The Non-confusing Travel Groupoids on a Finite Connected Graph

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Abstract. The notion of travel groupoids was introduced by L. Nebeský in 2006 in connection with a study on geodetic graphs. A travel groupoid is a pair of a set V and a binary operation $*$ on V satisfying two axioms. For a travel groupoid, we can associate a graph. We say that a graph G has a travel groupoid if the graph associated with the travel groupoid is equal to G. A travel groupoid is said to be non-confusing if it has no confusing pairs. Nebeský showed that every finite connected graph has at least one non-confusing travel groupoid.

In this note, we study non-confusing travel groupoids on a given finite connected graph and we give a one-to-one correspondence between the set of all non-confusing travel groupoids on a finite connected graph and a combinatorial structure in terms of the given graph.

Keywords: Travel groupoid · Confusing pair · Non-confusing travel groupoid · Geodetic graph · Spanning tree

2010 Mathematics Subject Classification: 20N02, 05C12, 05C05.

1 Introduction

A *groupoid* is the pair $(V, *)$ of a nonempty set V and a binary operation $*$ on V. The notion of travel groupoids was introduced by L. Nebeský [5] in 2006 in connection with his study on geodetic graphs [1–3] and signpost systems [4]. First, let us recall the definition of travel groupoids.

A *travel groupoid* is a groupoid $(V, *)$ satisfying the following axioms (t1) and (t2):

(t1) $(u * v) * u = u$ (for all $u, v \in V$),
(t2) if $(u * v) * v = u$, then $u = v$ (for all $u, v \in V$).

Jung Rae Cho—This research was supported for two years by Pusan National University Research Grant.
Yoshio Sano—This work was supported by JSPS KAKENHI grant number 25887007.

© Springer International Publishing Switzerland 2014
J. Akiyama et al. (Eds.): JCDCGG 2013, LNCS 8845, pp. 14–17, 2014.
DOI: 10.1007/978-3-319-13287-7_2

A *geodetic graph* is a connected graph in which there exists a unique shortest path between any two vertices. Let G be a geodetic graph, and let $V := V(G)$. For two vertices u and v of G, let $A_G(u, v)$ denote the vertex adjacent to u which is on the unique shortest path from u to v in G. Define a binary operation $*$ on V as follows: For all $u, v \in V$, let $u * v := A_G(u, v)$ if $u \neq v$ and $u * v := u$ if $u = v$. This groupoid $(V, *)$ is called the *proper groupoid* of the geodetic graph G. Remark that the proper groupoid of any geodetic graph is a travel groupoid.

Let $(V, *)$ be a travel groupoid, and let G be a graph. We say that $(V, *)$ *is on* G or that G *has* $(V, *)$ if $V(G) = V$ and $E(G) = \{\{u, v\} \mid u, v \in V, u \neq v, \text{ and } u * v = v\}$. Note that every travel groupoid is on exactly one graph.

Let $(V, *)$ be a travel groupoid. For $u, v \in V$, we define $u *^0 v := u$ and $u *^{i+1} v := (u *^i v) * v$ for every nonnegative integer i. It is clear that $(u *^j v) *^k v = u *^{j+k} v$ holds for any nonnegative integers j and k. Note that, for any two distinct elements u and v in V, it holds that $u * v \neq u$ (see [5, Proposition 2 (2)]) and that $u *^2 v \neq u$ by (t2). An ordered pair (u, v) of two distinct elements of V is called a *confusing pair* in $(V, *)$ if there exists an integer $i \geq 3$ such that $u *^i v = u$. A travel groupoid $(V, *)$ is said to be *non-confusing* if there is no confusing pair in $(V, *)$.

Nebeský gave a characterization of non-confusing travel groupoids on a finite graph.

Theorem 1 ([5, Theorem 3]). *Let $(V, *)$ be a travel groupoid on a finite graph G. Then, $(V, *)$ is non-confusing if and only if, for all distinct elements u and v in V, there exists a positive integer k such that the sequence $(u *^0 v, \dots, u *^{k-1} v, u *^k v)$ is a path from u to v in G.* □

Nebeský also showed a result on the existence of non-confusing travel groupoids. Note that a travel groupoid $(V, *)$ is said to be *simple* if it satisfies the following axiom (t3) if $v * u \neq u$, then $u * (v * u) = u * v$ (for all $u, v \in V$).

Theorem 2 ([5, Theorem 4]). *For every finite connected graph G, there exists a simple non-confusing travel groupoid on G.* □

By Theorem 2, we know that every finite connected graph has always at least one non-confusing travel groupoid.

In this note, we study non-confusing travel groupoids on a given finite connected graph and we give a one-to-one correspondence between the set of all non-confusing travel groupoids on a finite connected graph and a combinatorial structure in terms of the given graph.

2 Main Result

To describe the structure of non-confusing travel groupoids on a graph, we define the following.

Definition. For a vertex v of a graph G, a *v-tree* is a spanning tree of G which contains all the edges incident to v. We denote by $\mathcal{S}_G(v)$ the set of all v-trees in G. □

Lemma 3. *Let $(V,*)$ be a non-confusing travel groupoid on a finite connected graph G. For an element v of V, let T_v be the graph defined by*

$$V(T_v) := V \quad \text{and} \quad E(T_v) := \{\{u, u*v\} \mid u \in V \setminus \{v\}\}.$$

Then, T_v is a v-tree of G.

Proof. Since $(V,*)$ is on G, for any two distinct elements u and v in V, $\{u, u*v\}$ is an edge of G. Therefore, we have $E(T_v) \subseteq E(G)$. Moreover, $V(T_v) = V = V(G)$. So T_v is a spanning subgraph of G. By the definition of the graph T_v, we have $|E(T_v)| \leq |V| - 1$. Since $(V,*)$ is non-confusing, it follows from Theorem 1 that any vertex u of T_v distinct from v is connected to the vertex v by a path P_{uv} in G, where P_{uv} is of the form $(u*^0 v, \ldots, u*^k v)$ for some positive integer k. Since $u*^{i+1} v = (u*^i v)*v$, the edge $\{u*^i v, u*^{i+1} v\}$ of the path P_{uv} is also an edge of T_v for any $i \in \{0, 1, \ldots, k-1\}$. Therefore, the path P_{uv} from u to v in G is also a path in T_v. So we can conclude that T_v is connected. Since T_v is a connected graph with $|E(T_v)| \leq |V(T_v)| - 1$, T_v is a tree. Thus, T_v is a spanning tree of G. For each vertex x which is adjacent to the vertex v in G, we have $x * v = v$ and therefore $\{x, v\} \in E(T_v)$. So, all the edges incident to the vertex v in G are contained in T_v. Hence, the graph T_v is a v-tree of G. □

The following theorem is our main result.

Theorem 4. *Let G be a finite connected graph. Then, there exists a one-to-one correspondence between the set $\Pi_{v \in V(G)} \mathcal{S}_G(v)$ and the set of all non-confusing travel groupoids on G.*

Proof. Let $V := V(G)$, and let $\mathcal{N}_G$ be the set of all non-confusing travel groupoids on G. Note that, since G is finite, both the sets $\Pi_{v \in V} \mathcal{S}_G(v)$ and $\mathcal{N}_G$ are finite.

For any $(T_v)_{v \in V} \in \Pi_{v \in V(G)} \mathcal{S}_G(v)$, we define a groupoid on V as follows: For two distinct elements u and v in V, $u * v$ is defined to be the vertex adjacent to u which is on the unique path from u to v in the tree T_v. If $u = v$, then let $u * v = u$. We show that this groupoid $(V,*)$ is a non-confusing travel groupoid. First, we check that $(V,*)$ satisfies the axioms (t1) and (t2). Take any two distinct elements u and v in V. Let $w := u * v$. Then $\{u, w\} \in E(G)$. Therefore $\{w, u\} \in E(T_u)$ and we have $(u * v) * u = w * u = u$. Moreover, if $u = v$, then $(u * v) * u = (u * u) * u = u * u = u$. Thus the axiom (t1) holds. Again, take any two distinct elements u and v in V. If u and v are adjacent (i.e. $\{u, v\} \in E(G)$), then $\{u, v\} \in E(T_v)$ and therefore $(u * v) * v = v * v = v \neq u$. If u and v are not adjacent, then $u * v \neq v$ and the element $(u * v) * v$ is the third vertex of the path from u to v in the tree T_v and therefore $(u * v) * v$ is not the element u. Thus the axiom (t2) holds. So, $(V,*)$ is a travel groupoid. Second, we check that $(V,*)$ is non-confusing. Take any two distinct elements u and v in V. Let k be the length of the path from u to v in T_v. Then, it follows from the definition of

$(V, *)$ that $u *^k v = v$ and that $(u *^0 v, \ldots, u *^k v)$ is the path from u to v in T_v. Therefore, $(u*^0 v, \ldots, u*^k v)$ is a path from u to v in G. By Theorem 1, the travel groupoid $(V, *)$ is non-confusing. Now, we define a map $\Phi : \Pi_{v \in V} \mathcal{S}_G(v) \to \mathcal{N}_G$ by $\Phi((T_v)_{v \in V}) = (V, *)$, where $(V, *)$ is the non-confusing travel groupoid defined as above for $(T_v)_{v \in V}$.

Next, let $(V, *)$ be a non-confusing travel groupoid on a finite connected graph G. For each $v \in V(G)$, we define a graph T_v by $V(T_v) := V$ and $E(T_v) := \{\{u, u * v\} \mid u \in V \setminus \{v\}\}$. By Lemma 3, T_v is a v-tree of G, i.e., $T_v \in \mathcal{S}_G(v)$. Therefore, $(T_v)_{v \in V} \in \Pi_{v \in V} \mathcal{S}_G(v)$. Now, we define a map $\Psi : \mathcal{N}_G \to \Pi_{v \in V} \mathcal{S}_G(v)$ by $\Psi((V, *)) = (T_v)_{v \in V}$, where $(T_v)_{v \in V}$ are v-trees defined as above for $(V, *)$.

Then, we can check that $\Psi \circ \Phi((T_v)_{v \in V}) = (T_v)_{v \in V}$ holds for any $(T_v)_{v \in V} \in \Pi_{v \in V} \mathcal{S}_G(v)$ and that $\Phi \circ \Psi((V, *)) = (V, *)$ holds for any $(V, *) \in \mathcal{N}_G$. Hence the map Φ is a one-to-one correspondence between the sets $\Pi_{v \in V} \mathcal{S}_G(v)$ and $\mathcal{N}_G$. □

Corollary 5. *Let G be a finite connected graph. Then, the number of non-confusing travel groupoids on G is equal to $\Pi_{v \in V(G)} |\mathcal{S}_G(v)|$.*

Proof. It follows from Theorem 4. □

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http://www.springer.com/978-3-319-13286-0

Discrete and Computational Geometry and Graphs
16th Japanese Conference, JCDCGG 2013, Tokyo, Japan,
September 17-19, 2013, Revised Selected Papers
Akiyama, J.; Ito, H.; Sakai, T. (Eds.)
2014, VIII, 191 p. 104 illus., Softcover
ISBN: 978-3-319-13286-0