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The Problem of Iterates in Some Classes of Ultradifferentiable Functions

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On the occasion of the 60th birthday of Prof. Michael Oberguggenberger

Abstract. We consider the problem of iterates in some spaces of ultradifferentiable classes in the sense of Braun, Meise and Taylor. In particular, we obtain a microlocal version, in this setting of functions, of the "Theorem of the iterates of Kotake and Narasimhan".

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1. Introduction and notation

Let us first recall some classical results. Let $\{L_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers such that $L_0 = 1$, $N \leq L_N$ and $L_{N+1} \leq cL_N$ for some $c > 0$ and for every $N \in \mathbb{N}$. As in [H], for an open subset $\Omega \subseteq \mathbb{R}^n$ we denote by $C^L(\Omega)$ the set of all $u \in C^\infty(\Omega)$ such that for every compact set $K \subset \Omega$, there is a constant $C > 0$ with

$$|D^\alpha u(x)| \leq C^{|\alpha|+1} L_{|\alpha|}^{|\alpha|} \qquad \forall x \in K,\ \alpha \in \mathbb{N}_0^n, \tag{1}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

When $L_N = N + 1$ this is the space $\mathcal{A}(\Omega)$ of real analytic functions in Ω; when $L_N = (N + 1)^s$ for some $s > 1$, then $C^L(\Omega)$ is the space $G^s(\Omega)$ of Gevrey functions of order s in Ω.

The problem of substituting in (1) the derivatives of u by the iterates P^N of a fixed linear partial differential operator P, is known as the "problem of iterates".

It was first solved by Komatsu in [K] in the analytic class $\mathcal{A}(\Omega)$, for a generic elliptic operator $P(D)$ of order m with constant coefficients, proving that $u \in C^\infty(\Omega)$ is real analytic in Ω if and only if for every compact set $K \subset \Omega$ there is a constant $C > 0$ such that

$$\|P^N u\|_{L^2(K)} \le C^{N+1} N^{Nm} \qquad \forall N \in \mathbb{N}.$$

Then Kotake and Narasimhan extended, in [KN], this result to the case of an elliptic operator $P(x, D)$ with real analytic coefficients in Ω: this is well known as the "Theorem of the iterates of Kotake and Narasimhan".

Then Newberger and Zielezny considered in [NZ] the Gevrey case, for a hypoelliptic operator $P(D)$ with constant coefficients. The case of Denjoy–Carleman classes was considered by Lions and Magenes in [LM].

Later Bolley, Camus, Mattera and Rodino looked for a microlocal version of the problem of iterates in [BCM], [BC], [BCR]. More precisely, they considered, for $s \in \mathbb{R}$ and $P(x, D)$ a linear partial differential operator of order m with real analytic coefficients in Ω, the space $C^L_s(\Omega; P)$ of all distributions $u \in \mathcal{D}'(\Omega)$ such that for every compact set $K \subset \Omega$ there exists a constant $C > 0$ with

$$\|P^N u\|_{H^s(K)} \le C^{N+1} L^{mN}_{mN} \qquad \forall N \in \mathbb{N}_0.$$

They define then $C^L(\Omega; P) := \bigcup_{s\in\mathbb{R}} C^L_s(\Omega; P)$ and prove that $u \in C^L(V; P)$ for a neighborhood V of x_0 if and only if there exists a neighborhood U of x_0 and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that

$$\begin{aligned} &f_N = P^N u \qquad \text{in } U \\ &|\widehat{f}_N(\xi)| \le C^{N+1} L^{mN}_{mN} (1 + |\xi|)^M \qquad \forall N \in \mathbb{N},\ \xi \in \mathbb{R}^n, \end{aligned}$$

for some $C > 0$ and $M \in \mathbb{R}$, where $\widehat{f}_N$ is the Fourier transform of f_N.

Starting from this result they could define the wave front set $\mathrm{WF}_L(u; P)$ of $u \in \mathcal{D}'(\Omega)$ with respect to the iterates of P as the complement of all points $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ such that there are a neighborhood U of x_0, an open conic neighborhood Γ of ξ_0 and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ with

$$\begin{aligned} &f_N = P^N u \qquad \text{in } U \\ &|\widehat{f}_N(\xi)| \le C^{N+1} (L_{mN} + |\xi|)^{mN+M} \qquad \forall N \in \mathbb{N},\ \xi \in \mathbb{R}^n, \\ &|\widehat{f}_N(\xi)| \le C^{N+1} L^{mN}_{mN} (1 + |\xi|)^M \qquad \forall N \in \mathbb{N},\ \xi \in \Gamma, \end{aligned}$$

for some $C > 0$ and $M \in \mathbb{R}$.

Then

$$\mathrm{WF}_L(u; P) \subset \mathrm{WF}_L(Pu) \subset \mathrm{WF}_L(u) \subset \mathrm{WF}_L(u; P) \cup \Sigma, \tag{2}$$

where $\mathrm{WF}_L(u)$ is the classical wave front set as defined by Hörmander in [H], and Σ is the characteristic set of P defined by

$$\Sigma := \{(x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}) : \ P_m(x, \xi) = 0\} \tag{3}$$

for P_m the principal part of P.

Remark 1.1. If P is elliptic then $\Sigma = \emptyset$ and (2) gives a microlocal version of the "Theorem of the iterates of Kotake and Narasimhan": $\mathrm{WF}_L(u; P) = \mathrm{WF}_L(u)$.

Recently, the problem of iterates and the wave front set with respect to the iterates have been considered, in [J1], [J2] and [BJJ], also in some classes of non-quasianalytic ultradifferentiable functions in the sense of Braun, Meise and Taylor [BMT]. To be more precise, we recall from [BMT]:

Definition 1.2. A *weight function* is a continuous increasing function $\omega\colon [0,+\infty) \to [0,+\infty)$ with the following properties:

(α) $\exists L > 0$ s.t. $\omega(2t) \le L(\omega(t)+1)$ $\forall t \ge 0$;
(γ) $\log t = o(\omega(t))$ as $t \to +\infty$;
(δ) $\varphi:\ t \mapsto \omega(e^t)$ is convex.

We say that ω is *non-quasianalytic* if it also satisfies:

(β) $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{t^2}dt < +\infty.$

Assuming, without any loss of generality, that $\omega|_{[0,1]} \equiv 0$, the *Young conjugate* $\varphi^*:\ [0,+\infty) \to [0,+\infty)$ of φ is then defined by

$$\varphi^*(s) := \sup_{t\ge 0}\{st - \varphi(t)\}.$$

It is a convex function and $\varphi^*(s)/s$ is increasing and tends to infinity as $s \to +\infty$.

The space of ω-*ultradifferentiable functions of Roumieu type* is defined by

$$\mathcal{E}_{\{\omega\}}(\Omega) := \Big\{u \in C^\infty(\Omega):\ \ \forall K \subset\subset \Omega\ \exists k \in \mathbb{N},\ c > 0 \text{ s.t.}\ \sup_K |D^\alpha u| \le c e^{\frac{1}{k}\varphi^*(|\alpha|k)}\ \ \forall \alpha \in \mathbb{N}_0^n\Big\}.$$

The space of ω-*ultradifferentiable functions of Beurling type* is defined by

$$\mathcal{E}_{(\omega)}(\Omega) := \Big\{u \in C^\infty(\Omega):\ \ \forall K \subset\subset \Omega,\ \forall k \in \mathbb{N}\ \exists c_k > 0 \text{ s.t.}\ \sup_K |D^\alpha u| \le c_k e^{k\varphi^*\left(\frac{|\alpha|}{k}\right)}\ \ \forall \alpha \in \mathbb{N}_0^n\Big\}.$$

For a linear partial differential operator $P(D)$ with constant coefficients, we defined in [BJJ] (see also [J1]) the spaces

$$\mathcal{E}^P_{\{\omega\}}(\Omega) := \Big\{u \in C^\infty(\Omega):\ \ \forall K \subset\subset \Omega\ \exists k \in \mathbb{N},\ c > 0 \text{ s.t.}\ \|P(D)^N u\|_{L^2(K)} \le c e^{\frac{1}{k}\varphi^*(mkN)}\ \ \forall N \in \mathbb{N}_0\Big\}$$

and

$$\mathcal{E}^P_{(\omega)}(\Omega) := \Big\{u \in C^\infty(\Omega):\ \ \forall K \subset\subset \Omega,\ \forall k \in \mathbb{N}\ \exists c_k > 0 \text{ s.t.}\ \|P(D)^N u\|_{L^2(K)} \le c_k e^{k\varphi^*\left(\frac{mN}{k}\right)}\ \ \forall N \in \mathbb{N}_0\Big\}.$$

Assuming that P is hypoelliptic we proved in [BJJ]:

Proposition 1.3. *Let Ω be an open subset of $\mathbb{R}^n$, ω a non-quasianalytic weight function and $P(D)$ a hypoelliptic linear partial differential operator of order m with constant coefficients. Then, for $u \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$, we have that $u \in \mathcal{E}^P_{\{\omega\}}(V)$ (resp. $u \in \mathcal{E}^P_{(\omega)}(V)$) for some neighborhood V of x_0 if and only if there exist a* *neighborhood U of x_0 and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that the following conditions* (i) *and* (ii) (*resp.* (i) *and* (iii)) *hold:*

(i) $f_N = P(D)^N u$ *in* U*;*

(ii) (Roumieu) $\exists k \in \mathbb{N}$ *s.t.* $\forall M \in \mathbb{R}$ $\exists C_M > 0$*:*

$$|\widehat{f}_N(\xi)| \le C_M e^{\frac{1}{k}\varphi^*(kNm)}(1+|\xi|)^M \qquad \forall N \in \mathbb{N},\ \xi \in \mathbb{R}^n. \tag{4}$$

(iii) (Beurling) $\forall k \in \mathbb{N}$, $M \in \mathbb{R}$ $\exists$ $C_{k,M} > 0$*:*

$$|\widehat{f}_N(\xi)| \le C_{k,M} e^{k\varphi^*(Nm/k)}(1+|\xi|)^M \qquad \forall N \in \mathbb{N},\ \xi \in \mathbb{R}^n. \tag{5}$$

This led to the following:

Definition 1.4 (Roumieu). Let Ω, u and $P(D)$ as in Proposition 1.3. We say that a point $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is not in the $\{\omega\}$-wave front set $\mathrm{WF}^P_{\{\omega\}}(u)$ with respect to the iterates of P, if there are a neighborhood U of x_0, an open conic neighborhood Γ of ξ_0 and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that $f_N = P(D)^N u$ in U and satisfies:

(i) There are constants $k \in \mathbb{N}$, $M > 0$ and $C > 0$, such that

$$|\widehat{f}_N(\xi)| \le C^N\big(e^{\frac{1}{Nmk}\varphi^*(Nmk)} + |\xi|\big)^{Nm}(1+|\xi|)^M, \qquad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n;$$

(ii) There is a constant $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}_0$, there is $C_\ell > 0$ with the property

$$|\widehat{f}_N(\xi)| \le C_\ell e^{\frac{1}{k}\varphi^*(kNm)}(1+|\xi|)^{-\ell}, \qquad \forall N \in \mathbb{N}, \xi \in \Gamma.$$

Definition 1.5 (Beurling). Let Ω, u and $P(D)$ as in Proposition 1.3. We say that a point $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is not in the (ω)-wave front set $\mathrm{WF}^P_{(\omega)}(u)$ with respect to the iterates of P, if there are a neighborhood U of x_0, an open conic neighborhood Γ of ξ_0 and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that $f_N = P(D)^N u$ in U and satisfies:

(i) There are $M, C > 0$ such that for all $k \in \mathbb{N}$ there is $C_k > 0$:

$$|\widehat{f}_N(\xi)| \le C_k C^N\big(e^{\frac{k}{Nm}\varphi^*(\frac{Nm}{k})} + |\xi|\big)^{Nm}(1+|\xi|)^M, \qquad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n;$$

(ii) For all $\ell \in \mathbb{N}_0$ and $k \in \mathbb{N}$ there is $C_{k,\ell} > 0$ such that

$$|\widehat{f}_N(\xi)| \le C_{k,\ell} e^{k\varphi^*(Nm/k)}(1+|\xi|)^{-\ell}, \qquad \forall N \in \mathbb{N}, \xi \in \Gamma.$$

Denoting by $\mathcal{E}_*(\Omega)$, $\mathcal{E}^P_*(\Omega)$ and $\mathrm{WF}^P_*(u)$ the above-defined spaces and wave front sets, where $*$ can be replaced either by $\{\omega\}$ or (ω), we proved in [BJJ, Proposition 9 and Theorem 13]:

Theorem 1.6. *Let Ω be an open subset of $\mathbb{R}^n$, ω a non-quasianalytic weight function and $P(D)$ a hypoelliptic linear partial differential operator of order m with constant coefficients. Then, for $u \in \mathcal{D}'(\Omega)$:*

$$\mathrm{WF}^P_*(u) \subset \mathrm{WF}_*(u) \subset \mathrm{WF}^P_*(u) \cup \Sigma, \tag{6}$$

where Σ is the characteristic set of P defined by (3) *and $\mathrm{WF}_*(u)$ is the $*$-wave front set in the class $\mathcal{E}_*(\Omega)$ defined as in* [AJO, Definition 3.4].

For the sake of completeness, we recall here the above-mentioned [AJO, Definition 3.4]:

Definition 1.7. Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{D}'(\Omega)$. The $\{\omega\}$-wave front set $\mathrm{WF}_{\{\omega\}}(u)$ (resp. (ω)-wave front set $\mathrm{WF}_{(\omega)}(u)$) of u is the complement in $\Omega \times (\mathbb{R}^n \setminus 0)$ of the set of points (x_0, ξ_0) such that there exist an open neighborhood U of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $u_N \in \mathcal{E}'(\Omega)$ equal to u in U which satisfies that there are $k \in \mathbb{N}$ and $C > 0$ with the property

$$|\xi|^N |\widehat{u}_N(\xi)| \leq C e^{\frac{1}{k}\varphi^*(kN)}, \qquad \forall N \in \mathbb{N},\ \xi \in \Gamma. \tag{7}$$

(resp. which satisfies that for every $k \in \mathbb{N}$ there is $C_k > 0$ with the property

$$|\xi|^N |\widehat{u}_N(\xi)| \leq C_k e^{k\varphi^*(N/k)}, \qquad \forall N \in \mathbb{N},\ \xi \in \Gamma). \tag{8}$$

Remark 1.8. If P is elliptic then $\Sigma = \emptyset$ and (6) implies that $\mathrm{WF}_*^P(u) = \mathrm{WF}_*(u)$.

2. Wave front set for non-hypoelliptic operators

In this paper we want to remove the assumption of hypoellipticity on P (and also of non-quasianalyticity on ω). To this aim we need to change the space where we work; following the ideas of [BCM] we define:

Definition 2.1. Let Ω be an open subset of $\mathbb{R}^n$, ω a weight function and $P(D)$ a linear partial differential operator of order m with constant coefficients. Then, for $s \in \mathbb{R}$,

a) (Roumieu) we denote by $C_s^{\{\omega\}}(\Omega; P)$ the set of all $u \in \mathcal{D}'(\Omega)$ such that for every compact set $K \subset \Omega$ there exist $k \in \mathbb{N}$ and $c > 0$ with

$$\|P(D)^N u\|_{H^s(K)} \leq c e^{\frac{1}{k}\varphi^*(mNk)} \qquad \forall N \in \mathbb{N}_0;$$

b) (Beurling) we denote by $C_s^{(\omega)}(\Omega; P)$ the set of all $u \in \mathcal{D}'(\Omega)$ such that for every compact set $K \subset \Omega$ and for every $k \in \mathbb{N}$ there exists $c_k > 0$ with

$$\|P(D)^N u\|_{H^s(K)} \leq c_k e^{k\varphi^*\left(\frac{mN}{k}\right)} \qquad \forall N \in \mathbb{N}_0.$$

Finally, for $* = \{\omega\}$ or (ω), we define

$$C^*(\Omega; P) = \bigcup_{s \in \mathbb{R}} C_s^*(\Omega; P). \tag{9}$$

We can then prove the following:

Theorem 2.2. *Let Ω be an open subset of $\mathbb{R}^n$, ω a weight function and $P(D)$ a linear partial differential operator of order m with constant coefficients. Then, for $u \in \mathcal{D}'(\Omega)$, we have that $u \in C^{\{\omega\}}(V; P)$ (resp. $u \in C^{(\omega)}(V; P)$) for a neighborhood V of $x_0 \in \Omega$ if and only if there exist a neighborhood U of x_0 and a sequence $\{f_N\}_{N \in \mathbb{N}} \subset \mathcal{E}'(\Omega)$ that satisfies the following two conditions* (i) *and* (ii) (*resp.* (i) *and* (iii))*:*

(i) $f_N = P(D)^N u$ *in* U*;*
(ii) Roumieu*: there exist* $M, c > 0$*,* $k \in \mathbb{N}$ *such that*

$$|\widehat{f}_N(\xi)| \leq ce^{\frac{1}{k}\varphi^*(kNm)}(1+|\xi|)^M \qquad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n;$$

(iii) Beurling*: there exists* $M > 0$ *such that for all* $k \in \mathbb{N}$ *there is* $c_k > 0$ *with*

$$|\widehat{f}_N(\xi)| \leq c_k e^{k\varphi^*\left(\frac{mN}{k}\right)}(1+|\xi|)^M \qquad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n.$$

Proof. Necessity. Roumieu case. Let $u \in C^{\{\omega\}}(V;P)$ for some neighborhood V of x_0 and let $s \in \mathbb{R}$. Following the same ideas as in [BC], we choose $\varphi, \psi \in \mathcal{D}(V)$ with $\psi\varphi = \varphi$ and $\varphi \equiv 1$ in a neighborhood $U \subset V$ of x_0. Setting $f_N = \varphi P(D)^N u$, we have that $f_N \in \mathcal{E}'(V)$, $f_N = P(D)^N u$ in U and, denoting by $\widehat{f} = \mathcal{F}(f)$ the Fourier transform of f:

$$\begin{aligned}
|\widehat{f}_N(\xi)| &= \left| \int_{\mathbb{R}^n} \varphi(x)\psi(x)P(D)^N u(x) e^{-i\langle x,\xi\rangle} dx \right| \\
&= |\mathcal{F}(\varphi \cdot \psi P(D)^N u)| = (2\pi)^{-n} |\widehat{\varphi} * \mathcal{F}(\psi P(D)^N u)| \\
&= \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \widehat{\varphi}(\xi - \eta) \mathcal{F}(\psi P(D)^N u)(\eta) d\eta \right| \\
&= \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} (1+|\eta|)^{-s} \widehat{\varphi}(\xi-\eta)(1+|\eta|)^s \mathcal{F}(\psi P(D)^N u)(\eta) d\eta \right| \\
&\leq \|\psi P(D)^N u\|_{H^s(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} (1+|\eta|)^{-2s} |\widehat{\varphi}(\xi-\eta)|^2 d\eta \right)^{1/2} \\
&\leq c\|P(D)^N u\|_{H^s(\operatorname{supp}\psi)} (1+|\xi|)^{-s} \left(\int_{\mathbb{R}^n} (1+|\xi-\eta|)^{2|s|} |\widehat{\varphi}(\xi-\eta)|^2 d\eta \right)^{1/2} \\
&\leq c' e^{\frac{1}{k}\varphi^*(mNk)} (1+|\xi|)^{-s} \|\varphi\|_{H^{|s|}(\mathbb{R}^n)} \\
&\leq c'' e^{\frac{1}{k}\varphi^*(mNk)} (1+|\xi|)^M
\end{aligned}$$

for some $c, c', c'' > 0$, proving condition (ii).

The Beurling case is similar.

Sufficiency. Roumieu case.

Let $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ satisfying (i) in some neighborhood U of x_0 and (ii) for some $M > 0$.

Fix $s \leq -M - (n+1)/2$. Then, for every compact set $K \subset U$ we have that

$$\begin{aligned}
\|P(D)^N u\|_{H^s(K)} &= \|f_N\|_{H^s(K)} \leq \|f_N\|_{H^s(\mathbb{R}^n)} \\
&= \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\widehat{f}_N(\xi)|^2 d\xi \right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2s} c^2 e^{\frac{2}{k}\varphi^*(mNk)} (1+|\xi|)^{2M} d\xi \right)^{1/2}
\end{aligned}$$

$$\leq ce^{\frac{1}{k}\varphi^*(mNk)}\left(\int_{\mathbb{R}^n}(1+|\xi|)^{2(s+M)}d\xi\right)^{1/2}$$
$$\leq c'e^{\frac{1}{k}\varphi^*(mNk)}$$

for some $c' > 0$, because of the choice of s.

The Beurling case is similar. □

The above theorem lets us define the wave front set with respect to the iterates of an operator in the classes $C^*(\Omega; P)$:

Definition 2.3. Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$, ω a weight function and $P(D)$ a linear partial differential operator of order m with constant coefficients. We say that a point $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is not in the wave front set $\mathrm{WF}_{\{\omega\}}(u; P)$ (resp. $\mathrm{WF}_{(\omega)}(u; P)$) with respect to the iterates of P, if there are a neighborhood U of x_0, an open conic neighborhood Γ of ξ_0 and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ that satisfies the following conditions (i) and (ii) (resp. (i) and (iii)):

(i) $f_N = P(D)^N u$ in U;

(ii) Roumieu: There are constants $M, C > 0$, $k \in \mathbb{N}$ such that
  (a) $|\widehat{f}_N(\xi)| \leq Ce^{\frac{1}{k}\varphi^*(kNm)}(1+|\xi|)^{M+Nm}, \qquad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n$;
  (b) $|\widehat{f}_N(\xi)| \leq Ce^{\frac{1}{k}\varphi^*(kNm)}(1+|\xi|)^{M}. \qquad \forall N \in \mathbb{N}, \xi \in \Gamma$.

(iii) Beurling: There is $M > 0$ such that $\forall k \in \mathbb{N}\ \exists C_k > 0$ with
  (a) $|\widehat{f}_N(\xi)| \leq C_k e^{k\varphi^*(Nm/k)}(1+|\xi|)^{M+Nm}, \qquad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n$;
  (b) $|\widehat{f}_N(\xi)| \leq C_k e^{k\varphi^*(Nm/k)}(1+|\xi|)^{M}. \qquad \forall N \in \mathbb{N}, \xi \in \Gamma$.

Comparing the last definition with the one of $\mathrm{WF}_*(u)$ (for $* = \{\omega\}$ or (ω)) as in Definition 1.7, we have that the new wave front set gives more precise information about the propagation of singularities of a distribution, as the following Theorem shows:

Theorem 2.4. *Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$, ω a weight function and $P(D)$ a linear partial differential operator of order m with constant coefficients. Then, the following inclusion holds:*

$$\mathrm{WF}_{\{\omega\}}(u; P) \subset \mathrm{WF}_{\{\omega\}}(u).$$

Moreover, if $\omega(t) = o(t)$ as t tends to infinity, we have that

$$\mathrm{WF}_{(\omega)}(u; P) \subset \mathrm{WF}_{(\omega)}(u).$$

Proof. Roumieu case. Let $(x_0, \xi_0) \notin \mathrm{WF}_{\{\omega\}}\, u$. Then, by Definition 1.7, there exist a neighborhood U of x_0, an open conic neighborhood F of ξ_0 and a bounded sequence $\{u_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ such that $u_N = u$ in U and, for some $c > 0$ and $k \in \mathbb{N}$,

$$|\xi|^N|\widehat{u}_N(\xi)| \leq ce^{\frac{1}{k}\varphi^*(kN)}, \qquad \forall N \in \mathbb{N},\ \xi \in F. \tag{10}$$

By [H, Lemma 2.2] we can find a sequence $\chi_N \in \mathcal{D}(U)$ such that $\chi_N = 1$ in a neighborhood of x_0 and

$$|D^{\alpha+\beta}\chi_N| \leq C_\alpha(C_\alpha N)^{|\beta|}, \qquad \forall \alpha, \beta \in \mathbb{N}_0^n,\ |\beta| \leq N. \tag{11}$$

Set then $f_N = \chi_{Nm}P(D)^N u_{Nm}$. We want to prove (i) and (ii) of Definition 2.3. Condition (i) is trivial by the choice of χ_N, since $u_{Nm} = u$ in U. To prove (ii)(a) we first remark that, since $\{u_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ is a bounded sequence, there exist $c_1, M > 0$ such that $|\widehat{u}_N(\xi)| \le c_1(1+|\xi|)^M$ for all $N \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$. Moreover, by [AJO, Lemma 3.5],

$$|\widehat{\chi}_{Nm}(\eta)| \le c_2^{N+1} \frac{e^{\frac{1}{k}\varphi^*(Nmk)}}{(|\eta| + e^{\frac{1}{kNm}\varphi^*(Nmk)})^{Nm}} (1+|\eta|)^{-n-1-M}, \tag{12}$$

for some $c_2 > 0$. Also

$$|P(\xi-\eta)|^N \le c_3|\xi-\eta|^{Nm} \le c_3(1+|\xi|)^{Nm}(1+|\eta|)^{Nm}$$

for some $c_3 > 0$.

Therefore

$$\begin{aligned} |\widehat{f}_N(\xi)| &= \left| \frac{1}{(2\pi)^n} \mathcal{F}(\chi_{Nm}) * \mathcal{F}(P(D)^N u_{Nm})(\xi) \right| \\ &\le \int |\widehat{\chi}_{Nm}(\eta) P(\xi-\eta)^N \widehat{u}_{Nm}(\xi-\eta)| d\eta \qquad (13) \\ &\le c_2^{N+1} c_3 c_1 \int_{\mathbb{R}^n} \frac{e^{\frac{1}{k}\varphi^*(Nmk)}}{(|\eta|+1)^{Nm+n+1+M}} (1+|\xi|)^{Nm}(1+|\eta|)^{Nm} \\ &\qquad \cdot (1+|\xi|)^M (1+|\eta|)^M d\eta \\ &\le c_4^{N+1} e^{\frac{1}{k}\varphi^*(Nmk)} (1+|\xi|)^{Nm+M} \end{aligned}$$

for some $c_4 > 0$.

To prove (ii)(b) we split the integral (13) into the sum of $J_1(\xi) + J_2(\xi)$, with

$$\begin{aligned} J_1(\xi) &:= \int_{|\eta|\le c|\xi|} |\widehat{\chi}_{Nm}(\eta)||P(\xi-\eta)|^N |\widehat{u}_{Nm}(\xi-\eta)| d\eta \\ J_2(\xi) &:= \int_{|\eta|\ge c|\xi|} |\widehat{\chi}_{Nm}(\eta)||P(\xi-\eta)|^N |\widehat{u}_{Nm}(\xi-\eta)| d\eta, \end{aligned}$$

for some $0 < c < 1$ such that, if Γ is a conic neighborhood of ξ_0 with $\Gamma \subset F$, then for $\xi \in \Gamma$ and $|\xi - \zeta| \le c|\xi|$ we have $\zeta \in F$.

From (12) we have that $\|\widehat{\chi}_{Nm}\|_{L^1} \le A^N$ for some $A > 0$ and hence, from (10):

$$\begin{aligned} |J_1(\xi)| &\le \|\widehat{\chi}_{Nm}\|_{L_1} \cdot \sup_{|\xi-\zeta|\le c|\xi|} |P(\zeta)|^N |\widehat{u}_{Nm}(\zeta)| \\ &\le c_5^{N+1} e^{\frac{1}{k}\varphi^*(Nmk)} \qquad \forall \xi \in \Gamma \end{aligned} \tag{14}$$

for some $c_5 > 0$.

Moreover, from (12) and

$$|\widehat{u}_{Nm}(\xi-\eta)| \le c_1(1+|\xi-\eta|)^M \\ \le c_1(1+|\eta|+c^{-1}|\eta|)^M, \qquad \text{for } |\eta|\ge c|\xi|,$$

we have that

$$|J_2(\xi)| \le c_6^{N+1} e^{\frac{1}{k}\varphi^*(Nmk)} \int \frac{1}{(1+|\eta|)^{Nm+n+1+M}}(1+|\eta|)^{Nm+M}d\eta \\ \le c_7^{N+1} e^{\frac{1}{k}\varphi^*(Nmk)} \qquad \forall \xi\in\mathbb{R}^n \tag{15}$$

for some $c_6, c_7 > 0$.

Substituting (14) and (15) in (13), that we write as

$$|\widehat{f}_N(\xi)| \le J_1(\xi)+J_2(\xi),$$

we finally have (ii)(b) of Definition 2.3.

Beurling case. We argue similarly as in the Roumieu case. By Definition 1.7, if $(x_0,\xi_0)\notin \mathrm{WF}_{(\omega)}u$, then there exist a neighborhood U of x_0, an open conic neighborhood F of ξ_0 and a bounded sequence $\{u_N\}_{N\in\mathbb{N}}\subset\mathcal{E}'(\Omega)$ such that $u_N=u$ in U for every $N\in\mathbb{N}$ and for every $k\in\mathbb{N}$ there is $C_k>0$, with

$$|\xi|^N|\widehat{u}_N(\xi)| \le C_k e^{k\varphi^*(N/k)}, \qquad \forall N\in\mathbb{N},\ \xi\in F. \tag{16}$$

We take now χ_N and f_N as in the Roumieu case. Since $\omega(t)=o(t)$ by assumption, from [AJO, Remark 2.4] for every $k\in\mathbb{N}$ there is $c_k>0$ such that

$$N \le c_k e^{\frac{k}{N}\varphi^*(N/k)}. \tag{17}$$

Then (11) can be substituted by

$$|D^{\alpha+\beta}\chi_N| \le C_\alpha\left(C_\alpha c_k e^{\frac{k}{N}\varphi^*(N/k)}\right)^{|\beta|} \qquad \forall \alpha,\beta\in\mathbb{N}_0^n,\ |\beta|\le N$$

and hence (12) by (see also [AJO, Lemma 3.5]):

$$|\widehat{\chi}_{Nm}(\eta)| \le C_k^{N+1}\frac{e^{k\varphi^*(Nm/k)}}{(|\eta|+e^{\frac{k}{Nm}\varphi^*(Nm/k)})^{Nm}}(1+|\eta|)^{-n-1-M}, \tag{18}$$

for some $C_k>0$.

From (16) and (18) we can proceed exactly as in the Roumieu case to obtain (i) and (iii) of Definition 2.3. □

For the opposite inclusion of Theorem 2.4 we get:

Theorem 2.5. *Let Ω be an open subset of $\mathbb{R}^n$ and $u\in\mathcal{D}'(\Omega)$. Let $P(D)$ be a linear partial differential operator of order m with constant coefficients and Σ its characteristic set defined by* (3). *Let $*$ denote $\{\omega\}$ or (ω), for a weight function ω with $\omega(t)=o(t)$ for t that tends to infinity. Then*

$$\mathrm{WF}_*(u)\subset \mathrm{WF}_*(u;P)\cup\Sigma.$$

Proof. The proof is quite similar to that of Theorem 1.6 as in [BJJ, Theorem 13]. We take $(x_0, \xi_0) \notin \mathrm{WF}_*(u; P)$ with $P_m(\xi_0) \neq 0$; there are then a neighborhood U of x_0, a conic neighborhood Γ of ξ_0 and a sequence $\{f_N\}_{N\in\mathbb{N}} \subset \mathcal{E}'(\Omega)$ that verifies (i), and (ii) (Roumieu case) or (iii) (Beurling case) of Definition 2.3. We take $F \subset \Gamma$ such that $P_m(\xi) \neq 0$ for all $\xi \in F$, a compact neighborhood $K \subset U$ of x_0 and a sequence $\{\chi_N\}_{N\in\mathbb{N}} \subset \mathcal{D}(U)$ satisfying (11) with $\chi_N = 1$ on K. Then we set $u_N = \chi_{3m^2N} u$.

As in [BJJ, Theorem 13] (cf. also [BCM]), we have that

$$\begin{aligned}\widehat{u}_N(\xi) &= \int e^{-i\langle x,\xi\rangle} e_N(x,\xi)u(x)dx + \int e^{-i\langle x,\xi\rangle} P_m^{-N}(\xi) w_N(x,\xi) P(D)^N u(x)dx \\ &=: H_1(\xi) + H_2(\xi) \qquad (19)\end{aligned}$$

where

$$e_N := \sum_{h=1}^{N} \sum_{j=mN+1-h}^{mN} \binom{N}{h}\binom{-N}{j}(-1)^{h+j+1} R^{h+j} \chi_{3m^2N}$$

and

$$w_N := \sum_{j=0}^{mN} \binom{-N}{j} (-1)^j R^j \chi_{3m^2N}$$

for $R = R_1 + \cdots + R_m$, with $R_j = R_j(\xi, D)$ a differential operator of order $\leq j$, which depends on the parameter ξ, such that $R_j|\xi|^j$ is homogeneous of order 0.

As is [BJJ, Theorem 13],

$$|H_1(\xi)| \leq c^N (1 + |\xi|)^M N^{N+M} |\xi|^{-N}, \qquad \forall |\xi| > N, \qquad (20)$$

for some $c, M > 0$ and for every $N \in \mathbb{N}$. Moreover

$$H_2(\xi) = P_m^{-N}(\xi) \cdot \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{w}_N(\eta) \widehat{f}_N(\xi - \eta) d\eta := S_1(\xi) + S_2(\xi), \qquad (21)$$

where $S_1(\xi)$ is the integral on $|\eta| \leq c|\xi|$ and $S_2(\xi)$ on $|\eta| \geq c|\xi|$, with $c > 0$ to be chosen.

Let us separate now the Roumieu and the Beurling cases.

Roumieu case. From [BJJ, formula (90)] we have that

$$|D_x^\beta w_N| \leq A^N (mN)^{|\beta|}, \qquad |\beta| \leq 2m^2 N, \ |\xi| \geq mN, \qquad (22)$$

for some $A > 0$. Moreover, from condition (ii)(a) of Definition 2.3, we have that

$$|\widehat{f}_N(\xi)| \leq C' 2^{Nm} \big(e^{\frac{1}{kNm}\varphi^*(kNm)} + |\xi|\big)^{Nm} (1 + |\xi|)^{M'} \qquad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n$$

for some $C', M' > 0$ and $k \in \mathbb{N}$. From (22) and [AJO, Lemma 3.5] we have that:

$$|\widehat{w}_N(\eta)| \leq C^{N+1} \frac{e^{\frac{1}{k}\varphi^*(Nmk)}}{(|\eta| + e^{\frac{1}{Nmk}\varphi^*(Nmk)})^{Nm}} (1 + |\eta|)^{-n-1-M'}, \quad \forall \eta \in \mathbb{R}^n, \quad (23)$$

for some $C > 0$.

This implies, since $|\xi-\eta|\leq(1+c^{-1})|\eta|$ in $S_2(\xi)$, that

$$\begin{aligned}|S_2(\xi)| &\leq |P_m(\xi)|^{-N}\int_{|\eta|\geq c|\xi|}|\widehat{w}_N(\eta)|\cdot|\widehat{f}_N(\xi-\eta)|d\eta\\ &\leq \tilde{A}^{N+1}2^{Nm}e^{\frac{1}{k}\varphi^*(Nmk)}|\xi|^{-Nm}\int_{|\eta|\geq c|\xi|}(1+|\eta|)^{-n-1-M'}(1+|\eta|)^{M'}d\eta\\ &\leq B^{N+1}e^{\frac{1}{k'}\varphi^*(Nmk')}|\xi|^{-Nm}, \qquad \forall N\in\mathbb{N},\ |\xi|>N \end{aligned}\tag{24}$$

for some $\tilde{A}, B>0$ and $k'\in\mathbb{N}$, since $2^{Nm}e^{\frac{1}{k}\varphi^*(Nmk)}\leq De^{\frac{1}{k'}\varphi^*(Nmk')}$ for some $D>0$ and $k'\geq kL$ where L is the constant in Definition 1.2 (see proof of Lemma 3.1 in [AJO]).

On the other hand

$$|S_1(\xi)|\leq|P_m(\xi)|^{-N}\|\widehat{w}_N\|_{L_1}\cdot\sup\nolimits_{|\eta|\leq c|\xi|}|\widehat{f}_N(\xi-\eta)|.\tag{25}$$

Choosing $c>0$ as in the proof of Theorem 2.4 we have, from condition (ii)(b) of Definition 2.3, that there is a conic neighborhood $\Gamma'\subset\Gamma$ of ξ_0 such that

$$\sup_{|\eta|\leq c|\xi|}|\widehat{f}_N(\xi-\eta)|\leq De^{\frac{1}{k}\varphi^*(Nmk)}(1+|\xi|)^M \qquad \forall\xi\in\Gamma'$$

for some $D>0$.

Substituting in (25), since $\|\widehat{w}_N\|_{L^1}\leq E^N$ for some $E>0$ because of (23), we have that

$$|S_1(\xi)|\leq G^{N+1}e^{\frac{1}{k}\varphi^*(Nmk)}|\xi|^{M-Nm}\tag{26}$$

for some $G>0$.

Substituting (24) and (26) in (21), taking into account (20) and (17), and substituting in (19) we have, by the convexity of φ^*, that

$$\begin{aligned}|\widehat{u}_N(\xi)| &\leq c_1^{N+1}e^{\frac{1}{k''}\varphi^*(Nmk'')}|\xi|^{M-Nm}\\ &\leq c_1^{N+1}e^{\frac{1}{2k''}\varphi^*(2Nk'')+\frac{1}{2k''}\varphi^*(2N(m-1)k'')}|\xi|^{M-Nm}\\ &\leq c_1^{N+1}e^{\frac{1}{2k''}\varphi^*(2Nk'')}|\xi|^{M-N} \qquad \forall|\xi|\geq R_N\end{aligned}\tag{27}$$

where $R_N:=e^{\frac{1}{2N(m-1)k''}\varphi^*(2N(m-1)k'')}$, $k''\in\mathbb{N}$ and $c_1>0$.

However, for $|\xi|\leq R_N$, since $\{u_N\}_{N\in\mathbb{N}}$ is bounded in $\mathcal{E}'(\Omega)$ and $\varphi^*(x)/x$ is increasing,

$$\begin{aligned}|\widehat{u}_N(\xi)| &\leq c_2(1+|\xi|)^{M'}\\ &\leq c_3\left(e^{\frac{1}{2N(m-1)k}\varphi^*(2N(m-1)k)}\right)^{M'+N}|\xi|^{-N}\\ &\leq c_3\left(e^{\frac{1}{Nk'}\varphi^*(Nk')}\right)^{M'+N}|\xi|^{-N}\\ &\leq c_3\left(e^{\frac{1}{(N+M')k'}\varphi^*((N+M')k')}\right)^{M'+N}|\xi|^{-N}\\ &\leq c_4e^{\frac{1}{k''}\varphi^*(Nk'')}|\xi|^{-N} \qquad \forall|\xi|\leq R_N\end{aligned}\tag{28}$$

for some $c_2, c_3, c_4>0$.

From (27) and (28) we have (7) and so $(x_0, \xi_0) \notin \mathrm{WF}_*(u)$.

Beurling case. Since $\omega(t) = o(t)$, from (17) we deduce, from (22) and [AJO, Lemma 3.5], that for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that (see also [BJJ, Theorem 13]):

$$|\widehat{w}_N(\eta)| \le C_k^{N+1} \frac{e^{k\varphi^*(Nm/k)}}{(|\eta| + e^{\frac{k}{Nm}\varphi^*(Nm/k)})^{Nm}} (1+|\eta|)^{-n-1-M'}, \quad \forall \eta \in \mathbb{R}^n. \quad (29)$$

We can thus proceed as in the Roumieu case obtaining, from (29) and (iii) of Definition 2.3, via (19), (20) and (21), the desired estimate (8) for $\widehat{u}_N$. □

Remark 2.6. If $P(D)$ is elliptic and $\omega(t) = o(t)$ (for instance if ω is non-quasianalytic), then Theorems 2.4 and 2.5 prove that

$$\mathrm{WF}_*(u) = \mathrm{WF}_*(u; P),$$

i.e., a microlocal version of the "Theorem of the iterates of Kotake and Narasimhan" in the classes $C^*(\Omega; P)$.

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