# Chapter 2
# A Uniform Theoretical Model for Fluid Flow and Heat Transfer in Porous Media

**Abstract** To comprehensively study fluid flow and heat transfer in porous media, a uniform model that is valid for both a regular microscopic geometry, e.g., spheres and cylinders, and irregular microscopic geometries, such as metal foams or ceramic, is needed to connect the macroscopic and microscopic flow and heat transfer. In this chapter, the microscopic governing equations and volume averaged macroscopic governing equations for flow and natural convection heat transfer are presented. As defined in Chap. 1, the dimensionless geometry factor connects the macroscopic and microscopic drag and heat flux between the solid and fluid phases in a porous medium. Based on this geometry factor, the closure models are presented based on the microscopic drag coefficients and heat transfer correlations in a uniform form for porous media of arbitrary microscopic geometry. Also, relationships between the microscopic drag coefficients and permeability, Forchheimer coefficient, and Ergun constants are presented.

**Keywords** Drag coefficients · Geometry factor · Heat transfer coefficients

## 2.1 Microscopic Governing Equations

The microscopic continuity equation for an incompressible flow in porous media is,

$$\nabla \cdot \mathbf{v}_f = 0. \tag{2.1}$$

The microscopic momentum equation is,

$$\rho_f \left[ \frac{\partial \mathbf{v}_f}{\partial t} + \nabla \cdot \left( \mathbf{v}_f \mathbf{v}_f \right) \right] = -\nabla p_f + \mu_f \nabla^2 \mathbf{v}_f + \left( \rho_f - \rho_{f\infty} \right) \mathbf{g}. \tag{2.2}$$

The microscopic energy equations for the fluid and solid phases are,

$$\left( \rho C_p \right)_f \left[ \frac{\partial T_f}{\partial t} + \nabla \cdot \left( \mathbf{v}_f T_f \right) \right] = \nabla \cdot \left( k_f \nabla T_f \right), \tag{2.3}$$

and

$$\left( \rho C_p \right)_s \frac{\partial T_s}{\partial t} = \nabla \cdot \left( k_s \nabla T_s \right) + q, \tag{2.4}$$



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where $q$ is a heat source in the solid phase. When the interface is treated as a surface with zero thickness, the interface conditions that guarantee continuity of temperature and heat flux between fluid and solid phases are,

$$T_f = T_s \quad on \quad A_{fs}, \tag{2.5}$$

and

$$\mathbf{n}_{fs} \cdot k_f \nabla T_f = \mathbf{n}_{fs} \cdot k_s \nabla T_s \quad on \quad A_{fs}. \tag{2.6}$$

## 2.2 Macroscopic Governing Equations

Whitaker [88] applies a volume averaging method to derive the macroscopic governing equations in porous media. The volume average of a spatial derivative is related to the spatial derivative of the volume average. Recently, Vafai [82] and Hsu [29] simplified the volume averaging method of Whitaker [88] and provided closure models for the thermal dispersion.

By distinguishing the gradient operators in the microscopic and macroscopic coordinates, a simple form of the volume average is presented by Hsu and Cheng [29] for the fluid ($f$) and solid ($s$) phases in a porous medium. An intrinsic phase average of a quantity associated with the fluid phase is defined as,

$$\hat{W}_f = \frac{1}{V_f} \int_{V_f} W_f \, dV, \tag{2.7}$$

where $V_f$ is the volume occupied by the fluid phase in $V$ and $V_f + V_s = V$.

By introducing the method of volume averaging as given in Eq. (2.7) for the velocity and temperature deviations in the porous medium, we obtain the macroscopic continuity momentum and energy equations, as discussed in [29].

The macroscopic continuity equation is,

$$\hat{\nabla} \cdot \mathbf{v} = 0, \tag{2.8}$$

where $\mathbf{v} = \phi \hat{\mathbf{v}}_f$ is the Darcy velocity vector.

The macroscopic momentum equation with the Boussinesq assumption is,

$$\rho_f \left[ \frac{\partial \mathbf{v}}{\partial t} + \hat{\nabla} \cdot \left( \frac{\mathbf{v}\mathbf{v}}{\phi} \right) \right] = -\hat{\nabla} p + \mu_f \hat{\nabla}^2 \mathbf{v} + \mathbf{B} - \rho_f \phi \beta \left( \hat{T}_f - \hat{T}_\infty \right) \mathbf{g}, \tag{2.9}$$

where

$$\mathbf{B} = -\frac{1}{V} \int_{A_{fs}} p_f \, d\mathbf{S} + \frac{\mu_f}{V} \int_{A_{fs}} \left( \nabla \mathbf{v}_f \right) \cdot d\mathbf{S}, \tag{2.10}$$

and

$$p = \phi \hat{p}_f. \tag{2.11}$$

On the right-hand side of Eq. (2.9), the second term is the viscous shear in the fluid. The third term, represented by **B**, is the drag force per unit volume between solid and fluid surfaces, i.e., the pressure and viscous drag force per unit volume of the porous media.

The macroscopic energy equations for fluid and solid are respectively,

$$\phi \left(\rho C_p\right)_f \left[\frac{\partial \hat{T}_f}{\partial t} + \hat{\nabla} \cdot \left(\hat{\mathbf{v}}_f \hat{T}_f\right)\right] = \phi \hat{\nabla} \cdot \left[\left(k_f + k'\right) \hat{\nabla} \hat{T}_f\right] + q_{sf}, \tag{2.12}$$

$$(1-\phi) \left(\rho C_p\right)_s \frac{\partial \hat{T}_s}{\partial t} = (1-\phi) \left[\hat{\nabla} \cdot \left(k_s \hat{\nabla} \hat{T}_s\right)\right] - q_{sf} + q. \tag{2.13}$$

## 2.3 Closure Models for Macroscopic Equations

### *2.3.1 Closure Model for Drag*

With a matched asymptotic expansion similar to that used for tubes [69, 71] and spheres [29, 54], the drag coefficient, $C_d$, for an arbitrary microscopic geometry with the microscopic length scale $d$ can be expressed as,

$$C_d = c_{d_0} + c_{d_1} Re_d^{-1} + c_{d_2} Re_d^{-1/2} + O(Re_d^{-3/2}), \tag{2.14}$$

where,

$$Re_d = \frac{\left|\hat{\mathbf{v}}_f\right| d}{\nu_f}, \tag{2.15}$$

and $c_{d_0}$, $c_{d_1}$, and $c_{d_2}$ are constants. The zeroth order term is a correction associated with the inertial effect, the − 1 order term is the Stokes drag, the − 1/2 order term is due to the skin friction, and the − 3/2 order term is a negligible higher order term. Hence, the drag per unit volume of the porous medium can be expressed as,

$$\begin{aligned} \mathbf{B} &= \frac{Drag_{fs}}{V_s + V_f} = -\frac{\frac{1}{2}\rho_f |\hat{\mathbf{v}}_f| \hat{\mathbf{v}}_f A_{fs} C_d}{V_s/(1-\phi)} \\ &= -(1-\phi)\eta \frac{\rho_f}{2} \frac{\nu_f^2}{d^3} \left[c_{d_0} Re_d^2 + c_{d_1} Re_d^1 + c_{d_2} Re_d^{3/2}\right] \hat{\mathbf{e}}_f, \end{aligned} \tag{2.16}$$

where, $\hat{\mathbf{e}}_f$ is the unit vector in the direction of the macroscopic velocity, i.e., the Darcy velocity, $\hat{\mathbf{v}}_f / |\hat{\mathbf{v}}_f| = \mathbf{v}/|\mathbf{v}|$. From Eq. (2.16), it can be seen that the geometry factor, $\eta$, presented in Eq. (1.2), and the bulk porosity, $\phi$, connect the macroscopic drag force and the microscopic drag coefficient for arbitrary microscopic geometry.

### 2.3.2 Relation to the Darcy–Brinkman Model

The drag model presented in Eq. (2.16) is a uniform format, which is appropriate for arbitrary microscopic geometries. The frequently used Darcy–Brinkman's model presented in Eq. (1.14) in Chap. 1 is a special case of Eq. (2.16). The two models are equivalent when the skin friction term of Eq. (2.14) is negligible or when the fluid viscous shear stress effect is negligible compared to the viscous drag, i.e., when $\frac{L}{d}$ is large.

For viscous drag in packed bed of spheres, the drag force based on Brinkman's model [7] is,

$$\nabla \hat{p}_f = -\left[\frac{\mu_f}{K}\mathbf{v} + \frac{C_F \rho_f}{\sqrt{K}}\mathbf{v}|\mathbf{v}|\right]. \tag{2.17}$$

By multiplying Eq. (2.17) by $\phi$, the drag force per unit volume is,

$$\mathbf{B} = \nabla p = \nabla \phi \hat{p}_f = -\phi\left[\frac{\mu_f}{K}\mathbf{v} + \frac{C_F \rho_f}{\sqrt{K}}\mathbf{v}|\mathbf{v}|\right], \tag{2.18}$$

which can be expressed as,

$$\mathbf{B} = -\left[\frac{\mu_f}{K}\phi^2\left(\frac{\nu_f}{d}\right)Re_d + \frac{C_F \rho_f}{\sqrt{K}}\phi^3\left(\frac{\nu_f}{d}\right)^2 Re_d^2\right]\hat{\mathbf{e}}_f. \tag{2.19}$$

Comparing Eqs. (2.16) and (2.19), based on the coefficients of zero and −1 order, one obtains,

$$\frac{\mu_f}{K}\phi^2\left(\frac{\nu_f}{d}\right) = (1-\phi)\eta\frac{\rho_f}{2}\left(\frac{\nu_f}{d}\right)^2\frac{1}{d}c_{d_1}, \tag{2.20}$$

and

$$\frac{C_F \rho_f}{\sqrt{K}}\phi^3\left(\frac{\nu_f}{d}\right)^2 = (1-\phi)\eta\frac{\rho_f}{2}\left(\frac{\nu_f}{d}\right)^2\frac{1}{d}c_{d_0}, \tag{2.21}$$

respectively. From Eqs. (2.20) and (2.21), the permeability and the Forchheimer coefficient can be expressed in terms of the drag coefficient and geometry factor for an arbitrary structured porous medium as,

$$K = \frac{\phi^2}{(1-\phi)\eta}\frac{2d^2}{c_{d_1}}, \quad \text{and} \quad C_F = \frac{\sqrt{(1-\phi)\eta}}{\phi^2}\frac{c_0}{\sqrt{2c_{d_1}}}. \tag{2.22}$$

Thus the Darcy number can be recast as,

$$Da = \frac{K}{L^2} = \frac{\phi^2}{(1-\phi)\eta}\left(\frac{d}{L}\right)^2\frac{2}{c_{d_1}}. \tag{2.23}$$

Equations (2.22) and (2.23) show the effects of the microscopic geometry factor, porosity, and the microscopic drag coefficients on permeability, Forchheimer coefficient, and Darcy number. From Ergun's experimental study [16] of a bed of packed spheres, permeability and the Forchheimer coefficient are related to the porosity by,

$$K = \frac{\phi^3 d^2}{a(1-\phi)^2}, \quad \text{and} \quad C_F = \frac{b}{\sqrt{a}\phi^{3/2}}. \tag{2.24}$$

Thus the Ergun constants can be expressed as,

$$a = \frac{\phi}{(1-\phi)}\frac{\eta}{2}c_{d_1}, \quad \text{and} \quad b = \frac{\eta}{2}c_{d_0}. \tag{2.25}$$

Based on prior studies [29] for a packed bed of spheres, $\eta = d(\pi d^2/4)/(\pi d^3/24) = 6$, $c_{d_0} = 0.4/4 = 0.1$, $c_{d_1} = 24/4 = 6$, and $c_{d_2} = 6/4 = 1.5$. For loosely packed cylinders [69], $\eta = d(\pi d)/(\pi d^2/4) = 4$, $c_{d_0} = 1.18/\pi$, $c_{d_1} = 6.8/\pi$, and $c_{d_2} = 1.96/\pi$. Thus the Ergun constants, $a$ and $b$, differ a great deal depending on the structure of the porous matrix. From Eq. (2.25), it is seen that $b$ is independent of porosity while $a$ is dependent on it. Thus we see why experimental correlation constants for $a$ vary more than $b$ when the porosity changes, as discussed in [7, 10] and [16].

From Eq. (2.16), a normalized drag force is derived,

$$|\mathbf{B}|\frac{d^3}{\frac{1}{2}\rho_f \nu_f^2(1-\phi)} = \eta\left[c_0 Re_d^2 + c_1 Re_d^1 + c_2 Re_d^{3/2}\right] = \eta C_D Re_d^2. \tag{2.26}$$

This dimensionless quantity is independent of porosity and linear in $\eta$.

## 2.3.3 *Closure Models for Heat Transfer in Porous Media*

### 2.3.3.1 LTE Model

The governing energy equation of the LTE model is identical to Eq. (1.17),

$$\left(\rho C_p\right)_m \frac{\partial \hat{T}}{\partial t} + \bar{\nabla}\cdot\left[\mathbf{v}\hat{T}\right] = \alpha_f\left(\frac{k_m}{k_f} + \frac{k'}{k_f}\right)\hat{\nabla}^2\hat{T} + (1-\phi)q. \tag{2.27}$$

The effective thermal conductivity of the porous medium is expressed as,

$$\frac{k_m}{k_f} = \phi + (1-\phi)\frac{k_s}{k_f}, \tag{2.28}$$

and the effective heat capacity is expressed as,

$$\frac{(\rho c_p)_m}{(\rho c_p)_f} = \phi + (1-\phi)\frac{(\rho c_p)_s}{(\rho c_p)_f}. \tag{2.29}$$

Hence, the ratio of thermal diffusivities is,

$$\frac{\alpha_m}{\alpha_f} = \frac{\phi + (1-\phi)\frac{k_s}{k_f}}{(\phi + (1-\phi)\frac{(\rho c_p)_s}{(\rho c_p)_f}}. \tag{2.30}$$

The above mixture model has some limitations. For complex geometries like metal foams, the effective thermal conductivity may be represented by [5],

$$\frac{k_m}{k_f} = M\left[\phi + (1-\phi)\frac{k_s}{k_f}\right] + \frac{(1-M)}{\left(\phi + \frac{1-\phi}{k_s/k_f}\right)}, \tag{2.31}$$

where $M = 0.33$ based on recent experiments of natural convection in an open cell metal foam [85]. (In [5], $M = 0.35$ was recommended for one-dimensional conduction.)

Furthermore, when the convective effects of the fluid motion are introduced, thermal dispersion becomes important. Hsu [52] extended his earlier work of interfacial heat transfer for pure conduction [29] to incorporate the effect of forced convection for both low and high Reynolds number flows.

The solution of the energy equation requires a closure model for thermal dispersion. For forced convection in packed cylinders [29],

$$\frac{\alpha'}{\alpha_f} = \frac{k'}{k_f} = \begin{cases} \varepsilon\frac{1-\phi}{\phi}Re_d Pr_f, & \text{if} Re_d \gg 10, \\ \varepsilon\frac{1-\phi}{\phi^2}(Re_d Pr_f)^2, & \text{if} Re_d \ll 10, \end{cases} \tag{2.32}$$

where $\alpha'$ is the thermal dispersion diffusivity, $k'$ is the the thermal dispersion conductivity, and $\varepsilon$ is the thermal dispersivity. Similarity analysis has demonstrated a closure model for dispersion in natural convection [69, 71]. For natural convection, the local velocity is estimated to be $\sqrt{g\beta\Delta T d}$, and thus the local Reynolds number is,

$$Re_d \sim \frac{\sqrt{g\beta\Delta T d}d}{\nu_f} = \sqrt{\frac{Ra_d}{Pr_f}}. \tag{2.33}$$

Equations (2.32) and (2.33) are combined to yield,

$$\frac{\alpha'}{\alpha_f} = \frac{k'}{k_f} = \begin{cases} \varepsilon\frac{1-\phi}{\phi}(Ra_d Pr_f)^{1/2}, & \text{if } \frac{Ra_d}{Pr_f} \geq 100, \\ \varepsilon\frac{1-\phi}{\phi^2}(Ra_d Pr_f), & \text{if } \frac{Ra_d}{Pr_f} < 100. \end{cases} \tag{2.34}$$

The quantities $\alpha'$ and $k'$ are tensors in anisotropic porous medium, but it is common to treat them as isotropic scalars. For forced convection of water and air through heated packed channels and cylindrical packed tubes, $\varepsilon \sim 0.04$ [29]. In our previous work [69, 74], we found that the enhancement of thermal conductivity due to thermal dispersion with value 0.04 is 1–3% for both cylinders and metal foams, and thus heat transfer enhancement due to thermal dispersion is insignificant.

#### 2.3.3.2 NLTE Model

The governing energy equations of the NLTE model are,

$$\phi \left(\rho C_p\right)_f \left[\frac{\partial \hat{T}_f}{\partial t} + \hat{\nabla} \cdot \left(\hat{\mathbf{v}}_f \hat{T}_f\right)\right] = \phi \hat{\nabla} \cdot \left[\left(k_f + k'\right) \hat{\nabla} \hat{T}_f\right] + q_{sf}, \tag{2.35}$$

$$(1-\phi) \left(\rho C_p\right)_s \frac{\partial \hat{T}_s}{\partial t} = (1-\phi)\, \hat{\nabla} \cdot \left(k_s \hat{\nabla} \hat{T}_s\right) - q_{sf} + q, \tag{2.36}$$

where $q_{sf}$ is the heat flux from solid to fluid phase and $q$ is a heat source in solid phase, such as electrical heaters, or as will be shown in Chap. 4 for tube bundle heat exchangers, treated as porous medium.

The heat flux between the solid and fluid phases per unit volume of porous media $q_{sf}$ can be expressed as,

$$q_{sf} = \frac{A_{sf} h_{sf} (\hat{T}_s - \hat{T}_f)}{\frac{V_s}{(1-\phi)}} = h_{sf} (\hat{T}_s - \hat{T}_f) ss = (1-\phi)\eta \frac{1}{d} h_{sf} (\hat{T}_s - \hat{T}_f). \tag{2.37}$$

In a manner similar to that for microscopic drag, an asymptotic expansion for the microscopic heat transfer coefficient is,

$$h_{sf} A_{fs} (T_s - T_f) = \oint_{REV} k_f \left(\frac{\partial T}{\partial \mathbf{n}}\right)_{sf} dA_{fs}. \tag{2.38}$$

With Eq. (2.38), the microscopic Nusselt number can be expressed as,

$$Nu_d = \frac{h_{sf} d}{k_f} = c_{ho} + c_{h1} Re_d^{c_{h2}} Pr_f^{c_{h3}}. \tag{2.39}$$

For packed spheres, $c_{h0} = 2$, $c_{h1} = 0.6$, $c_{h2} = 1/2$, and $c_{h3} = 1/3$ [88], and for loosely packed cylinders, $c_{h0} = 0.3$, $c_{h1} = 0.62$, $c_{h2} = 1/2$, and $c_{h3} = 1/3$ [71]. For the saturated metal foam investigated by Wade [85], we observe that there are many wedge shapes, or nearly so, in the microscopic structures. We therefore compare the wedge flow heat transfer coefficients (with $c_{h0} = 0.3$, $c_{h1} = 0.51$, $c_{h2} = 0.5$, and $c_{h3} = 1/3$ [89]) to widely used correlations for fibrous metal foams [97], and find that they show reasonably good agreement over a wide range of $Re_d$.

Substituting Eq. (2.39) into Eq. (2.37), the heat flux per unit volume is,

$$q_{sf} = (1-\phi)\eta \left[c_{ho} + c_{h1} Re_d^{c_{h2}} Pr_f^{c_{h3}}\right] \frac{k_f}{d^2} (\hat{T}_s - \hat{T}_f), \tag{2.40}$$

and the normalized heat flux is,

$$\frac{q_{sf} d^2}{k_f (\hat{T}_s - \hat{T}_f)(1-\phi)} = \eta \left[c_{ho} + c_{h1} Re_d^{c_{h2}} Pr_f^{c_{h3}}\right] = \eta Nu_d, \tag{2.41}$$

which, like the microscopic drag coefficients, are independent of $\phi$ and linear in $\eta$. The grouping $\eta Nu_d$ demonstrates the importance of the microscopic interface on heat transfer.

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