm{Z},$$

so

$$\delta(\cos x) = \sum_k \delta\Big(x - \frac{(2k+1)\pi}{2}\Big).$$

Exercise 2.11 Compute $\delta(x^4 - 1)$.

Answer $\frac{\delta(x-1)+\delta(x+1)}{4}$.

2.3 Sequences

Definition 2.6 The sequence $\Big\{u_n\Big\}_{n=1}^{\infty}$ of elements of $D'(X)$ tends to the distribution u defined on X if

$$\lim_{n \longrightarrow \infty} u_n(\phi) = u(\phi) \quad \forall \phi \in \mathscr{C}_0^\infty(X).$$

If so we write

$$\lim_{n\longrightarrow\infty} u_n = u \qquad \text{or} \qquad u_n \longrightarrow_{n\longrightarrow\infty} u.$$

If $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are two sequences of distributions on X that converge to the distributions u and v respectively, then $\{\alpha u_n + \beta v_n\}_{n=1}^{\infty}$ converges to $\alpha u + \beta v$ on X. Here $\alpha, \beta \in \mathbb{C}$. Indeed, let $\phi \in \mathscr{C}_0^{\infty}(X)$ be arbitrary. Then

$$u_n(\phi) \longrightarrow_{n\longrightarrow\infty} u(\phi), \qquad v_n(\phi) \longrightarrow_{n\longrightarrow\infty} v(\phi).$$

Hence,

$$(\alpha u_n + \beta v_n)(\phi) = (\alpha u_n)(\phi) + (\beta v_n)(\phi)$$

$$= \alpha u_n(\phi) + \beta v_n(\phi) \longrightarrow_{n\longrightarrow\infty} \alpha u(\phi) + \beta v(\phi).$$

Example 2.5 Let $x \in \mathbb{R}^1$ and

$$f_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon} & \text{for} \quad |x| \le \epsilon, \\ 0 & \text{for} \quad |x| > \epsilon. \end{cases}$$

We will compute

$$\lim_{\epsilon\longrightarrow+0} f_\epsilon(x)$$

in $\mathscr{D}'(\mathbb{R}^1)$. Let $\phi \in \mathscr{C}_0^{\infty}(\mathbb{R}^1)$ be arbitrary. Then

$$\lim_{\epsilon\longrightarrow+0} f_\epsilon(\phi)(x) = \lim_{\epsilon\longrightarrow+0} \int_{|x|\le\epsilon} \frac{1}{2\epsilon}\phi(x)dx \qquad (x = \epsilon y)$$

$$= \frac{1}{2} \lim_{\epsilon\longrightarrow+0} \int_{|y|\le1} \phi(\epsilon y)dy$$

$$= \phi(0) = \delta(\phi)(x).$$

Consequently

$$\lim_{\epsilon\longrightarrow+0} f_\epsilon(x) = \delta(x)$$

in $\mathscr{D}'(\mathbb{R}^1)$.

Exercise 2.12 Find

$$\lim_{\epsilon \longrightarrow +0} \frac{2\epsilon}{\pi(x^2+\epsilon^2)}.$$

Answer $2\delta(x)$.

2.4 Support

Definition 2.7 A distribution $u \in \mathscr{D}'(X)$ is said to vanish on an open set $X_1 \subset X$ if its restriction to X_1 is the zero functional in $\mathscr{D}'(X_1)$, i.e., $u(\phi) = 0$ for all $\phi \in \mathscr{C}_0^\infty(X_1)$. This is written $u(x) = 0, x \in X_1$.

Suppose a distribution $u \in \mathscr{D}'(X)$ vanishes on X. Then it vanishes on the neighbourhood of every point in X. Conversely, let $u \in \mathscr{D}'(X)$ vanish on a neighbourhood $U(x) \subset X$ of every point x in X. Consider the cover $\{U(x), x \in X\}$ of X. We will construct a locally finite cover $\{X_k\}$ such that X_k is contained in some $U(x)$. Let

$$X_1^1 \subset\subset X_2^1 \subset\subset \ldots, \qquad \bigcup_{k\geq 1} X_k^1 = X.$$

By the Heine-Borel lemma, the compact set $\overline{X}_1^1$ is covered by a finite number of neighbourhoods $U(x)$, say $U(x_1)$, $U(x_2)$, …, $U(x_{N_1})$. Similarly, the compact set $\overline{X}_2^1 \backslash X_1^1$ is covered by a finite number of neighbourhoods $U(x_{N_1+1})$, …, $U(x_{N_1+N_2})$, and so on. We set

$$X_k = U(x_k) \bigcap X_1^1, \qquad k = 1, 2, \ldots, N_1,$$
$$X_k = U(x_k) \bigcap (\overline{X}_2^1 \backslash X_1^1), \qquad k = N_1+1, \ldots, N_1+N_2,$$

and so forth. In this way we obtain the required cover $\{X_k\}$. Let $\{e_k\}$ be the partition of unity corresponding to the cover $\{X_k\}$ of X. Then

$$\mathrm{supp}(\phi e_k) = 0$$

for every $\phi \in \mathscr{C}_0^\infty(X)$. This implies

$$u(\phi) = u\Big(\sum_{k\geq 1} \phi e_k\Big) = \sum_{k\geq 1} u(\phi e_k) = 0.$$

Consequently the distribution u vanishes on the whole X.

The union of all neighbourhoods where a distribution $u \in \mathscr{D}'(X)$ vanishes forms an open set X_u, called the zero set of the distribution u. Therefore $u = 0$ on X_u, and X_u is the largest open set where u vanishes.

Definition 2.8 The support of a distribution $u \in \mathscr{D}'(X)$ is the complement $\mathrm{supp} u = X \backslash X_u$ of X_u in X.

Note that $\mathrm{supp} u$ is a closed subset in X.

Definition 2.9 The distribution $u \in \mathscr{D}'(X)$ is said to have compact support if $\mathrm{supp} u \subset\subset X$.

Example 2.6 $\mathrm{supp} H = [0, \infty)$.

Exercise 2.13 Find $\mathrm{supp} 1$.

Let A be a closed set in X. With $\mathscr{D}'(X, A)$ we denote the subset of distributions on X whose supports are contained in A, endowed with the following notion of convergence: $u_k \longrightarrow 0$ in $\mathscr{D}'(X, A)$ as $k \longrightarrow \infty$, if $u_k \longrightarrow 0$ in $\mathscr{D}'(X)$ as $k \longrightarrow \infty$ and $\mathrm{supp} u_k \subset A$ for every $k = 1, 2, \ldots$. For simplicity $\mathscr{D}'(A)$ will denote $\mathscr{D}'(\mathrm{R}^n, A)$. Now suppose that for every point $y \in X$ there is a neighbourhood $U(y) \subset\subset X$ on which a given distribution u_y is defined. Assume further that $u_{y_1}(x) = u_{y_2}(x)$ if $x \in U(y_1) \cap U(y_2) \neq \emptyset$. Then there exists a unique distribution $u \in \mathscr{D}'(X)$ so that $u = u_y$ in $U(y)$ for every $y \in X$. To see this we construct, starting as previously with the cover $\{U(y), y \in X\}$, the locally finite cover $\{X_k\}$, $X_k \subset U(y_k)$, and the corresponding partition of unity $\{e_k\}$. We also set

$$u(\phi) = \sum_{k \geq 1} u_{y_k}(\phi e_k), \quad \phi \in \mathscr{C}_0^\infty(X). \tag{2.4}$$

The number of summands in the right-hand side of (2.4) is finite and does not depend on $\phi \in \mathscr{C}_0^\infty(X')$, for any $X' \subset\subset X$. By definition (2.4) u is linear and continuous on $\mathscr{C}_0^\infty(X)$, i.e., $u \in \mathscr{D}'(X)$. Furthermore if $\phi \in \mathscr{C}_0^\infty(U(y))$, then $\phi e_k \in \mathscr{C}_0^\infty(U(y_k))$. From (2.4),

$$u(\phi) = u_y\Big(\phi \sum_{k \geq 1} e_k\Big) = u_y(\phi),$$

i.e., $u = u_y$ on $U(y)$. If we suppose there are two distributions u and $\tilde{u}$ such that $u = u_y$ and $\tilde{u} = u_y$ on $U(y)$ for every $y \in X$, then $u - \tilde{u} = 0$ on $U(y)$ for every $y \in X$. Therefore $u - \tilde{u} = 0$ in X, showing that the distribution u is unique.

The set of distributions with compact support in X will be denoted by $\mathscr{E}'(X)$, and we set $\mathscr{E}'^k(X) = \mathscr{E}'(X) \cap \mathscr{D}'^k(X)$.

2.5 Singular Support

Definition 2.10 The set of points of X not admitting neighbourhoods where $u \in \mathscr{D}'(X)$ coincides with a $\mathscr{C}^\infty$ function is called the singular support of u, written $\mathrm{singsupp}u$.

Hence u coincides with a $\mathscr{C}^\infty$ function on $X\backslash\mathrm{singsupp}u$.

Example 2.7 Let $f \in \mathscr{C}^\infty(X)$. We define the functional u in the following manner:

$$u(\phi) = \int_X f(x)\phi(x)dx, \quad \phi \in \mathscr{C}_0^\infty(X).$$

For $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X)$ and $\alpha_1, \alpha_2 \in \mathrm{C}$, we have

$$\begin{aligned} u(\alpha_1\phi_1 + \alpha_2\phi_2) &= \textstyle\int_X f(x)(\alpha_1\phi_1(x) + \alpha_2\phi_2(x))dx \\ &= \textstyle\int_X (\alpha_1 f(x)\phi_1(x) + \alpha f(x)\phi_2(x))dx \\ &= \alpha_1 \textstyle\int_X f(x)\phi_1(x)dx + \alpha_2 \int_X f(x)\phi_2(x)dx \\ &= \alpha_1 u(\phi_1) + \alpha_2 u(\phi_2). \end{aligned}$$

Therefore u is a linear functional on $\mathscr{C}_0^\infty(X)$. For $\phi \in \mathscr{C}_0^\infty(X)$, moreover, there exists a compact subset K of X such that $\mathrm{supp}\phi \subset K$ and

$$\begin{aligned} |u(\phi)| &= \left|\textstyle\int_X f(x)\phi(x)dx\right| = \left|\int_K f(x)\phi(x)dx\right| \\ &\le \textstyle\int_K |f(x)||\phi(x)|dx \le \int_K |f(x)|dx \sup_{x\in K}|\phi(x)| < \infty. \end{aligned}$$

Consequently the linear functional $u : \mathscr{C}_0^\infty(X) \longrightarrow \mathrm{C}$ is well defined. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ such that $\phi_n \longrightarrow \phi$, $n \longrightarrow \infty$, $\phi \in \mathscr{C}_0^\infty(X)$, in $\mathscr{C}_0^\infty(X)$. Then

$$u(\phi_n) = \int_X f(x)\phi_n(x)dx \longrightarrow_{n\longrightarrow\infty} u(\phi) = \int_X f(x)\phi(x)dx.$$

Therefore $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$ is a linear continuous functional, i.e., $u \in \mathscr{D}'(X)$. Note that $u \equiv f \in \mathscr{C}^\infty(X)$ and therefore $\mathrm{singsupp}u = \emptyset$.

Exercise 2.14 Find $\mathrm{singsupp}P\frac{1}{x}$ for $x \in \mathrm{R}^1\backslash\{0\}$.

Exercise 2.15 Determine $\mathrm{singsupp}P\frac{1}{x}$ for $x \in \mathrm{R}^1$.

Exercise 2.16 Compute $\mathrm{singsupp}P\frac{1}{x^2}$ for $x \in \mathrm{R}^1\backslash\{0\}$.

Exercise 2.17 Find singsupp$P\frac{1}{x^2}$ for $x \in \mathrm{R}^1$.

Definition 2.11 The distribution $u \in \mathscr{D}'(X)$ is called regular if there exists $f \in L^1_{\text{loc}}(X)$ such that

$$u(\phi) = \int_X f(x)\phi(x)dx \quad \text{for} \quad \forall \phi \in \mathscr{C}_0^\infty(X).$$

In this case we will write $u = u_f$. If no such f exists, u is called singular.

Example 2.8 Let $f = \frac{1}{1+x^2}$, $x \in \mathrm{R}^1$. The map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathrm{C}$,

$$u(\phi) = \int_{\mathrm{R}^1} f(x)\phi(x)dx, \quad \phi \in \mathscr{C}_0^\infty(\mathrm{R}^1),$$

is a regular distribution.

Example 2.9 Consider $\delta(x)$, $x \in \mathrm{R}^1$, and suppose that δ is a regular distribution. Then there exists $f \in L^1_{\text{loc}}(\mathrm{R}^1)$ such that $u_f = \delta$. Choose $\rho \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ for which $\text{supp}(\rho) \subset \overline{B_1(0)}$, $\rho(0) = 1$. Define the sequence $\{\rho_n\}_{n=1}^\infty$ by

$$\rho_n(x) = \rho(nx).$$

Then $\text{supp}(\rho_n) \subset \overline{B_{\frac{1}{n}}(0)}$ and $\rho_n(0) = 1$. In addition,

$$\delta(\rho_n) = \rho_n(0) = 1$$

and

$$1 = |\delta(\rho_n)| = \left|\int_{\overline{B_{\frac{1}{n}}(0)}} f(x)\rho(nx)dx\right| \leq \int_{\overline{B_{\frac{1}{n}}(0)}} |f(x)||\rho(nx)|dx$$

$$\leq \sup_{x\in\mathrm{R}^1} |\rho(x)| \int_{\overline{B_{\frac{1}{n}}(0)}} |f(x)|dx \longrightarrow_{n\longrightarrow\infty} 0,$$

which is a contradiction. Therefore $\delta \in \mathscr{D}'(\mathrm{R}^1)$ is a singular distribution.

Exercise 2.18 Let $u_1, u_2 \in \mathscr{D}'(X)$ be regular distributions. Prove that $\alpha_1 u_1 + \alpha_2 u_2$ is a regular distribution for every $\alpha_1, \alpha_2 \in \mathrm{C}$.

Exercise 2.19 Show that singular distributions form a vector subspace of $\mathscr{D}'(X)$ over C.

2.6 Measures

Definition 2.12 A measure on a Borel set A is a complex-valued additive function

$$\mu(E) = \int_E \mu(dx),$$

that is finite ($|\mu(E)| < \infty$) on any bounded Borel subset E of A.

The measure $\mu(E)$ of A can be represented in a unique way in terms of four nonnegative measures $\mu_i(E) \geq 0$, $i = 1, 2, 3, 4$, on A in the following way

$$\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$$

and

$$\int_E \mu(dx) = \int_E \mu_1(dx) - \int_E \mu_2(dx) + i\int_E \mu_3(dx) - i\int_E \mu_4(dx).$$

The measure $\mu(E)$ on the open set X determines a distribution μ on X as follows

$$\mu(\phi) = \int_X \phi(x)\mu(dx), \quad \phi \in \mathscr{C}_0^\infty(X),$$

where $\int$ is the Lebesgue-Stieltjes integral. From the integral's properties it follows that $\mu \in \mathscr{D}'(X)$. Every measure μ of X for which $\mu(dx) = f(x)dx, f \in L^1_{\mathrm{loc}}(X)$, defines a regular distribution.
Let $u \in \mathscr{D}'(X)$ define a measure μ of X. Then

$$|u(\phi)| = \left|\int_{X_1} \phi(x)\mu(dx)\right| \leq \int_{X_1} \mu(dx) \sup_{x \in X_1} |\phi(x)|$$

for every $X_1 \subset\subset X$ and every $\phi \in \mathscr{C}_0^\infty(X_1)$. Hence $u \in \mathscr{D}'^0(X)$.

Now we suppose $u \in \mathscr{D}'^0(X)$, i.e., for every $X_1 \subset\subset X$

$$|u(\phi)| \leq C(X_1) \sup_{x \in X_1} |\phi(x)|$$

where $C(X_1)$ is a constant which depends on X_1. Let $\{X_k\}_{k=1}^\infty$ be a family of open sets such that $X_k \subset\subset X_{k+1}$, $\cup_k X_k = X$. Since $\mathscr{C}_0^\infty(X_k)$ is dense in $\mathscr{C}_0(\overline{X_k})$, the Riesz-Radon theorem implies that there exists a measure μ_k of $\overline{X_k}$ such that

$$u(\phi) = \int_{X_k} \phi(x)\mu_k(dx), \quad \phi \in \mathscr{C}_0(\overline{X_k}).$$

Therefore the measures μ_k and μ_{k+1} coincide on X_k. From this we conclude that there is a measure μ on X which coincides with μ_k on X_k and with the distribution u on X.

Definition 2.13 The distribution $u \in \mathscr{D}'(X)$ is called nonnegative on X if $u(\phi) \geq 0$ for every $\phi \in \mathscr{C}_0^\infty(X)$, $\phi(x) \geq 0$, $x \in X$.

Example 2.10 The distribution 1 is nonnegative.

Exercise 2.20 Prove that the distribution H is nonnegative.

Exercise 2.21 Prove that the distribution 1 is a measure.

2.7 Multiplying Distributions by $\mathscr{C}^\infty$ Functions

Definition 2.14 The product of a distribution $u \in \mathscr{D}'(X)$ by a function $b \in \mathscr{C}^\infty(X)$ is defined by

$$bu(\phi) = u(b\phi) \quad \text{for} \quad \phi \in \mathscr{C}_0^\infty(X).$$

We have

$$\begin{aligned} bu(\alpha_1\phi_1 + \alpha_2\phi_2) &= u(b(\alpha_1\phi_1 + \alpha_2\phi_2)) \\ &= u(\alpha_1 b\phi_1 + \alpha_2 b\phi_2) = \alpha_1 u(b\phi_1) + \alpha_2 u(b\phi_2) \\ &= \alpha_1 bu(\phi_1) + \alpha_2 bu(\phi_2) \end{aligned}$$

for $\alpha_1, \alpha_2 \in \mathrm{C}$, $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X)$, i.e., bu is a linear map on $\mathscr{C}_0^\infty(X)$. Let $\{\phi_n\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ such that $\phi_n \longrightarrow_{n\longrightarrow\infty} \phi$, $\phi \in \mathscr{C}_0^\infty(X)$, in $\mathscr{C}_0^\infty(X)$. Then $b\phi_n \longrightarrow_{n\longrightarrow\infty} b\phi$ in $\mathscr{C}_0^\infty(X)$. Since $u \in \mathscr{D}'(X)$, we have

$$u(b\phi_n) \longrightarrow_{n\longrightarrow\infty} u(b\phi),$$

so

$$bu(\phi_n) \longrightarrow_{n\longrightarrow\infty} bu(\phi).$$

Consequently bu is a continuous functional on $\mathscr{C}_0^\infty(X)$ and $bu \in \mathscr{D}'(X)$.

Example 2.11 Take $x^2\delta$. Then

$$x^2\delta(\phi) = \delta(x^2\phi) = 0^2\phi(0) = 0$$

for $\phi \in \mathscr{C}_0^\infty(X)$. Therefore $x^2\delta = 0$.

Exercise 2.22 Compute $(x^2 + 1)\delta$.

Answer δ.

Let $\alpha_1, \alpha_2 \in$ C, $b_1, b_2 \in \mathscr{C}^\infty(X)$ and $u_1, u_2 \in \mathscr{D}'(X)$. Then

1. $(\alpha_1 b_1(x) + \alpha_2 b_2(x))u_1 = \alpha_1 b_1(x)u_1 + \alpha_2 b_2(x)u_1$,
2. $b_1(x)(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 b_1(x)u_1 + \alpha_2 b_1(x)u_2$.

Let us prove that this multiplication is neither associative nor commutative. Suppose the contrary, so

$$x\delta(\phi) = \delta(x\phi) = 0\phi(0) = 0(\phi),$$

$$xP\tfrac{1}{x}(\phi) = P\tfrac{1}{x}(x\phi) = P.V.\int_{\mathrm{R}^1} \phi(x)dx = 1(\phi)$$

for $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Hence

$$0 = 0P\tfrac{1}{x} = (x\delta(x))P\tfrac{1}{x} = (\delta(x)x)P\tfrac{1}{x} = \delta(x)(xP\tfrac{1}{x}) = \delta(x)1 = \delta(x),$$

a contradiction.

2.8 Exercises

Problem 2.1 Let α be a multi-index and set $u(\phi) = D^\alpha\phi(x_0)$, $\phi \in \mathscr{C}_0^\infty(X)$ for a given $x_0 \in X$. Prove that u is a distribution of order $|\alpha|$.

Proof Let $\phi_1, \phi_2 \in \mathscr{C}_0^\infty(X)$ and $a, b \in$ C. Then

$$u(a\phi_1 + b\phi_2) = D^\alpha(a\phi_1 + b\phi_2)(x_0) = aD^\alpha\phi_1(x_0) + bD^\alpha\phi_2(x_0) = au(\phi_1) + bu(\phi_2).$$

Consequently u is a linear map on $\mathscr{C}_0^\infty(X)$. Let K be a compact subset of X and $\phi \in \mathscr{C}_0^\infty(K)$. Since supp $\phi \subset K$ we have to consider two cases: $x_0 \in K$ and $x_0 \notin K$. If $x_0 \in K$,

$$|u(\phi)| \leq C \sum_{|\beta| \leq |\alpha|} \sup_K \left| D^\beta u(\phi)(x) \right| \tag{2.5}$$

for $C \geq 1$. If $x_0 \notin K$, then $u(\phi) = 0$. Therefore inequality (2.5) holds, and then $u \in \mathscr{D}'(X)$. Using the definition of u and (2.5) we conclude that u has order $|\alpha|$.

Problem 2.2 Take $f \in \mathscr{C}(\mathrm{R}^n)$ and a multi-index α. Let $D^\alpha f$ be defined on $\mathscr{C}_0^\infty(\mathrm{R}^n)$ as follows:

$$D^\alpha f(\phi) = (-1)^{|\alpha|} \int_{\mathrm{R}^n} f(x) D^\alpha \phi(x) dx.$$

Prove that $D^\alpha f$ is a distribution of order $|\alpha|$.

Problem 2.3 Show $\delta_a \in D'^0(\mathrm{R}^n)$.

Problem 2.4 Let $P\frac{1}{x^2}$ be defined on $\mathscr{C}_0^\infty(\mathrm{R}^1)$ by

$$P\frac{1}{x^2}(\phi) = P.V. \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(0)}{x^2} dx.$$

Prove that $P\frac{1}{x^2}$ is a distribution.

Problem 2.5 Define u by

$$u(\phi) = \int_{|x| \le 1} \phi(x) dx \quad \forall \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n).$$

Prove that $u \in D'(\mathrm{R}^n)$.

Problem 2.6 Define

$$u(\phi) = \int_{|x| \le 1} D^\alpha \phi(x) dx \quad \forall \phi \in \mathscr{C}_0^\infty(\mathrm{R}^n),$$

where α is a multi-index. Show that $u \in D'(\mathrm{R}^n)$.

Problem 2.7 Prove that $H \in D'^0(\mathrm{R}^1)$.

Problem 2.8 Let

$$u(\phi) = \sum_{q=0}^{\infty} \phi^{(q)}\left(\frac{1}{q}\right) \quad \forall \phi \in \mathscr{C}_0^\infty(0,1).$$

Prove that u belongs to $D'(0,1)$ but not to $D'_F(0,1)$.

Problem 2.9 Let $P(x, D) = \sum_{|\alpha| \le q} a_\alpha(x) D^\alpha$, where $q \in \mathrm{N} \cup \{0\}$ is fixed, and $a \in \mathscr{C}(\mathrm{R}^n)$. Let u be defined on $\mathscr{C}_0^\infty(\mathrm{R}^n)$ by

$$u(\phi) = \int_{\mathrm{R}^n} u(x) P(x, D) \phi(x) dx.$$

Prove that $u \in D'^q(\mathrm{R}^n)$.

Problem 2.10 Let $u \in D'(X)$ and suppose $u(\phi) \geq 0$ for every nonnegative function $\phi \in \mathscr{C}_0^\infty(X)$. Prove that u is a measure, i.e., a distribution of order 0.

Proof Let $K \subset X$ be a compact set. Then there exists a function $\chi \in \mathscr{C}_0^\infty(X)$ such that $0 \leq \chi(x) \leq 1$ on X and $\chi = 1$ on K. Then

$$\chi \sup_K |\phi| \pm \phi \geq 0$$

for every $\phi \in \mathscr{C}_0^\infty(K)$, and therefore

$$u(\chi \sup_K |\phi| \pm \phi) \geq 0. \tag{2.6}$$

On the other hand,

$$u(\chi \sup_K |\phi| \pm \phi) = \sup_K |\phi| u(\chi) \pm u(\phi).$$

Consequently, using (2.6),

$$\pm u(\phi) \leq u(\chi) \sup_K |\phi|.$$

Therefore $u \in D'^0(X)$, i.e., u is a measure.

Problem 2.11 Take $\phi(x, y) \in \mathscr{C}^\infty(X \times Y)$, where Y is an open set in R^m, $m \geq 1$. Suppose there is a compact set $K \subset X$ such that $\phi(x, y) = 0$ for every $x \notin K$. Prove that the map

$$y \longmapsto u(\phi(\cdot, y))$$

is a $\mathscr{C}^\infty$ function for every $u \in D'(X)$ and

$$D_y^\alpha u(\phi(\cdot, y)) = u(D_y^\alpha \phi(\cdot, y))$$

for every multi-index α.

Proof Since $u \in \mathscr{D}'(X)$ and $\phi \in \mathscr{C}_0^\infty(X \times Y)$, we have that $u(\phi(x, y))$ is continuous in the variable y. We will prove

$$\frac{\partial}{\partial y_j} u(\phi(x, y)) = u\Big(\frac{\partial}{\partial y_j} \phi(x, y)\Big) \quad \text{for} \quad x \in K$$

and $j \in \{1, \ldots, m\}$. For $y \in Y$ given,

$$\phi(x, y + h) = \phi(x, y) + \sum_{j=1}^{m} h_j \frac{\partial \phi}{\partial y_j}(x, y) + o(|h|^2)$$

for $\phi \in \mathscr{C}_0^\infty(K \times Y)$. Let $h = (0, \ldots, 0, h_j, 0, \ldots, 0)$. Then

$$\frac{\phi(x, y + h) - \phi(x, y)}{h} = \frac{\partial \phi}{\partial y_j}(x, y) + \frac{1}{h} o(h_j^2).$$

Since u is linear, we have

$$u\Big(\frac{\phi(x, y + h) - \phi(x, y)}{h}\Big) = u\Big(\frac{\partial \phi}{\partial y_j}(x, y)\Big) + \frac{1}{h} u\Big(o(h_j^2)\Big).$$

From this equality we obtain

$$u\Big(\frac{\partial}{\partial y_j}\phi(x, y)\Big) = \frac{\partial}{\partial y_j} u(\phi(x, y))$$

as $h \longrightarrow 0$. By induction

$$u\Big(D_y^\alpha \phi(x, y)\Big) = D_y^\alpha u(\phi(x, y)).$$

Problem 2.12 Prove that a linear map $u : \mathscr{C}_0^\infty(X) \longrightarrow \mathrm{C}$ is a distribution if and only if $u(\phi_j) \longrightarrow_{j \longrightarrow \infty} 0$ for every sequence $\{\phi_j\}_{j=1}^\infty$ of elements of $\mathscr{C}_0^\infty(X)$ with $\phi_j \longrightarrow_{j \longrightarrow \infty} 0$ in $\mathscr{C}_0^\infty(X)$.

Proof Let $u \in D'(X)$ and $\Big\{\phi_n\Big\}_{n=1}^\infty$ be a sequence in $\mathscr{C}_0^\infty(X)$ tending to 0 in $\mathscr{C}_0^\infty(X)$. There is a compact subset K of X such that supp $\phi_n \subset K$ for every natural number n and $D^\alpha \phi_n \longrightarrow_{n \longrightarrow \infty} 0$ for every multi-index α. Hence using (2.1) there exist constants C and k for which

$$\Big|u(\phi_n)\Big| \leq C \sum_{|\alpha| \leq k} \sup_K \Big|D^\alpha \phi_n\Big| \longrightarrow_{n \longrightarrow \infty} 0.$$

Now suppose that for every sequence $\Big\{\phi_n\Big\}_{n=1}^\infty$ in $\mathscr{C}_0^\infty(X)$ tending to 0 in $\mathscr{C}_0^\infty(X)$, we have $u(\phi_n) \longrightarrow_{n \longrightarrow \infty} 0$. Let us assume there exists a compact subset K of X such that

$$|u(\phi_n)| > C \sum_{|\alpha| \leq k} \sup_K \Big|D^\alpha \phi_n\Big|$$

for every constants $C > 0$ and $k \in \mathbb{N} \cup \{0\}$. When $C = n$ and $k = n$, we get

$$|u(\phi_n)| > n \sum_{|\alpha| \leq n} \sup_K \left| D^\alpha(\phi_n) \right|.$$

Let

$$\psi_n = \frac{\phi_n}{\sum_{|\alpha|\leq n} \left| D^\alpha \phi_n \right|}.$$

Since u is linear on $\mathscr{C}_0^\infty(X)$, we obtain

$$\left| u(\psi_n) \right| = \frac{|u(\phi_n)|}{\sum_{|\alpha|\leq n} \left| D^\alpha \phi_n \right|} > n,$$

which is a contradiction because

$$\psi_j \longrightarrow_{j\longrightarrow\infty} 0$$

in $\mathscr{C}_0^\infty(X)$ and $u(\psi_j) \longrightarrow_{j\longrightarrow\infty} 0$.

Problem 2.13 Prove that a linear map $u : \mathscr{C}_0^\infty(X) \longmapsto \mathbb{C}$ is a distribution if and only if there exist functions $\rho_\alpha \in \mathscr{C}(X)$ such that

$$|u(\phi)| \leq \sum_\alpha \sup_K \left| \rho_\alpha D^\alpha \phi \right| \quad \forall \phi \in \mathscr{C}_0^\infty(K), \tag{2.7}$$

for every compact set $K \subset X$, and only a finite number of the ρ_α vanish identically.

Proof

1. Let u be a linear map from $\mathscr{C}_0^\infty(X)$ to $\mathbb{C}$ and $\rho_\alpha \in \mathscr{C}(X)$ be such that inequality (2.7) holds for every $\phi \in \mathscr{C}_0^\infty(X)$ and every compact K. Since $\rho_\alpha \in \mathscr{C}(X)$, there exists a constant C such that

   $$\sup_K |\rho_\alpha| \leq C.$$

   From this and (2.7) it follows that

   $$|u(\phi)| \leq C \sum_\alpha \sup_K \left| D^\alpha \phi \right|.$$

   As only finitely many ρ_α are zero, there is a constant k such that

   $$|u(\phi)| \leq C \sum_{|\alpha|\leq k} \sup_K \left| D^\alpha \phi(x) \right|,$$

   i.e., $u \in \mathscr{D}'(X)$.

2. Let $u \in D'(X)$ and $\{K_j\}$ be compact subsets of X such that any compact subset is contained in some K_j. Take maps $\chi_j \in \mathscr{C}_0^\infty(X)$ with $\chi_j \equiv 1$ on K_j and define

$$\begin{aligned} \psi_j &= \chi_j - \chi_{j-1} \quad j > 1, \\ \psi_1 &= \chi_1. \end{aligned}$$

Any $\phi \in \mathscr{C}_0^\infty(X)$ satisfies

$$\phi = \sum_{j=1}^{\infty} \psi_j \phi. \tag{2.8}$$

Note that only a finite number of summands in (2.8) vanish identically. Moreover,

$$\begin{aligned} &\psi_j \neq 0 \quad \text{on} \quad K_j \setminus K_{j-1} \quad \text{for} \quad j > 1, \\ &\psi_1 \neq 0 \quad \text{on} \quad K_1. \end{aligned}$$

Consequently

$$\text{supp}(\psi_j \phi) \subset \text{supp}\psi_j.$$

As $\psi_j \phi$ has compact support, for every compact K there are constants C and k_j such that

$$|u(\psi_j \phi)| \leq C \sum_{|\alpha| \leq k_j} \sup_K \left| D^\alpha (\psi_j \phi) \right|.$$

From this and (2.8) we obtain

$$\begin{aligned} |u(\phi)| &= \left| \textstyle\sum_j u(\psi_j \phi) \right| \leq \textstyle\sum_j |u(\psi_j \phi)| \\ &\leq C \textstyle\sum_j \sum_{|\alpha| \leq k_j} \sup_K \left| D^\alpha (\psi_j \phi) \right| \\ &\leq C \textstyle\sum_j \sum_{|\alpha| \leq k_j} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sup_K \left| D^\beta \psi_j \right| \sup_K \left| D^{\alpha - \beta} \phi \right|. \end{aligned}$$

If we set

$$\rho_\beta = \sum_j \sum_{|\alpha| \leq k_j} \binom{\alpha}{\beta} D^\beta \psi_j$$

we obtain

$$|u(\phi)| \leq C \sum_{\beta \leq \alpha} \sup_K \left| \rho_\beta D^{\alpha-\beta} \phi \right|.$$

Problem 2.14 Prove that $u \in D'^k(X)$ can be extended in a unique way to a linear map on $\mathscr{C}_0^k(X)$ so that inequality (2.1) holds for every $\phi \in \mathscr{C}_0^k(X)$.

Proof Since the space $\mathscr{C}_0^\infty(X)$ is everywhere dense in $\mathscr{C}_0^k(X)$, for every $\phi \in \mathscr{C}_0^k(X)$ there exists a sequence $\left\{\phi_n\right\}_{n=1}^\infty$ in $\mathscr{C}_0^\infty(X)$ for which $\phi_n \longrightarrow_{n \longrightarrow \infty} \phi$ in $\mathscr{C}_0^k(X)$. Hence

$$|u(\phi_n) - u(\phi_l)| \leq C \sum_{|\alpha| \leq k} \sup_K \left| D^\alpha \phi_n - D^\alpha \phi_l \right| \longrightarrow_{n,l \longrightarrow \infty} 0.$$

Therefore $\left\{u(\phi_n)\right\}_{n=1}^\infty$ is a cauchy sequence in R^1, and as such it converges to, say,

$$u(\phi) = \lim_{n \longrightarrow \infty} u(\phi_n). \qquad (2.9)$$

The claim is that (2.9) is consistent. In fact, let $\left\{\phi_n\right\}_{n=1}^\infty$, $\left\{\psi_n\right\}_{n=1}^\infty$ be two sequences in $\mathscr{C}_0^\infty(X)$ for which

$$\lim_{n \longrightarrow \infty} \phi_n = \lim_{n \longrightarrow \infty} \psi_n = \phi$$

in $\mathscr{C}_0^k(X)$. Then $u(\phi) = \lim_{n \longrightarrow \infty} u(\gamma_n) = \lim_{n \longrightarrow \infty} u(\phi_n) = \lim_{n \longrightarrow \infty} u(\psi_n)$, where $\left\{\gamma_n\right\}_{n=1}^\infty = \left\{\phi_n\right\}_{n=1}^\infty \cup \left\{\psi_n\right\}_{n=1}^\infty$. For the sequence γ_n we have

$$\left| u(\gamma_n) \right| \leq C \sum_{|\alpha| \leq k} \sup_K |D^\alpha \gamma_n \Big|,$$

so

$$\left| u(\phi) \right| \leq C \sum_{|\alpha| \leq k} \sup_K \left| D^\alpha \phi \right|$$

when $n \longrightarrow \infty$.

Problem 2.15 Let $u_n \in D'(X)$, $u_n(\phi) \geq 0$ for every nonnegative $\phi \in \mathscr{C}_0^\infty(X)$ and $u_n \longrightarrow_{n \longrightarrow \infty} u$ in $D'(X)$. Prove that $u \geq 0$ and $u_n(\phi) \longrightarrow_{n \longrightarrow \infty} u(\phi)$ for every $\phi \in \mathscr{C}_0^0(X)$.

Problem 2.16 Prove that the functions

1. $f = e^{\frac{1}{x}}$,
2. $f = e^{\frac{1}{x^2}}$,
3. $f = e^{\frac{1}{x^m}}$, $m \in \mathrm{N}$

do not define distributions, i.e. $f \notin D'(\mathrm{R}^1\backslash\{0\})$ in all cases.

1. *Proof* Take $f(x) = e^{\frac{1}{x}}$ and suppose—by contradiction—that $f \in D'(\mathrm{R}^1\backslash\{0\})$. Pick $\phi_0 \in \mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$ such that $\phi_0(x) \geq 0$ for every $x \neq 0$, $\phi_0(x) = 0$ for $x < 1$ and $x > 2$, and

$$\int_{-\infty}^{\infty} \phi_0(x)dx = 1.$$

Define the sequence $\left\{\phi_k\right\}_{k=1}^{\infty}$ by

$$\phi_k(x) = e^{-\frac{k}{2}} k\phi_0(kx).$$

It satisfies

$$\phi_k(x) \longrightarrow_{k\longrightarrow\infty} 0$$

in $\mathscr{C}_0^\infty(\mathrm{R}^1\backslash\{0\})$, so

$$f(\phi_k) \longrightarrow_{k\longrightarrow\infty} 0.$$

On the other hand,

$$f(\phi_k(x)) = \int_{-\infty}^{\infty} e^{\frac{1}{x}} \phi_k(x)dx$$
$$= \int_1^2 e^{k\left(\frac{1}{y}-\frac{1}{2}\right)} \phi_0(y)dy \geq \int_1^{\frac{3}{2}} e^{k\left(\frac{1}{y}-\frac{1}{2}\right)} \phi_0(y)dy \geq e^{\frac{k}{6}} \int_1^{\frac{3}{2}} \phi_0(y)dy.$$

By this and the definition of $\phi_0(x)$ we conclude

$$\lim_{k\longrightarrow\infty} f(\phi_k(x)) = \infty,$$

which is a contradiction.

2. **Hint.** Use

$$\phi_k(x) = e^{-\frac{k^2}{4}} k\phi_0(kx).$$

3. **Hint.** Use

$$\phi_k(x) = e^{-\left(\frac{k}{2}\right)^m} k\phi_0(kx).$$

Problem 2.17 Given constants $m \in \mathrm{N}$, a_i, $i = 1, 2, \ldots, m$, prove that

$$f = a_1 e^{\frac{1}{x}} + a_2 e^{\frac{1}{x^2}} + \cdots + a_m e^{\frac{1}{x^m}} \notin D'(\mathrm{R}^1 \backslash \{0\}).$$

Hint. Use the previous problem.

Problem 2.18 Show that

1. $\lim_{\epsilon \longrightarrow 0} \int_{\mathrm{R}^1} \frac{\phi(x)}{x - i\epsilon} dx = i\pi\phi(0) + P.V. \int_{\mathrm{R}^1} \frac{\phi(x)}{x} dx$, $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$,
2. $\lim_{\epsilon \longrightarrow 0} \int_{\mathrm{R}^1} \frac{\phi(x)}{x + i\epsilon} dx = -i\pi\phi(0) + P.V. \int_{\mathrm{R}^1} \frac{\phi(x)}{x} dx$, $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$.

1. *Proof* Take $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ with supp $\phi \subset [-R, R]$. Then

$$\int_{\mathrm{R}^1} \frac{\phi(x)}{x - i\epsilon} dx = \int_{-R}^{R} \frac{(x + i\epsilon)\phi(x)}{x^2 + \epsilon^2} dx$$
$$= \int_{-R}^{R} \frac{(x + i\epsilon)(\phi(x) - \phi(0))}{x^2 + \epsilon^2} dx + \int_{-R}^{R} \frac{(x + i\epsilon)\phi(0)}{x^2 + \epsilon^2} dx.$$

From this

$$\lim_{\epsilon \longrightarrow 0} \int_{-R}^{R} \frac{(x + i\epsilon)(\phi(x) - \phi(0))}{x^2 + \epsilon^2} dx = P.V. \int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx.$$

What is more,

$$\lim_{\epsilon \longrightarrow 0} \int_{-R}^{R} \frac{(x + i\epsilon)\phi(0)}{x^2 + \epsilon^2} dx = 2i\phi(0) \lim_{\epsilon \longrightarrow 0} \operatorname{arctg} \frac{R}{\epsilon} = i\pi\phi(0) = i\pi\delta(\phi).$$

2. **Hint.** Use the solution of part 1.

Problem 2.19 Prove that

$$\frac{1}{x - i0} = i\pi\delta + P\left(\frac{1}{x}\right), \qquad \frac{1}{x + i0} = -i\pi\delta + P\left(\frac{1}{x}\right).$$

Hint. Use the previous problem.

Problem 2.20 Prove that

1. $\lim_{\epsilon \longrightarrow +0} \frac{1}{2\sqrt{\pi\epsilon} \exp^{-\frac{x^2}{4\epsilon}}} = \delta(x)$, $\lim_{\epsilon \longrightarrow +0} \frac{1}{\pi x} \sin \frac{x}{\epsilon} = \delta(x)$,
2. $\lim_{\epsilon \longrightarrow +0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} = \delta(x)$, $\lim_{\epsilon \longrightarrow +0} \frac{\epsilon}{\pi x^2} \sin^2 \frac{x}{\epsilon} = \delta(x)$,
3. $\lim_{t \longrightarrow \infty} \frac{\exp^{ixt}}{x - i0} = 2\pi i \delta(x)$, $\lim_{t \longrightarrow \infty} \frac{\exp^{-ixt}}{x - i0} = 0$,

4. $\lim_{t\longrightarrow\infty}\frac{\exp^{ixt}}{x+i0}=0, \lim_{t\longrightarrow\infty}\frac{\exp^{-ixt}}{x+i0}=-2\pi i\delta(x),$
5. $\lim_{t\longrightarrow\infty} t^m\exp^{ixt}=0, m\geq 0, \lim_{t\longrightarrow\infty} P\Big(\frac{\cos tx}{x}\Big)=0,$
6. $\lim_{\epsilon\longrightarrow+0}\frac{1}{\epsilon}\omega\Big(\frac{x}{\epsilon}\Big)=\delta(x), \lim_{n\longrightarrow\infty}\frac{2n^3x^2}{\pi(1+n^2x^2)^2}=\delta(x),$
7. $\lim_{n\longrightarrow\infty}\frac{n}{\pi(1+n^2x^2)}=\delta(x), \lim_{n\longrightarrow\infty}\frac{1}{n\pi}\frac{\sin^2 nx}{x^2}=\delta(x),$
8. $\lim_{n\longrightarrow\infty} f_n(x)=\delta(x)$, where $f_n(x)=\begin{cases}\frac{n}{2} & \text{for} \quad |x|\leq\frac{1}{n}\\ 0 & \text{otherwise;}\end{cases}$
9. $\lim_{n\longrightarrow\infty}\frac{n}{\sqrt{2\pi}}\exp^{-\frac{n^2x^2}{2}}=\delta(x), \lim_{n\longrightarrow\infty}\frac{\sin nx}{\pi x}=\delta(x),$
10. $\lim_{n\longrightarrow\infty}\frac{1}{2}n\exp^{-n|x|}=\delta(x), \lim_{n\longrightarrow\infty}\frac{1}{\pi}\frac{n}{\exp^{nx}+\exp^{-nx}}=\delta(x),$
11. $\lim_{n\longrightarrow\infty}\sqrt{\frac{n}{\pi}}\exp^{-nx^2}=\delta(x), \lim_{n\longrightarrow\infty}\frac{n}{n^2x^2+1}=\delta(x)\pi.$

1. *Proof* Take $\phi(x)\in\mathscr{C}_0^\infty(\mathrm{R}^1)$. Then there exists $R>0$ such that $\mathrm{supp}\phi\subset[-R,R]$. Now,

$$\Big(\frac{1}{2\sqrt{\pi\epsilon}}e^{-\frac{x^2}{4\epsilon}},\phi(x)\Big)=\int_{-R}^R\frac{e^{-\frac{x^2}{4\epsilon}}}{2\sqrt{\pi\epsilon}}\phi(x)dx$$

$$=\frac{1}{\sqrt{\pi}}\int_{-R}^R\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}\Big[\phi(x)-\phi(0)\Big]dx+\frac{\phi(0)}{\sqrt{\pi}}\int_{-R}^R\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}dx$$

$$=\frac{1}{\sqrt{\pi}}\int_{-R}^R\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}x\frac{\phi(x)-\phi(0)}{x}dx+\frac{\phi(0)}{\sqrt{\pi}}\int_{-R}^R e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}d\Big(\frac{x}{2\sqrt{\epsilon}}\Big)$$

$$=\frac{1}{\sqrt{\pi}}\int_{-R}^R\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}x\frac{\phi(x)-\phi(0)}{x}dx+\frac{\phi(0)}{\sqrt{\pi}}\int_{-\frac{R}{2\sqrt{\epsilon}}}^{\frac{R}{2\sqrt{\epsilon}}}e^{-y^2}dy.$$

Therefore

$$\lim_{\epsilon\longrightarrow 0}\Big(\frac{1}{2\sqrt{\pi\epsilon}}e^{-\frac{x^2}{4\epsilon}},\phi(x)\Big)$$

$$=\lim_{\epsilon\longrightarrow 0}\frac{1}{\sqrt{\pi}}\int_{-R}^R\frac{e^{-\left(\frac{x}{2\sqrt{\epsilon}}\right)^2}}{2\sqrt{\epsilon}}x\frac{\phi(x)-\phi(0)}{x}dx+\frac{\phi(0)}{\sqrt{\pi}}\lim_{\epsilon\longrightarrow 0}\int_{-\frac{R}{2\sqrt{\epsilon}}}^{\frac{R}{2\sqrt{\epsilon}}}e^{-y^2}dy$$

$$=\frac{\phi(0)}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-y^2}dy=\phi(0)=\delta(\phi).$$

Problem 2.21 Let $\{X_i\}_{i\in I}$ be an open cover of R^n, and suppose $u_i\in D'(X_i)$ satisfy $u_i=u_j$ on $X_i\bigcap X_j$. Prove that there exists a unique distribution $u\in D'(X)$ such that $u_{|X_i}=u_i$ for every $i\in I$.

Proof Take $\phi \in \mathscr{C}_0^\infty(X)$ and $\phi_i \in \mathscr{C}_0^\infty(X_i)$ and define

$$\phi = \sum_i \phi_i$$

and

$$u(\phi) = \sum_i u_i(\phi_i). \tag{2.10}$$

We claim that definition (2.10) does not depend on the choice of the sequence $\{\phi_i\}$. For this purpose it is enough to prove that

$$\sum_i \phi_i = 0$$

implies

$$u\Big(\sum_i \phi_i\Big) = 0.$$

Set

$$K = \bigcup_i \mathrm{supp}\phi_i,$$

clearly a compact set. There exist functions $\psi_k \in \mathscr{C}_0^\infty(X_k)$ such that $0 \leq \psi_k \leq 1$ and

$$\sum_k \psi_k = 1 \quad \text{on} \quad K.$$

By compactness only a finite number of the above summands are different from zero. Moreover,

$$\psi_k\phi_i \in \mathscr{C}_0^\infty(X_k \bigcap X_i)$$

and

$$u_k(\psi_k\phi_i) = u_i(\psi_k\phi_i).$$

Therefore

$$\sum_i u_i(\phi_i) = \sum_i u_i\Big(\sum_k \psi_k\phi_i\Big) = \sum_i \sum_k u_i(\psi_k\phi_i) = \sum_i \sum_k u_k(\psi_k\phi_i)$$
$$= \sum_k \sum_i u_k(\psi_k\phi_i) = \sum_k u_k\Big(\psi_k \sum_i \phi_i\Big) = \sum_k u_k(0) = 0.$$

Consequently definition (2.10) is consistent.
Let $\phi \in \mathscr{C}_0^\infty(K)$. Then

$$\phi = \sum_k \phi\psi_k,$$

and

$$\Big|u(\phi)\Big| = \Big|\sum_i u_i(\psi_i\phi)\Big| \le \sum_i \Big|u_i(\phi\psi_i)\Big|$$
$$\le \sum_i C_i \sum_{|\alpha|\le k} \sup\Big|\partial^\alpha(\phi\psi_i)\Big| \le \sum_i C_i \sum_{|\alpha|\le k} \sup\Big|\partial^\alpha\phi\Big|$$
$$\le C \sum_{|\alpha|\le k} \sup\Big|\partial^\alpha\phi\Big|,$$

showing u is a distribution. We also have

$$u = u_i \quad \text{on} \quad X_i.$$

Now we will prove the uniqueness of u. Suppose there are two distributions u and $\tilde{u}$ with the previous properties. We conclude

$$u_{|X_i} = u_i, \quad \tilde{u}_{|X_i} = u_i,$$

so

$$(u - \tilde{u})_{|X_i} = 0 \quad \forall i.$$

Since X is open in R^n, it follows that

$$u \equiv \tilde{u} \quad \text{on} \quad X,$$

proving uniqueness.

Problem 2.22 Take $u \in D'(X)$ and let F be a relatively open subset of X with $\text{supp} u \subset F$. Prove there exists a unique linear map $\tilde{u}$ on

$$\left\{\phi : \phi \in \mathscr{C}^{\infty}(X), F \cap \text{supp}\phi \subset X\right\}$$

such that

1. $\tilde{u}(\phi) = u(\phi) \quad \text{for} \quad \phi \in \mathscr{C}_0^{\infty}(X)$,
2. $\tilde{u}(\phi) = 0 \quad \text{for} \quad \phi \in \mathscr{C}^{\infty}(X), F \cap \text{supp}\phi = \emptyset$.

Proof

1. (uniqueness) Let $\phi \in \mathscr{C}^{\infty}(X)$ and $F \cap \text{supp}\phi = K$. As K is compact, there exists $\psi \in \mathscr{C}_0^{\infty}(X)$ such that $\psi \equiv 1$ on a neighbourhood of K. Let

$$\phi_0 = \psi\phi,$$
$$\phi_1 = (1 - \psi)\phi$$

so

$$\phi = \phi_0 + \phi_1. \tag{2.11}$$

Therefore

$$\tilde{u}(\phi) = \tilde{u}(\phi_0) + \tilde{u}(\phi_1).$$

Note $\tilde{u}(\phi_1) = 0$, so

$$\tilde{u}(\phi) = \tilde{u}(\phi_0) = u(\phi_0).$$

Now suppose that there are two such distributions $\tilde{u}$, $\tilde{\tilde{u}}$. Then

$$\tilde{u}(\phi) = \tilde{u}(\phi_0),$$
$$\tilde{\tilde{u}}(\phi) = \tilde{\tilde{u}}(\phi_0),$$

and consequently

$$\tilde{u}(\phi) = \tilde{\tilde{u}}(\phi)$$

for every $\phi \in \mathscr{C}^{\infty}(X)$ so that $F \cap \text{supp}\phi = \emptyset$. Therefore $\tilde{u} = \tilde{\tilde{u}}$.

2. (existence) Let

$$\phi = \phi_0' + \phi_1'$$

be another decomposition of kind (2.11) and define

$$\chi = \phi_0 - \phi_0'.$$

Then

$$\chi \in \mathscr{C}_0^\infty(X), \quad F \cap \operatorname{supp}\chi = F \cap \operatorname{supp}(\phi_1 - \phi_1') = \emptyset$$

and so

$$u(\chi) = u(\phi_0) - u(\phi_0') = 0.$$

Define $\tilde{u}(\phi)$ by

$$\tilde{u}(\phi) = u(\phi_0).$$

This is makes sense since

$$\tilde{u}(\phi) = u(\phi) = u(\phi_0),$$
$$\tilde{u}(\phi) = 0 \quad \text{if} \quad \phi \in \mathscr{C}^\infty(X), \quad F \cap \operatorname{supp}\phi = \emptyset.$$

Problem 2.23 Prove that $\operatorname{supp}\delta = \{0\}$.

Problem 2.24 Let $\phi \in \mathscr{C}_0^\infty(X)$ and $\operatorname{supp}(u) \bigcap \operatorname{supp}(\phi) = \emptyset$. Prove that $u(\phi) = 0$.

Proof Since $\operatorname{supp}(u) \bigcap \operatorname{supp}(\phi) = \emptyset$, we have $\phi \in \mathscr{C}_0^\infty(X \backslash \operatorname{supp}(u))$. If $x \in \operatorname{supp}(u)$, then $\phi(x) = 0$, so $u(\phi) = 0$. If $x \in X \backslash \operatorname{supp}(u)$, then $u(\phi)(x) = 0$.

Problem 2.25 Prove that the set of distributions on X with compact support coincides with the dual space of $\mathscr{C}^\infty(X)$ with the topology

$$\phi \mapsto \sum_{|\alpha| \le k} \sup_K \left|\partial^\alpha \phi\right|,$$

where K is a compact set in X.

Proof Let u be a distribution with compact support and take $\phi \in \mathscr{C}^\infty(X)$ and $\psi \in \mathscr{C}_0^\infty(X)$, $\psi \equiv 1$ on a neighbourhood of $\operatorname{supp} u$. Then

$$\phi = \psi\phi + (1 - \psi)\phi$$

and

$$u(\phi) = u(\psi\phi + (1 - \psi)\phi) = u(\psi\phi) + u((1 - \psi)\phi) = u(\psi\phi).$$

Define u on $\mathscr{C}^\infty(X)$ via

$$u(\phi) = u(\psi\phi)$$

for $\phi \in \mathscr{C}^\infty(X)$. Since u is a distribution and $\psi\phi \in \mathscr{C}_0^\infty(X)$, we have

$$|u(\phi)| = |u(\psi\phi)| \le C \sum_{|\alpha|\le k} \sup_K \left|\partial^\alpha(\phi\psi)\right| \le C_1 \sum_{|\alpha|\le k} \left|\partial^\alpha \phi\right|.$$

Now we suppose that v is a linear operator on $\mathscr{C}^\infty(X)$ for which

$$|v(\phi)| \le C \sum_{|\alpha|\le k} \sup_K \left|\partial^\alpha \phi\right|$$

for $\phi \in \mathscr{C}^\infty(X)$ and K a compact set. Then

$$v(\phi) = 0$$

when $\mathrm{supp}\phi \cap K = \emptyset$. If $\phi \in \mathscr{C}_0^\infty(X) \subset \mathscr{C}^\infty(X)$, v is a distribution. Therefore there exists a unique distribution $u \in D'(X)$ such that

$$u(\phi) = v(\phi)$$

for every $\phi \in \mathscr{C}^\infty(X)$.

Problem 2.26 Let u be a distribution with a compact support of order $\le k$, ϕ a $\mathscr{C}^k$ map with $\partial^\alpha\phi(x) = 0$ for $|\alpha| \le k$, $x \in \mathrm{supp}\phi$. Prove that $u(\phi) = 0$.

Proof Let $\chi_\epsilon \in \mathscr{C}_0^\infty(X)$, $\chi_\epsilon \equiv 1$ on a neighbourhood U of supp$u$, while $\chi_\epsilon = 0$ on $X\backslash U$. Define the set M_ϵ, $\epsilon > 0$ by

$$M_\epsilon = \left\{y : |x - y| \le \epsilon, \quad x \in \mathrm{supp}u\right\},$$

making M_ϵ an ϵ-neighbourhood of supp$u$. Moreover,

$$\left|\partial^\alpha \chi_\epsilon\right| \le C\epsilon^{-|\alpha|}, \quad |\alpha| \le k,$$

for some positive constant C. Since

$$\mathrm{supp}u \cap \mathrm{supp}(1 - \chi_\epsilon)\phi = \emptyset,$$

we have

$$u(\phi) = u(\phi\chi_\epsilon) + u((1-\chi_\epsilon)\phi) = u(\phi\chi_\epsilon),$$

$$|u(\phi)| \le C\left|\sum_{|\alpha|\le k}\sup\Big(\partial^\alpha(\phi\chi_\epsilon)\Big)\right|$$

$$\le C_1\sum_{|\alpha|+|\beta|\le k}\sup\left|\partial^\alpha\phi\right|\left|\partial^\beta\chi_\epsilon\right|$$

$$\le C_2\sum_{|\alpha|+|\beta|\le k}\sup\left|\partial^\alpha\phi\right|\epsilon^{|\alpha|-k}\longrightarrow_{\epsilon\longrightarrow 0},\quad |\alpha|\le k.$$

Consequently $u(\phi) = 0$.

Problem 2.27 Let u be a distribution of order k with support $\{y\}$. Prove that $u(\phi) = \sum_{|\alpha|\le k} a_\alpha\partial^\alpha\phi(y)$, $\phi\in\mathscr{C}^k$.

Proof For $\phi\in\mathscr{C}^k$ we have

$$\phi(x) = \sum_{|\alpha|\le k}\partial^\alpha\phi(y)\frac{(x-y)^\alpha}{\alpha!} + \psi(x),$$

where

$$\partial^\alpha\psi(y) = 0 \quad \text{for} \quad |\alpha|\le k.$$

Hence,

$$u(\psi) = 0.$$

Therefore

$$u(\phi(x)) = u\Big(\sum_{|\alpha|\le k}\partial^\alpha\phi(y)\tfrac{(x-y)^\alpha}{\alpha!} + \psi(x)\Big)$$

$$= u\Big(\sum_{|\alpha|\le k}\partial^\alpha\phi(y)\tfrac{(x-y)^\alpha}{\alpha!}\Big) + u(\psi(x))$$

$$= \sum_{|\alpha|\le k} u\Big(\tfrac{(x-y)^\alpha}{\alpha!}\Big)\partial^\alpha\phi(y).$$

Let

$$a_\alpha = u\Big(\frac{(x-y)^\alpha}{\alpha!}\Big).$$

Then

$$u(\phi) = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \phi(y).$$

Problem 2.28 Write $x = (x', x'') \in \mathrm{R}^n$. Prove that for every distribution $u \in D'(\mathrm{R}^n)$ of order k with compact support contained in the plane $x' = 0$, we have

$$u(\phi) = \sum_{|\alpha| \le k} u_\alpha(\phi_\alpha), \tag{2.12}$$

where $\alpha = (\alpha', 0)$, u_α is a distribution in the variables x'', of order $k - |\alpha|$, with compact support with and $\phi_\alpha(x'') = \partial^\alpha \phi(x', x'')|_{x'=0}$.

Proof For $\phi \in \mathscr{C}^\infty$ we have

$$\phi(x) = \sum_{|\alpha'| \le k, \alpha''=0} \partial^\alpha \phi(0, x'') \frac{x'^\alpha}{\alpha!} + \Phi(x),$$

where

$$\partial^\alpha \Phi(x)|_{x'=0} = 0 \quad \text{for} \quad |\alpha| \le k.$$

This implies

$$u(\Phi) = 0.$$

Since u is a distribution,

$$u(\phi) = \sum_{|\alpha'| \le k, \alpha''=0} u\Big(\partial^\alpha \phi(0, x'') \frac{x'^\alpha}{\alpha!}\Big).$$

Now let

$$u_\alpha(\phi) = u\Big(\partial^\alpha \phi(0, x'') \frac{x'^\alpha}{\alpha!}\Big).$$

We want to show u_α is a distribution of order $k - |\alpha|$. Set

$$\psi(x) = \phi(0, x'') \frac{x'^\alpha}{\alpha!} + O(|x'|^{k+1}) \quad \text{for} \quad x' \longrightarrow 0.$$

Then

$$u(\psi) = u_\alpha(\phi) \quad \text{for} \quad \psi \in \mathscr{C}^\infty \tag{2.13}$$

and

$$\sum_{|\gamma|\le k}\sup_K\left|\partial^\gamma\phi\right| \le C\sum_{|\beta|\le k-|\alpha|}\sup_K\left|\partial^\beta\psi\right|,$$

so

$$\sup_K\left|\partial^\alpha\phi\right| \le C\sum_{|\beta|\le k-|\alpha|}\sup_K\left|\partial^\beta\psi\right|.$$

Consequently

$$u_\alpha(\psi) \le C'\sum_{|\beta|\le k-|\alpha|}\sup_K\left|\partial^\beta\psi\right|$$

for every $\psi\in\mathscr{C}_0^\infty$, proving u_α is a distribution of order $k-|\alpha|$ in the variable x''. From (2.13) it follows that u_α has compact support.

Problem 2.29 Let K be a compact set in R^n which cannot be written as union of finitely many compact connected domains. Prove that there exists a distribution $u\in\mathscr{E}'(K)$ of order 1 that does not satisfy

$$u(\phi) \le C\sum_{|\alpha|\le k}\sup_K\left|\partial^\alpha\phi\right|,\quad \phi\in\mathscr{C}^\infty(X)$$

for any constants C and k.

Problem 2.30 Let K be a compact set in R^n and u_α, $|\alpha|\le k$, continuous functions on K. For $|\alpha|\le k$ we set

$$U_\alpha(x,y) = \left|u_\alpha(x) - \sum_{|\beta|\le k-|\alpha|}u_{\alpha+\beta}(y)\frac{(x-y)^\beta}{\beta!}\right||x-y|^{|\alpha|-k},$$

for $x,y\in K, x\neq y$, and $U_\alpha(x,x)=0$ for $x\in K$. Supposing every function U_α, $|\alpha|\le k$, is continuous on $K\times K$, prove that there exists $v\in\mathscr{C}^k(\mathrm{R}^n)$ such that $\partial^\alpha v(x) = u_\alpha(x)$ for $x\in K$, $|\alpha|\le k$. Then prove that v can be chosen so that

$$\sum_{|\alpha|\le k}\sup\left|\partial^\alpha v\right| \le C\Big(\sum_{|\alpha|\le k}\sup_{K\times K}U_\alpha + \sum_{|\alpha|\le k}\sup_K u_\alpha\Big),$$

where C is a constant depending on K only.

Problem 2.31 Prove that

$$|u(\phi)| \leq C\Big(\sum_{|\alpha|\leq k} \sup_{x,y\in K, x\neq y}\Big|\partial^\alpha \phi(x) - \sum_{|\beta|\leq k-|\alpha|} \partial^{\alpha+\beta}\phi(y)\frac{(x-y)^\beta}{\beta!}\Big|$$

$$\times |x-y|^{|\alpha|-k} + \sum_{|\alpha|\leq k} \sup_K \Big|\partial^\alpha \phi\Big|\Big), \quad \phi \in \mathscr{C}^\infty(\mathrm{R}^n),$$

for every distribution u of order k with compact support $K \subset \mathrm{R}^n$.

Problem 2.32 Let K be a compact set in R^n with finitely many connected components, such that every two points x and y in the same component can be joined by a rectifiable curve in K of length $\leq C|x-y|$. Prove that for every distribution u of order k with $\mathrm{supp}\, u \subset K$ the estimate

$$|u(\phi)| \leq C \sum_{|\alpha|\leq k} \sup_K \Big|\partial^\alpha \phi\Big|, \quad \phi \in \mathscr{C}^k(\mathrm{R}^n)$$

holds.

Problem 2.33 Let $a \in \mathrm{C}^n$. Prove that $\delta_a(x)$, $x \in \mathrm{R}^n$, is a singular distribution.

Problem 2.34 Let $u_1, u_2 \in \mathscr{D}'(X)$ with u_1 regular and u_2 singular. Prove that

$$\alpha_1 u_1 + \alpha_2 u_2$$

is singular for every $\alpha_1, \alpha_2 \in \mathrm{C}$.

Problem 2.35 Let $f_n, f \in L^1_{\mathrm{loc}}(X)$ and

$$\int_K |f_n(x) - f(x)|dx \longrightarrow_{n\longrightarrow\infty} 0$$

for every compact subset K of X. Prove that

$$f_n \longrightarrow_{n\longrightarrow\infty} f$$

in $\mathscr{D}'(X)$.

Problem 2.36 Prove that

1. $\delta(-x) = \delta(x)$,
2. $(\delta(ax - x_0), \phi) = \phi\Big(\frac{x_0}{a}\Big)$, for any $\phi \in \mathscr{C}_0^\infty(X)$ and any constant $a \neq 0$.

Proof

1. Let $\phi \in \mathscr{C}_0^\infty(X)$. Then

$$\Big(\delta(-x), \phi(x)\Big) = \Big(\delta(x), \phi(-x)\Big) = \phi(0) = \Big(\delta(x), \phi(x)\Big).$$

Consequently

$$\delta(-x) = \delta(x).$$

2. Let $\phi \in \mathscr{C}_0^\infty(X)$. Then

$$\Big(\delta(ax - x_0), \phi(x)\Big) \qquad (ax = y + x_0)$$

$$= \Big(\delta(y), \phi\Big(\tfrac{y+x_0}{a}\Big)\Big) = \phi\Big(\tfrac{x_0}{a}\Big).$$

Problem 2.37 Prove that

1. $\delta(x^2 - a^2) = \frac{1}{2a}\Big[\delta(x-a) + \delta(x+a)\Big], a \neq 0,$
2. $\delta(\sin x) = \sum_{k=-\infty}^{\infty} \delta(x - k\pi).$

Problem 2.38 Prove that $\delta(x)$, $x \in \mathrm{R}^1$, is a measure.

Problem 2.39 Prove that $H(x)$, $x \in \mathrm{R}^1$, is a measure.

Problem 2.40 Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathscr{D}'(X)$ such that $|f_n(\phi)| \leq c_\phi$ for every $\phi \in \mathscr{C}_0^\infty(X)$, and $\{\phi_n\}_{n=1}^\infty \subset \mathscr{C}_0^\infty(X)$ a sequence converging to 0 in $\mathscr{C}_0^\infty(X)$ as $n \longrightarrow \infty$. Prove that $f_n(\phi_n) \longrightarrow 0$, $n \longrightarrow \infty$.

Proof We suppose the contrary. Then there exists a constant $c > 0$ such that

$$|f_n(\phi_n)| \geq c > 0,$$

for n large enough. Since $\phi_n \longrightarrow 0$ in $\mathscr{C}_0^\infty(X)$ as $n \longrightarrow \infty$, there exists a compact set X' such that $\mathrm{supp}\phi_n \subset X'$ for every n and

$$D^\alpha \phi_n \longrightarrow_{n\longrightarrow\infty} 0,$$

for every $x \in X$ and every $\alpha \in \mathrm{N}^n \cup \{0\}$. Hence

$$|D^\alpha \phi_n(x)| \leq \frac{1}{4^n}, \quad |\alpha| \leq n = 0, 1, 2, \ldots,$$

for n large enough and every $x \in X'$. We set

$$\psi_n = 2^n \phi_n.$$

We have $\mathrm{supp}\psi_n \subset X'$ and

$$|D^\alpha \psi_n(x)| \leq \frac{1}{2^n}, \quad |\alpha| \leq n = 0, 1, 2, \ldots, \tag{2.14}$$

$$|f_n(\psi_n)| = 2^n |f_n(\phi_n)| \geq 2^n c \longrightarrow_{n\longrightarrow\infty} \infty. \tag{2.15}$$

Let us find subsequences $\{f_{k_\nu}\}_{\nu=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and $\{\psi_{k_\nu}\}_{\nu=1}^{\infty}$ of $\{\psi_n\}_{n=1}^{\infty}$ so that $|f_{k_\nu}(\psi_{k_\nu})| \geq 2^\nu$ for $\nu = 1, 2, \ldots$. As $\psi_k \longrightarrow_{k\longrightarrow\infty} 0$ in $\mathscr{C}_0^\infty(X)$, we have $f_{k_j}(\psi_k) \longrightarrow_{k\longrightarrow\infty} 0$ for $j = 1, 2, \ldots, \nu - 1$. Therefore there exists $N \in \mathrm{N}$ such that for every $k \geq N$

$$|f_{k_j}(\psi_k)| \leq \frac{1}{2^{\nu-j}}, \quad j = 1, 2, \ldots, \nu - 1. \tag{2.16}$$

We note that $|f_k(\psi_{k_j})| \leq c_{k_j}, j = 1, 2, \ldots, \nu-1$. From (2.15), we can choose $k_\nu \geq N$ so that

$$|f_{k_\nu}(\psi_{k_\nu})| \geq \sum_{1\leq j\leq \nu-1} c_{k_j} + \nu + 1. \tag{2.17}$$

From (2.16) and (2.17) we have

$$|f_{k_j}(\psi_{k_\nu})| \leq \frac{1}{2^{\nu-j}}, \quad j = 1, 2, \ldots, \nu - 1, \tag{2.18}$$

$$|f_{k_\nu}(\psi_{k_\nu})| \geq \sum_{1\leq j\leq \nu-1} |f_{k_\nu}(\psi_{k_j})| + \nu + 1. \tag{2.19}$$

We set

$$\psi = \sum_{j\geq 1} \psi_{k_j}.$$

From (2.14) it follows that ψ is a convergent series, $\psi \in \mathscr{C}_0^\infty(X)$ and

$$f_{k_\nu}(\psi) = f_{k_\nu}(\psi_{k_\nu}) + \sum_{j\geq 1, j\neq \nu} f_{k_\nu}(\psi_{k_j}).$$

Therefore

$$|f_{k_\nu}(\psi)| \geq |f_{k_\nu}(\psi_{k_\nu})| - \textstyle\sum_{1\leq j\leq \nu-1} |f_{k_\nu}(\psi_{k_j})| - \sum_{j\geq \nu+1} |f_{k_\nu}(\psi_{k_j})|$$

$$\geq \nu + 1 - \textstyle\sum_{j\geq\nu+1} \frac{1}{2^{j-\nu}} = \nu,$$

and then

$$(f_{k_\nu}, \psi) \longrightarrow_{\nu\longrightarrow\infty} \infty,$$

which contradicts $|f_{k_\nu}(\psi)| \leq c_\psi$.

Problem 2.41 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{D}'(X)$ such that $\{f_n(\phi)\}_{n=1}^{\infty}$ converges for every $\phi \in \mathscr{C}_0^{\infty}(X)$. Prove that the functional

$$f(\phi) = \lim_{n\longrightarrow\infty} f_n(\phi), \quad \phi \in \mathscr{C}_0^{\infty}(X)$$

is an element of $\mathscr{D}'(X)$.

Proof Let $\alpha_1, \alpha_2 \in \mathrm{C}$ and $\phi_1, \phi_2 \in \mathscr{C}_0^{\infty}(X)$. Then

$$f(\alpha_1\phi_1 + \alpha_2\phi_2) = \lim_{n\longrightarrow\infty} f_n(\alpha_1\phi_1 + \alpha_2\phi_2) = \lim_{n\longrightarrow\infty}(\alpha_1 f_n(\phi_1) + \alpha_2 f_n(\phi_2))$$
$$= \alpha_1 \lim_{n\longrightarrow\infty} f_n(\phi_1) + \alpha_2 \lim_{n\longrightarrow\infty} f_n(\phi_2) = \alpha_1 f(\phi_1) + \alpha_2 f(\phi_2).$$

Therefore f is a linear map on $\mathscr{C}_0^{\infty}(X)$. Now we will prove that f is a continuous functional on $\mathscr{C}_0^{\infty}(X)$. Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{C}_0^{\infty}(X)$ such that $\phi_n \longrightarrow_{n\longrightarrow\infty}$ 0 in $\mathscr{C}_0^{\infty}(X)$. We claim $f(\phi_n) \longrightarrow_{n\longrightarrow\infty} 0$, so suppose the contrary. There exists a constant $a > 0$ such that

$$|f(\phi_\nu)| \geq a,$$

for every $\nu = 1, 2, \ldots$. Since

$$f(\phi_\nu) = \lim_{k\longrightarrow\infty} f_k(\phi_\nu),$$

there is $k_\nu \in \mathrm{N}$ such that

$$|f_{k_\nu}(\phi_\nu)| \geq a$$

for every $\nu = 1, 2, \ldots$, which is in contradiction with the result of the previous problem. Consequently $f(\phi_n) \longrightarrow_{n\longrightarrow\infty} 0$ and $f \in \mathscr{D}'(X)$.

Problem 2.42 Let $u \in \mathscr{D}'(X)$ and $b \in \mathscr{C}^{\infty}(X)$ be such that $b(x) \equiv 1$ on a neighbourhood of supp$u$. Show

$$u = b(x)u.$$

Proof For the function $1 - b(x)$ we have that $1 - b(x) \equiv 0$ on supp$u$. Then for $\phi \in \mathscr{C}_0^{\infty}(X)$ we have

$$0 = u((1 - b(x))\phi) = u(\phi - b(x)\phi) = u(\phi) - u(b(x)\phi) = u(\phi) - b(x)u(\phi),$$

so

$$u(\phi) = b(x)u(\phi)$$

for every $\phi \in \mathscr{C}_0^{\infty}(X)$. Therefore $u = b(x)u$.

Problem 2.43 Compute

$$(x^4 + x^2 + 3)\delta(x) + xP\frac{1}{x}, \quad x \in \mathrm{R}^1.$$

Answer $3\delta + 1$.

Problem 2.44 Let $b \in \mathscr{C}^\infty(\mathrm{R}^1)$. Compute

$$b(x)\delta(x), \quad x \in \mathrm{R}^1.$$

Answer $b(0)\delta$.

Problem 2.45 Let $a \in \mathscr{C}^\infty(X)$, $u \in D'(X)$. Prove that $\mathrm{supp}(au) \subset \mathrm{supp}\, a \cap \mathrm{supp}\, u$.

Problem 2.46 Let $f, u \in D'(X)$ and $\mathrm{singsupp}\, u \cap \mathrm{singsupp} f = \emptyset$. Prove that $f \circ u \in D'(X)$.

Problem 2.47 Let $f \in \mathscr{C}^\infty(X)$, $u \in D'(X)$ and $\mathrm{supp}\, u \cap \mathrm{supp} f \subset\subset X$. Prove that $u(f)$ can be defined by $u(f) = (fu)(1)$.

Problem 2.48 Let $f \in \mathscr{C}^k(X)$, $u \in D'^k(X)$. Prove that $fu \in D'^k(X)$.

Problem 2.49 Solve the equation

$$(x-3)u = 0$$

in $\mathscr{D}'(X)$.

Solution Let $\phi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$. Then we have

$$(x-3)u(\phi) = 0 \quad \text{or} \quad u((x-3)\phi) = 0. \tag{2.20}$$

Let now $\psi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$, and choose $\eta \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ so that $\eta \equiv 1$ on $[3-\epsilon, 3+\epsilon]$ and $\eta \equiv 0$ on $\mathrm{R}^1 \backslash [3-\epsilon, 3+\epsilon]$, for a small enough $\epsilon > 0$. Then the function $\frac{\psi(x)-\eta(x)\psi(3)}{(x-3)}$ belongs in $\mathscr{C}_0^\infty(\mathrm{R}^1)$. From this and (2.20) we have that

$$u\Big((x-3)\frac{\psi(x)-\eta(x)\psi(3)}{(x-3)}\Big) = 0.$$

Hence

$$\begin{aligned} u(\psi) &= u\Big((x-3)\tfrac{\psi(x)-\eta(x)\psi(3)}{(x-3)} + \eta(x)\psi(3)\Big) \\ &= u\Big((x-3)\tfrac{\psi(x)-\eta(x)\psi(3)}{(x-3)}\Big) + u(\eta(x)\psi(3)) \\ &= \psi(3)u(\eta) = C\psi(3) = C\delta(x-3)(\psi). \end{aligned}$$

Here $C = u(\eta) = \text{const}$. Since $\psi \in \mathscr{C}_0^\infty(\mathrm{R}^1)$ was chosen arbitrarily, $u = C\delta(x-3)$.

Problem 2.50 Solve the equation

$$(x-3)u = P\frac{1}{x-3}$$

in $\mathscr{D}'(\mathrm{R}^1)$.

Solution By using the previous problem the corresponding homogeneous equation $(x-3)u = 0$ is solved by $u = C\delta(x-3)$, $C = \text{const}$, and a particular solution is $P\frac{1}{(x-3)^2}$. Therefore

$$u = C\delta(x-3) + P\frac{1}{(x-3)^2}.$$

Problem 2.51 Solve the equations

1. $(x-1)(x-2)u = 0$,
2. $x^2 u = 2$,
3. $(\sin x)u = 0$.

Answer

1. $u = C_1\delta(x-1) + c_2\delta(x-2)$, $C_1, C_2 = \text{const}$,
2. $u = C_0\delta(x) + C_1\delta'(x) + 2P\frac{1}{x^2}$, $C_0, C_1 = \text{const}$,
3. $\sum_{k=-\infty}^{\infty} C_k\delta(x-k\pi)$, $C_k = \text{const}$.

Springer

http://www.springer.com/978-3-319-18731-0

Modeling and Stochastic Learning for Forecasting in High Dimensions
Antoniadis, A.; Poggi, J.-M.; Brossat, X. (Eds.)
2015, X, 339 p. 105 illus., 49 illus. in color., Softcover
ISBN: 978-3-319-18731-0