Chapter 2
Basic Counting

2.1 The Multiplication Principle

Suppose that we are ordering dinner at a small restaurant. We must first order our drink, the choices being Soda, Tea, Water, Coffee, and Wine (respectively S, T, W, C, and I). Then, we order our appetizer, either Soup or Salad (respectively O and A). Next we order our entree from Beef, Chicken, Fish, and Vegetarian. Finally, we order dessert from Pie, Cake, and Ice Cream.

When ordering dinner, we can think of each choice as a separate independent event. In other words, our choice of an appetizer does not limit our choice of an entree. In reality, we may prefer certain choices of drink and appetizer with our choice of entree. However, we do not limit ourselves to these considerations for this example. So, our possibilities for drink and appetizer are:

SO	TO	WO	CO	IO
SA	TA	WA	CA	IA

Note that regardless of our choice of drink, we still have two choices for an appetizer. Hence, we have $5(2) = 10$ choices for drink and appetizer. For each of these ten choices, we then have four possibilities for our entree (Beef, Chicken, Fish, and Vegetarian). This gives us a total of $5(2)(4) = 40$ possibilities. Finally, for each of these 40 possibilities, we have three possibilities for dessert (Pie, Cake, and Ice Cream). Thus we have a total of $5(2)(4)(3) = 120$ possibilities for dinner at this restaurant. The Multiplication Principle is a generalization of the above example.

Theorem 2.1.1 *(The Multiplication Principle) Suppose that there are n sets denoted* $A_1, \ldots, A_n$*. If elements can be selected from each set independently, then the number of ways to select one element from each set is given by* $|A_1| \cdots |A_n|$.

Proof Note that this problem is equivalent to selecting an element from the set $A_1 \times \cdots \times A_n$. The cardinality of this set is $|A_1| \cdots |A_n|$ by Proposition 1.3.10. ■

Example 2.1.2 Suppose that we wish to go shopping. There are shopping districts in the north, east, west, and south side of town. We can take a car, bus, or train to any one of these destinations. Further, we may choose to take a scenic or a direct route.

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While in the shopping district, we may shop for any one of clothing, groceries, or movies. While we are out, we may either go to the park, a restaurant, or neither. How many different shopping trips are possible?

Solution Let A_1 be the set of directions we can travel (N, E, W, S). Let A_2 denote the set of transportation options (car, bus, or train). We let A_3 be the set {Scenic, Direct}. A_4 will denote the shopping options (clothes, groceries, or movies). Finally, A_5 will be the set of "side trips" (park, restaurant, or no side trip). Thus the number of different shopping trips is given by $|A_1||A_2||A_3||A_4||A_5| = 4(3)(2)(3)(3) = 216$ by the Multiplication Principle. □

Suppose that we want to make a string of n colored beads. Each bead may be one of m colors and we have unlimited beads of each color. As usual, we want to determine the number of visually distinct strings. Usually, we would only be interested in visually distinct strings, in other words, those that cannot be obtained from another by flipping the string. So, if we are using the colors 1, 2, and 3, then 1223 and 3221 would be considered the same string. However, this is a more difficult problem and will wait until Chap. 8. For now, we will consider reflections to be different. In this case, we can think of the string as an n-tuple where the entries come from the set $[m]$.

Example 2.1.3 How many ways are there to make an n-tuple from the elements of $[m]$ in such a way that no two adjacent elements of the n-tuple are the same?

Solution There are m choices for the first element of the n-tuple. The ith element of the n-tuple ($i = 2, ..., n$) can be anything other than what was used for $(i - 1)$th element. Thus the number of acceptable strings is

$$m \prod_{i=2}^{n} (m-1) = m(m-1)^{n-1}.$$ □

You may notice that the particular sets change for each of the choices in Example 2.1.3. For instance, suppose that $m = 3$ and we select 1 for the first entry in the n-tuple. Our set of options for the second entry is $\{2, 3\}$. However, if we select 2 as the first entry, then our set of options for the second entry is $\{1, 3\}$. While these sets are different, the cardinality of each set *is* the same. For this reason, the Multiplication Principle still applies. Further note that in Example 2.1.3, exponentiation appears as part of our solution. This is also the case in the following corollary.

Corollary 2.1.4 *Suppose that we have k distinguishable trophies to distribute to n people (who are by definition distinguishable). Each person may receive more than one trophy and some people may not receive a trophy. The number of ways to distribute the trophies is given by* n^k.

Proof Let A_i be the set of ways to distribute the ith trophy. Since there are n people, it follows that $|A_i| = n$ for all i. Since there are k trophies, there are n^k ways to distribute them by the Multiplication Principle. ■

Recall that in algebra 0^0 is undefined. Similarly, the limit form 0^0 is considered an *indeterminant form* in calculus. However, in this book we will *always* assume that $0^0 = 1$. By Corollary 2.1.4, there are $0^0 = 1$ ways to distribute 0 trophies to 0 people, the way in which no one gets anything.

The case of 0^0 is a special case of an *empty product*. In an empty product, no terms are being multiplied. Because an empty product should not change the value of a product, algebraically an empty product should equal one. Analogously, an *empty sum* should be zero.

Corollary 2.1.4 is often referred to as *sampling with replacement*. Consider the problem of selecting 5 cards from a standard poker deck of 52 cards. Each time a card is selected, it is placed back into the deck. Thus there are 52^5 possible ways to sample five cards from the deck with replacement, if the order of the cards is important.

Example 2.1.5 The Henry Classification System for fingerprints classifies the print for each of the ten digits. The possible classifications are plain arch, tented arch, radial loop, ulnar loop, plain whorl, accidental whorl, double loop whorl, peacock's eye whorl, composite whorl, and central packet whorl. How many possible finger print patterns are possible? Based on this, should we believe that no two people have the same fingerprints?

Solution There are ten classifications for each of the ten fingers. Hence there are 10^{10} possible patterns by Corollary 2.1.4. This means that there are 10 billion possible patterns under this classification system. As this exceeds the almost 7 billion people currently alive, it is plausible to assume that no two individuals have the same pattern. ■

A necessary condition for a challenging or stimulating game is that there is a large number of different games. If there is a relatively small number of possible games, then eventually the player has seen every possibility. For this reason, Tic-tac-toe (or Naughts and crosses) which has only 26830 possible different games (up to symmetries on the board) has little appeal except to school children. We will consider chess. While an exact computation of the number of games of chess is untractable, we will be satisfied by an approximation. Our approximation will be based on the estimates used by Shannon [38] and improved upon by Allis [2].

Example 2.1.6 Suppose that on average there are 80 moves made in chess (in other words, both players make 40 moves). For each player's first move, there are 20 possibilities (namely, each of the eight pawns can move either one or two spaces and each of the two knights can jump to the left or the right). For the remaining moves, each player has an average of 35 choices available at each move. Approximate the number of possible games of chess.

Solution Each player has 20 moves on their first turn. Thus there are $20^2 = 400$ possibilities for the first two moves. Assuming 35 moves for each of the 78 remaining moves, there are 35^{78} possibilities for the remaining turns by Corollary 2.1.4. Thus, there are $400 * 35^{78}$ possible games of chess by the Multiplication Principle. ■

Note that $400 * 35^{78}$ is approximately 10^{123}. For dedicated chess players, this may be very comforting. If only one in 10^{23} games is a "good game," then there are over 10^{100} "good games" of chess. Therefore, it is very likely that many "good games" of chess are yet to be played. To put this number in perspective, there are only 5×10^{20} possible games of checkers and less than 10^{81} atoms in the universe.

When using the Multiplication Principle, it is important to consider any restrictions on our choices.

Example 2.1.7 At a particular company, any valid password consists of six lower-case letters. Further, any valid password must end in a vowel (in other words, 'a,' 'e,' 'i,' 'o,' or 'u') and cannot contain the same letter twice. Find the number of valid passwords.

Solution As a first attempt at a solution, we might try to put the letters in order from left to right. It is easy to see we have 26 possibilities for the first letter, 25 for the second (anything but the first letter used), and so on. However, placing the last letter is more difficult, as we do not know how many vowels have been used.

To find a solution, we begin with the most restrictive selection we have to make. Namely, we begin by selecting the last letter. There are five ways of doing this. There are then 25 ways to select the first letter (anything but the letter used in the last slot). Similarly, there are 24 ways to select the second letter, 23 ways to select the third, 22 ways for the forth letter, and 21 ways for the fifth letter. Hence the number of valid passwords is given by $5(25)(24)(23)(22)(21) = 31878000$. □

Example 2.1.8 Areas must contain a large number of valid telephone numbers in order to accommodate their customer base. Further, it is common to restrict which telephone numbers are assigned to customers. How many seven-digit telephone numbers:

(i) Are there?
(ii) Do not start with 0 (such a number would dial the operator)?
(iii) Do not contain 911 (such a number would immediately dial emergency services)?
(iv) Do not contain 911 nor do they start with 0?

Solution

(i) Note that there are 10 possible digits that can be used. Hence there 10^7 possible telephone numbers by Corollary 2.1.4.
(ii) First find how many telephone numbers start with 0. There are six remaining numbers, so there are 10^6 possibilities. Thus there are $10^7 - 10^6$ telephone numbers that do not start with 0 by the Subtraction Principle.
Alternatively, there are nine choices for the first digit, namely anything but zero. There are then 10^6 choices for the remaining digits. Hence, there are $9 * 10^6 = 9000000$ telephone numbers that do not start with zero.
(iii) If a number contains 911, then there are 10^4 choices for the remaining digits. There are five ways to place the sequence 911 within the number, so $5 * 10^4$ telephone numbers contain 911. Thus by the Subtraction Principle there are $10^7 - 5 * 10^4 = 9950000$ acceptable numbers.
(iv) We must find the numbers that start with 0 and contain 911. If a number contains both, then there are 10^3 choices for the remaining digits. There are four choices as to where to place the 911. Hence there are $4 * 10^3$ numbers that start with

0 and contain 911. By the Principle of Inclusion and Exclusion, the numbers of ways that a telephone number can start with 0 and contain 911 is given by $10^6 + 5 * 10^4 - 4 * 10^3$. Hence by the Subtraction Principle, the number of telephone numbers that do not start with 0 nor contain 911 is given by:

$$10^7 - (10^6 + 5 * 10^4 - 4 * 10^3) = 10^7 - 10^6 - 5 * 10^4 + 4 * 10^3 = 8954000.$$

□

A different kind of restriction is one that considers certain configurations to be identical. For example, suppose that we want to label the faces of a six -sided die with distinct elements of [6]. Since a die will be rolled around on a surface, we will consider two labelings to be identical if one can be obtained from another by rotating or rolling the die.

Example 2.1.9 Find the number of distinguishable ways to label a six-sided die with the elements of [6].

Solution Label the top face with 1. This breaks one of the symmetries on the faces of the cube. The remaining faces are labeled as follows:

(i) Choose one of the remaining numbers to place on the bottom face. There are 5 possibilities.
(ii) Place one of the remaining numbers on the front face. This breaks the final symmetry.
(iii) Label the right face. There are 3 possibilities.
(iv) Label the back face. There are 2 possibilities.
(v) The final number is the forced on the left face.

Thus by the Multiplication Principle, there are $5 * 3 * 2 = 30$ possibilities.

Proposition 2.1.10 *The cardinality of the power set of A is* $2^{|A|}$.

Proof For each element x of A, x must be included in a subset or excluded from the subset. Thus there are two choices for each element of A. Since there are $|A|$ such elements, there are $2^{|A|}$ subsets of A by the Multiplication Principle. ■

Example 2.1.11 A particular deli offers sandwiches with a choice of 5 meats, 3 cheeses, 12 vegetables, and 4 condiments. A sandwich may consist of any combination of these "toppings." How many possible different sandwiches does the deli offer?

Solution Let A be the set of sandwich toppings. We note that $|A| = 24$. The contents of the sandwich can be thought of as a subset of the available toppings (the empty set can be thought of a sandwich consisting of only bread). The number of subsets of A is $2^{24} = 16777216$. Hence, there are 1677216 distinct sandwiches. □

An effective tool in examining problems involving the Multiplication Principle is the *tree diagram*. In a tree diagram, each "branch" of the "tree" corresponds to a choice or event taking place. For example, consider flipping a coin (either heads or tails), drawing a suit from a deck of cards (either spades, hearts, clubs, or diamonds), and rolling a six-sided die. The tree diagram corresponding to this chain of events is given in Fig. 2.1. At the first branch, the coin lands either heads or tails. For each of these outcomes, the diagram branches further. In particular, for each outcome of the coin, either a heart, spade, club, or diamond is drawn from the deck. Each of these branches into six possible leaves, one for each of the possible roll of the die.

We can also use these principles to prove other, more surprising results.

Theorem 2.1.12 *If $F_0 = 1$, $F_1 = 2$, and $F_n = F_{n-1} + F_{n-2}$, then*

$$F_{m+n+1} = F_m F_n + F_{m-1} F_{n-1}.$$

Proof Note that under this indexing, F_{m+n+1} counts the number of words of length $m + n + 1$ from the alphabet {a,b} that have no adjacent a's (see Theorem 1.6.1). The right side also counts this by counting two disjoint, exhaustive sets:

(i) The set of words in which the $(m + 1)$st letter is 'b.' The first m letters comprise a word of length m from the alphabet {a,b} in which there are no adjacent a's. There are F_m such words. Similarly, the remaining n letters comprise a word of length n from the alphabet {a,b} with no adjacent a's. There are F_n such words. Thus by the Multiplication Principle, there are $F_m F_n$ words in the first set.
(ii) The set of words in which the $(m + 1)$st letter is 'a.' Note that the mth letter and the $(m + 2)$nd letter must both be 'b.' The first $m - 1$ letters comprise a word of length $m - 1$ from the alphabet {a,b} in which there are no adjacent a's. There are F_{m-1} such words. Similarly, the remaining n letters comprise a word of length $n - 1$ from the alphabet {a,b}. There are F_{n-1} such words. Thus by the Multiplication Principle there are $F_{m-1} F_{n-1}$ words in the second set.

The result then follows by the Addition Principle. ■

The above theorem gives an efficient method for computing large Fibonacci numbers. This method is actually more efficient than using the closed form discussed in the previous chapter.

Recall that in linear algebra, an $n \times n$ matrix is invertible if and only if

(i) No row is composed entirely of zeros *and*
(ii) The rows are *linearly independent*. In other words, if $r_1,...,r_n$ are the rows of this matrix and $a_1 r_1 + \cdots + a_n r_n = 0$, then $a_1 = \cdots = a_n = 0$ [5].

Example 2.1.13 Find the number of invertible $n \times n$ matrices in which the entries are from the field of order q.

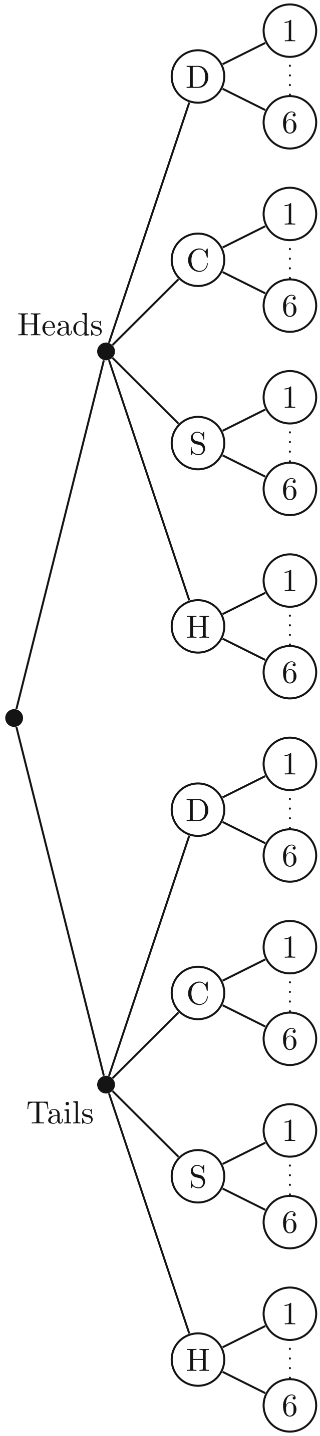
Solution Note that if the entries are from the field of order q, then there are q^n possible rows by the Multiplication Principle. However, one of these rows is the zero row, hence there are $q^n - 1$ choices for the first row.

Further note that there are q rows that are linearly dependent on this one. Hence there are $q^n - q$ choices for the second row.

In general, if we determined the first k rows, any linearly dependent $(k + 1)$st row will be of the form:

$$r_{k+1} = a_1 r_1 + \cdots + a_k r_k,$$

Fig. 2.1 A tree diagram



where the a_i are elements of the field of order q. Since there are q choices for each a_i, there are q^k rows that are linearly dependent on $r_1,\ldots, r_k$. So there are $q^n - q^k$ choices for the $(k+1)$st row.

Thus, the number of invertible $n \times n$ matrices from the field of order q is given by:

$$(q^n - 1)(q^n - q)(q^n - q^2)\cdots(q^n - q^{n-1}).$$ □

Exercise 2.1.14 The city council contains 8 clergy, 4 scientists, 5 lawyers, 3 doctors, and 10 lay persons. How many possible five person committees are there if the committee must contain one person from each group and no one is in two groups?

Exercise 2.1.15 A particular pizza parlor advertises that they have over a million possible pizza combinations. How many different toppings must they offer if their advertisement is true?

Exercise 2.1.16 Show that the number of bit strings of length n is 2^n.

Exercise 2.1.17 *The* snake cube *puzzle consists of 17 joints, each of which can take four positions. How many configurations are possible?*

Exercise 2.1.18 Suppose that we assume that the game of *Go* has an average of 150 moves. Further suppose that at each move, the player has an average of 250 choices per move. Approximately how many games of Go are there?

Exercise 2.1.19 A *palindrome* is a word that is spelled the same forward and backwards (for example, "civic" or "radar"). Find the number of n letter words that *are not* palindromes. Hint: Consider two cases, one where n is odd and the second when n is even.

Exercise 2.1.20 In a particular state, the standard license plates for personal vehicles consists of three numbers followed by three letters.

(i) How many possible license plates are there?
(ii) In this state, no license plate may begin with 9 as these plates are to be reserved for emergency vehicles. How many plates do not start with 9?
(iii) Many people complain that certain plates contain offensive words. To minimize this, the state is considering discontinuing plates that have a vowel (in other words, 'a,' 'e,' 'i,' 'o,' and 'u') as the second letter. How many plates do not have a vowel as the second letter?
(iv) How many plates do not start with 9 nor have a vowel as the second letter?

Exercise 2.1.21 In a particular state, the standard license plates for personal vehicles consists of three numbers followed by three letters. Specialty plates consists of two letters followed by four numbers. How many specialty plates are possible? How many license plates are possible?

Exercise 2.1.22 At a particular company, any valid password starts with six lower case letters followed by two digits.

(i) How many valid passwords are there?
(ii) How many valid passwords do not start with a?
(iii) How many valid passwords do not end with 88?
(iv) How many valid passwords do not start with a nor end with 88?

Exercise 2.1.23 A five digit number cannot start with zero.

(i) Find the number of five digit numbers.
(ii) Find the number of five digit numbers in which no digit appears twice.
(iii) Find the number of five digit odd numbers in which no digit appears twice.
(iv) Find the number of five digit even numbers in which no digit appears twice.

Exercise 2.1.24 Let $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Show that $F_{m+n} = F_m F_n + F_{m-1} F_{n-1}$. Hint: See Exercise 1.6.5, Exercise 1.6.6, and Theorem 2.1.12.

2.2 The Addition Principle

We have already looked at two rules dealing with the cardinality of the union of two sets. In particular, Proposition 1.3.6 states that if A and B are disjoint sets, then $|A \cup B| = |A| + |B|$. As a generalization of this, Theorem 1.3.8 states that if A and B are any sets then $|A \cup B| = |A| + |B| - |A \cap B|$. In this section, we expand Proposition 1.3.6 to an arbitrary number of sets. We will not generalize Theorem 1.3.8 until Chap. 7.

Throughout the remainder of this book, we will use the notation

$$\cup_{i=1}^{n} A_i = A_1 \cup \cdots \cup A_n$$

to denote the union of the sets $A_1, ..., A_n$.

Theorem 2.2.1 *(The Generalized Addition Principle) If $A_1, ..., A_n$ are mutually disjoint sets, then:*

$$| \cup_{i=1}^{n} A_i | = \sum_{i=1}^{n} |A_i|.$$

Proof We proceed by induction on n. If $n = 1$, then the claim is obvious. Assume that for some n, the mutually disjoint sets $A_1, ..., A_n$ satisfy:

$$| \cup_{i=1}^{n} A_i | = \sum_{i=1}^{n} |A_i|.$$

Let $A_1, ..., A_n$, and A_{n+1} be mutually disjoint sets. Thus, $\cup_{i=1}^{n} A_i$ and A_{n+1} are disjoint sets. Hence, by Proposition 1.3.6, we have that

$$| \cup_{i=1}^{n+1} A_i | = | \cup_{i=1}^{n} A_i | + |A_{n+1}|.$$

Applying the inductive hypothesis yields

$$|\cup_{i=1}^{n+1} A_i| = |\cup_{i=1}^{n} A_i| + |A_{n+1}|$$
$$= \sum_{i=1}^{n} |A_i| + |A_{n+1}| = \sum_{i=1}^{n+1} |A_i|.$$

Alternatively, if $x \in A_i$, then $x \notin A_j$ for $j \neq i$. Therefore, x is counted once by $| \cup_{i=1}^{n} A_i|$ and once by $\sum_{i=1}^{n} |A_i|$. Similarly, if $x \notin A_i$ for all i, then x is counted zero times by both sides of the equation. ■

Theorem 2.2.1 can be summarized as follows: Suppose that there are n events. The ith event can occur in a_i ways. If no two events can occur simultaneously (in other words, they are disjoint), then there are $a_i + \cdots + a_n$ ways that exactly one of the events can occur.

Example 2.2.2 Suppose that you can go to one of five small restaurants for dinner. No two restaurants offer the same menu items. The first restaurant offers 8 menu items, the second restaurant offers 6 items, the third restaurant offers 11 items, the fourth restaurant offers 4 items, and the final restaurant offers 20 items. How many meals are possible?

Solution Let A_i be the set of all menu choices in the ith restaurant. Since no two restaurants offer the same menu items, these sets are disjoint. So by the Addition Principle, we have

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5|$$
$$= 8 + 6 + 11 + 4 + 20 = 49.$$ □

Example 2.2.3 Find the number of non-negative integer solutions to $x + y \leq 6$.

Solution Let A_i denote the set of non-negative integer solutions to $x + y = i$, where $i = 0, 1, ..., 6$. Note that the A_i are disjoint sets. So we have:

$$A_0 = \{(0,0)\}.$$
$$A_1 = \{(1,0), (0,1)\},$$
$$A_2 = \{(2,0), (1,1), (0,2)\},$$
$$A_3 = \{(3,0), (2,1), (1,2), (0,3)\},$$
$$A_4 = \{(4,0), (3,1), (2,2), (1,3), (0,4)\},$$
$$A_5 = \{(5,0), (4,1), (3,2), (2,3), (1,4), (0,5)\},$$
$$A_6 = \{(6,0), (5,1), (4,2), (3,3), (2,4), (1,5), (0,6)\}.$$

Thus by the Addition Principle, we have:

$$|A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6|$$

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