Analysis, Control and Synchronization of a Nine-Term 3-D Novel Chaotic System

Sundarapandian Vaidyanathan and Ahmad Taher Azar

Abstract This research work describes a nine-term 3-D novel chaotic system with four quadratic nonlinearities. First, this work describes the dynamic analysis of the novel chaotic system and qualitative properties of the novel chaotic system are derived. The Lyapunov exponents of the nine-term novel chaotic system are obtained as $L_1 = 9.45456$, $L_2 = 0$ and $L_3 = -30.50532$. Since the maximal Lyapunov exponent (MLE) of the novel chaotic system is $L_1 = 9.45456$, which is a high value, the novel chaotic system exhibits strong chaotic properties. Next, this work describes the adaptive control of the novel chaotic system with unknown system parameters. Also, this work describes the adaptive synchronization of the identical novel chaotic systems with unknown system parameters. The adaptive control and synchronization results are proved using Lyapunov stability theory. MATLAB simulations are given to demonstrate and validate all the main results derived in this work for the nine-term 3-D novel chaotic system.

Keywords Chaos · Chaotic system · Chaos control · Chaos synchronization

1 Introduction

Chaotic systems are nonlinear dynamical systems which are sensitive to initial conditions, topologically mixing and with dense periodic orbits. Sensitivity to initial conditions of chaotic systems is popularly known as the *butterfly effect* [1].

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A.T. Azar and S. Vaidyanathan (eds.), *Chaos Modeling and Control Systems Design*, Studies in Computational Intelligence 581, DOI 10.1007/978-3-319-13132-0_2

The Lyapunov exponent is a measure of the divergence of phase points that are initially very close and can be used to quantify chaotic systems. A positive maximal Lyapunov exponent and phase space compactness are usually taken as defining conditions for a chaotic system.

In 1963, Lorenz modelled a 3-D chaotic system to study convection in the atmosphere and experimentally verified that a very small difference in the initial conditions resulted in very large changes in his deterministic weather model [28].

In the last four decades, there is a great deal of interest in the chaos literature in modelling and analysis of new chaotic systems. Some well-known paradigms of 3-D chaotic systems in the literature are [2, 3, 5, 6, 21, 26, 29, 40, 48, 54, 56, 65–67, 79, 80].

Chaotic systems have several important applications in science and engineering such as oscillators [18, 46], lasers [22, 75], chemical reactions [11, 35], crypto-systems [39, 59], secure communications [9, 31, 76], biology [8, 20], ecology [12, 50], robotics [30, 32, 69], cardiology [36, 72], neural networks [15, 17, 24], finance [13, 49], etc.

The problem of control of a chaotic system is to find a state feedback control law to stabilize a chaotic system around its unstable equilibrium [47, 71].

Some popular methods for chaos control are active control [53, 61], adaptive control [52, 60], sliding mode control [62], etc.

Major works on synchronization of chaotic systems deal with the complete synchronization [38, 55, 68] which has the goal of using the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Pecora and Carroll pioneered the research on synchronization of chaotic systems with their seminal papers in 1990s [4, 34]. The active control method [27, 37, 51, 58, 64, 70] is commonly used when the system parameters are available for measurement and the adaptive control method [14, 25, 41–43, 73] is commonly used when some or all the system parameters are not available for measurement and estimates for unknown parameters of the systems.

Other popular methods for chaos synchronization are the sampled-data feedback method [10, 23, 74, 77], time-delay feedback method [7, 16, 44, 45], backstepping method [33, 57, 63, 78], etc.

This research work is organized as follows. Section 2 introduces the nine-term 3-D novel chaotic system with four quadratic nonlinearities. In this section, the phase portraits of the novel chaotic system are also displayed using MATLAB. Section 3 details the qualitative properties of the 3-D novel chaotic system. The Lyapunov exponents of the novel chaotic system are obtained as $L_1 = 9.45456, L_2 = 0$ and $L_3 = -30.50532$. The Lyapunov dimension of the novel chaotic system is obtained as $D_L = 2.30993$. As the maximal Lyapunov exponent (MLE) of the novel chaotic system is $L_1 = 9.45456$, which is a high value, it is noted that the 3-D novel chaotic system exhibits strong chaotic properties. Section 4 describes new results for the adaptive controller design for stabilizing the 3-D novel chaotic system with unknown parameters. Section 5 describes new results for the design of adaptive synchronization of the identical 3-D novel chaotic systems with unknown parameters. MATLAB simulations are shown to validate and illustrate all the main adaptive results derived for the control and synchronization of the 3-D novel chaotic system. Section 6 contains a summary of the main results derived in this research work.

2 A Nine-Term 3-D Novel Chaotic System

This section describes the equations and phase portraits of a nine-term 3-D novel chaotic system. It is shown that the nine-term novel chaotic system exhibits an attractor, which may be named as *umbrella attractor*.

The novel chaotic system is described by the 3-D dynamics

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + 30x_2x_3, \\ \dot{x}_2 &= bx_1 + cx_2 - x_1x_3, \\ \dot{x}_3 &= 0.5x_1x_3 - dx_3 + x_1^2, \end{aligned} \tag{1}$$

where x_1, x_2, x_3 are the states and a, b, c, d are constant, positive, parameters.

The system (1) is a nine-term polynomial chaotic system with four quadratic nonlinearities.

The system (1) depicts a strange chaotic attractor when the constant parameter values are taken as

$$a = 25, \quad b = 33, \quad c = 11, \quad d = 6 \tag{2}$$

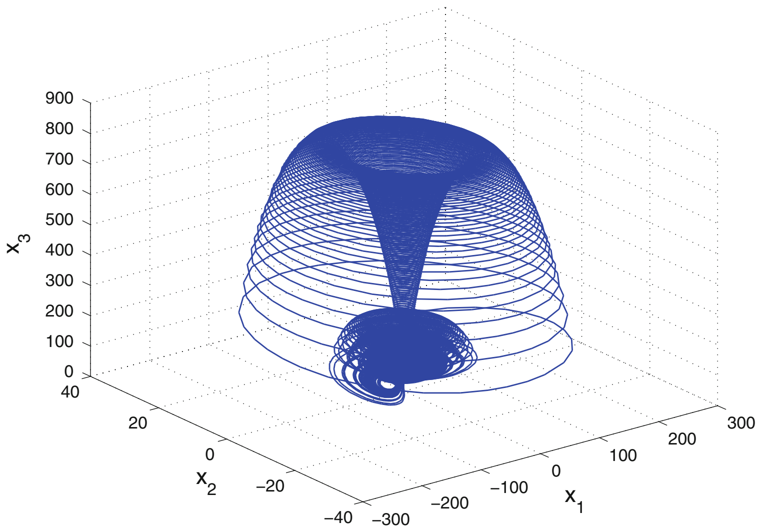

Fig. 1 Strange attractor of the novel chaotic system in $\mathbb{R}^3$

For simulations, the initial values of the novel chaotic system (1) are taken as

$$x_1(0) = 1.2, \quad x_2(0) = 0.6, \quad x_3(0) = 1.8 \tag{3}$$

Figure 1 describes the strange chaotic attractor of the novel chaotic system (1) in 3-D view. It is easily seen that the strange chaotic attractor is strongly chaotic and it has the shape of an "umbrella". In view of this observation, the strange chaotic attractor obtained for the novel chaotic system (1) may be also called as an "umbrella attractor".

3 Analysis of the Novel Chaotic System

This section gives the qualitative properties of the nine-term novel 3-D chaotic system proposed in this research work.

3.1 Dissipativity

We write the system (1) in vector notation as

$$\dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}, \tag{4}$$

where

$$\begin{aligned} f_1(x) &= a(x_2 - x_1) + 30x_2x_3 \\ f_2(x) &= bx_1 + cx_2 - x_1x_3 \\ f_3(x) &= 0.5x_1x_3 - dx_3 + x_1^2 \end{aligned} \tag{5}$$

We take the parameter values as

$$a = 8, \quad b = 55, \quad c = 4, \quad d = 5 \tag{6}$$

The divergence of the vector field f on $\mathbb{R}^3$ is obtained as

$$\operatorname{div} f = \frac{\partial f_1(x)}{\partial x_1} + \frac{\partial f_2(x)}{\partial x_2} + \frac{\partial f_3(x)}{\partial x_3} = c - (a + d) = -\mu \tag{7}$$

where

$$\mu = a + d - c = 25 + 6 - 11 = 20 > 0 \tag{8}$$

Let $\boldsymbol{\Omega}$ be any region in $\mathbb{R}^3$ with a smooth boundary. Let $\boldsymbol{\Omega}(t) = \boldsymbol{\Phi}_t(\boldsymbol{\Omega})$, where $\boldsymbol{\Phi}_t$ is the flow of the vector field f.

Let $V(t)$ denote the volume of $\boldsymbol{\Omega}(t)$.

By Liouville's theorem, it follows that

$$\frac{dV(t)}{dt} = \int\limits_{\Omega(t)} (\mathrm{div} f) dx_1 dx_2 dx_3 \tag{9}$$

Substituting the value of $\mathrm{div} f$ in (9) leads to

$$\frac{dV(t)}{dt} = -\mu \int\limits_{\boldsymbol{\Omega}(t)} dx_1 dx_2 dx_3 = -\mu V(t) \tag{10}$$

Integrating the linear differential Eq. (10), $V(t)$ is obtained as

$$V(t) = V(0) exp(-\mu t), \quad \text{where} \quad \mu = 20 > 0. \tag{11}$$

From Eq. (11), it follows that the volume $V(t)$ shrinks to zero exponentially as $t \to \infty$.

Thus, the novel chaotic system (1) is dissipative. Hence, any asymptotic motion of the system (1) settles onto a set of measure zero, i.e. a strange attractor.

3.2 Invariance

It is easily seen that the x_3-axis is invariant for the flow of the novel chaotic system (1). Hence, all orbits of the system (1) starting from the x_3 axis stay in the x_3 axis for all values of time.

3.3 Equilibria

The equilibrium points of the novel chaotic system (1) are obtained by solving the nonlinear equations

$$\begin{aligned} f_1(x) &= a(x_2 - x_1) + 30x_2x_3 = 0 \\ f_2(x) &= bx_1 + cx_2 - x_1x_3 = 0 \\ f_3(x) &= 0.5x_1x_3 - dx_3 + x_1^2 = 0 \end{aligned} \tag{12}$$

We take the parameter values as in the chaotic case, viz.

$$a = 25, \quad b = 33, \quad c = 11, \quad d = 6 \tag{13}$$

Solving the nonlinear system of Eq. (12) with the parameter values (13), we obtain three equilibrium points of the novel chaotic system (1) as

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 8.0776 \\ 0.1974 \\ 33.2688 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} -24.7120 \\ -0.6039 \\ 33.2688 \end{bmatrix}. \tag{14}$$

The Jacobian matrix of the novel chaotic system (1) at $(x_1^{\star}, x_2^{\star}, x_3^{\star})$ is obtained as

$$J(x^{\star}) = \begin{bmatrix} -25 & 25 + 30x_3^{\star} & 30x_2^{\star} \\ 33 - x_3^{\star} & 11 & -x_1^{\star} \\ 0.5x_3^{\star} + 2x_1^{\star} & 0 & 0.5x_1^{\star} - 6 \end{bmatrix} \tag{15}$$

The Jacobian matrix at E_0 is obtained as

$$J_0 = J(E_0) = \begin{bmatrix} -25 & 25 & 0 \\ 33 & 11 & 0 \\ 0 & 0 & -6 \end{bmatrix} \tag{16}$$

The matrix J_0 has the eigenvalues

$$\lambda_1 = -6, \quad \lambda_2 = -40.8969, \quad \lambda_3 = 26.8969 \tag{17}$$

This shows that the equilibrium point E_0 is a saddle-point, which is unstable.
The Jacobian matrix at E_1 is obtained as

$$J_1 = J(E_1) = \begin{bmatrix} -25.00 & 1023.06 & 5.92 \\ -0.27 & 11.00 & -8.08 \\ 32.79 & 0 & -1.96 \end{bmatrix} \tag{18}$$

The matrix J_1 has the eigenvalues

$$\lambda_1 = -71.5851, \quad \lambda_{2,3} = 27.8120 \pm 55.1509i \tag{19}$$

This shows that the equilibrium point E_1 is a saddle-focus, which is unstable.
The Jacobian matrix at E_2 is obtained as

$$J_2 = J(E_2) = \begin{bmatrix} -25.00 & 1023.06 & -18.12 \\ -0.27 & 11.00 & 24.71 \\ -32.79 & 0 & -18.36 \end{bmatrix} \tag{20}$$

The matrix J_2 has the eigenvalues

$$\lambda_1 = -107.62, \quad \lambda_{2,3} = 37.63 \pm 79.67i \tag{21}$$

This shows that the equilibrium point E_2 is a saddle-focus, which is unstable.

Hence, E_0, E_1, E_2 are all unstable equilibrium points of the 3-D novel chaotic system (1), where E_0 is a saddle point and E_1, E_2 are saddle-focus points.

3.4 Lyapunov Exponents and Lyapunov Dimension

We take the initial values of the novel chaotic system (1) as in (3) and the parameter values of the novel chaotic system (1) as (2).

Then the Lyapunov exponents of the novel chaotic system (1) are numerically obtained as

$$L_1 = 9.45456, \quad L_2 = 0, \quad L_3 = -30.50532 \tag{22}$$

Since $L_1 + L_2 + L_3 = -21.05076 < 0$, the system (1) is dissipative.

Also, the Lyapunov dimension of the system (1) is obtained as

$$D_L = 2 + \frac{L_1 + L_2}{|L_3|} = 2.1095 \tag{23}$$

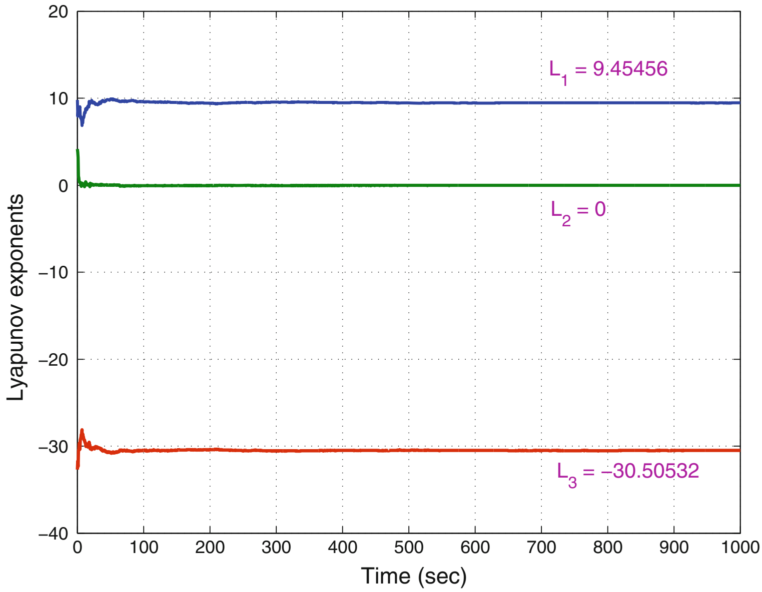

Fig. 2 Dynamics of the Lyapunov exponents of the novel chaotic system

Figure 2 depicts the dynamics of the Lyapunov exponents of the novel chaotic system (1). From this figure, it is seen that the maximal Lyapunov exponent of the novel chaotic system (1) is $L_1 = 9.45456$, which is a large value. Thus, the novel chaotic system (1) exhibits strong chaotic properties.

4 Adaptive Control of the Novel Chaotic System

This section derives new results for the adaptive controller to stabilize the unstable novel chaotic system with unknown parameters for all initial conditions.

The controlled novel 3-D chaotic system is given by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + 30x_2x_3 + u_1 \\ \dot{x}_2 &= bx_1 + cx_2 - x_1x_3 + u_2 \\ \dot{x}_3 &= 0.5x_1x_3 - dx_3 + x_1^2 + u_3 \end{aligned} \tag{24}$$

where x_1, x_2, x_3 are state variables, a, b, c, d are constant, unknown, parameters of the system and u_1, u_2, u_3 are adaptive controls to be designed.

An adaptive control law is taken as

$$\begin{aligned} u_1 &= -A(t)(x_2 - x_1) - 30x_2x_3 - k_1x_1 \\ u_2 &= -B(t)x_1 - C(t)x_2 + x_1x_3 - k_2x_2 \\ u_3 &= -0.5x_1x_3 + D(t)x_3 - x_1^2 - k_3x_3 \end{aligned} \tag{25}$$

where $A(t), B(t), C(t), D(t)$ are estimates for the unknown parameters a, b, c, d, respectively, and k_1, k_2, k_3 are positive gain constants.

The closed-loop control system is obtained by substituting (25) into (24) as

$$\begin{aligned} \dot{x}_1 &= (a - A(t))(x_2 - x_1) - k_1x_1 \\ \dot{x}_2 &= (b - B(t))x_1 + (c - C(t))x_2 - k_2x_2 \\ \dot{x}_3 &= -(d - D(t))x_3 - k_3x_3 \end{aligned} \tag{26}$$

To simplify (26), we define the parameter estimation error as

$$\begin{aligned} e_a(t) &= a - A(t) \\ e_b(t) &= b - B(t) \\ e_c(t) &= c - C(t) \\ e_d(t) &= d - D(t) \end{aligned} \tag{27}$$

Using (27), the closed-loop system (26) can be simplified as

$$
\begin{aligned}
\dot{x}_1 &= e_a(x_2 - x_1) - k_1 x_1 \\
\dot{x}_2 &= e_b x_1 + e_c x_2 - k_2 x_2 \\
\dot{x}_3 &= -e_d x_3 - k_3 x_3
\end{aligned}
\tag{28}
$$

Differentiating the parameter estimation error (27) with respect to t, we get

$$
\begin{aligned}
\dot{e}_a &= -\dot{A} \\
\dot{e}_b &= -\dot{B} \\
\dot{e}_c &= -\dot{C} \\
\dot{e}_d &= -\dot{D}
\end{aligned}
\tag{29}
$$

Next, we find an update law for parameter estimates using Lyapunov stability theory.

Consider the quadratic Lyapunov function defined by

$$
V(x_1, x_2, x_3, e_a, e_b, e_c, e_d) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right), \tag{30}
$$

which is positive definite on $\mathbb{R}^7$.

Differentiating V along the trajectories of (28) and (29), we get

$$
\begin{aligned}
\dot{V} = &-k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 + e_a\left[x_1(x_2 - x_1) - \dot{A}\right] + e_b\left(x_1 x_2 - \dot{B}\right) \\
&+ e_c\left(x_2^2 - \dot{C}\right) + e_d\left(-x_3^2 - \dot{D}\right)
\end{aligned}
\tag{31}
$$

In view of Eq. (31), an update law for the parameter estimates is taken as

$$
\begin{aligned}
\dot{A} &= x_1(x_2 - x_1) \\
\dot{B} &= x_1 x_2 \\
\dot{C} &= x_2^2 \\
\dot{D} &= -x_3^2
\end{aligned}
\tag{32}
$$

Theorem 1 *The novel chaotic system* (24) *with unknown system parameters is globally and exponentially stabilized for all initial conditions* $x(0) \in \mathbb{R}^3$ *by the adaptive control law* (25) *and the parameter update law* (32)*, where* $k_i, (i = 1, 2, 3)$ *are positive constants.*

Proof The result is proved using Lyapunov stability theory [19]. We consider the quadratic Lyapunov function V defined by (30), which is positive definite on $\mathbb{R}^7$.

Substitution of the parameter update law (32) into (31) yields

$$
\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2, \tag{33}
$$

which is a negative semi-definite function on $\mathbb{R}^7$.

Therefore, it can be concluded that the state vector $x(t)$ and the parameter estimation error are globally bounded, i.e.

$$[x_1(t) \quad x_2(t) \quad x_3(t) \quad e_a(t) \quad e_b(t) \quad e_c(t) \quad e_d(t)]^T \in L_\infty. \tag{34}$$

Define

$$k = \min\{k_1, k_2, k_3\} \tag{35}$$

Then it follows from (33) that

$$\dot{V} \leq -k\|x\|^2 \quad \text{or} \quad k\|x\|^2 \leq -\dot{V} \tag{36}$$

Integrating the inequality (36) from 0 to t, we get

$$k \int_0^t \|x(\tau)\|^2 d\tau \leq -\int_0^t \dot{V}(\tau) d\tau = V(0) - V(t) \tag{37}$$

From (37), it follows that $x(t) \in L_2$.

Using (28), it can be deduced that $\dot{x}(t) \in L_\infty$.

Hence, using Barbalat's lemma, we can conclude that $x(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $x(0) \in \mathbb{R}^3$.

This completes the proof. □

For numerical simulations, the parameter values of the novel system (24) are taken as in the chaotic case, viz.

$$a = 25, \quad b = 33, \quad c = 11, \quad d = 6 \tag{38}$$

The gain constants are taken as

$$k_1 = 5, \quad k_2 = 5, \quad k_3 = 5 \tag{39}$$

The initial values of the parameter estimates are taken as

$$A(0) = 7, \quad B(0) = 12, \quad C(0) = 25, \quad D(0) = 15 \tag{40}$$

The initial values of the novel system (24) are taken as

$$x_1(0) = 12.5, \quad x_2(0) = -5.6, \quad x_3(0) = 9.4 \tag{41}$$

Figure 3 shows the time-history of the controlled states $x_1(t), x_2(t), x_3(t)$.

From Fig. 3, it is seen that the states $x_1(t), x_2(t)$ and $x_3(t)$ are stabilized in 2 s (MATLAB). This shows the efficiency of the adaptive controller defined by (25).

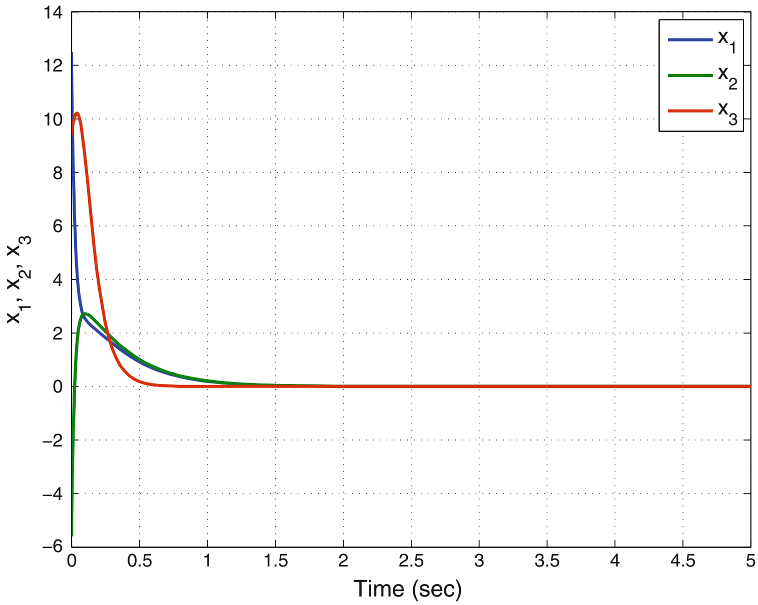

Fig. 3 Time-history of the states $x_1(t), x_2(t), x_3(t)$

5 Adaptive Synchronization of the Identical Novel Chaotic Systems

This section derives new results for the adaptive synchronization of the identical novel chaotic systems with unknown parameters.

The master system is given by the novel chaotic system

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + 30x_2x_3 \\ \dot{x}_2 &= bx_1 + cx_2 - x_1x_3 \\ \dot{x}_3 &= 0.5x_1x_3 - dx_3 + x_1^2 \end{aligned} \tag{42}$$

where x_1, x_2, x_3 are state variables and a, b, c, d are constant, unknown, parameters of the system.

The slave system is given by the controlled novel chaotic system

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + 30y_2y_3 + u_1 \\ \dot{y}_2 &= by_1 + cy_2 - y_1y_3 + u_2 \\ \dot{y}_3 &= 0.5y_1y_3 - dy_3 + y_1^2 + u_3 \end{aligned} \tag{43}$$

where y_1, y_2, y_3 are state variables and a, b, c, d are constant, unknown, parameters of the system.

The synchronization error is defined as

$$\begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 - x_2 \\ e_3 &= y_3 - x_3 \end{aligned} \tag{44}$$

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) + 30(y_2 y_3 - x_2 x_3) + u_1 \\ \dot{e}_2 &= b e_1 + c e_2 - y_1 y_3 + x_1 x_3 + u_2 \\ \dot{e}_3 &= -d e_3 + 0.5(y_1 y_3 - x_1 x_3) + y_1^2 - x_1^2 + u_3 \end{aligned} \tag{45}$$

An adaptive control law is taken as

$$\begin{aligned} u_1 &= -A(t)(e_2 - e_1) - 30(y_2 y_3 - x_2 x_3) - k_1 e_1 \\ u_2 &= -B(t) e_1 - C(t) e_2 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\ u_3 &= D(t) e_3 - 0.5(y_1 y_3 - x_1 x_3) - y_1^2 + x_1^2 - k_3 e_3 \end{aligned} \tag{46}$$

where $A(t), B(t), C(t), D(t)$ are estimates for the unknown parameters a, b, c, d, respectively, and k_1, k_2, k_3 are positive gain constants.

The closed-loop control system is obtained by substituting (46) into (45) as

$$\begin{aligned} \dot{e}_1 &= (a - A(t))(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= (b - B(t)) e_1 + (c - C(t)) e_2 - k_2 e_2 \\ \dot{e}_3 &= -(d - D(t)) e_3 - k_3 e_3 \end{aligned} \tag{47}$$

To simplify (26), we define the parameter estimation error as

$$\begin{aligned} e_a(t) &= a - A(t) \\ e_b(t) &= b - B(t) \\ e_c(t) &= c - C(t) \\ e_d(t) &= d - D(t) \end{aligned} \tag{48}$$

Using (48), the closed-loop system (47) can be simplified as

$$\begin{aligned} \dot{e}_1 &= e_a(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= e_b e_1 + e_c e_2 - k_2 e_2 \\ \dot{e}_3 &= -e_d e_3 - k_3 e_3 \end{aligned} \tag{49}$$

Differentiating the parameter estimation error (48) with respect to t, we get

$$
\begin{aligned}
\dot{e}_a &= -\dot{A} \\
\dot{e}_b &= -\dot{B} \\
\dot{e}_c &= -\dot{C} \\
\dot{e}_d &= -\dot{D}
\end{aligned}
\tag{50}
$$

Next, we find an update law for parameter estimates using Lyapunov stability theory.

Consider the quadratic Lyapunov function defined by

$$
V(e_1, e_2, e_3, e_a, e_b, e_c, e_d) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right), \tag{51}
$$

which is positive definite on $\mathbb{R}^7$.

Differentiating V along the trajectories of (49) and (50), we get

$$
\begin{aligned}
\dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a\left[e_1(e_2 - e_1) - \dot{A}\right] + e_b\left(e_1 e_2 - \dot{B}\right) \\
&+ e_c\left(e_2^2 - \dot{C}\right) + e_d\left(-e_3^2 - \dot{D}\right)
\end{aligned}
\tag{52}
$$

In view of Eq. (31), an update law for the parameter estimates is taken as

$$
\begin{aligned}
\dot{A} &= e_1(e_2 - e_1) \\
\dot{B} &= e_1 e_2 \\
\dot{C} &= e_2^2 \\
\dot{D} &= -e_3^2
\end{aligned}
\tag{53}
$$

Theorem 2 *The identical novel chaotic systems* (42) *and* (43) *with unknown system parameters are globally and exponentially synchronized for all initial conditions* $x(0), y(0) \in \mathbb{R}^3$ *by the adaptive control law* (46) *and the parameter update law* (53)*, where* $k_i, (i = 1, 2, 3)$ *are positive constants.*

Proof The result is proved using Lyapunov stability theory [19].

We consider the quadratic Lyapunov function V defined by (51), which is positive definite on $\mathbb{R}^7$.

Substitution of the parameter update law (53) into (52) yields

$$
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2, \tag{54}
$$

which is a negative semi-definite function on $\mathbb{R}^7$.

Therefore, it can be concluded that the synchronization error vector $e(t)$ and the parameter estimation error are globally bounded, i.e.

$$
\left[e_1(t)\; e_2(t)\; e_3(t)\; e_a(t)\; e_b(t)\; e_c(t)\; e_d(t)\right]^T \in L_\infty. \tag{55}
$$

Define

$$k = \min\{k_1, k_2, k_3\} \tag{56}$$

Then it follows from (54) that

$$\dot{V} \leq -k\|e\|^2 \quad \text{or} \quad k\|e\|^2 \leq -\dot{V} \tag{57}$$

Integrating the inequality (36) from 0 to t, we get

$$k\int_0^t \|e(\tau)\|^2 d\tau \leq -\int_0^t \dot{V}(\tau)d\tau = V(0) - V(t) \tag{58}$$

From (58), it follows that $e(t) \in L_2$.

Using (49), it can be deduced that $\dot{e}(t) \in L_\infty$.

Hence, using Barbalat's lemma, we can conclude that $e(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $e(0) \in \mathbb{R}^3$.

This completes the proof. □

For numerical simulations, the parameter values of the novel systems (42) and (43) are taken as in the chaotic case, viz.

$$a = 25, \quad b = 33, \quad c = 11, \quad d = 6 \tag{59}$$

The gain constants are taken as $k_i = 5$ for $i = 1, 2, 3$.

The initial values of the parameter estimates are taken as

$$A(0) = 16, \quad B(0) = 8, \quad C(0) = 4, \quad D(0) = 7 \tag{60}$$

The initial values of the master system (42) are taken as

$$x_1(0) = 6.8, \quad x_2(0) = 3.7, \quad x_3(0) = -9.1 \tag{61}$$

The initial values of the slave system (43) are taken as

$$y_1(0) = 3.4, \quad y_2(0) = -12.5, \quad y_3(0) = 1.8 \tag{62}$$

Figures 4, 5 and 6 show the complete synchronization of the identical chaotic systems (42) and (43).

Figure 4 shows that the states $x_1(t)$ and $y_1(t)$ are synchronized in 2 s (MATLAB). Figure 5 shows that the states $x_2(t)$ and $y_2(t)$ are synchronized in 2 s (MATLAB). Figure 6 shows that the states $x_3(t)$ and $y_3(t)$ are synchronized in 2 s (MATLAB).

Figure 7 shows the time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$. From Fig. 7, it is seen that the errors $e_1(t), e_2(t)$ and $e_3(t)$ are stabilized in 2 s (MATLAB).

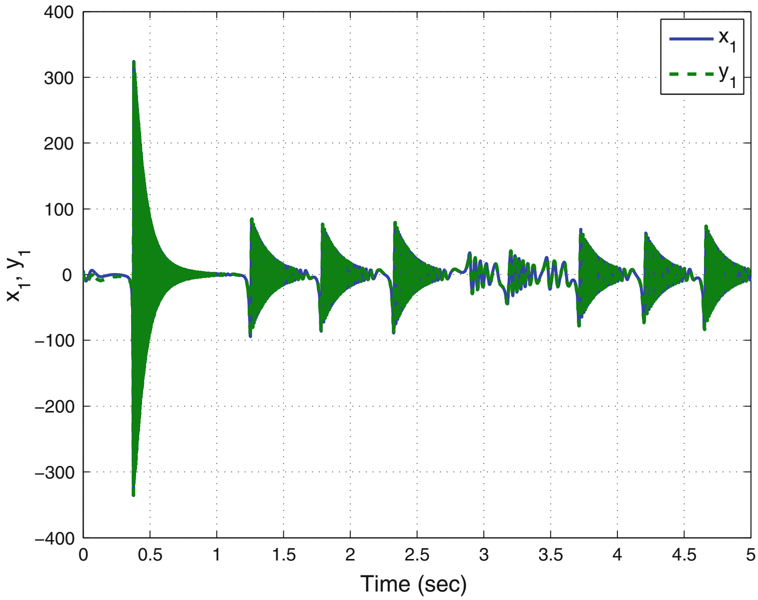

Fig. 4 Synchronization of the states x_1 and y_1

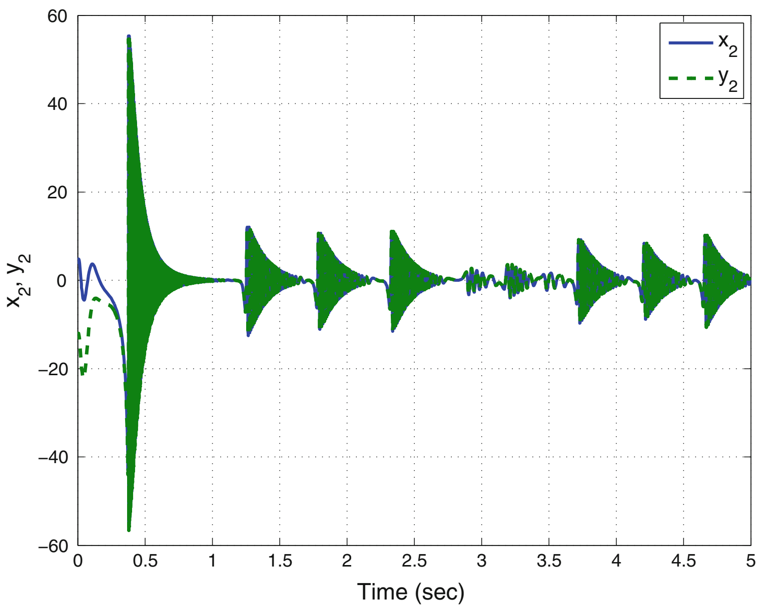

Fig. 5 Synchronization of the states x_2 and y_2

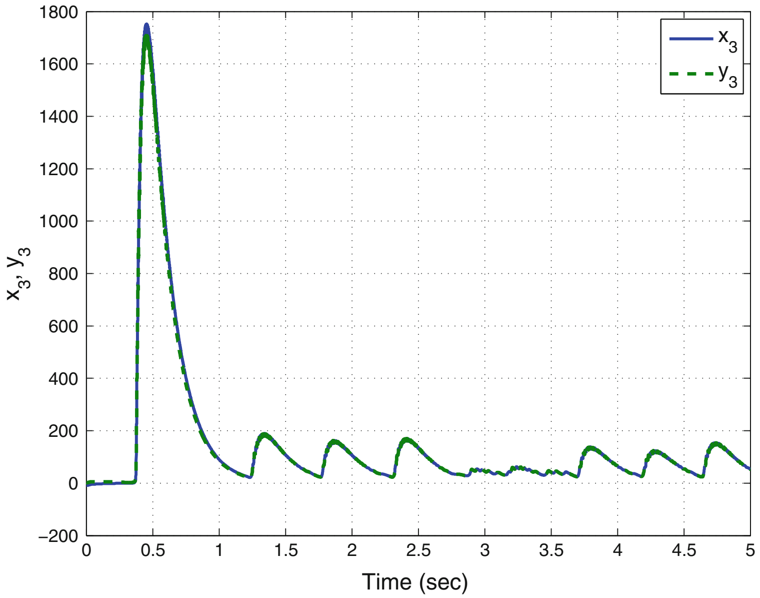

Fig. 6 Synchronization of the states x_3 and y_3

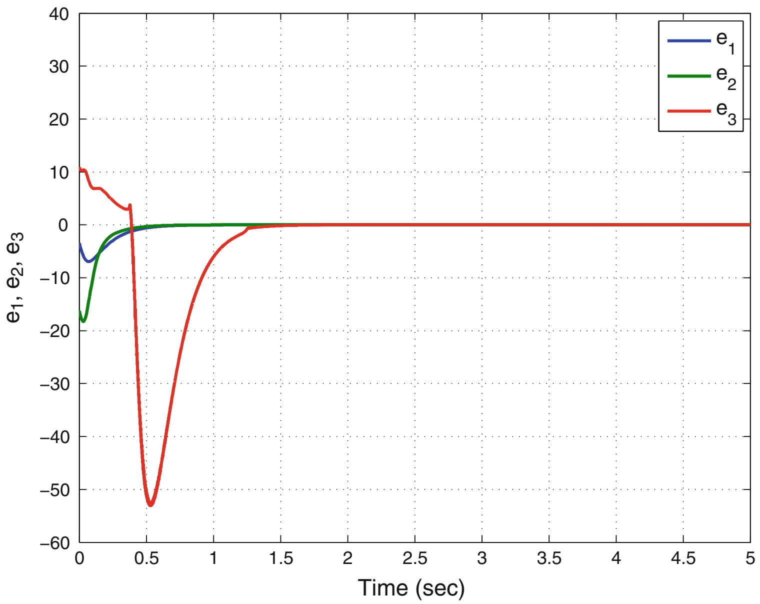

Fig. 7 Time-history of the synchronization errors e_1, e_2, e_3

6 Conclusions

In this research work, a nine-term 3-D novel chaotic system with four quadratic nonlinearities has been proposed and its qualitative properties have been derived. The Lyapunov exponents of the nine-term novel chaotic system have been obtained as $L_1 = 9.45456, L_2 = 0$ and $L_3 = -30.50532$. Since the maximal Lyapunov exponent (MLE) of the novel chaotic system is $L_1 = 9.45456$, which is a high value, the novel chaotic system exhibits strong chaotic properties. The novel chaotic system has three unstable equilibrium points. Next, an adaptive controller has been derived for globally stabilizing the novel chaotic system with unknown system parameters. Furthermore, an adaptive synchronizer has been derived for completely and globally synchronizing the identical novel chaotic systems with unknown system parameters. The adaptive control and synchronization results were proved using Lyapunov stability theory. MATLAB simulations were shown to demonstrate and validate all the main results derived in this work for the nine-term 3-D novel chaotic system. As future research directions, new control techniques like sliding mode control or backstepping control may be considered for stabilizing the novel chaotic system with three unstable equilibrium points or synchronizing the identical novel chaotic systems for all initial conditions.

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http://www.springer.com/978-3-319-13131-3

Chaos Modeling and Control Systems Design
Azar, A.T.; Vaidyanathan, S. (Eds.)
2015, X, 416 p. 193 illus., 15 illus. in color., Hardcover
ISBN: 978-3-319-13131-3