Chapter 2
Introduction to Analytic Curves

The study of *analytic curves*, which at first sight appears to be unrelated to the stability analysis of time-delay systems, will be extremely helpful for addressing the stability problem.

In this book, we will see that the mathematical properties concerning the singularities of analytic curves provide us with a new angle (called the analytic curve perspective or point of view in this book) to study the stability of time-delay systems. New insights for the complete stability problem will be developed based on this analytic curve perspective. To be more precise, two aspects are essential. First, it will be used for studying the asymptotic behavior of the critical pairs. Second, the analytic curve perspective will be used to improve the classical frequency-sweeping approach. Moreover, as we will discuss later, the analytic curve perspective may be applied to many other important problems.

In this chapter, we start by presenting some fundamentals concerning *analytic curves*. Especially, as an important tool for studying analytic curves, the Puiseux series will be introduced and discussed in detail.

In Sect. 2.1, we will first present the related concepts on analytic curves and show that an analytic curve can be understood in an intuitive manner. In Sect. 2.2, the *Puiseux series* will be introduced for describing and analyzing an analytic curve. The convergence of the Puiseux series will be discussed in Sect. 2.3. In Sect. 2.4, we will briefly review a classical method, the *Newton diagram*, for computing the Puiseux series. In Sect. 2.5, we will explain how to analyze the asymptotic behavior of an analytic curve by means of the Puiseux series. Finally, some notes and comments will be given in Sect. 2.6.

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2.1 Introductory Remarks to Singularities of Analytic Curves

Consider a power series $\Phi(y, x)$ in two variables $x \in \mathbb{C}$ and $y \in \mathbb{C}$:

$$\Phi(y, x) = \sum_{\alpha,\beta \geq 0} \phi_{\alpha,\beta} y^{\alpha} x^{\beta}, \tag{2.1}$$

where $\phi_{\alpha,\beta}$ ($\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$) are complex coefficients.

We suppose that $\Phi(0, 0) = 0$ (that is, the constant term $\phi_{0,0} = 0$) and that the power series $\Phi(y, x)$ is convergent in a small neighborhood of the point $(x = 0, y = 0)$.

Remark 2.1 If there exists a point (y^*, x^*) other than $(0, 0)$ such that $\Phi(y^*, x^*) = 0$, we may obtain a new power series with a zero constant term. More precisely, we may define two new variables $\widetilde{x} = x - x^*$ and $\widetilde{y} = y - y^*$. As a result, we obtain a new power series $\widetilde{\Phi}(\widetilde{y}, \widetilde{x})$ satisfying that $\widetilde{\Phi}(0, 0) = 0$ from the original power series equation $\Phi(y^*, x^*) = 0$ and the local behavior of the original equation $\Phi(y, x) = 0$ as $y \to y^*$ and $x \to x^*$ is reflected by that of the new one $\widetilde{\Phi}(\widetilde{y}, \widetilde{x}) = 0$ as $\widetilde{y} \to 0$ and $\widetilde{x} \to 0$.

Remark 2.2 One may have a question why we are now considering a power series. The reason is related to the fact that for many stability problems in the control area, we need to study characteristic functions of the form $\rho(\lambda, \xi)$, where λ and ξ denote, respectively, the characteristic root and the system parameter under consideration, and $\rho(\lambda, \xi)$ is usually analytic. One may notice that in the case of time-delay system (1.1), the corresponding characteristic function $f(\lambda, \tau)$ falls in this class. Next, near a critical pair (λ^*, ξ^*) such that $\rho(\lambda^*, \xi^*) = 0$, we may expand $\rho(\lambda, \xi)$ as a two-variable Taylor series, which is exactly a power series of the $\Phi(y, x)$ type.

From the algebraic geometry point of view, in e.g., [15, 121], the equation $\Phi(y, x) = 0$ defines an *analytic curve* in the $\mathbb{C}^2$ plane.[1] Instead of studying the whole curve, we are interested in a small neighborhood of the origin O (i.e., the point $(x = 0, y = 0)$) in the $\mathbb{C}^2$ plane. In other words, we study how y varies near "0" with respect to an infinitesimal variation of x near "0". Such a local study will be extremely useful in the subsequent study of the asymptotic behavior of time-delay systems.

Throughout this book, we define the notation $\mathrm{ord}(\cdot)$ as follows.

Definition 2.1 For a function $\varphi(x)$, $\mathrm{ord}(\varphi(x)) = \kappa$ for $x = x^*$ denotes that $\frac{d^i\varphi(x)}{dx^i} = 0$ ($i = 0, \ldots, \kappa - 1$) and that $\frac{d^\kappa\varphi(x)}{dx^\kappa} \neq 0$ when $x = x^*$.

Furthermore, for simplicity, we denote by ord_y and ord_x, respectively, the values of $\mathrm{ord}(\Phi(y, 0))$ when $y = 0$ and $\mathrm{ord}(\Phi(0, x))$ when $x = 0$. If $\mathrm{ord}_x = 1$ and/or

[1] Note that we cannot explicitly draw such a curve since there are two complex variables.

$\text{ord}_y = 1$, the curve defined by $\Phi(y, x) = 0$ is called *non-singular* at the origin O and the origin O is called a *non-singular point* of the curve. If both ord_x and ord_y are larger than 1, the curve defined by $\Phi(y, x) = 0$ is called *singular* at the origin O and the origin O is called a *singular point* of the curve.

In order to have a better understanding of the above notions and notations, consider now two simple examples.

Example 2.1 Consider $\Phi(y, x) = y^3 + yx + x$ (polynomials represent a specific type of power series). At the point $(0, 0)$, it follows that $\Phi(0, 0) = 0$, $\text{ord}_y = 3$ ($\frac{d\Phi(y,0)}{dy} = \frac{d^2\Phi(y,0)}{dy^2} = 0$, $\frac{d^3\Phi(y,0)}{dy^3} \neq 0$), and $\text{ord}_x = 1$ ($\frac{d\Phi(0,x)}{dx} \neq 0$). The curve defined by $\Phi(y, x) = y^3 + yx + x = 0$ is non-singular at the origin O ($\text{ord}_x = 1$). □

Example 2.2 Consider $\Phi(y, x) = y^3 + yx + x^2$. At the point $(0, 0)$, it follows that $\Phi(0, 0) = 0$, $\text{ord}_y = 3$ ($\frac{d\Phi(y,0)}{dy} = \frac{d^2\Phi(y,0)}{dy^2} = 0$, $\frac{d^3\Phi(y,0)}{dy^3} \neq 0$), and $\text{ord}_x = 2$ ($\frac{d\Phi(0,x)}{dx} = 0$, $\frac{d^2\Phi(0,x)}{dx^2} \neq 0$). The curve defined by $\Phi(y, x) = y^3 + yx + x^2 = 0$ is singular at the origin O (both ord_y and ord_x are larger than 1). □

As we will show later in the book, a critical pair for the time-delay system (1.1) can be viewed as a non-singular (singular) point if $n = 1$ and/or $g = 1$ (both n and g are greater than 1). As expected, the singular case is much more complicated than the non-singular case.

For simplicity, we will only study the case where both ord_y and ord_x are bounded. In fact, we will see that this case corresponds to the complete stability problem under consideration in this book.

The study of singularities of analytic curves is a meeting point for various mathematical fields such as algebra, geometry, topology, and function theory. The first systematic contribution on curve singularities is due to Isaac Newton. Later on, some theoretical framework (for analysis and classification of curve singularities) was established by many geometers such as Puiseux, Smith, Noether, Halphen, Enriques, and Zariski. A detailed introduction to this subject can be found in e.g., [2, 15, 121]. It is worth mentioning that the analytic curve perspective to be introduced in this book is at an *elementary* level at present.

Intuitively speaking, we may view $y = 0$ as a root for $\Phi(y, x) = 0$ when $x = 0$, whose multiplicity is ord_y. Clearly, the equation $\Phi(y, x) = 0$ determines the corresponding ord_y root loci near the origin O. Such an angle (we interpret the relation between y and x as local root loci in the $\mathbb{C}^2$ plane) is easy to follow and will be frequently used in the sequel.

We now recall the classical *Weierstrass preparation theorem* (see, e.g., [15, 60, 91, 121]). It states that in a small neighborhood of O, $\Phi(y, x)$ can be decomposed as

$$\Phi(y, x) = G(y, x)Q(y, x), \tag{2.2}$$

where $G(y, x)$ is a convergent power series with $G(0, 0) \neq 0$ and $Q(y, x)$ is a polynomial in y

$$Q(y,x) = y^{\text{ord}_y} + \sum_{i=0}^{\text{ord}_y - 1} q_i(x) y^i,$$

where for $i = 0, \ldots, \text{ord}_y - 1$, $q_i(x)$ are convergent power series at $x = 0$ such that $q_i(0) = 0$. This polynomial $Q(y,x)$ is called a *Weierstrass polynomial*.

In other words, in a small neighborhood of O, the root loci of y with respect to x governed by the equation $\Phi(y,x) = 0$ coincide with those for the equation $Q(y,x) = 0$.

Now we know that in a small neighborhood of O, for each x there are ord_y continuous solutions for y, denoted by $y(x)$, such that $\Phi(y(x),x) = 0$ (since a polynomial equation with degree ord_y always has ord_y solutions in $\mathbb{C}$).

In addition, it is not hard to anticipate that the solutions of $y(x)$ can be expressed by some appropriate convergent series.

Two questions arise here. First, which class of series do the solutions of $y(x)$ belong to? Second, how to obtain the corresponding series? In the following two sections, we will give some answers. It should be pointed out that the factorization (2.2) is in general difficult to find since all $q_i(x)$ are power series.

2.2 Puiseux Series

In this section we will introduce an effective tool, the Puiseux series, to describe the local behavior of power series $\Phi(y,x)$ (i.e., the solutions $y(x)$ in a small neighborhood of O). We start with a specific case. If $\frac{\partial \Phi(y,x)}{\partial y} \neq 0$ at O (i.e., $\text{ord}_y = 1$), we may apply the well-known implicit function theorem (see Appendix A). In this particular case (corresponding to the case where the linear time-delay system with commensurate delays has a simple critical imaginary root), $y(x)$ corresponds to a Taylor series, and we can calculate the derivatives of y with respect to x (based on the implicit function theorem) to determine the coefficients of the Taylor series.

However, in the general case, i.e., ord_y is allowed to be greater than 1 (corresponding to the general case where the time-delay system is allowed to have a critical imaginary root with any multiplicity), the implicit function theorem does not allow to conclude. For this reason, the analysis of $y(x)$ calls for a different mathematical tool.

In mathematics, the local variation of $y(x)$ can be well studied by using the *Puiseux' theorem*, see, e.g., [91, 121]. Actually, this theorem has multiple versions. In the sequel, we briefly recall some results closely related to the objective of our study.

According to the Puiseux' theorem, the general solutions of $y(x)$ such that $\Phi(y(x),x) = 0$ are some series "s" of the form

$$s = \sum_{i=1}^{\infty} C_i x^{\frac{i}{\text{N}}}, \tag{2.3}$$

where C_i are complex coefficients and N is a positive integer.

The fractional power series of the form (2.3) are called the *Puiseux series*. The concept of Puiseux series is not new in mathematics. It was first introduced by Issac Newton in his correspondence with Leibniz and Oldenburg in 1676 [90] and further developed by Victor Puiseux in 1850 [101]. The naming of the series after Puiseux rather than Newton is based upon the fact that Puiseux investigated this series expansion more thoroughly. The above information can be found in [13].

Remark 2.3 Unlike the well-known Taylor series, the exponents of a Puiseux series are allowed to be positive fractional numbers.

A Puiseux series s is called a y-root for $\Phi(y, x) = 0$ if $\Phi(s, x) = 0$. In Sect. 2.4, we will introduce an effective tool for obtaining such y-roots.

Remark 2.4 It should be stressed that a Puiseux series has an infinite number of terms and, hence, we are unable to entirely obtain a Puiseux series by calculation. Fortunately, for the stability problem, we only need to invoke finitely many terms of a Puiseux series (see Chap. 4). In particular, we only need to obtain the first-order term of a Puiseux series in the nondegenerate case. Of course, the more terms we obtain, a more elaborate picture of the root loci we have.

At the end of this section, we borrow two examples from the literature on solving polynomial[2] equations.

Example 2.3 Consider a polynomial equation $y^3 - 3xy + x^3 = 0$, where $y = 0$ is a root when $x = 0$. Following the discussions in Sect. 2.1, there exist three $y(x)$ solutions near the origin O as $\text{ord}_y = 3$. The solutions, which can be found in [115], are the Puiseux series $y = \frac{1}{3}x^2 + o\left(x^2\right)$ and $y = \pm\sqrt{3}x^{\frac{1}{2}} + o(x^{\frac{1}{2}})$. □

Remark 2.5 It shall be noticed that solving a polynomial equation generally cannot be accomplished by radicals (for a power series equation, it is obviously more difficult). It has been proved that the general equation of the fifth degree is not solvable by radicals [53].

Example 2.4 Consider a polynomial equation $y^5 + 2xy^4 - xy^2 - 2x^2y - x^3 + x^4 = 0$, for which $y = 0$ is a root when $x = 0$. As $\text{ord}_y = 5$, the equation has five $y(x)$ solutions near the origin O. The solutions, reported in [120], are as follows: Two solutions are of the form $y = -x + o\,(x)$ and the other three ones are $y = x^{\frac{1}{3}} + o(x^{\frac{1}{3}})$, $y = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)x^{\frac{1}{3}} + o(x^{\frac{1}{3}})$, and $y = (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)x^{\frac{1}{3}} + o(x^{\frac{1}{3}})$. □

In Sect. 2.4, we will provide some details on how to acquire the above Puiseux series solutions.

[2] For simplicity, we here give two examples where $\Phi(y, x)$ are polynomials, which represent a specific form of power series. The approach applies to the general power series equations. Historically, the study of the singularities of analytic curves stemmed from solving the polynomial equations.

2.3 Convergence of Puiseux Series

Before discussing deeper the way to derive the Puiseux series, it is necessary to pay attention to the corresponding convergence property. Needless to say, a divergent series will not be useful for the problem studied in this book. A property regarding the convergence of a Puiseux series is given as follows, see [15].

Property 2.1 *A Puiseux series* $\sum_{i=1}^{\infty} C_i x^{\frac{i}{\mathrm{N}}}$ *is a convergent series if and only if the power series* $\sum_{i=1}^{\infty} C_i \chi^i$ *is convergent.*

We see from Property 2.1 that the convergence of a Puiseux series $\sum_{i=1}^{\infty} C_i x^{\frac{i}{\mathrm{N}}}$ depends only on the coefficients C_i, $i = 1, \ldots, \infty$ (it does not depend on the integer N).

As the Puiseux series considered in this chapter are derived from the power series $\Phi(y, x)$, a nice result for the convergence property is available from [15] and given below.

Property 2.2 *If the power series* $\Phi(y, x)$ *are convergent, all the* y*-roots for* $\Phi(y, x) = 0$ *are convergent series.*

In light of property 2.2, the convergence of all the Puiseux series used in this book associated with the complete stability problem for time-delay systems with commensurate delays (including the Puiseux series for studying the asymptotic behavior of the critical imaginary roots as well as the dual Puiseux series, to be proposed later in this book, for studying the asymptotic behavior of the frequency-sweeping curves) can be guaranteed.

2.4 Newton Diagram

The Newton diagram (or Newton polygon) is a geometrical approach proposed by Newton in order to obtain the y-roots for the equation $\Phi(y, x) = 0$ in terms of the Puiseux series. In this section, we briefly review this approach.

Consider power series $\Phi(y, x)$ described by (2.1), where both ord_y and ord_x are bounded. As we just mentioned, according to the Puiseux' Theorem, all the y-root solutions are in the form of Puiseux series.

In the sequel, we demonstrate how to find the initial terms of the corresponding Puiseux series by using the classical Newton diagram. More precisely, we will determine the solutions of the form

$$y = Cx^{\mu} + o(x^{\mu}), \tag{2.4}$$

where C is the complex coefficient and μ is a rational number. Obviously, C and μ may have multiple values.

We mark the point (α, β) by a "dot" in a coordinate plane if there is a nonzero coefficient $\phi_{\alpha,\beta}$ in (2.1). In this way, we obtain a discrete set of points with non-negative integral coordinates in the coordinate plane, called the *Newton diagram* of $\Phi(y, x)$.

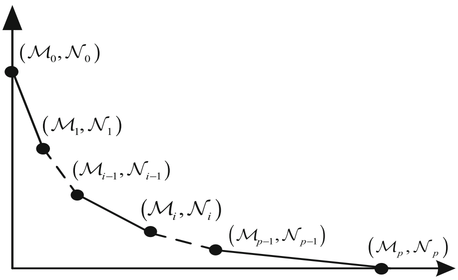
We draw a line through the point $(0, \text{ord}_x)$ (this point belongs to the Newton diagram) coinciding with the ordinate axis and we rotate this line counterclockwise around the point $(0, \text{ord}_x)$ until it touches other points from the Newton diagram. Among the touched points from the Newton diagram, we select the one with the greatest abscissa, say $(\mathscr{M}_1, \mathscr{N}_1)$. We now have a segment linking the two points $(0, \text{ord}_x)$ and $(\mathscr{M}_1, \mathscr{N}_1)$. We next rotate the line counterclockwise around the point $(\mathscr{M}_1, \mathscr{N}_1)$ until it touches new points from the Newton diagram. We also select the one with the greatest abscissa, say $(\mathscr{M}_2, \mathscr{N}_2)$, among the touched points. We have a new segment linking two points $(\mathscr{M}_1, \mathscr{N}_1)$ and $(\mathscr{M}_2, \mathscr{N}_2)$. We continue this procedure till the segment ending at the point $(\text{ord}_y, 0)$ (this point belongs to the Newton diagram) is found.

As a result, we obtain the so-called *Newton polygon* which consists of all the segments found by the above procedure (referred to as the *rotating ruler method*). Without any loss of generality, suppose that the Newton polygon of $\Phi(y, x)$ consists of $p \in \mathbb{N}_+$ segments. The starting point and the ending point of the ith segment are denoted by $(\mathscr{M}_{i-1}, \mathscr{N}_{i-1})$ and $(\mathscr{M}_i, \mathscr{N}_i)$ (it is easy to see that $\mathscr{M}_0 = 0, \mathscr{N}_0 = \text{ord}_x, \mathscr{M}_p = \text{ord}_y, \mathscr{N}_p = 0$), respectively. The Newton polygon is depicted in Fig. 2.1.

Note that on a segment of the Newton polygon, say, the ith segment with the endpoints $(\mathscr{M}_{i-1}, \mathscr{N}_{i-1})$ and $(\mathscr{M}_i, \mathscr{N}_i)$, there may exist other points from the Newton diagram. Without loss of generality, suppose there are q points other than the endpoints lying on the ith segment: $(\widetilde{\mathscr{M}}_{i1}, \widetilde{\mathscr{N}}_{i1}), \ldots, (\widetilde{\mathscr{M}}_{iq}, \widetilde{\mathscr{N}}_{iq})$, with $\mathscr{M}_i > \widetilde{\mathscr{M}}_{i1} > \cdots > \widetilde{\mathscr{M}}_{iq} > \mathscr{M}_{i-1}$.

Each segment of the Newton polygon determines a set of solutions of C and μ. More precisely, from the ith segment linking points $(\mathscr{M}_{i-1}, \mathscr{N}_{i-1})$ and $(\mathscr{M}_i, \mathscr{N}_i)$, we have $\mathscr{M}_i - \mathscr{M}_{i-1}$ roots in the form (2.4) with $\mu = \frac{\mathscr{N}_{i-1} - \mathscr{N}_i}{\mathscr{M}_i - \mathscr{M}_{i-1}}$ (note that $-\mu$ is the slope of the segment). The coefficient C associated with this exponent μ has

Fig. 2.1 Newton polygon



$\mathscr{M}_i - \mathscr{M}_{i-1}$ (note that this value is equal to the length of the ith segment's projection on the abscissa axis) solutions, which are given by the solutions of the polynomial equation.

$$\phi_{\mathscr{M}_i,\mathscr{N}_i} C^{\mathscr{M}_i - \mathscr{M}_{i-1}} + \phi_{\widetilde{\mathscr{M}}_{i1},\widetilde{\mathscr{N}}_{i1}} C^{\widetilde{\mathscr{M}}_{i1} - \mathscr{M}_{i-1}} + \cdots + \phi_{\mathscr{M}_{i-1},\mathscr{N}_{i-1}} = 0. \quad (2.5)$$

A rigorous proof of the above results can be found in e.g., [115]. In summary, a segment of the Newton polygon gives rise to some initial terms of the Puiseux series with the same exponent. To be more precise, the number of the obtained Puiseux series equals to the length of the projection of this segment on the abscissa and the exponent is the negative slope of this segment.

One can see that the p sets of Puiseux series derived from the p segments of the Newton polygon include all the ord_y y-roots (expressed by the first-order terms of the Puiseux series) for $\Phi(y, x) = 0$.

We now give the Newton polygons, for Examples 2.3 and 2.4, respectively, in Fig. 2.2a, b, from which one may obtain the Puiseux series solutions by employing the Newton diagram introduced above.

2.5 A Direct Application of Puiseux Series

It should be pointed out that, to the best of the authors' knowledge, there are at least two ways to express the Puiseux series solutions. The expression given in the sequel is relatively simple to understand.[3]

Without any loss of generality, the Newton polygon for the power series $\Phi(y, x)$ is supposed to have p segments.

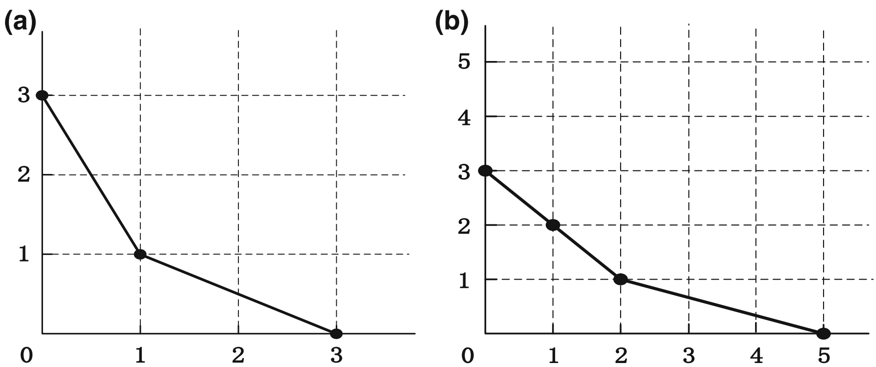


Fig. 2.2 Newton polygons for Examples 2.3 and 2.4. **a** Example 2.3. **b** Example 2.4

[3] In Chap. 4, the expression of the Puiseux series will be simplified. However, some additional algebraic properties (mainly concerning the concept of the *conjugacy class*) will be required.

Following Sect. 2.4, the ith segment determines a set of Puiseux series

$$y = \widetilde{C}_{\mu_i,l} x^{\mu_i} + o(x^{\mu_i}), l = 1, \ldots, \mathscr{M}_i - \mathscr{M}_{i-1}, \tag{2.6}$$

where μ_i is the negative slope of the ith segment, $\widetilde{C}_{\mu_i,l}$ are the corresponding coefficients calculated according to (2.5), and $\mathscr{M}_i - \mathscr{M}_{i-1}$ equals to the length of the segment's projection on the abscissa axis.

Totally, the p segments give rise to the following Puiseux series

$$\begin{cases} y = \widetilde{C}_{\mu_1,l} x^{\mu_1} + o(x^{\mu_1}), l = 1, \ldots, \mathscr{M}_1 - \mathscr{M}_0, \\ \vdots \\ y = \widetilde{C}_{\mu_p,l} x^{\mu_p} + o(x^{\mu_p}), l = 1, \ldots, \mathscr{M}_p - \mathscr{M}_{p-1}. \end{cases} \tag{2.7}$$

With the expression (2.7), we may consider each x^{μ_i} as a single-valued number[4] in $\mathbb{C}$. As a result, the $\mathscr{M}_i - \mathscr{M}_{i-1}$ Puiseux series corresponding to the ith segment as described by (2.6) have $\mathscr{M}_i - \mathscr{M}_{i-1}$ values for $y(x)$. The total p sets of Puiseux series corresponding to all the p segments (i.e., the total ord_y Puiseux series) as described by (2.7) present all the ord_y solutions $y(x)$.

Remark 2.6 In (2.7), we only present the first-order terms (also called the initial terms) of the Puiseux series. As we will see later in this book, the first-order terms are sufficient for the stability analysis in the nondegenerate case. However, in the degenerate case, we need to obtain higher order terms. We will see in Sect. 4.3 that the Newton diagram can be used in an iterative manner such that higher order terms of the Puiseux series can be obtained.

Remark 2.7 On may notice that invoking the Puiseux series (by using the Newton diagram) is not a trivial work, even if only the first-order terms are required. Some representative examples will be given in Chap. 4. Fortunately, the calculation of the Puiseux series may be bypassed. It will be interesting to see that we can accomplish the complete stability analysis for time-delay systems with commensurate delays (by adopting the frequency-sweeping approach to be proposed in this book) without explicitly employing the Newton diagram.

2.6 Notes and Comments

In this chapter, we introduced some useful results for analytic curves including the basic concepts, the Puiseux series, and the Newton diagram. More precisely, we followed the ideas proposed by [15, 60, 91, 121] in order to introduce some of the

[4] In fact, each x^{μ_i} may have multiple values. We may choose any one among them. As will be illustrated by the examples in Chap. 4, the value set of all the Puiseux series is identical for any choice.

notions and properties needed in the forthcoming chapters. From the next chapter, we will apply these results to study the complete stability problem of time-delay systems.

In our opinion, the analytic curve idea in fact may be used for a broader range of stability and stabilization problems in the area of control, as it is applicable to both continuous-time and discrete-time systems.

For continuous-time systems (including the time-delay systems considered in the forthcoming chapters), we are concerned with the variation of the critical roots with respect to the imaginary axis $\mathbb{C}_0$ as some system parameters vary. Recall that for a continuous-time system a critical root refers to a characteristic root located on the imaginary axis $\mathbb{C}_0$. We may perform a qualitative stability analysis through the real parts of the corresponding Puiseux series.

For discrete-time systems (e.g., the state transition expression of a networked control system is a discrete-time model [80]), we are concerned with the variation of the critical roots (note that for discrete-time systems a critical root refers to a characteristic root located on the unit circle $\partial\mathbb{D}$) with respect to the unit circle $\partial\mathbb{D}$, as some system parameters vary. In this case, the stability analysis requires to know the variation directions of the critical roots with respect to the unit circle $\partial\mathbb{D}$, based on the Puiseux series. For instance, if for a critical root its variation direction points to the outside (inside) of the unit circle $\partial\mathbb{D}$, it implies that the critical root becomes an unstable (stable) root.

It was already pointed out that we only adopt some preliminary results on the singularities of analytic curves and one will find that they are not hard to follow. The studies from a decidedly geometrical viewpoint (e.g., resolution of singularities and classification of singularities) are generally much more complicated and can be found in [2, 15, 121].

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