Reconstructing Point Set Order Types from Radial Orderings

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Abstract. We consider the problem of reconstructing the combinatorial structure of a set of n points in the plane given partial information on the relative position of the points. This partial information consists of the radial ordering, for each of the n points, of the $n-1$ other points around it. We show that this information is sufficient to reconstruct the chirotope, or labeled order type, of the point set, provided its convex hull has size at least four. Otherwise, we show that there can be as many as $n-1$ distinct chirotopes that are compatible with the partial information, and this bound is tight. Our proofs yield polynomial-time reconstruction algorithms. These results provide additional theoretical insights on previously studied problems related to robot navigation and visibility-based reconstruction.

1 Introduction

Many properties of point sets in the plane do not depend on the exact coordinates of the points but only on their relative positions. The *order type*, or *chirotope*, of a point set $P \subset \mathbb{R}^2$ is the orientation (clockwise or counterclockwise) of every ordered triple of P [1]. More precisely, a chirotope χ associates a sign

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$\chi(a,b,c) \in \{0,+1,-1\}$ with each ordered triple (a,b,c) of points, indicating whether the three points a,b,c make a left turn ($+1$), a right turn (-1), or are collinear (0). When $\chi(a,b,c) \neq 0$ for all triples (a,b,c), the order type is said to be *uniform* or to be in *general position*. We consider only uniform order types.

Chirotopes must satisfy a collection of well-studied axioms which define the *abstract order types*. For details on the axioms, we refer the reader to a book by Knuth [2], who refers to chirotopes as CC-systems. These axioms form one of the several axiom systems that define *uniform acyclic rank-3 oriented matroids* [3]. An abstract order type χ is *realizable* if there exists a point set in $\mathbb{R}^2$ with order type χ. An abstract order type χ is typically identified with its opposite $-\chi$, where all signs are reversed, and we follow this convention in this paper. Abstract order types correspond exactly to arrangements of pseudolines, as a consequence of the Folkman-Lawrence topological representation theorem [4]. The smallest non-realizable order type corresponds to the well-known Pappus arrangement of nine pseudolines; all smaller order types are realizable. The convex hull $h_1,\ldots,h_t$ of χ is uniquely defined (also for non-realizable order types) by the property[1] that $\chi(h_i,h_{i+1},v) = +1$ for all $v \in V \setminus \{h_i,h_{i+1}\}$ and all $1 \leq i \leq t$.

Unlike most other publications on order types, we consider *labeled* order types, not order type isomorphism classes. For instance, whereas there is only one order type isomorphism class for four points in convex position, there are actually three such labeled order types. More precisely, given two order types χ_1 and χ_2 on a set V, we define $\chi_1 = \chi_2$ if and only if either (i) for all $u,v,w \in V$: $\chi_1(u,v,w) = \chi_2(u,v,w)$ or (ii) for all $u,v,w \in V$: $\chi_1(u,v,w) = -\chi_2(u,v,w)$.

Radial Orderings and Radial Systems. We next introduce the *clockwise radial system* R_χ of an abstract order type χ (in general position) on a set V. For an element u of V, let $R_\chi(u)$ be the *clockwise radial ordering* of u, defined as the unique *cyclic* ordering $v_1,\ldots,v_{n-1}$ of all elements other than u, sorted clockwise around u. Figure 1 shows a point set and the clockwise radial orderings of one of its points.

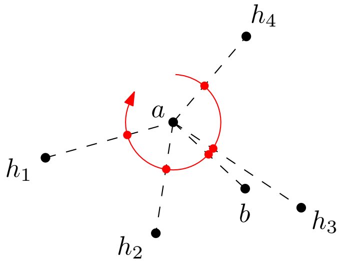

Fig. 1. A point set with $R_\chi(a) = h_4,h_3,b,h_2,h_1$

When given only the abstract order type χ, we can compute $R_\chi(u)$ as follows. Let v be any vertex other than u. Now sort $V \setminus \{u\}$ radially around u by using $w < w'$ iff $\chi(u,v,w) > \chi(u,v,w')$, or $\chi(u,v,w) = \chi(u,v,w')$ and $\chi(w,u,w') = +1$ (where $\chi(u,v,v) := 0$).

We write $U \sim R_\chi$ and say that U and R_χ are *equivalent* if U can be obtained from R_χ by reversing of some of the clockwise radial orderings of R_χ. Thus the relation $\sim$ forgets about the directions of the radial orderings. We call U an *undirected radial system*, and each $U(v)$ an *undirected radial ordering*.

While χ uniquely determines the equivalence class of R_χ, the converse is not necessarily true. We define $T(U)$ as the set of labeled order types χ for which

[1] Index additions and substractions are always modulo the length of the sequence.

$U \sim R_\chi$. In this paper we investigate the properties of $T(U)$. We show that in many cases $T(U) = \{\chi\}$ for $U \sim R_\chi$: in other words, that χ can be reconstructed uniquely from one of its undirected radial systems. However, this is not true in general, as we will discuss below.

Local Sequences. Radial orderings are similar in flavor, but different than *local sequences* defined by Goodman and Pollack [5]. The radial ordering around a point p can be thought of as the order of the intersections of a ray of origin p with the other points. If instead of a ray, we consider the successive intersections of a rotating *line* through p with the other points, we get what Goodman and Pollack call the local sequences. The order type (up to projective transformations) can be recovered from the local sequences. Felsner [6] and Felsner and Valtr [7] study simplified encodings of local sequences to prove upper bounds on the number of pseudoline arrangements.

Examples. Figure 2 shows three point sets with different (labeled) order types. Figure 2(a) and 2(b) have equivalent radial systems, but Figure 2(c) has a different radial system. Conversely, Figure 2(a) and 2(c) have equivalent local sequences (the sequence for point 1 is reversed), but Figure 2(b) has different local sequences. It follows that local sequences and radial orderings are incomparable in the sense that neither can be computed from the other in general. Figure 2(b) is obtained from Figure 2(a) by cyclically shifting the labels $2, 3, \ldots, n$ once. Each such cyclic shift in this example preserves the undirected radial system U, and hence $|T(U)| \geq n - 1$. We show in what follows that this is the worst case in the sense that $|T(U)| \leq n - 1$ for all radial systems U. Figure 3(a-b) shows another example of two point sets with different order types but the same radial system U. In this case, a discussion later in the paper shows that $|T(U)| = 2$.

In the preceeding examples, the labeled ordered types were distinct, but isomorphic in the sense they differ only by a relabeling of the points. Figure 3(c-d) shows that this is not always the case: the two point sets have the same radial system and distinct and non-isomorphic order types (see [8]). This construction can be generalized to obtain examples with an arbitrary number of points.

Related Work. Concepts similar to radial systems have been studied in a wide variety of contexts. Tovar, Freda and LaValle [9] considered the problem of exploring an unknown environment using a robot that is able to sense the

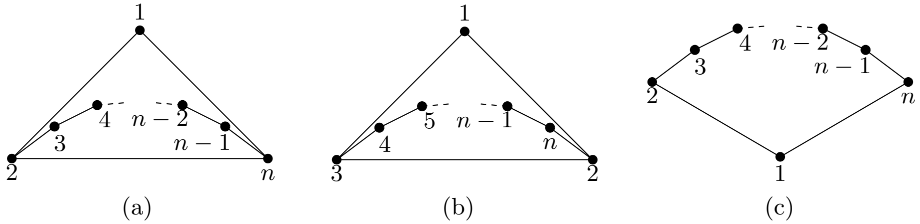

Fig. 2. An example to illustrate the difference between local sequences and radial systems

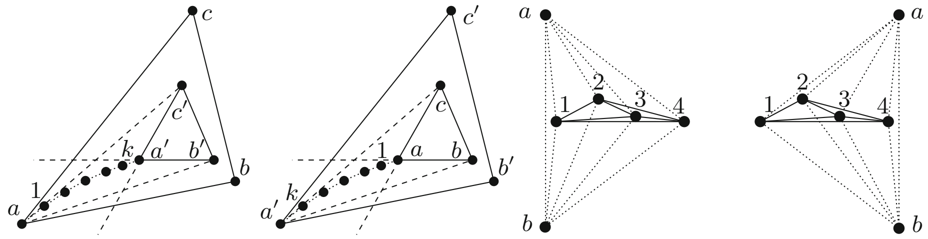

Fig. 3. (a-b) Two point sets with equivalent radial systems. The points $a, 1, \ldots, k, a'$ lie on a convex arc in both sets. (c-d) Two point sets with the same radial system but nonisomorphic order types.

radial orderings of landmarks around it. They use the order type machinery as well, and consider robots with operations like moving towards a landmark to accomplish several recognition tasks. Wismath [10] considered related reconstruction problems involving partial visibility information. He mentions the fact that radial orderings are not always sufficient to reconstruct order types, and solves a related reconstruction problem where, additionally, the x-coordinate of every point is given. Another similarly flavored problem, the *polygon reconstruction problem from angles*, has been tackled by Disser et al. [11], and Chen and Wang [12]. There they reconstruct a polygon given, for each vertex v, the sequence of angles formed by the vertices visible from v. The results developed in this paper will hopefully lay the ground for a complete theoretical treatment of the relation between observed radial orderings and the structure of point sets, and could be useful in such applications.

Some other problems involving radial orderings have been studied in several previous publications. For instance, Devillers et al. [13] considered the problem of maintaining the radial ordering associated with a moving point. Díaz-Báñez, Fabila, and Pérez-Lantero [14] study the number of distinct radial orderings that can be obtained from a point set, and introduce a colored version of the problem. Durocher et al. [15] propose algorithms for realizing radial orderings in point sets. The notion of radial ordering has been used previously by a subset of the current authors in the context of graph drawing. More precisely, it is instrumental in an elementary proof of the $\exists\mathbb{R}$-completeness of the general simultaneous geometric graph embedding problem [16]. Pilz and Welzl [8] consider crossing-preserving mappings between order types. Non-isomorphic order types having the same radial system form an equivalence class in their hierarchy.

Our Results. In Section 2, we give a preliminary analysis of radial systems on five points, which will serve as a building block for later sections. In Section 3, we show that $T(U)$ can be computed from U in polynomial time. The main procedure involved in the recognition algorithm consists of repeatedly considering five-point configurations, and removing the points that are inside the convex hull of four others. As a byproduct, we can show that if the convex hull has at least four vertices, then there is at most one compatible order type, that is, $|T(U)| = 1$. In Section 4, we prove that $|T(U)| \leq |V| - 1$ for all undirected rotation systems U on the set V. As a consequence of Section 3, this can happen only when the convex hull of the reconstructed order type is a triangle. This bound is tight, as shown by the example of Figure 2(a)-2(b).

For the sake of readability, the proofs involve Euclidean point sets, but we are careful to use only those properties of point sets that hold also for arbitrary order types (realizable or not). An easy way to verify this is to use the representation of abstract order types as *generalized configurations*, discussed in detail in [5]. A generalized configuration in general position is a pair (P, L) where $P \subset \mathbb{R}^2$ and L is a pseudoline arrangement such that every pseudoline in L contains exactly two points of P. Note that for realizable order types, such a generalized configuration is obtained simply by taking a point set realization of the order type and its set of supporting lines. Whereas for point sets P, every triple $p_1, p_2, p_3 \in P$ defines a cone at p_2, every triple defines a pseudocone at p_2 (an infinite region bounded by two curves that intersect only at p_2) in a generalized configuration, and these have all the properties required for the proofs. Hence, our results extend to abstract order types.

2 Bootstrapping

First, we define *signature graphs*, which will prove to be a useful tool in the analysis of undirected radial systems. Given a vertex set V and some $U \sim R_\chi$ on V for some labeled abstract order type χ, we construct a labeling of the complete digraph D_U on V as follows. For each directed edge (u, v) in D_U, label (u, v) with the set of vertices that are not equal to v and not directly before or after v in the undirected radial ordering around u. For example, if $U(u) = v_1, v_2, v_3, v_4$ with $v = v_2$, then label (u, v) with $\{v_4\}$. Next, we construct a coloring of the complete undirected graph G_U on V by coloring each edge $\{u, v\}$ green if (u, v) and (v, u) have the same label in D_U and red otherwise. We call G_U the signature graph of U. Figure 4 shows several examples.

Lemma 1. *Consider an abstract labeled order type χ on a set V with $|V| = 5$ and let $U \sim R_\chi$.*

(i) The abstract labeled order types in $T(U)$ all have the same convex hull size and this size can be computed from U in constant time.
(ii) If χ has convex hull size 4 or 5 then $T(U) = \{\chi\}$ and χ can be computed from U in constant time.

Proof. Figure 4(a-c) shows the signature graphs of the undirected radial systems of each of the three order type isomorphism classes on five elements. Note that the number of green edges is different for each isomorphism class. This proves (i). For (ii), recall that we want to recover the labeled order type, not just its equivalence class. We perform a case distinction on the isomorphism class of χ (which we identify by the number of vertices on the convex hull of χ).

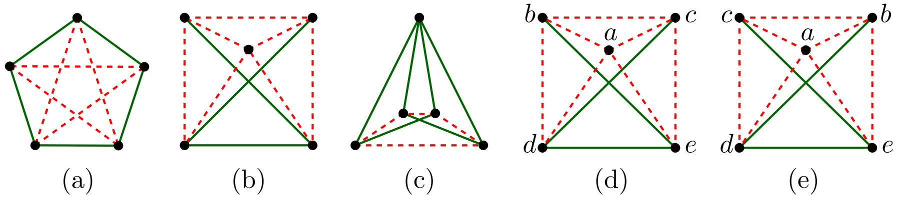

Fig. 4. Green edges are solid and red edges are dashed. (a-c) The undirected radial systems of each of the three order type isomorphism classes on five elements. (d-e) The two labeled order types of size five with four vertices on the convex hull, where vertex a has no incident green edges and b and c have one incident green edge.

Suppose that there are five vertices on the convex hull of χ. An edge $\{u, v\}$ is green if and only if $\{u, v\}$ is on the convex hull. We assume without loss of generality that $\{a, b\}$ is green. There are six labeled order types of size five with five vertices on the convex hull, under the assumption that $\{a, b\}$ is on the convex hull. Those order types correspond to sequences starting with a, b and ending with all six permutations of the three remaining points. The green neighbors of a and b thus completely identify the labeled order type.

Suppose now that there are four vertices on the convex hull of χ. Referring again to Figure 4(b), we see that there is one vertex with no incident green edges, two vertices with one incident green edge and two vertices with two incident green edges. Without loss of generality, we may assume that the vertex with no incident green edges is vertex a and the vertices with one incident green edge are b and c. This leaves the two labeled order types shown in Figure 4(d-e), which are easily distinguished by the green neighbor of vertex b. $\square$

Figure 2(a)-2(b) show that (ii) does not always hold for triangular convex hulls.

3 Reconstruction Algorithms

In this section we develop an algorithm to compute $T(U)$ from an undirected rotation system $U \sim R_\chi$. The general approach is the following. We first show, in two steps, that the convex hull H of χ and U together uniquely determine χ. Then we repeatedly apply Lemma 1 to compute $|H|$ from U. We show that U uniquely determines H if $|H| \geq 4$. In that case, we can compute $T(U) = \{\chi\}$ from U. Otherwise, if $|H| = 3$, we compute $T(U)$ by trying each possible convex hull. Given an order type χ on the vertex set V, let $\chi[V']$ be the restriction of χ to $V' \subseteq V$. We define $U[V']$ analogously for an undirected radial system U.

Lemma 2. *Consider an abstract labeled order type χ on a set V and let $U \sim R_\chi$. Let $H \subseteq V$ be the set of vertices on the convex hull of χ. The pair (H, U) uniquely determines the cyclic order $h_1, \ldots, h_k$ of the vertices on the convex hull and the clockwise radial system R_χ (up to complete reversal of both). Furthermore, there is a polynomial-time algorithm that takes (H, U) as input and returns $h_1, \ldots, h_k$ and R_χ.*

Proof. We first give an algorithm to recover the sequence $h_1, \ldots, h_k$. If $|H| = 3$ then any ordering of H will do. If $4 \leq |H| \leq 5$ then choose any $H \subseteq V_5 \subset V$ with $|V_5| = 5$ and use Lemma 1 with V_5 to recover the order type of H in polynomial time. If $|H| > 5$, then let $h_1, \ldots, h_k$ be a cyclic order of H and consider the signature graph $G_{U[H]}$. Note that we can compute the signature graph in polynomial time using only $U[H]$. In the digraph $D_{U[H]}$, the edges (h_i, h_{i+1}) and (h_{i+1}, h_i) will both be labeled $H \setminus \{h_{i-1}, h_i, h_{i+1}, h_{i+2}\}$ and thus $\{h_i, h_{i+1}\}$ is green in $G_{U[H]}$ for all $1 \leq i \leq k$. On the other hand, the edge (h_i, h_j) will be labeled $H \setminus \{h_i, h_{j-1}, h_j, h_{j+1}\}$, whereas (h_j, h_i) will be labeled $H \setminus \{h_{i-1}, h_i, h_{i+1}, h_j\}$ for $|i-j| > 1$. Hence, $\{h_i, h_j\}$ is red in G_u for all remaining edges. It follows that the green edges in $G_{U[H]}$ form a hamiltonian cycle which reveals the order of the vertices of H along the convex hull.

To recover R_χ, we assume that $h_1, \ldots, h_k$ is the counterclockwise order and recover the corresponding clockwise radial system R_χ (recall that we defined $\chi = -\chi$). For $|H| \geq 4$, every $U(v)$ contains at least three vertices from the convex hull, and hence we can recover the clockwise direction by setting $R_\chi(v)$ to $U(v)$ if $h_1, \ldots, h_k$ (without v if v is on the convex hull) appear in this order in $U(v)$ and setting $R_\chi(v)$ to the reverse of $U(v)$ otherwise. For $|H| = 3$ the same procedure works except when v is on the convex hull. If $v = h_1$ then the two possible directions are of the form $h_2, h_3, v_1, v_2, \ldots$ and $h_2, v_1, v_2, \ldots, h_3$. The second one is the correct clockwise order and is easy to recognize (note that if $V = H$ both orders are identical). The cases $v = h_2$ and $v = h_3$ are analogous. This procedure takes polynomial time. □

We omit the proof of the following lemma due to space limitations; it is essentially an application of Lemma 2, followed by some case analysis to recover χ.

Lemma 3. *Consider an abstract labeled order type χ on a set V with $|V| \geq 5$ and let $U \sim R_\chi$. Let $H \subseteq V$ be the set of vertices on the convex hull of χ. Then the pair (H, U) uniquely determines χ, i.e., $\{\chi' \in T(U) \mid \chi'$ has convex hull $H\} = \{\chi\}$. Furthermore, there is a polynomial-time algorithm that takes (H, U) as input and returns χ.*

Theorem 1. *Consider an abstract labeled order type χ on a set V with $|V| \geq 5$ and let $U \sim R_\chi$. There is a polynomial-time algorithm that takes U as input and returns $T(U)$. Furthermore, let H be the vertices of the convex hull of χ. Then*

(i) all elements of $T(U)$ have convex hull size $|H|$; and
(ii) if $|H| \geq 4$, then $T(U) = \{\chi\}$.

Proof. The algorithm begins by computing a set $V' \subseteq V$ that contains (at least) all vertices that appear on the convex hull of an order type in $T(U)$. Initially, let $V' := V$. For each subset $V_5 \subseteq V$ with $|V_5| = 5$, we do the following. By Lemma 1, the elements of $T(U[V_5])$ all have the same convex hull size s, and we can compute s from U in constant time. If $s \neq 4$, we do nothing. If $s = 4$, then the algorithm from Lemma 1 in addition returns $\chi[V_5]$, and there must be some vertex $v \in V_5$ that is not on the convex hull of $\chi[V_5]$. Note that v is not on the convex hull of any order type in $T(U)$ either. Hence, we delete v from V'. After running this procedure for all subsets $V_5 \subseteq V$ of size 5, we are left with a $V' \subseteq V$ that contains (at least) all vertices of the convex hulls of all order types in $T(U)$. Every 5-element subset of V' has convex hull size 3 or 5.

We perform a case analysis depending on the size of the set V'. First suppose that $|V'| \leq 5$. If necessary, add back previously deleted vertices to V' until $|V'| = 5$. Use the algorithm from Lemma 1 to recover $|H|$ from V'. If $|H| = 3$, then continue with the procedure described in the paragraph at the end of this proof. If $|H| = 4$ or $|H| = 5$, then Lemma 1 in addition returns $\chi[V']$ and thereby H. Then, by Lemma 3, $T(U) = \{\chi\}$ and we can compute $T(U)$ in polynomial time. This shows that (i) and (ii) hold in that case.

Now suppose that $|V'| > 5$ and note that this implies $|H| \neq 4$. If $|H| = 3$, then there is a $V_5 \subset V$ with convex hull size 3. If $|H| \geq 5$, then we claim that $H = V'$. For the sake of obtaining a contradiction, suppose that there exists a vertex $v \in V'$ that is not in H. Fix any triangulation of $\chi[H]$. Let $h_i h_j h_k$ be the cell of the triangulation that contains v and let h_ℓ be any other vertex of H. Then $V_5 = \{h_i, h_j, h_k, h_\ell, v\}$ is a set of five vertices with convex hull size four and $V_5 \subseteq V'$, which is a contradiction. We conclude that if $|H| \geq 5$ then $H = V'$ and in particular, every $V_5 \subset V'$ is in convex position. Our algorithm proceeds as follows. If there is a $V_5 \subset V'$ with convex hull size 3, then we conclude $|H| = 3$ and continue with the procedure described in the last paragraph below. Otherwise, we conclude that $H = V'$. Then $T(U) = \{\chi\}$ by Lemma 3 and we can compute $T(U)$ in polynomial time. This finishes the proof of (ii).

It remains to consider the case where the algorithm has established $|H| = 3$. If some order type in $T(U)$ would have convex hull size larger than 3, then the algorithm would already have terminated by the discussion above. Hence, all order types in $T(U)$ have convex hull size 3, which completes the proof of (i).

Finally, we describe what the algorithm does when $|H| = 3$. Consider all subsets $H_3 \subseteq V$ of size 3. For each such H_3, run the algorithm from Lemma 3 with (H_3, U), which returns a function χ. If H_3 is the convex hull of an order type in $T(U)$ then $\chi \in T(U)$ and χ is the only order type in $T(U)$ with convex hull H_3. If no order type in $T(U)$ has convex hull H_3, then the output χ is undefined. Hence, it is sufficient to check for each H_3 whether χ is an order type (in polynomial time, using the order type axioms) and if so, whether $U \sim R_\chi$. If and only if both conditions hold, then $\chi \in T(U)$ and hence $T(U)$ can be computed. Since there are $O(|V|^3)$ subsets of size 3 in V, the algorithm runs in polynomial time. $\square$

Given a set V and for each $v \in V$ a permutation of $V \setminus \{v\}$, we can decide in polynomial time whether this is a radial system corresponding to an actual order type. This is done by running the algorithm above until either an inconsistency is detected or an output is produced. If one of the chirotopes in the output has radial system U then the answer to the decision problem is yes, and no otherwise.

Corollary 1. *Given a set V and for each $v \in V$ a permutation of $V \setminus \{v\}$, we can decide in polynomial time whether this is the radial system of some order type.*

4 Triangular Convex Hulls

Theorem 1 only guarantees the trivial bound $|T(U)| \leq |V|^3$ when a radial system U has a triangular convex hull. As discussed in the introduction, there are radial systems U with $|T(U)| \geq |V| - 1$. We next prove the matching upper bound.

Recall that U and the convex hull together uniquely determine the labeled order type (Lemma 3). We also know that if U is the undirected radial system of a labeled order type with a triangular convex hull, then all order types in $T(U)$ have a triangular convex hull (Theorem 1). If a triangle $a, b, c \in V$ is the convex hull for some order type in $T(U)$, we say that abc is *important* (with respect to U). Note that if abc is important, then b and c must appear consecutively in the radial ordering of a (and the analogous statements for b and c also hold). We capture the relations between important triangles with the following four propositions. In each proposition, we consider an abstract labeled order type χ on a set V with $|V| \geq 5$ and a triangular convex hull and a $U \sim R_\chi$.

Proposition 1. *U has at most two disjoint important triangles. If U has exactly two disjoint important triangles, then these are the only important triangles and hence $|T(U)| \leq 2$.*

Proof. Suppose that U has disjoint important triangles abc and $a'b'c'$. We now argue that without loss of generality, c', a', b' appear consecutivly and in this order in $U(a)$. Figure 5(a) depicts the order type where abc forms the convex hull. Since b' and c' must appear consecutively in $U(a')$ and since a' is not on the convex hull, the cone $b'a'c'$ must not contain any other vertices. The same argumentation for b' and c' shows that the dark gray region in Figure 5(a) must be empty. We wish to show that all remaining vertices must be in the light green regions. So suppose there is a vertex x outside both the dark gray and light green regions. By symmetry we may assume that it is in the position indicated by Figure 5(a). In the order type where $a'b'c'$ forms the convex hull, $U(a')$ and $U(b')$ force x to be in region R_1 in Figure 5(b). But $U(c')$ forces x to be in region R_2, which is disjoint from R_1 (except vertex b). Hence, $a'b'c'$ cannot form the convex hull, which is a contradiction. We conclude that all remaining vertices must be in the light green regions in Figure 5(a). We call the complement of the light green regions the *forbidden region* of $a'b'c'$.

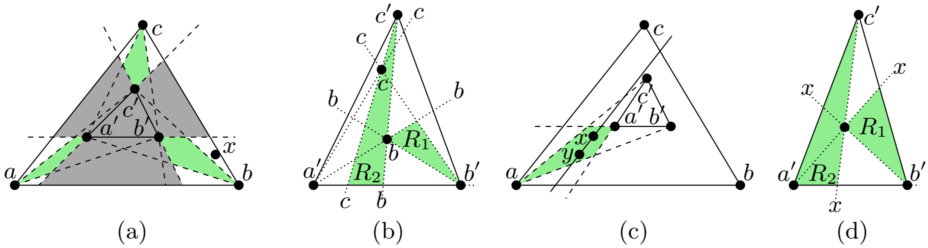

Fig. 5. Two disjoint important triangles. (a-b) Vertex x cannot be in the indicated position. (c-d) The supporting line of xy cannot avoid the segment $b'c'$.

We claim that the supporting line $\overline{xy}$ of two vertices x and y in a light green region in Figure 5(a) must separate the other two green regions. Note that this holds trivially if one of x and y is a or a'. Otherwise, suppose without loss of generality that x and y are in the light green region incident to a, that a clockwise sweep from c to b around a encounters x before y, that $\overline{xy}$ does not intersect $c'b'$ and that c' is below $\overline{xy}$. See Figure 5(c). Looking at the order type where $a'b'c'$ forms the outer face (Figure 5(d)), we see that $U(a')$ and $U('b)$ force y to be in region R_1. But $U(c')$ forces y to be in region R_2, which is a contradiction. Hence, the supporting line of x and y in Figure 5(a) must intersect c' and b' and thus separate the light green regions incident to b and c.

Finally, we argue that there are no other important triangles. Consider again the order type depicted in Figure 5(a). Suppose that there is another important triangle Δ. Suppose that Δ is completely inside one light green region, say the one incident to a. Since all three supporting lines of Δ separate the other two light green regions, either b or c must be in the forbidden region of Δ, which is a contradiction. Similarly, if Δ has one vertex in every light green region, then at least one of a', b' and c' is strictly inside Δ and hence in Δ's forbidden region. Hence, Δ must have two vertices a'' and b'' in one light green region, say the one incident to a, and one vertex c'' in another light green region, say the one incident to c. We must have $c'' = c$: otherwise c is in the forbidden region of Δ. But then c' is in the forbidden region of Δ, which is a contradiction. Hence, there are only two important triangles and thus $|T(U)| \leq 2$ by Lemma 3. $\square$

Proposition 2. *If there is a vertex v^* that is common to all important triangles in U, then $|T(U)| \leq |V| - 1$.*

Proof. For every important triangle v^*uw we know that u and w must be consecutive in $U(v^*)$. Since there are only $|V|-1$ consecutive pairs in $U(v^*)$, the proposition follows immediately by Lemma 3. $\square$

We omit the proof of the following proposition due to space limitations; it is similar to the proof of Proposition 1.

Proposition 3. *If every pair of important triangles has exactly one vertex in common, then all important triangles must all have the same vertex in common.*

Proposition 4. *If there exists a pair of important triangles with two vertices in common, then all important triangles must have the same vertex in common.*

Proof. Let abc and abd be the important triangles from the statement. Suppose for the sake of obtaining a contradiction that not all important triangles share the same vertex, i.e., that there is an important triangle Δ_1 that does not contain a and an important triangle Δ_2 that does not contain b, with possibly $\Delta_1 = \Delta_2$. If $\Delta := \Delta_1 = \Delta_2$, then by Proposition 1 we have $\Delta = cde$ with $e \neq a, b$. See Figure 6(a). The forbidden region of Δ contains a if e is in the light green region A and it contains b otherwise. It follows that $\Delta_1 \neq \Delta_2$.

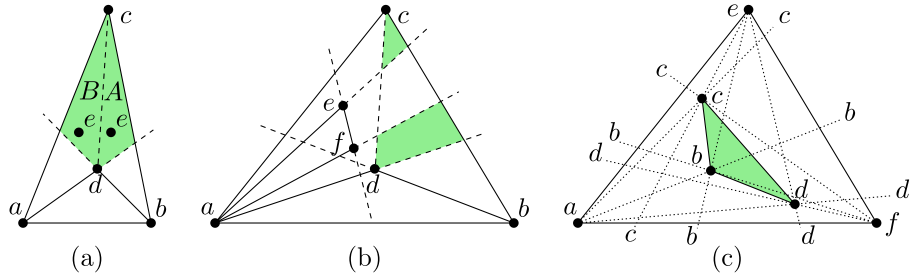

Fig. 6. Two important triangles that share two vertices. (a) Triangle cde cannot be important. (b) Possible locations for the vertex x. (c) Contradiction to (b) when aef forms the convex hull.

Since Δ_1 is not disjoint from abc and abd by Proposition 1 (since we have four important triangles), Δ_1 must contain b or both c and d. Similarly, Δ_2 must contain a or both c and d. Suppose that Δ_1 contains both c and d and let e be the third vertex of Δ_1. By the argument in the previous paragraph, we must have $e = b$. But then the forbidden regions of abd and $\Delta_1 = bcd$ together cover all of abc. This is a contradiction since $|V| \geq 5$. Symmetrically, Δ_2 cannot contain both c and d. Hence, Δ_1 must contain b and Δ_2 must contain a. Furthermore, neither triangle can intersect cd since c or d would be in the forbidden region otherwise. Let $\Delta_2 = aef$ such that a clockwise sweep from c to d around a encounters e and f in this order (with possibly $e = c$ or $f = d$ but not both). Let x be a vertex of Δ_1 different from b, c and d. The light green region in Figure 6(b) shows the allowed locations for x. The supporting line of ef cannot intersect cd since c or d would be in the forbidden region of Δ_2 otherwise. Figure 6(c) shows the resulting order type where aef forms the convex hull. The radial orderings of b, c and d force x to be in the light green region. Referring to Figure 6(b), we see that d, b and c appear consecutively in $U(x)$. But in Figure 6(c), this certainly cannot be the case, even if $c = e$ or $d = f$, which contradicts our assumption. We conclude that we cannot have such Δ_1 and Δ_2 and therefore that all important triangles must share a vertex. □

It now follows from Propositions 1,2,3, and 4 that:

Theorem 2. *Consider an abstract labeled order type χ on a set V with $|V| \geq 5$ and let $U \sim R_\chi$. Then $|T(U)| \leq |V| - 1$.*

5 Discussion and Open Problems

Theorem 2 cannot be improved by considering clockwise radial systems instead of undirected ones. For $|H| \geq 4$, the undirected radial system is already sufficient to reconstruct the order type. For $|H| = 3$, the worst case example from Figure 2(a)-2(b) applies even for clockwise radial systems.

In terms of future work, an axiomatic characterization of radial systems could lead to a simpler recognition algorithm. Our algorithms are obtained as byproducts of the proofs and their running time can undoubtedly be improved. Finally, one could think of generalizing the problem to higher dimensions. Instead of a cyclic ordering of points, every point p of a set in $\mathbb{R}^3$ could be associated with a rank-3 oriented matroid obtained by projecting all other points on a small sphere around p. The higher-dimensional counterparts of local sequences were defined for instance by Bokowski et al. [4] and are called *hyperline sequences*.

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