Chapter 2

Dynamical Characterizations

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2.1 Definitions and statements of results

Before discussing the relationship between the Haagerup property and group actions on von Neumann algebras, we present a proof of the equivalence of the four characterizations of this property stated in Chapter 1. The equivalences are spread over [AW81], [BCV95], [Jol00] and [Jul98], and it may be useful to gather them all together in the same place.

Theorem 2.1.1. *For a locally compact, second countable noncompact group G, the following conditions are equivalent:*

(1) *[AW81] there exists a proper, continuous function $\psi\colon G \to \mathbb{R}^+$ which is conditionally negative definite, that is, $\psi(g^{-1}) = \psi(g)$ for all $g \in G$, and for all $g_1, \ldots, g_n \in G$ and all $a_1, \ldots, a_n \in \mathbb{C}$ with $\sum a_i = 0$,*

$$\sum_{i,j} \overline{a}_i a_j \psi(g_i^{-1} g_j) \leq 0;$$

(2) *the abelian C^*-algebra $C_0(G)$ possesses an approximate unit of normalized, positive definite functions, that is, there exists a sequence $(\varphi_n)_{n\geq 1}$ of functions in $C_0(G)$ such that $\varphi_n(e) = 1$ for all n, $\varphi_n \to 1$ uniformly on compact subsets of G and which are positive definite, that is,*

$$\sum_{i,j} \overline{a}_i a_j \varphi_n(g_i^{-1} g_j) \geq 0$$

for all $g_1, \ldots, g_n \in G$ and all $a_1, \ldots, a_n \in \mathbb{C}$;

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(3) *[Jol00] there exists a C_0 unitary representation $(\pi, \mathcal{H})$ of G, that is, all matrix coefficients $\varphi_{\xi,\eta}\colon g \mapsto \langle \pi(g)\xi, \eta\rangle$ belong to $C_0(G)$, which weakly contains the trivial representation 1_G (denoted $1_G \prec \pi$);*

(4) *[BCV95] G is **a-T-menable**: there exists a Hilbert space $\mathcal{H}$ and an isometric affine action α of G on $\mathcal{H}$ which is proper in the sense that, for all pairs of bounded subsets B and C of $\mathcal{H}$, the set of elements $g \in G$ such that $\alpha_g(B) \cap C \neq \emptyset$ is relatively compact.*

Moreover, if these conditions hold, one can choose in (1) a proper, continuous, conditionally negative definite function ψ such that $\psi(g) > 0$ for all $g \neq e$, and similarly the representation π in condition (3) may be chosen such that for all $g \neq e$, there exists a unit vector $\xi \in \mathcal{H}$ such that $|\langle \pi(g)\xi, \xi\rangle| < 1$. In particular, π is faithful.

Proof. First we prove the equivalence of (1) and (2). If ψ satisfies (1), Schoenberg's theorem ([HV89, p. 66]) states that $\exp(-t\psi)$ is positive definite for all positive t. Hence (1) implies (2).

Conversely, if G satisfies (2), let $(K_n)_{n\geq 1}$ be an increasing sequence of compact subsets of G whose union is G. Choose an unbounded increasing positive sequence $(\alpha_n)_{n\geq 1}$ and a decreasing sequence $(\varepsilon_n)_{n\geq 1}$, tending to 0, such that $\sum \alpha_n \varepsilon_n$ converges. For all n, choose a continuous, positive definite function φ_n on G such that $\varphi_n \in C_0(G)$ and

$$\sup_{g\in K_n} |\varphi_n(g) - 1| \leq \varepsilon_n.$$

Replacing φ_n by $|\varphi_n|^2$ if necessary, we assume further that $0 \leq \varphi_n \leq 1$ for all n. Set, for $g \in G$,

$$\psi(g) = \sum_{n\geq 1} \alpha_n (1 - \varphi_n(g)),$$

which defines a conditionally negative definite function on G. As the series converges uniformly on compact sets, ψ is continuous. To check that it is proper, take $R > 0$, and fix an integer n so large that $\alpha_n \geq 2R$. As φ_n belongs to $C_0(G)$, there exists a compact subset L of G such that $|\varphi_n(g)| < 1/2$ for all $g \notin L$. Then

$$\{g \in G : \psi(g) \leq R\} \subseteq \{g \in G : 1 - \varphi_n(g) \leq 1/2\} \subseteq L.$$

Now we prove the equivalence of (1) and (4). Let ψ be a (not necessarily proper) continuous, conditionally negative definite function on G. Then, by [HV89, p. 63], there exists an essentially unique triple $(\mathcal{H}, \pi, b)$ where $\mathcal{H}$ is a real Hilbert space, π is an orthogonal representation of G on $\mathcal{H}$ and b is a

π-cocycle (that is, $b(gh) = b(g) + \pi(g)b(h)$ for all $g, h \in G$), $\mathcal{H}$ is topologically generated by the range of b, and finally

$$\psi(g) = \|b(g)\|^2$$

for all g. The associated affine action is defined by

$$\alpha_g(\xi) = \pi(g)\xi + b(g).$$

Conversely, if b is a cocycle as above, the function $g \mapsto \|b(g)\|^2$ defines a continuous, conditionally negative definite function. The equivalence of (1) and (4) follows from the fact that ψ is proper if and only if α is.

Now we show that (2) and (3) are equivalent. If $(\varphi_n)_{n\geq 1}$ satisfies condition (2), let $(\pi_n, \mathcal{H}_n, \xi_n)$ be the Gel'fand–Naimark–Segal triple associated with φ_n, and set

$$\pi = \bigoplus_n \pi_n.$$

Then π is a C_0-representation and $1_G \prec \pi$. Conversely, if $(\pi, \mathcal{H})$ satisfies condition (3), let $(\eta_n)_{n\geq 1}$ be a sequence of unit vectors such that, for all compact subsets K of G,

$$\lim_{n\to\infty} \sup_{g\in K} \|\pi(g)\eta_n - \eta_n\| = 0.$$

Set $\varphi_n = \langle \pi(\cdot)\eta_n, \eta_n \rangle$. Then the sequence (φ_n) satisfies condition (2).

In order to prove the additional property of ψ in condition (1), choose a sequence $(V_n)_{n\geq 1}$ of relatively compact neighbourhoods of the identity e such $\bigcap_n V_n = \{e\}$. Next, choose nonnegative continuous functions f_n for all n such that $\int_G f_n(g)^2 dg = 1$ and such that the associated matrix coefficient function

$$\omega_n(g) = \langle \lambda_G(g) f_n, f_n \rangle$$

is supported in V_n. Replacing ψ by

$$\psi' = \psi + \sum_{n\geq 1} \frac{1}{2^n}\,(1 - \omega_n),$$

it is easy to check that $\psi'(g) > 0$ for all $g \neq e$.

A similar argument works for the representation π in condition (3). □

Remark. In Definition 2.4 of [BR88], V. Bergelson and J. Rosenblatt proved that if G satisfies condition (2) of Theorem 2.1.1, then it has the following interesting property: fix an infinite dimensional, separable Hilbert space $\mathcal{H}$, and denote by $\mathrm{Rep}(\mathcal{H})$ the set of all unitary representations of G on $\mathcal{H}$, endowed with a suitable natural topology. Then the subset of C_0-representations is dense in $\mathrm{Rep}(\mathcal{H})$.

In the 1980s, Rosenblatt [Ros81], K. Schmidt [Sch81] and A. Connes and B. Weiss [CW80] found characterizations of amenability and of property (T) for countable groups in terms of measure preserving ergodic actions. For instance, it follows from [Sch81] and [CW80] that a countable group Γ is amenable if and only if no measure preserving ergodic action of Γ is strongly ergodic, and on the other hand, Γ has property (T) if and only if every measure preserving ergodic action of Γ is strongly ergodic (this means that there are no nontrivial asymptotically invariant sequences: see the remark following the proof of Theorem 2.2.2 and Proposition 2.2.3). The main results of this chapter fit into this circle of ideas, since they characterize the Haagerup property in terms of suitable measure preserving ergodic actions on the one hand, and on some approximately finite dimensional factors on the second. In order to state them, we need to fix notation and give some definitions.

Assume that G is a locally countable second countable group that acts (on the right) on a standard probability space (S, μ) by measure-preserving Borel automorphisms.

Definition 2.1.2. The action of G on (S, μ) is said to be **strongly mixing** if, for all Borel subsets A and B of S,

$$\lim_{g \to \infty} \mu(Ag \cap B) = \mu(A)\mu(B),$$

that is, for all positive ε, there exists a compact subset K of G such that

$$|\mu(Ag \cap B) - \mu(A)\mu(B)| < \varepsilon \qquad \forall g \in G \setminus K.$$

A sequence of Borel subsets $(A_n)_{n \geq 1}$ of S is said to be **asymptotically invariant** if, for all compact subsets K of G,

$$\sup_{g \in K} \mu(A_n g \triangle A_n) \to 0 \quad \text{as } n \to \infty.$$

It is said to be **nontrivial** if moreover

$$\inf_n \mu(A_n)(1 - \mu(A_n)) > 0.$$

A sequence of nonnull Borel subsets $(A_n)_{n \geq 1}$ of S is said to be a **Følner sequence** if $\mu(A_n) \to 0$ as $n \to \infty$ and if for all compact subsets K of G,

$$\sup_{g \in K} \frac{\mu(A_n g \triangle A_n)}{\mu(A_n)} \to 0 \quad \text{as } n \to \infty.$$

Finally, we define the unitary representation $\pi_S \colon G \to \mathcal{U}(L^2(S, \mu))$ by

$$(\pi_S(g)\xi)(s) = \xi(sg),$$

for all $\xi \in L^2(S,\mu)$, $g \in G$ and $s \in S$, and the subspace $L^2_0(S,\mu)$ to be

$$\left\{\xi \in L^2(S,\mu) : \int_S \xi \, d\mu = 0\right\}.$$

Then $L^2_0(S,\mu)$ is closed and G-invariant, and we denote by ρ_S the restriction of π_S to $L^2_0(S,\mu)$.

Remark. Similar sequences have already been used several times: see [CW80], [Rin88], [Ros81], [Sch81] and [Sch80]. For instance, a Følner sequence in our sense is called an **I-sequence** in [Sch81].

There are relationships between the existence of nontrivial asymptotically invariant sequences and the existence of Følner sequences; for simplicity, assume that G is countable. If (S,μ) is an ergodic G-space which has a nontrivial asymptotically invariant sequence, then for all $c \in]0,1[$, there exists an asymptotically invariant sequence $(C_n)_{n\geq 1}$ such that $\mu(C_n) = c$ for all n (see [Sch81]), and this obviously implies the existence of Følner sequences. However, the converse fails: Example 2.7 of [Sch81] exhibits an action of the nonabelian free group F_3 on a probability space (S,μ) with no nontrivial asymptotically invariant sequences, but with a Følner sequence.

Here is our first main result.

Theorem 2.1.3. *Let G be a locally compact second countable group. Then G has the Haagerup property if and only if there exists a measure preserving G-action on a standard probability space (S,μ) such that*

(1) *the action of G on (S,μ) is strongly mixing, and*

(2) *(S,μ) contains a Følner sequence for the G-action.*

Moreover, (S,μ) contains a nontrivial asymptotically invariant sequence, and S may be taken to be a compact metrizable space with a continuous, essentially free action of G.

This will be proved in Section 2.2, where the construction of (S,μ) is taken from [Sch96].

Remark. We will see in the next section that conditions (1) and (2) translate into properties of the representation ρ_S, namely, condition (1) holds if and only if ρ_S is of class C_0, and condition (2) is equivalent to the condition that $1_G \prec \rho_S$ by [Rin88, Prop. 4]. This means that the representation π in condition (3) of Theorem 2.1.1 may be chosen to be of the form ρ_S.

It turns out that Theorem 2.1.3 has a noncommutative analogue. In order to state it, we need more notation and definitions. Let N be a von Neumann algebra with separable predual, and let φ be a faithful normal state on N. We denote by $\|x\|_\varphi$ the associated Hilbert norm $\varphi(x^*x)^{1/2}$. By completing N with respect to $\|\cdot\|_\varphi$ and extending the left multiplication on N, we obtain a Hilbert space $L^2(N,\varphi)$ on which N acts. Moreover, we assume that there is an action $\alpha\colon G \to \mathrm{Aut}(N)$ such that φ is α-invariant: $\varphi \circ \alpha_g = \varphi$ for all $g \in G$.

The following is the noncommutative analogue of Definition 2.1.2.

Definition 2.1.4. With the same assumptions as above, we say that the action α is **strongly mixing** for φ if

$$\lim_{g\to\infty} \varphi(\alpha_g(x)y) = \varphi(x)\varphi(y)$$

for all $x, y \in N$. A sequence of projections $(e_k)_{k\geq 1}$ on N is said to be a **nontrivial asymptotically invariant sequence** for α and φ if

$$\lim_{k\to\infty} \sup_{g\in K} \|\alpha_g(e_k) - e_k\|_\varphi = 0$$

for all compact subsets K of G and

$$\lim_{k\to\infty} \varphi(e_k)(1-\varphi(e_k)) > 0.$$

The sequence of projections is said to be a **Følner sequence** if

$$\lim_{k\to\infty} \sup_{g\in K} \frac{\|\alpha_g(e_k) - e_k\|_\varphi}{\|e_k\|_\varphi} = 0$$

for all compact subsets K of G and

$$\lim_{k\to\infty} \varphi(e_k) = 0.$$

In Section 2.3, we observe first that if α is strongly mixing for φ and if N contains a Følner sequence, then G has the Haagerup property. Our second main result is a converse.

Theorem 2.1.5. *Let G be a locally compact second countable group with the Haagerup property. Then for each factor N listed below, there exist an action α of G on N and an α-invariant state φ for which α is strongly mixing, and N contains a Følner sequence and a nontrivial asymptotically invariant sequence for α and φ:*

(1) *N is the hyperfinite factor R of type* II_1 *and φ is the canonical trace τ.*

(2) *N is the approximately finite dimensional factor $R_{0,1} = R \otimes B$ of type* II_∞ *and $\varphi = \tau \otimes \omega$, where B is the type* I_∞ *factor and ω is a suitable normal state on B.*

(3) *N is the Powers factor R_λ of type* III_λ *and $\varphi = \varphi_\lambda$ is the associated Powers state.*

This will be proved in Section 2.3. The idea of the proof of Theorem 2.1.5 is as follows: given a C_0-representation $(\pi, \mathcal{H})$ of G such that $1_G \prec \pi$, one defines an action of G on the canonical anticommutation relation C*-algebra CAR($\mathcal{H}$), and every factor listed above is obtained from the Gel'fand–Naimark–Segal construction starting from a suitable state on CAR($\mathcal{H}$).

Remark. As in the commutative case, the existence and the mixing properties of the asymptotically invariant sequences and the Følner sequences in Theorem 2.1.5 may be expressed in terms of a representation ρ of G on a suitable subspace of $L^2(N)$. See Section 2.3 for a precise statement.

Finally, we focus on actions of a *discrete* group G with the Haagerup property on the hyperfinite II$_1$-factor R and improve Theorem 2.1.5: there exists an action α of G on R which has many nontrivial asymptotically invariant sequences. See the end of Section 2.3, where we discuss centralizing sequences, the asymptotic centralizer R_ω, where ω is a free ultrafilter on $\mathbb{N}$, and the induced action α^ω on R_ω (see also [Con75] and [Ocn85]).

Theorem 2.1.6. *Let G be a countable group with the Haagerup property. Then there exists an action α of G on R with the following properties:*

(1) *α is strongly mixing and (centrally) free;*
(2) *the fixed point algebra $(R_\omega)^\alpha$, that is, the set of all $x \in R_\omega$ such that $\alpha^\omega_g(x) = x$ for all $g \in G$, is a type* II$_1$ *subalgebra of R_ω.*

This result merits some comment. In [Ocn85], A. Ocneanu classified (centrally) free actions of countable *amenable* groups on R, and to do that, he used the fact that the fixed point algebra $(R_\omega)^\alpha$ is always of type II$_1$ for such an action. On the other hand, for any countable *nonamenable* group G, V.F.R. Jones [Jon83] studied the Bernoulli shift action β of G on the space R, realized as the infinite tensor product

$$R = \bigotimes_{g \in G} M_2(\mathbb{C}).$$

It is obvious that β is strongly mixing, and Jones proved that the fixed point algebra $(R_\omega)^\beta$ is trivial. Theorem 2.1.6 shows that groups with the Haagerup property behave in some sense like amenable groups.

2.2 Actions on measure spaces

The following result is a special case of Proposition 2.3.1 in the next section.

Proposition 2.2.1. *Assume that the locally compact second countable group G has a measure-preserving action on a standard probability space (S, μ), that the* *action is strongly mixing and that there exists a Følner sequence in S. Then G has the Haagerup property.*

We simply observe that the strong mixing property is equivalent to the fact that the representation ρ_S is a C_0-representation and that the existence of a Følner sequence implies $1_G \prec \rho_S$.

The main result of this section is the following converse.

Theorem 2.2.2. *Suppose that G has the Haagerup property. Then there exists a measure-preserving Borel action of G on a standard probability space (S, μ), with the following properties:*

(1) *the action is strongly mixing;*

(2) *there exists a sequence of Borel sets $(A_n)_{n\geq 1}$ in S such that $\mu(A_n) = 1/2$ for all n and for all compact subsets K of G,*

$$\lim_{n\to\infty} \sup_{g\in K} \mu(A_n g \triangle A_n) = 0;$$

(3) *the action is essentially free: the subset of all $s \in S$ for which the stabilizer $G_s \neq \{e\}$ is of measure zero.*

Before proving Theorem 2.2.2, we introduce more notation. Let $(\pi, \mathcal{H})$ be a representation of G, and set $\mathcal{H}^\sigma = \sum_{k\geq 0}^{\oplus} \mathcal{H}^{\times k}$, where $\mathcal{H}^{\times 0} = \mathbb{C}$, and for $k > 0$, $\mathcal{H}^{\times k}$ is the k^{th} symmetric tensor product of $\mathcal{H}$, that is, the closed subspace of the Hilbert tensor product space $\mathcal{H}^{\otimes k}$ generated by the vectors of the form

$$\sum_{\varepsilon\in S_k} \xi_{\varepsilon(1)} \otimes \ldots \otimes \xi_{\varepsilon(k)},$$

where S_k denotes the usual permutation group. Then the representation π extends in a natural way to a representation π^σ of G on $\mathcal{H}^\sigma$ which leaves the subspace $\mathcal{H}_0^\sigma = \mathcal{H}^\sigma \ominus \mathcal{H}^{\times 0}$ invariant. Finally, we denote by π_0^σ the restriction of π^σ to $\mathcal{H}_0^\sigma$.

Proof of Theorem 2.2.2. Fix a proper, continuous, conditionally negative definite function $\psi\colon G \to [0, +\infty[$, nonvanishing except at e. For $n \geq 1$, set $\varphi_n = \exp(-\psi/n)$, and denote by $(\pi_n, \mathcal{H}_n, \xi_n)$ the associated Gel'fand–Naimark–Segal triple. Since φ_n is real-valued, there is a real Hilbert subspace $\mathcal{H}_n'$ of $\mathcal{H}_n$, containing ξ_n, such that

$$\mathcal{H}_n = \mathcal{H}_n' \oplus i\mathcal{H}_n' \quad \text{and} \quad \pi_n(g)\mathcal{H}_n' = \mathcal{H}_n'$$

for all n and g. Write $\mathcal{H}$ for $\bigoplus_{n\geq 1} \mathcal{H}_n$ and π for $\bigoplus_{n\geq 1} \pi_n$, and observe that $\mathcal{H} = \mathcal{H}' \oplus i\mathcal{H}'$, where $\mathcal{H}' = \bigoplus_{n\geq 1} \mathcal{H}_n'$. To simplify notation, we denote by ξ_n

the corresponding vector $0 \oplus \ldots \oplus \xi_n \oplus \ldots \in \mathcal{H}'$, and we observe that $\xi_n \perp \xi_m$ when $n \neq m$.

We first define a G-space (Ω, ν), as in [Sch96, Sec. 3], satisfying conditions (1) and (2), and then we indicate how to obtain a G-space (S, μ) which also satisfies condition (3), using the techniques of S. Adams, G.A. Elliot and T. Giordano [AEG94, Prop. 1.2] (faithfulness of π is required here).

Choose a countable orthonormal basis $\mathcal{B}$ of $\mathcal{H}'$ containing $\{\xi_n : n \geq 1\}$, and set

$$(\Omega, \nu) = \prod_{b \in \mathcal{B}} \Big(\mathbb{R}, \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big)\, dx\Big).$$

Define $\eta' \colon \mathcal{H}' \to \Omega$ by $\eta'(\xi) = (\langle \xi, b\rangle)_{b \in \mathcal{B}}$ and $X_b \colon \Omega \to \mathbb{R}$ by $X_b((\omega_{b'})_{b' \in \mathcal{B}}) = \omega_b$ for every $b \in \mathcal{B}$. The random variables X_b are independent Gaussians with mean 0 and variance 1. In particular, if $b_1, \ldots b_k \in \mathcal{B}$ are distinct and $l_1, \ldots, l_k$ are nonnegative integers, then

$$\int_\Omega X_{b_1}^{l_1} \ldots X_{b_k}^{l_k}\, d\nu = \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{l_j} \exp\Big(-\frac{x^2}{2}\Big)\, dx.$$

This allows us to define $\eta \colon \mathcal{H}' \to L^2(\Omega, \nu, \mathbb{R})$ by

$$\eta\Big(\sum_{b \in \mathcal{B}} \xi_b b\Big) = \sum_{b \in \mathcal{B}} \xi_b X_b.$$

Thus η is an isometry that extends to a unitary operator $u \colon \mathcal{H}^\sigma \to L^2(\Omega, \nu)$ which sends $\mathcal{H}^{\times 0}$ onto the subspace of constant functions on Ω and such that $u\left(\sum_{\varepsilon \in S_n} b_{\varepsilon(1)} \otimes \ldots \otimes b_{\varepsilon(n)}\right) = X_{b_1} \ldots X_{b_n}$ for all $b_1, \cdots, b_n \in \mathcal{B}$. Moreover there exists a ν-preserving action of G on (Ω, ν) such that $u^* \pi_\Omega(g) u = \pi^\sigma(g)$ and $u^* \rho_\Omega(g) u = \pi_0^\sigma(g)$ for all $g \in G$. It is straightforward to check that π_0^σ is a C_0-representation, which shows that condition (1) is satisfied.

As a second step, we construct a sequence (B_n) of Borel subsets of Ω satisfying condition (2); our construction is inspired by E. Glasner and B. Weiss [GW97, Thm 2]. For $n \geq 1$, set $X_n = X_{\xi_n}$ (recall that $\xi_n \in \mathcal{B}$), and $X_n^g = \pi_\Omega(g) X_n$ for all $g \in G$. Observe that

$$X_n^g = \rho_\Omega(g) X_n = u \pi_0^\sigma(g) \xi_n = u \pi(g) \xi_n = \sum_{b \in \mathcal{B}} \langle \pi(g)\xi_n, b\rangle\, X_b.$$

Define $B_n = \{\omega \in \Omega : X_n(\omega) \geq 0\}$; by symmetry, we have $\nu(B_n) = 1/2$ for all n. Now fix $g \neq e$. Then X_n^g is a Gaussian random variable with zero mean and variance 1 because it has the same probability distribution as X_n; indeed, for all Borel subsets E of $\mathbb{R}$,

$$\nu(\{X_n^g \in E\}) = \nu(\{X_n \in E\}\, g^{-1}) = \nu(\{X_n \in E\}).$$

Write $X_n^g = \cos(\alpha_n(g))X_n + \sin(\alpha_n(g))Y_n$, where $\alpha_n(g) = \arccos(\langle\pi(g)\xi_n, \xi_n\rangle)$ and

$$Y_n = \frac{1}{\sin(\alpha_n(g))} \sum_{b \neq \xi_n} \langle \pi(g)\xi_n, b\rangle\, X_b$$

which is a Gaussian random variable with mean 0 and variance 1, and it is independent of X_n since all linear combinations of $\{X_b : b \neq \xi_n\}$ are. As in the proof of [GW97, Thm 2],

$$\nu(B_n g \triangle B_n) = \frac{\alpha_n(g)}{\pi}\ ,$$

which tends to 0 uniformly on compact sets. Thus (Ω, ν) satisfies conditions (1) and (2), but perhaps not (3), the essential freeness of the action. But, arguing as in [AEG94], define

$$(S, \mu) = \prod_{m \geq 1} (\Omega, \nu)$$

with the action $(\omega_m)g = (\omega_m g)$. Then the action is strongly mixing and essentially free, and the sequence (A_n) defined by

$$A_n = B_n \times \prod_{m \geq 2} \Omega$$

satisfies condition (3). □

Remark. The construction of the G-space (S, μ) in the proof of Theorem 2.2.2 provides another proof of the theorem of Connes and Weiss [CW80] which works even when G is not countable. Theorem $2'$ of [GW97] also provides a proof of the theorem of Connes and Weiss.

Proposition 2.2.3 ([CW80], [GW97]). *Let G be a locally compact second countable group which does not have Kazhdan's property (T). There exists a measure-preserving G-action on a probability space (S, μ) such that*

(1) *the action is ergodic and essentially free, and*
(2) *there exists a nontrivial asymptotically invariant sequence of Borel subsets of S.*

Proof. Since G does not have property (T), by [AW81], there exists a continuous, unbounded, conditionally negative definite function $\psi\colon G \to \mathbb{R}^+$; we assume that $\psi(g) > 0$ for all $g \neq e$, and we construct (S, μ) and a sequence (A_n) of Borel subsets of S as in the proof of Theorem 2.2.2. The action is essentially free and (A_n) is a nontrivial asymptotically invariant sequence. Since ψ is unbounded, it follows (for instance, from [Jol93, Lem. 4.4]) that Godement's mean $M(e^{-t\psi}) = 0$ for all positive t. By Theorem A.1 of [GW97], this implies that the G-action on S is ergodic (and even weakly mixing). □

2.3 Actions on factors

Let N and φ be as at the end of Section 2.1. Let $(L^2(N), J, L^2(N)^+)$ be the standard form of N, and denote by $\xi_0 \in L^2(N)^+$ the associated cyclic vector, so that $\varphi(x) = \langle x\xi_0, \xi_0\rangle$ for all $x \in N$. An action α of G on N gives rise to a unique representation $\pi\colon G \to \mathcal{U}(L^2(N))$ such that

$$\pi(g)x\pi(g^{-1}) = \alpha_g(x) \qquad \forall g \in G \quad \forall x \in N,$$

and

$$\pi(g)J = J\pi(g) \quad \text{and} \quad \pi(g)L^2(N)^+ = L^2(N)^+ \qquad \forall g \in G.$$

Moreover, if φ is α-invariant, ξ_0 is invariant under π, and the subspace $L^2_0(N)$ of all $\xi \in L^2(N)$ which are orthogonal to ξ_0 is invariant under π, and we denote by ρ the restriction of π to $L^2_0(N)$.

Proposition 2.3.1. *Assume that $\alpha\colon G \to \mathrm{Aut}(N)$ is a strongly mixing action for φ and that N contains a Følner sequence for α and φ. Then the representation ρ weakly contains the trivial representation of G and is C_0. In particular, G has the Haagerup property.*

Proof. The set of vectors $\{x\xi_0 - \varphi(x)\xi_0 : x \in N\}$ is total in $L^2_0(N)$, and for all $x, y \in N$,

$$\langle \rho(g)(x\xi_0 - \varphi(x)\xi_0), y\xi_0 - \varphi(y)\xi_0\rangle = \varphi(\alpha_g(x)y) - \varphi(x)\varphi(y) \to 0$$

as $g \to \infty$. This proves that ρ is of class C_0. Moreover, if $(e_k)_{k\geq 1}$ is a Følner sequence for α and φ, set

$$\xi_k = \frac{e_k - \varphi(e_k)}{\sqrt{\varphi(e_k) - \varphi(e_k)^2}}\,\xi_0.$$

Then $\xi_k \in L^2_0(N)$ and $\|\xi_k\|_\varphi = 1$. As $\sup\varphi(e_k) < 1$, there exists a positive constant c such that $1 - \varphi(e_k) \geq c$ for all k. Hence, if K is a compact subset of G,

$$\sup_{g\in K} \|\rho(g)\xi_k - \xi_k\|_\varphi \leq c^{-1/2} \sup_{g\in K} \frac{\|\alpha_g(e_k) - e_k\|_\varphi}{\|e_k\|_\varphi} \to 0$$

as $k \to \infty$. This proves that $1_G \prec \rho$. □

We describe now our realizations of the factors, states and actions listed in Theorem 2.1.5. Let $(\pi, \mathcal{H})$ be a separable unitary representation of G; we assume that the scalar product on $\mathcal{H}$ is antilinear in the first variable. Following Chapters 7 and 8 of [HJ], let A be the Fermion C*-algebra or CAR-algebra CAR($\mathcal{H}$) over $\mathcal{H}$; it is the C*-algebra generated by $\{a(\xi) : \xi \in \mathcal{H}\}$, where $a\colon \mathcal{H} \to A$ is a linear isometry, and the following **canonical anticommutation relations** hold:

$$a^*(\xi)a(\eta) + a(\eta)a^*(\xi) = \langle \xi, \eta \rangle \tag{2.1}$$

$$a(\xi)a(\eta) + a(\eta)a(\xi) = 0 \tag{2.2}$$

for all ξ and η. It is well known that A is a uniformly hyperfinite algebra, and that the representation π induces an action α of G on A characterized by

$$\alpha_g(a(\xi)) = a(\pi(g)\xi) \qquad \forall g \in G \quad \forall \xi \in \mathcal{H}.$$

Our constructions rest on the following fact (see [HJ, Thm 8.2]): for all $b \in \mathcal{L}(\mathcal{H})$ such that $0 \leq b \leq 1$, there exists a unique state ϕ_b on A such that

$$\phi_b(a^*(\xi_m) \dots a^*(\xi_1)a(\eta_1) \dots a(\eta_n)) = \delta_{n,m} \det(\langle \xi_j, b\eta_k \rangle), \tag{2.3}$$

for all $\xi_1, \dots, \xi_m$ and $\eta_1, \dots, \eta_n \in \mathcal{H}$. Such a state is called **quasifree**. We denote by N_b the von Neumann algebra obtained by applying the Gel'fand–Naimark–Segal construction to the pair (A, ϕ_b); it is known that N_b is always a factor. If $b = 1/2$ then $\tau = \phi_{1/2}$ is the unique normalized trace on A and the associated factor is the hyperfinite II_1 factor R. If $t \in]0, 1/2[$ and $\lambda = t/(1-t)$, then N_t is the Powers factor R_λ of type III_λ and ϕ_t is the Powers state, henceforth denoted by φ_λ. Finally, if $b = 1$, then ϕ_1 is the vacuum state ω; it is pure, and thus the associated factor is the type I_∞ factor B. Whenever b belongs to the commutant $\pi(G)'$ of $\pi(G)$, ϕ_b is α-invariant and the action α extends to an action of G on N_b, still denoted α. Notice that this is the case for all the values of b above.

The following result proves that there are actions of groups with the Haagerup property on the factors R and R_λ, as stated in Theorem 2.1.5.

Theorem 2.3.2. *Let G be a noncompact, locally compact second countable group.*

1. *If $(\pi, \mathcal{H})$ is a C_0-representation of G, then for all $b \in \pi(G)'$ such that $0 \leq b \leq 1$, the associated action α on N_b is strongly mixing for ϕ_b.*
2. *Assume further that G has the Haagerup property. Then there exist sequences of projections $(e_n)_{n\geq 1}$ and $(f_n)_{n\geq 1}$ in the Fermion C^*-algebra A such that, for all $t \in]0, 1/2]$,*

$$\phi_t(e_n) = t \quad \textit{and} \quad \phi_t(f_n) = t^n$$

for all $n \geq 1$, and for all compact subsets K of G,

$$\lim_{n\to\infty} \sup_{g\in K} \|\alpha_g(e_n) - e_n\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \sup_{g\in K} \frac{\|\alpha_g(f_n) - f_n\|}{\phi_t(f_n)} = 0.$$

Proof. To prove (1), it suffices to prove that $\phi_b(\alpha_g(x)y) \to \phi_b(x)\phi_b(y)$ as $g \to \infty$ for x and y in the total subset

$$\{a^*(\xi_n)\dots a^*(\xi_1)a(\eta_1)\dots a(\eta_m) : \xi_i, \eta_j \in \mathcal{H}\}$$

of N_b. Let us fix $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m, \zeta_1, \dots, \zeta_k$ and $\omega_1, \dots, \omega_l$ in $\mathcal{H}$, and set

$$\begin{aligned} x &= a^*(\xi_n)\dots a^*(\xi_1)a(\eta_1)\dots a(\eta_m) \\ y &= a^*(\zeta_k)\dots a^*(\zeta_1)a(\omega_1)\dots a(\omega_l). \end{aligned}$$

Then $\phi_b(\alpha_g(x)y)$ is equal to

$$\phi_b\Big(a^*(\pi(g)\xi_n)\dots a^*(\pi(g)\xi_1)a(\pi(g)\eta_1)\dots a(\pi(g)\eta_m) \\ a^*(\zeta_k)\dots a^*(\zeta_1)a(\omega_1)\dots a(\omega_l)\Big).$$

We shift $a^*(\zeta_k)\dots a^*(\zeta_1)$ to the left between $a^*(\pi(g)\xi_1)$ and $a(\pi(g)\eta_1)$, using the canonical anticommutation relation

$$a(\pi(g)\eta_i)a^*(\zeta_j) = \langle \zeta_j, \pi(g)\eta_i\rangle - a^*(\zeta_j)a(\pi(g)\eta_i).$$

This requires mk transpositions. Next, we shift $a(\pi(g)\eta_1)\dots a(\pi(g)\eta_m)$ to the right of $a(\omega_l)$, using the relation

$$a(\pi(g)\eta_i)a(\omega_j) = -a(\omega_j)a(\pi(g)\eta_i).$$

This, in turn, requires ml transpositions. Hence, $\phi_b(\alpha_g(x)y) = \sigma(g) + \psi(g)$, where $\sigma(g)$ is a linear combination of mk terms of the type

$$\langle \zeta_i, \pi(g)\eta_j\rangle\, \phi_b(x(i,j,g)),$$

where

$$\|x(i,j,g)\| \le \max(1, \|\xi_n\| \dots \|\omega_l\|)$$

for all g and all i, j, and where $\psi(g)$ is equal to

$$\pm\phi_b\Big(a^*(\pi(g)\xi_n)\dots a^*(\pi(g)\xi_1)a^*(\zeta_k)\dots a^*(\zeta_1) \\ a(\omega_1)\dots a(\omega_l)a(\pi(g)\eta_1)\dots a(\pi(g)\eta_m)\Big).$$

For arbitrary values of n, m, k and l, $\sigma(g) \to 0$ as $g \to \infty$ since π is a C_0-representation. If $n+k \neq l+m$, then $n \neq m$ or $k \neq l$ and $\phi_b(x)\phi_b(y) = 0$ using formula (2.3). Moreover, $\psi(g) = 0$, and $\phi_b(\alpha_g(x)y) \to 0$ as $g \to \infty$. Finally, if $n + k = m + l$, using (2.3) again,

$$\psi(g) = \pm \det \begin{pmatrix} A & B(g) \\ C(g) & D \end{pmatrix}$$

where

$$\begin{aligned} A &= (\langle \zeta_i, b\omega_j \rangle) \in M_{k\times l}(\mathbb{C}) \\ B(g) &= (\langle \zeta_i, \pi(g) b\eta_j \rangle) \in M_{k\times m}(\mathbb{C}) \\ C(g) &= (\langle \pi(g)\xi_i, b\omega_j \rangle) \in M_{n\times l}(\mathbb{C}) \\ D &= (\langle \xi_i, b\zeta_j \rangle) \in M_{n\times m}(\mathbb{C}). \end{aligned}$$

As $g \to \infty$, $B(g) \to 0$ and $C(g) \to 0$, and

$$\lim_{g\to\infty} \phi_b(\alpha_g(x)y) = \pm \det \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

If $n \neq m$ or $k \neq l$, then necessarily $n \neq m$ and $k \neq l$. Moreover, the rank of the matrix

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

is not maximal, and

$$\lim_{g\to\infty} \phi_b(\alpha_g(x)y) = 0 = \phi_b(x)\phi_b(y).$$

If $n = m$ and $k = l$, the signs of $\psi(g)$ and $\phi_b(a^*(\pi(g)\xi_n)\dots a(\pi(g)\eta_n))$ are the same, because $2nk$ sign changes occurred. From formula (2.3),

$$\lim_{g\to\infty} \phi_b(\alpha_g(x)y) = \det \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \phi_b(x)\phi_b(y).$$

Now we prove (2). Fix $t \in]0, 1/2]$, and let $(K_n)_{n\geq 1}$ be an increasing sequence of compact subsets of G whose union is G, and $(\sigma, \mathcal{H}_\sigma)$ be a separable representation of G satisfying condition (3) of Theorem 2.1.1. Let $(\eta_k)_{k\geq 1}$ be a sequence of unit vectors in $\mathcal{H}_\sigma$ such that

$$\sup_{g\in K} \|\sigma(g)\eta_k - \eta_k\| \to 0$$

as $k \to \infty$, for all compact subsets K of G, set $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathcal{H}_\sigma$ and $\pi = 1 \otimes \sigma$. Then π is still a C_0-representation and $1_G \prec \pi$. For positive integers k, n, set

$$\xi_k^n = \delta_n \otimes \eta_k,$$

where $(\delta_n)_{n\geq 1}$ is the natural basis of $l^2(\mathbb{N})$, so that for all compact subsets K of G,

$$\lim_{k\to\infty} \sup_{\substack{n\geq 1\\ g\in K}} \|\pi(g)\xi_k^n - \xi_k^n\| = 0,$$

and $\xi_k^n \perp \xi_l^m$ for all k, l and all $n \neq m$. For $n \geq 1$, choose an integer $k(n)$ so large that

$$\sup_{g\in K_n} \left\|\pi(g)\xi_{k(n)}^j - \xi_{k(n)}^j\right\| \leq \frac{t^{n+1}}{n^2}$$

when $1 \leq j \leq n$. Set $e_n^j = a^*(\xi_{k(n)}^j)a(\xi_{k(n)}^j)$ and $e_n = e_n^1$, and write f_n for $\prod_{j=1}^n e_n^j$. Then, using relations (2.1) and (2.2) above, $e_n^1, \dots, e_n^n$ are pairwise commuting projections of A, while e_n and f_n are projections in A satisfying $\phi_t(e_n) = t$ and $\phi_t(f_n) = t^n$ for all $t \in]0, 1/2]$. Moreover, for all $g \in K_n$ and $1 \leq j \leq n$,

$$\left\|\alpha_g(e_n^j) - e_n^j\right\| \leq 2\left\|\pi(g)\xi_{k(n)}^j - \xi_{k(n)}^j\right\| \leq \frac{2t^{n+1}}{n^2} \leq \frac{t^n}{n^2}\ ,$$

where $\left\|\alpha_g(e_n^j) - e_n^j\right\|$ means the operator norm on A. It follows that

$$\sup_{g\in K_n} \|\alpha_g(f_n) - f_n\| \leq n \sup_{\substack{g\in K_n\\ 1\leq j\leq n}} \left\|\alpha_g(f_n^j) - f_n^j\right\| \leq \frac{1}{n}\, t^n = \frac{1}{n}\, \phi_t(f_n),$$

and similarly for (e_n). □

Now, let ω be the vacuum state on A and let B be the type I_∞ factor arising from the Gel'fand–Naimark–Segal construction on (A, ω). It follows from Theorem 2.3.2 that the action of G on B is strongly mixing for ω. Set $R_{0,1} = R \otimes B$ and $\varphi = \tau \otimes \omega$; then $R_{0,1}$ is the approximately finite dimensional factor of type II_∞ and the action of G is the tensor product action. Then part (2) of Theorem 2.1.5 follows from the following result whose proof is straightforward.

Proposition 2.3.3. *Let φ and ψ be normal states on von Neumann algebras M and N respectively. Assume that there are strongly mixing actions $\alpha\colon G \to \mathrm{Aut}(M)$ and $\beta\colon G \to \mathrm{Aut}(N)$ for φ and ψ respectively. If M contains a nontrivial asymptotically invariant or Følner sequence of projections for α and φ, then the action $\alpha \otimes \beta$ on $M \otimes N$ is strongly mixing for $\varphi \otimes \psi$, and $M \otimes N$ contains a nontrivial asymptotically invariant or Følner sequence for $\alpha \otimes \beta$ and $\varphi \otimes \psi$.*

Remark. Actions of groups with property (T) on the hyperfinite II_1 factor R using the CAR-algebra were also considered by H. Araki and M. Choda in [AC83].

We assume for the rest of this section that G is countable and that it has the Haagerup property. In order to state our last result, we need to recall some definitions (see for instance [Ocn85]). Let M be a type II_1 factor with separable predual; we denote by τ its normal, normalized, faithful trace and by $\|a\|_2$ the associated Hilbertian norm $\tau(a^*a)^{1/2}$. Let ω be a free ultrafilter on $\mathbb{N}$. Then

$$I_\omega = \left\{(a_n)_{n\geq 1} \in l^\infty(\mathbb{N}, M) : \lim_{n\to\omega} \|a_n\|_2 = 0\right\}$$

is a closed two-sided ideal of the von Neumann algebra $l^\infty(\mathbb{N}, M)$ and the corresponding quotient algebra is denoted by M^ω.

We will write $[(a_n)] = (a_n) + I_\omega$ for the equivalence class of (a_n) in M^ω, and we recall that M embeds naturally into M^ω, where the image of $a \in M$ is the class of the constant sequence $(a, a, a, \dots)$. A sequence $(a_n) \in l^\infty(\mathbb{N}, M)$ is said to be a **central sequence** if

$$\lim_{n\to\infty} \|[a, a_n]\|_2 = 0$$

for all $a \in M$, where $[a, b] = ab - ba$; two central sequences (a_n) and (b_n) are said to be **equivalent** if

$$\lim_{n\to\infty} \|a_n - b_n\|_2 = 0.$$

The relative commutant of M in M^ω is denoted by M_ω; every element of M_ω is represented by a bounded sequence (x_n) such that $\lim_{n\to\omega} \|[x, x_n]\|_2 = 0$ for all $x \in M$. Any automorphism θ of M extends naturally to an automorphism θ^ω of M^ω, whose restriction to M_ω is an automorphism of M_ω; θ is said to be **centrally trivial** if $\|\theta(x_n) - x_n\|_2 \to 0$ as $n \to \infty$ for all central sequences (x_n). A discrete group action $\alpha \colon G \to \mathrm{Aut}(M)$ is said to be **centrally free** if α_g is not centrally trivial for every $g \neq e$. Recall also that every centrally trivial automorphism of the hyperfinite factor R is inner, so that every outer action on R is centrally free.

Theorem 2.3.4. *There exists an action α of G on the hyperfinite type II_1 factor R with the following properties:*

(1) *α is strongly mixing and outer;*

(2) *the fixed point algebra $(R_\omega)^\alpha$, that is, the set of all $x \in R_\omega$ such that $\alpha_g^\omega(x) = x$ for all $g \in G$, is of type II_1.*

Proof. Let $(\sigma, \mathcal{H}_\sigma)$ be a representation of G satisfying condition (3) of Theorem 2.1.1 with the additional property that for all $g \neq e$, there exists a unit vector $\eta \in \mathcal{H}_\sigma$ such that

$$|\langle \sigma(g)\eta, \eta\rangle| < 1.$$

Let $(\pi, \mathcal{H})$ be as in the proof of Theorem 2.3.2. We again realize R as the factor obtained from the Gel'fand–Naimark–Segal construction of $\mathrm{CAR}(\mathcal{H})$ with respect to the normalized trace τ, and the action α is still determined by

$$\alpha_g(a(\xi)) = a(\pi(g)\xi)$$

for all $g \in G$ and $\xi \in \mathcal{H}$, so that α is strongly mixing.

We prove that α is centrally free. Fix an element $g \in G \setminus \{e\}$; then there exists a unit vector $\eta \in \mathcal{H}$ such that $|\langle \sigma(g)\eta, \eta\rangle| < 1$. For all $n \geq 1$, set $e_n = a^*(\delta_n \otimes \eta)a(\delta_n \otimes \eta)$, so that e_n is a projection with trace 1/2. Then $[(e_n)]$ belongs to R_ω, and by formulae (2.1) and (2.2) we see that

$$\|\alpha_g(e_n) - e_n\|_2^2 = \frac{1}{2} - \frac{1}{2}\,|\langle \eta, \sigma(g)\eta\rangle|\,,$$

which is positive and independent of n.

We now prove that $(R_\omega)^\alpha$ is of type II_1. Arguing as in the proof of Theorem 2.2.1 of [Con75], it suffices to prove that $(R_\omega)^\alpha$ is noncommutative. Let $e \in F_1 \subset F_2 \subset \dots$ be an increasing sequence of finite subsets of G whose union is G, and let $(\eta_n)_{n\geq 1}$ be a sequence of unit vectors in $\mathcal{H}_\sigma$ such that

$$\max_{g\in F_n} \|\sigma(g)\eta_n - \eta_n\| \leq \frac{1}{n}\,.$$

Set $\xi_n = \delta_n \otimes \eta_n$ and $\zeta_n = 2^{-1/2}\,(\delta_n + \delta_{n+1}) \otimes \eta_n$, so that

$$\max_{g\in F_n} \|\pi(g)\xi_n - \xi_n\| = \max_{g\in F_n} \|\pi(g)\zeta_n - \zeta_n\| \leq \frac{1}{n}\,.$$

Define $e_n = a^*(\xi_n)a(\xi_n)$ and $f_n = a^*(\zeta_n)a(\zeta_n)$, so that both $e = [(e_n)_n]$ and $f = [(f_n)_n]$ are projections of $(R_\omega)^\alpha$ of traces 1/2. Moreover, $ef \neq fe$ because

$$\|[e_n, f_n]\|_2^2 = \frac{1}{8}$$

for all n. □

Remark. If a group G admits an action α on a type II_1 factor M with properties (1) and (2) of Theorem 2.3.4, then M is a McDuff factor (that is, M_ω is nonabelian), by [Con75, Thm 2.2.1], since M_ω is automatically of type II_1, and of course G has the Haagerup property.

Further, it is easy to construct an action of such a group G on some nonhyperfinite type II_1 McDuff factor: let N be a type II_1 nonhyperfinite factor with separable predual. Set $N_g = N$ for all $g \in G$ and

$$M = \left(\bigotimes_{g\in G} N_g\right) \otimes R,$$

which is a nonhyperfinite type II_1 McDuff factor. The action α of G is defined by $\alpha = \beta \otimes \gamma$, where β is the Bernoulli shift action as in [Jon83] and γ is the action on R given by Theorem 2.3.4. Then it is straightforward to check that α is strongly mixing and that $(M_\omega)^\alpha$ is of type II_1.

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