

# Chapter 2

## Double Counting

### 2.1 Introduction

Often mathematicians want to develop formulas for sums of terms. As we have seen, induction is one way to establish the truth of such formulas. However, induction relies on foreknowledge of the formula which has to be derived or guessed in some other way first. Induction also does not assist with recall. An alternate way to find and derive many counting formulas is *double counting*. With this technique, we divide up a collection of objects in two different ways. The results of the two different divisions must agree and this can yield a formula for the more complicated one. Such approaches have the advantage that the technique for finding the formula is often more memorable than the formula itself.

### 2.2 Summing Numbers

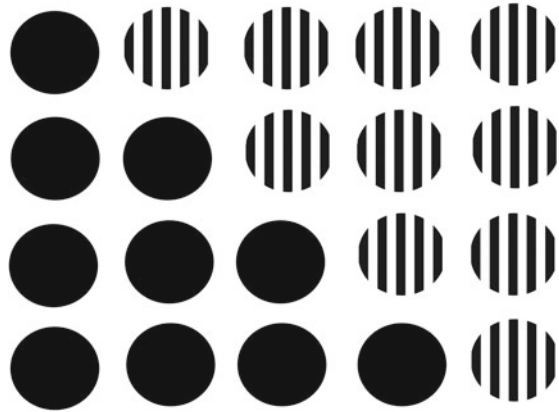
We wish to find the sum of the first  $N$  numbers. Call this  $x_N$ . So

$$x_N = \sum_{j=1}^N j.$$

We can regard the sum as a triangle with 1 stone in the first row, two stones in the second, three in the third and so on. See Fig. 2.1. Now if we make a copy of the triangle and rotate it through  $180^\circ$ , we get  $N$  stones in the first row,  $N - 1$  in the second,  $N - 2$  in the third and so on. Joining the two triangles together, we have  $N$  rows of  $N + 1$  stones. The total number of stones is  $N(N + 1)$ . So

$$X_N = \frac{1}{2}N(N + 1).$$

**Fig. 2.1** Stones arranged in rows with one more stone in each row together with the same stones rotated



We could also give this proof algebraically. Before proceeding to the algebra, we look at a special case. If we have 5 stones, then

$$\begin{aligned} X_5 &= 1 + 2 + 3 + 4 + 5, \\ X_5 &= 5 + 4 + 3 + 2 + 1, \\ 2X_5 &= (1 + 5) + (2 + 4) + (3 + 3) + (4 + 2) + (5 + 1). \end{aligned}$$

Now the algebra, if we reverse the order of the sum, we get

$$x_N = \sum_{j=1}^N (N + 1 - j).$$

Adding the two expressions for  $x_N$  together,

$$2x_N = \sum_{j=1}^N (N + 1) = N(N + 1),$$

and the result follows.

A similar argument can be used for the sum of the first  $N$  odd numbers

$$y_N = \sum_{j=1}^N (2j - 1).$$

Either by rotating the triangle or by reversing the order of the sum, we have

$$y_N = \sum_{j=1}^N (2N + 1 - 2j).$$

So

$$2y_N = \sum_{j=1}^N 2N.$$

We conclude that

$$y_N = N^2.$$

As well as being the sum of the first  $N$  odd numbers,  $N^2$  is also the sum of the first  $N$  numbers plus the first  $N - 1$  numbers. One way to see this is take an  $N \times N$  grid of stones and see how many stones lie on each upwards sloping diagonal. For the first  $N$  diagonals, the  $j$ th diagonal has  $j$  stones. These contribute the sum of the first  $N$  numbers. After  $N$ , the length of the diagonals goes down by one each time. The second set therefore contribute the sum of the first  $N - 1$  numbers and the result follows.

### 2.3 Vandermonde's Identity

Suppose we have  $m + n$  jewels of varying sizes. There are  $m$  rubies and  $n$  sapphires. How many different ways can we select  $r$  jewels to be placed on a bracelet? Clearly, the answer is

$$\binom{m+n}{r}.$$

Note that since the jewels are all of different sizes, two different selections of  $r$  jewels are essentially different. However, if we first think in terms of using  $k$  rubies and  $r - k$  sapphires, we see that the answer is also

$$\sum_{k=0}^r \binom{m}{k} \binom{n}{r-k}.$$

(We take the binomial coefficient to be zero when the inputs are out of their natural range. For example, if  $r > m$ , there are zero ways to choose  $r$  rubies.) In conclusion, we have *Vandermonde's identity*:

$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{k} \binom{n}{r-k}.$$

## 2.4 Fermat's Little Theorem

Fermat's little theorem states

**Theorem 2.1** *If  $a$  is a positive integer and  $p$  is prime then  $p$  divides  $a^p - a$ .*

An elementary proof can be made using double counting.

*Proof* Consider strings of letters of length  $p$ . The letters are from the first  $a$  letters in the alphabet. (If  $a > 26$ , we add extra letters to the alphabet.) How many such strings are there? Order matters, so we have  $a$  choices in each slot and there are  $p$  slots, so we get  $a^p$  different strings.

Now consider the operation on these strings of chopping off an element at the end and reinserting it at the front. Call this  $T^1$ . We define  $T^j$  to be the result of applying  $T^1$   $j$  times. Clearly,  $T^p$  is the identity map. We give some examples when  $p = 3$  and  $a = 2$ .

$$T^1(AAA) = AAA,$$

$$T^1(ABA) = AAB,$$

$$T^2(ABA) = BAA.$$

Two strings,  $x$  and  $y$ , are said to be in the same *orbit* if there exists  $j$  such that

$$T^j x = y.$$

Note that then

$$T^{p-j} y = x.$$

Also note that if  $x$  and  $y$  are in the same orbit, and  $y$  and  $z$  are in the same orbit then  $x$  and  $z$  are too. So being in the same orbit is an equivalence relation. (See Appendix B for further discussion of equivalence relations.) This implies that every string is in exactly one orbit.

If a string is all one letter, e.g. "AAA", then it is the only string in its orbit. There are  $a$  such strings and so  $a$  orbits of size 1.

Now suppose a string  $x$  has more than one letter in it. Consider the strings

$$x, Tx, T^2x, \dots, T^{p-1}x.$$

These will all be in the same orbit and everything in  $x$ 's orbit is of this form. If we keep going we just get the same strings over again since  $T^p x = x$ . There are at most  $p$  elements in these orbits then. We show that when  $p$  is prime there are exactly  $p$  elements. If there were less than  $p$  then for some  $k < p$ , we would have

$$T^k x = x.$$

This means that cutting off the last  $k$  elements and sticking them at the front does not change the string. We also have

$$x = T^k x = T^{2k} x = T^{3k} x = T^{4k} x = \dots$$

This implies that  $x$  is made of  $p/k$  copies of the first  $k$  elements. However,  $p$  is prime so its only divisors are 1 and  $p$ . If  $k = 1$  then we have a string of elements the same which is the case we already discussed. If  $k = p$  then we are just saying that  $T^p x = x$  which is always true.

So we have two sorts of orbits, those with  $p$  elements and those with 1 element. We showed that there are  $a$  of the second sort. Let there be  $m$  of the first sort. Since there are  $a^p$  strings in total, we have

$$a^p = mp + a,$$

So

$$pm = a^p - a.$$

This says precisely that  $p$  divides  $a^p - a$ . □

This proof is due to Golomb (1956).

## 2.5 Icosahedra

An icosahedron is a polyhedron with 20 faces. When working with three-dimensional solids, we can divide their surfaces into *vertices*, *edges* and *faces*. A vertex is a corner, a face is a flat two-dimensional side and an edge is the line defining the side of two faces.

A regular icosahedron is a Platonic solid. Every face is a triangle and the same number of faces meet at each vertex. How many edges does an icosahedron have? Call this number  $E$ . We know that there are 20 faces and that each face is a triangle. Define an edge-face pair to be a face together with one of the sides of the face which is, of course, an edge of the icosahedron. There are 60 such pairs since each face has 3 sides.

Each edge of the icosahedron lies in precisely two sides. The number of edge-face pairs is therefore double the number of edges. That is

$$2E = 60$$

and so  $E = 30$ .

## 2.6 Pythagoras's Theorem

The reader will already be familiar with the theorem that for a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. So if the sides are  $a$ ,  $b$  and  $c$ , with  $c$  the longest side, we have

$$c^2 = a^2 + b^2.$$

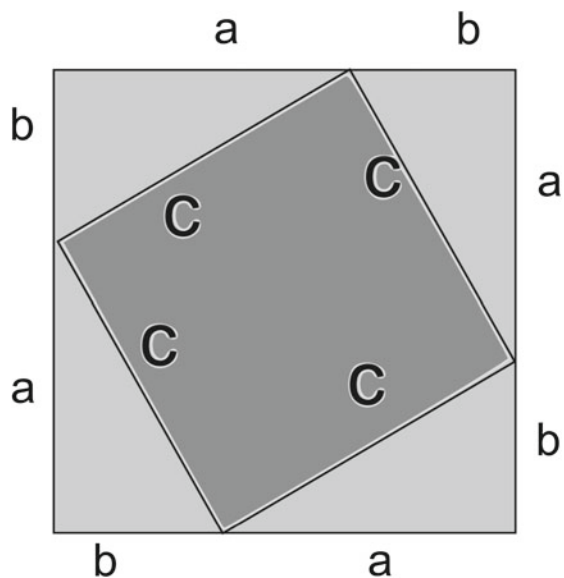
We can use an extension of the double counting pattern to prove this theorem. Instead of using equal numbers of objects, we use equal areas. We take the triangle and fix a square with side length  $c$  to its side of that length. We then fix a copy of the triangle to each of the square's other sides. See Fig. 2.2. At each vertex of the square, we get 3 angles, each of which is one of the angles of the triangle so these add up to  $180^\circ$  and make a straight line. We now have two squares: the big one has side  $a + b$  and the small one  $c$ . The former's area is

$$(a + b)^2 = a^2 + 2ab + b^2.$$

We can also regard the big square as the small one plus 4 copies of the triangle and so it has area

$$c^2 + 4 \times 0.5ab = c^2 + 2ab.$$

**Fig. 2.2** Four right-angled triangles with sides  $a$ ,  $b$ ,  $c$ , placed on a square of side  $c$



Equating these two, we get

$$a^2 + b^2 = c^2,$$

as required. Our proof is complete.

## 2.7 Problems

**Exercise 2.1** Let  $a$  and  $b$  be integers. Develop a formula for

$$\sum_{j=1}^n a + jb.$$

**Exercise 2.2** How many vertices does an icosahedron have?

**Exercise 2.3** If  $p$  is a prime and not 2, and  $a$  is an integer, show that 2 divides into  $a^p - a$ . (Try to construct a bijection on the set of size  $p$  orbits which pairs them.)

**Exercise 2.4** Suppose we have three sorts of jewels and apply the arguments for Vandermonde's identity, what formula do we find?



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Proof Patterns

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