Chapter 2
Introduction to Continuity

In this chapter the algebra of continuous functions is established. A function that is continuous at each irrational number and discontinuous at each rational number is constructed. This function is know as the Riemann function, the Thomae function, the ruler function, or the raindrop function.

Establishing some of the more subtle properties of continuous functions requires properties of the set of real numbers that do not follow from the ordered field axioms. The relevant properties of the set of real numbers are contained in Chap. 3. We revisit continuity in Chap. 5, where we establish global properties of continuous functions.

2.1 Definition and Algebra

Let D be a subset of $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is *continuous* at a point $a \in D$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon. \tag{2.1}$$

Using neighborhoods this can also be written as

$$\forall \varepsilon > 0, \exists \delta > 0, f\left(D \cap B_\delta(a)\right) \subseteq B_\varepsilon\left(f(a)\right).$$

If a is not an accumulation point of D, then there is a $\delta > 0$, such that $\forall x \in D$, $|x - a| < \delta \implies x = a$. Such points are *isolated*. If a is isolated, the limit of $f(x)$ as $x \to a$ does not exist, but clearly, f is continuous at a. However, if a is an accumulation point of D, Eq. (2.1) means

$$f(x) \to f(a) \text{ as } x \to a.$$

Consequently, the algebra of limits leads to:

- if f is continuous at a and $k \in \mathbb{C}$, then kf is continuous at a;
- if f, g are continuous at a, then $f + g$ is continuous at a;

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- if f, g are continuous at a, then fg is continuous at a;
- if $f.g$ are continuous at a and $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

The first two of these are summarized by linearity, if f, g are continuous at x_0 and a, b are complex numbers, then $af + bg$ is continuous at x_0.

Theorem 2.1.1 (Composition Rule). *If f is continuous at a and g is continuous at $b := f(a)$, then $g \circ f$ is continuous at a.*

Proof. Let $\varepsilon > 0$ be given. Since g is continuous at b, there is a $\gamma > 0$, such that

$$|y - b| < \gamma \implies |g(y) - g(b)| < \varepsilon. \tag{2.2}$$

Since f is continuous at a and $\gamma > 0$, there is a $\delta > 0$, such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \gamma,$$

since $b = f(a)$, Eq. (2.2) with $y = f(x)$ implies

$$|g(f(x)) - g(f(a))| = |g(y) - g(b)| < \varepsilon.$$

Thus $g \circ f$ is continuous at a. ☺

We say f is continuous *on* D, if f is continuous at every point in D. We showed in Sect. 1.4 that any rational function is continuous on the set of points where it is defined.

Example 2.1.2. $f(z) := |z|$ is continuous on $\mathbb{C}$.

Proof. Let $a \in \mathbb{C}$. Let $\varepsilon > 0$ be given. Let $\delta := \varepsilon$. Then $|z - a| < \delta$ implies

$$|f(z) - f(a)| = ||z| - |a|| \leq |z - a| < \delta = \varepsilon.$$

Hence Eq. (2.1) holds. ☺

Exercise 2.1.3. Explain why

$$f(x) := \frac{|3 + 7|x|| - |3x^2 - 8|^9}{5 - |1 - x^2|}$$

is continuous at 2.

Example 2.1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be determined by $f(x) := \begin{cases} 1 & \text{if } x \geq 3 \\ -1 & \text{if } x < 3 \end{cases}$. Then f is *discontinuous*, that is not continuous, at 3.

Proof. Since

$$\lim_{x \nearrow 3} f(x) = -1 \neq 1 = \lim_{x \searrow 3} f(x)$$

f does not have a limit at 3, in particular, f is not continuous at 3. ☺

Example 2.1.5. Let $f : \mathbb{C} \to \mathbb{R}$ be determined by $f(x) := \begin{cases} 8 & \text{if } x \neq 5 \\ 2 & \text{if } x = 5 \end{cases}$. Then

$$\lim_{x \to 5} f(x) = 8 \neq 2 = f(5),$$

hence f is discontinuous at 5.

In preparation for the next example we need to know more about approximating irrational numbers by rational numbers:

Exercise 2.1.6. Let a be some irrational number. Given any $M \in \mathbb{N}$, there is a $\gamma > 0$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $q \leq M \implies \left|a - \frac{p}{q}\right| \geq \gamma$.

Writing the contrapositive of the implication gives: Given any $M \in \mathbb{N}$, there is a $\gamma > 0$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $\left|a - \frac{p}{q}\right| < \gamma \implies q > M$.

Remark 2.1.7. Exercise 2.1.6 should be compared to Theorem 1.8.5.

The function in the following exercise is a modification of the Dirichlet function.

Exercise 2.1.8 (Riemann Function). Let $f : \mathbb{R} \to \mathbb{R}$ be determined by

$$f(x) := \begin{cases} 1/q & \text{when } x = p/q \text{ for some } p \in \mathbb{Z}, q \in \mathbb{N} \text{ in lowest terms} \\ 0 & \text{when } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Show f is discontinuous at every point in $\mathbb{Q}$ and continuous at every point in $\mathbb{R} \setminus \mathbb{Q}$.

We named this function after Georg Friedrich Bernhard Riemann (17 September 1826, Breselenz to 20 July 1866, Selasca). It is also called the Thomae function, after Carl Johannes Thomae (11 December 1840, Laucha an der Unstrut to 1 April 1921, Jena), the ruler function, the raindrop function among many other names.

Remark 2.1.9. Vito Volterra (3 May 1860, Ancona to 11 October 1940, Rome) showed that we cannot have a function that is continuous at the rational numbers and discontinuous at the irrationals. In fact, he showed that we cannot have two functions for which the points of discontinuity of one are the points of continuity of the other and vice versa (See Sect. 3.4). Thus, the roles of the rationals and irrationals in the previous exercise cannot be reversed by some clever choice of f.

2.2 Removable Discontinuity

Suppose f is discontinuous at a. Then, f has a *removable discontinuity* at a, if there is a function g, such that g is continuous at a and $g(x) = f(x)$ for all $x \neq a$. In the definition of removable discontinuity, it does not matter whether or not f is defined at a.

Exercise 2.2.1. Suppose f is discontinuous at a. Then $\lim_{x\to a} f(x)$ exists iff f has a removable discontinuity at a.

Example 2.2.2. $f(x) := \frac{x^2-1}{x-1}$ has a removable discontinuity at $a = 1$. Because, if $g(x) := x+1$, then $f(x) = g(x)$ for all $x \neq 1$ and g is continuos at $a = 1$.

2.3 One-Sided Continuity

In this section we assume $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{C}$.

Let $a \in D$. If

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, 0 < x-a < \delta \implies |f(x) - f(a)| < \varepsilon,$$

then we say f is *continuous from the right* at a. In particular, if a is an accumulation point of D, then $\lim_{x\searrow a} f(x)$ exists and equals $f(a)$ if and only if f is continuous from the right at a.

Similarly, f is *continuous from the left* at a, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, -\delta < x-a < 0 \implies |f(x) - f(a)| < \varepsilon.$$

If a is an accumulation point of D, this means $\lim_{x\nearrow a} f(x)$ exists and equals $f(a)$.

Example 2.3.1. Let $f(x) := \begin{cases} 2 & \text{when } x > 3 \\ 5 & \text{when } x \le 3 \end{cases}$. Then f is continuous from the left at 3, since $\lim_{x\nearrow 3} f(x) = 5 = f(3)$. And f is not continuous from the right at 3 because $\lim_{x\searrow 3} f(x) = 2 \neq 5 = f(3)$.

Exercise 2.3.2. f is continuous at a iff f is both continuous from the right and left at a.

Problems

Problems for Sect. 2.1

1. If g is continuous at L and $f(x) \to L$ as $x \to a$, prove $g(f(x)) \to g(L)$ as $x \to a$.
2. Let $f : [0,1] \to [0,1]$ be determined by $f(0) = 0$, and for any $n \in \mathbb{N}$, $f(x) = 1/n$, when $\frac{1}{n+1} < x \le \frac{1}{n}$. Since $\bigcup_{n=1}^{\infty} \left]\frac{1}{n+1}, \frac{1}{n}\right] =]0,1]$ and the union is disjoint, f is a function defined on the closed interval $[0,1]$.

   a. Prove that f is increasing, i.e., $x < y \implies f(x) \le f(y)$.
   b. Prove that f is continuous at 0.
   c. Prove that f is continuous at 1.

d. Prove that f is continuous on $\left]\frac{1}{n+1}, \frac{1}{n}\right[$ for all $n \in \mathbb{N}$.
e. Prove that f is discontinuous at $x = 1/n$ for all $n \in \mathbb{N}$ with $n \geq 2$.

3. Let $D := [-1,1] \cup \{3\} \cup [5,7]$ and let $f : D \to \mathbb{R}$. Then f is continuous at 3.
4. If $f : \mathbb{R} \to \mathbb{C}$ is continuous on $\mathbb{R}$ and $f(x) = x^2$ for every rational x, show $f(x) = x^2$ for every real x.
5. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R},$$

then there is constant $c \in \mathbb{R}$, such that $f(x) = cx$, for all x in $\mathbb{R}$. [*Hint*: $f(2) = f(1+1) = 2f(1)$, and $f(1) = f(1/2) + f(1/2) = 2f(1/2)$, so $f(1/2) = \frac{1}{2}f(1)$].

6. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous at a. Suppose (x_n) is a sequence of complex numbers converging to a. Prove the sequence $(f(x_n))$ converges to $f(a)$.
7. Why does the composition rule for limits (Theorem 1.4.14) not imply the composition rule for continuity (Theorem 2.1.1)?

Problems for Sect. 2.2

1. Let σ be the pseudo-sine function. Let $f(x) := \sigma(1/x)$, when $x \neq 0$ and let $f(0) := 0$. Show that f is discontinuous at 0.
2. Let σ be the pseudo-sine function. Let $g(x) := x\sigma(1/x)$ for $x \neq 0$. Prove g has a removable discontinuity at 0.

Problems for Sect. 2.3

1. Prove the the function in Problem 2 for Sect. 2.1 is continuous from the left at every point in the half-open interval $]0,1]$.

Solutions and Hints for the Exercises

Exercise 2.1.6. For a fixed q, there are only finitely many p such that $a - \gamma \leq p/q \leq a + \gamma$. Alternatively, for any integer $k \geq 1$, the two integers closest to ka are $\lfloor ka \rfloor$ and $\lfloor ka \rfloor + 1$, in fact $\lfloor ka \rfloor < ka < \lfloor ka \rfloor + 1$. Hence, the largest γ satisfying the desired conclusion is the smallest of the numbers $a - \frac{\lfloor ka \rfloor}{k}$, $\frac{\lfloor ka \rfloor + 1}{k} - a$, $k = 1, 2, \ldots, M$.

Exercise 2.1.8. This is a consequence of Exercise 2.1.6 and Corollary 1.4.20.

Exercise 2.2.1. Let $L := \lim_{x\to a} f(x)$ exists, then

$$g(x) := \begin{cases} f(x) & \text{when } x \neq a \\ L & \text{when } x = a \end{cases}$$

is continuous at a.

Exercise 2.3.2. Similar to the corresponding result for one-sided limits.

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