Burning a Graph as a Model of Social Contagion

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Abstract. We introduce a new graph parameter called the burning number, inspired by contact processes on graphs such as graph bootstrap percolation, and graph searching paradigms such as Firefighter. The burning number measures the speed of the spread of contagion in a graph; the lower the burning number, the faster the contagion spreads. We provide a number of properties of the burning number, including characterizations and bounds. The burning number is computed for several graph classes, and is derived for the graphs generated by the Iterated Local Transitivity model for social networks.

1 Introduction

The spread of social influence is an active topic in social network analysis; see, for example, [3,8,13,14,18,19]. A recent study on the spread of emotional contagion in Facebook [16] has highlighted the fact that the underlying network is an essential factor; in particular, in-person interaction and nonverbal cues are not necessary for the spread of the contagion. Hence, agents in the network spread the contagion to their friends or followers, and the contagion propagates over time. If the goal was to minimize the time it took for the contagion to reach the entire network, then which agents would you target with the contagion, and in which order?

As a simplified, deterministic approach to these questions, we consider a new approach involving a graph process which we call *burning*. Burning is inspired by graph theoretic processes like Firefighting [4,7,10], graph cleaning [1], and graph bootstrap percolation [2]. There are discrete time-steps or rounds. Each node is either *burned* or *unburned*; if a node is burned, then it remains in that state until the end of the process. Every round, we choose a node to burn. Once a node is burned in round t, in round $t+1$, each of its unburned neighbours becomes burned. In every round, we choose one additional unburned node to burn (if such a node is available). The process ends when all nodes are burned. The *burning number* of a graph G, written by $b(G)$, is the minimum number of rounds needed for the process to end. For example, it is straightforward to see that $b(K_n) = 2$. However, even for a relatively simple graph such as the path P_n on n nodes, computing the burning number is more complex; in fact, $b(P_n) = \lceil n^{1/2} \rceil$ as stated below in Theorem 3 (and proven in [6]).

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Burning may be viewed as a simplified model for the spread of social contagion in a social network such as Facebook or Twitter. The lower the value of $b(G)$, the easier it is to spread such contagion in the graph G. Suppose that in the process of burning a graph G, we eventually burned the whole graph G in k steps, and for each i, $1 \leq i \leq k$, we denote the node that we burn in the i-th step by x_i. We call such a node simply a *source of fire*. The sequence $(x_1, x_2, \ldots, x_k)$ is called a *burning sequence* for G. With this notation, the burning number of G is the length of a shortest burning sequence for G; such a burning sequence is referred to as *optimal*. For example, for the path P_4 with nodes v_1, v_2, v_3, v_4, the sequence (v_2, v_4) is an optimal burning sequence (See Figure 1). Note that for a graph G with at least two nodes, we have that $b(G) \geq 2$.

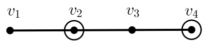

Fig. 1. Burning the path P_4 (the open circles represent burned nodes)

The goal of the current paper is to introduce the burning number and explore its core properties. A characterization of burning number via a decomposition into trees is given in Theorem 1. As proven in [6], computing the burning number of a graph is **NP**-complete, even for planar, disconnected, or bipartite graphs. As such, we provide sharp bounds on the burning number for connected graphs, which are useful in many cases when computing the burning number. See Theorem 2.2 for bounds on the burning number. We compute the burning number on the Iterated Local Transitivity model for social networks (introduced in [5]) and grids; see Theorem 8 and Theorem 9, respectively. In the final section, we summarize our results and present open problems for future work.

2 Properties of the Burning Number

In this section, we collect a number of results on the burning number, ranging from characterizations, bounds, to computing the burning number on certain kinds of graphs. We first need some terminology. If G is a graph and v is a node of G, then the *eccentricity* of v is defined as $\max\{d(v, u) : u \in G\}$. The *radius* of G is the minimum eccentricity over the set of all nodes in G. The *center* of G consists of the nodes in G with minimum eccentricity. Given a positive integer k, the *k-th closed neighborhood* of v is defined to be the set $\{u \in V(G) : d(u, v) \leq k\}$ and is denoted by $N_k[v]$; we denote $N_1[v]$ simply by $N[v]$.

We first make the following observation. Suppose that $(x_1, x_2, \ldots, x_k)$, where $k \geq 3$, is a burning sequence for a given graph G. For $1 \leq i \leq k$, the fire spread from x_i will burn only all the nodes within distance $k - i$ from x_i by the end of the k-th step. On the other hand, every node $v \in V(G)$ must be either a source of fire, or burned from at least one of the sources of fire by the end of the k-th step. In other words, any node of G that is not a source of fire must be an element of

$N_{k-i}[x_i]$, for some $1 \leq i \leq k$. Therefore, we can see that $(x_1, x_2, \ldots, x_k)$ forms a burning sequence for G if and only if the following set equation holds:

$$N_{k-1}[x_1] \cup N_{k-2}[x_2] \cup \ldots \cup N_0[x_k] = V(G). \tag{1}$$

Here is another simple observation. For each pair i and j, with $1 \leq i < j \leq k$, $d(x_i, x_j) \geq j-i$. Since, otherwise, if $d(x_i, x_j) = l < j-i$, then x_j will be burned at stage $l+i$ $(< j)$ which is a contradiction. Hence, we have the following corollary.

Corollary 1. *Suppose that* $(x_1, x_2, \ldots, x_k)$ *is a burning sequence for a graph* G*. If for some node* $x \in V(G) \setminus \{x_1, \ldots, x_k\}$ *and* $1 \leq j \leq k-1$*, we have that* $N[x] \subseteq N[x_j]$*, and for every* $i \neq j$*,* $d(x, x_i) \geq |i-j|$*, then* $(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_k)$ *is also a burning sequence for* G*.*

2.1 Characterizations of Burning Number via Trees

The following theorem provides an alternative characterization of the burning number. Note that through the rest of this paper we consider the burning problem for connected graphs. The *depth* of a node in a rooted tree is the number of edges in a shortest path from the node to the tree's root. The *height* of a rooted tree T is the greatest depth in T. A *rooted tree partition* of G is a collection of rooted trees which are subgraphs of G, with the property that the node sets of the trees partition $V(G)$.

Theorem 1. *Burning a graph* G *in* k *steps is equivalent to finding a rooted tree partition into* k *trees* $T_1, T_2, \ldots, T_k$*, with heights at most* $(k-1), (k-2), \ldots, 0$*, respectively such that for every* $1 \leq i, j \leq k$ *the distance between the roots of* T_i *and* T_j *is at least* $|i-j|$*.*

Proof. Assume that $(x_1, x_2, \ldots, x_k)$ is a burning sequence for G. For all $1 \leq i \leq k$, after x_i is burned, in each round $t > i$ those unburned nodes of G in the $(t-i)$-neighborhood of x_i will burn. Hence, any node v is burned by receiving fire via a shortest path of burned nodes from a fire source like x_i (this path can be of length zero in the case that $v = x_i$). Hence, we may define a surjective function $f : V(G) \to \{x_1, x_2, \ldots, x_k\}$, with $f(v) = x_i$ if v receives fire from x_i, where i is chosen with the smallest index. Now $\{f^{-1}(x_1), f^{-1}(x_2), \ldots, f^{-1}(x_k)\}$ forms a partition of $V(G)$ such that $G[f^{-1}(x_i)]$ (the subgraph induced by $f^{-1}(x_i)$) forms a connected subgraph of G. Since every node v in $f^{-1}(x_i)$ receives the fire spread from x_i through a shortest path between x_i and v, by deleting extra edges in $G[f^{-1}(x_i)]$ we can make a rooted subtree of G, called T_i with root x_i. Since every node is burned after k steps, the distance between each node on T_i and x_i is at most $k-i$. Therefore, the height of T_i is at most $k-i$.

Conversely, suppose that we have a decomposition of the nodes of G into k rooted subtrees $T_1, T_2, \ldots, T_k$, such that for each $1 \leq i \leq k$, T_i is of height at most $k-i$. Assume that $x_1, x_2, \ldots, x_k$ are the roots of $T_1, T_2, \ldots, T_k$, respectively, and for each pair i and j, with $1 \leq i < j \leq k$, $d(x_i, x_j) \geq j-i$. Then $(x_1, x_2, \ldots, x_k)$ is a burning sequence for G, since the distance between any node in T_i and x_i is at most $k-i$. Thus, after k steps the graph G will be burned. □

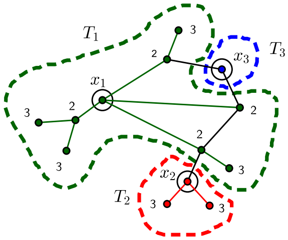

Fig. 2. A rooted tree partition

Figure 2 illustrates Theorem 1. The burning sequence is (x_1, x_2, x_3). We have shown the decomposition of G into subgraphs T_1, T_2, and T_3 based on this burning sequence by drawing dashed curves around the corresponding subgraphs. Each node has been indexed by a number corresponding to the step that it is burned.

The following corollary is useful for determining the burning number of a graph, as it reduces the problem of burning a graph to burning its spanning trees. First, note that for a spanning subgraph H of G, it is evident that $b(G) \leq b(H)$ (since every burning sequence for H is also a burning sequence for G).

Corollary 2. *For a graph G we have that*

$$b(G) = \min\{b(T) : T \text{ is a spanning subtree of } G\}.$$

Proof. By Theorem 1, we assume that $T_1, T_2, \ldots, T_k$ is a rooted tree partition of G, where $k = b(G)$, derived from an optimal burning sequence for G. If we take T to be a spanning subtree of G obtained by adding edges sequentially between the T_i's which do not induce a cycle in G, then $b(T) \leq k = b(G) \leq b(T)$, where the second inequality holds since T is a spanning subgraph of G. $\square$

2.2 Bounds

A subgraph H of a graph G is called an *isometric subgraph* if for every pair of nodes u, v in H, we have that $d_H(u, v) = d_G(u, v)$. For example, a subtree of a tree is an isometric subgraph. As another example, if G is a connected graph and P is a shortest path connecting two nodes of G, then P is an isometric subgraph of G. The following theorem (with proof omitted) shows that the burning number is monotonic on isometric subgraphs.

Theorem 2. *For any isometric subgraph H of a graph G, we have that $b(H) \leq b(G)$.*

However, this inequality may fail for non-isometric subgraphs. For example, let H be a path of order 5, and form G by adding a universal node to H. Then $b(H) = 3$, but $b(G) = 2$. The following corollary is an immediate consequence of Theorem 2.

Corollary 3. *If T is a tree and H is a subtree of T, then we have that $b(H) \leq b(T)$.*

The burning number of paths is derived in the following result (with proof omitted).

Theorem 3. *For a path P_n on n nodes, we have that $b(P_n) = \lceil n^{1/2} \rceil$.*

We have the following immediate corollaries.

Corollary 4. *1. For a cycle C_n, we have that $b(C_n) = \lceil n^{1/2} \rceil$.*
2. For a graph G of order n with a Hamiltonian (that is, spanning) path, we have that $b(G) \leq \lceil n^{1/2} \rceil$.

The following theorem gives sharp bounds on the burning number. For $s \geq 3$, let $K_{1,s}$ denotes a *star*; that is, a complete bipartite graph with parts of order 1 and s. We call a graph obtained by a sequence of subdivisions starting from $K_{1,s}$ a *spider graph*. In a spider graph G, any path which connects a leaf to the node with maximum degree is called an *arm* of G. If all the arms of a spider graph with maximum degree s are of the same length r, we denote such a spider graph by $SP(s, r)$.

Lemma 1. *For any graph G with radius r and diameter d, we have that*

$$\lceil (d+1)^{1/2} \rceil \leq b(G) \leq r + 1.$$

Proof. Assume that c is a central node of G with eccentricity r. Since every node in G is within distance r from c, the fire will spread to all nodes after $r+1$ steps. Hence, $r+1$ is an upper bound for $b(G)$.

Now, let P be a path connecting two nodes u and v in G with $d(u, v) = d$. Since P is an isometric subgraph of G, and $|P| = d + 1$, by Theorem 2 and Theorem 3 we conclude that $b(G) \geq b(P) = \lceil (d+1)^{1/2} \rceil$. □

As proven in [6], the lower bound is achieved by paths, and the right side bound is achieved by spider graphs $SP(r, r)$. Note that when proving $b(G) \leq r + 1$ in Theorem 1, we viewed G as covered by a ball with radius r, with a central node chosen as a center of the ball. Hence, by burning a central node, after $r+1$ steps every node in G will be burned. A *covering* of G is a set of subsets of the nodes of G whose union is $V(G)$. We may generalize this idea to the case that there is a covering of G by a collection of balls with a specified radius.

Theorem 4. *Let $\{C_1, C_2, \ldots, C_t\}$ be a covering of the nodes of a graph G, in which each C_i is a connected subgraph of radius at most k. Then we have that $b(G) \leq t + k$.*

We finish this section by providing some bounds on the burning number in terms of certain domination numbers. A *k-distance dominating set* like D_k for G is a subset of nodes such that for every node $u \in V(G) \setminus D_k$, there exists a node $v \in D_k$, with $d(u, v) \leq k$. The number of the nodes in a minimum k-distance dominating set of G is denoted by $\gamma_k(G)$ and we call it the *k-distance domination number* of G. We have the following result (proof omitted).

Theorem 5. *For any graph G with burning number k we have, $\gamma_{k-1}(G) \leq k$.*

We now give bounds on the burning number in terms of distance domination numbers.

Theorem 6. *If G is a connected graph, then we have that*

$$\frac{1}{2}\left(\min_{i\geq 1}\{\gamma_i(G)+i\}+1\right) \leq b(G) \leq \min_{i\geq 1}\{\gamma_i(G)+i\}.$$

Proof. The upper bound is an immediate corollary of Theorem 4. For the lower bound, let $k = b(G)$, and let $(x_1, \ldots, x_k)$ be a burning sequence. Then we have that

$$\begin{aligned} V(G) &\subseteq N_{k-1}[x_1] \cup \ldots \cup N_0[x_k] \\ &\subseteq N_{k-1}[x_1] \cup \ldots \cup N_{k-1}[x_k]. \end{aligned}$$

Hence, $\{x_1, \ldots, x_k\}$ is a k-distance dominating set of G. Since by Theorem 5 we have that $\gamma_{k-1}(G) \leq k$, and $\gamma_{k-1}(G) + (k-1) \leq 2k-1 = 2b(G)-1$, we derive that $\min_{i\geq 1}\{\gamma_i(G)+i\} \leq \gamma_{k-1}(G) + (k-1) \leq 2b(G)-1$. □

We have the following fact about the k-distance domination number of graphs.

Theorem 7. *[17] If G is a connected graph of order n with $n \geq k+1$, then we have that*

$$\gamma_k(G) \leq \frac{n}{k+1}.$$

Now we use the bound in Theorem 7 for k-distance domination number which provides another upper bound for the burning number.

Corollary 5. *If G is a connected graph of order n, then we have that*

$$b(G) \leq 2n^{1/2} - 1.$$

We conjecture that for any connected graph G of order n, $b(G) \leq \lceil n^{1/2} \rceil$.

3 Burning in the ILT Model

The *Iterated Local Transitivity* (ILT) model [5], simulates on-line social networks (or OSNs). The central idea behind the ILT model is what sociologists call *transitivity*: if u is a friend of v, and v is a friend of w, then u is a friend of w. In its simplest form, transitivity gives rise to the notion of *cloning*, where u is joined to all of the neighbours of v. In the ILT model, given some initial graph as a starting point, nodes are repeatedly added over time which clone *each* node, so that the new nodes form an independent set. The only parameter of the model is the initial graph G_0, which is any fixed finite connected graph. Assume that for a fixed $t \geq 0$, the graph G_t has been constructed. To form G_{t+1}, for each node

$x \in V(G_t)$, add its *clone* x', such that x' is joined to x and all of its neighbours at time t. Note that the set of new nodes at time $t+1$ form an independent set of cardinality $|V(G_t)|$.

The ILT model shares many properties with OSNs such as low average distance, high clustering coefficient densification, and bad spectral expansion; see [5]. The ILT model has also been studied from the viewpoint of competitive diffusion which is one model of the spread of influence; see [20].

We have the following theorem about the burning number of graphs obtained based on ILT model. Even though the graphs generated by the ILT model grow exponentially in order with t, we see that the burning number of such networks remains constant.

Theorem 8. *Let G_t be the graph generated at time $t \geq 1$ based on the ILT model with initial graph G_0. If G_0 has an optimal burning sequence $(x_1, \ldots, x_k)$ in which x_k has a neighbor that is burned in the $(k-1)$-th step, then $b(G_t) = b(G_0)$. Otherwise, $b(G_t) = b(G_0) + 1$.*

Proof. It is straightforward to see that G_0 is an isometric subgraph of G_t. Therefore, by Theorem 2, $b(G_t) \geq b(G_0)$. On the other hand, assume that $(x_1, \ldots, x_k)$ is an optimal burning sequence for G_0. Since every node $x' \in V(G_t) \setminus V(G_0)$ is adjacent to a node in G_0, we have that $(x_1, \ldots, x_k)$ is a burning sequence for the subgraph of G_t induced by $V(G_t) \setminus (N_{G_t}[x_k] \setminus N_{G_0}[x_k])$. Thus, $b(G_t) \leq b(G_0)+1$. Hence, we conclude that always either we have that $b(G_t) = b(G_0)$, or $b(G_t) = b(G_0) + 1$.

Suppose that for every optimal burning sequence of G_0 all the neighbours of x_k are burned in the k-th step. We claim that $b(G_1) = b(G_0) + 1$. Assume not; that is, $b(G_1) = b(G_0)$. Let $(y_1, y_2, \ldots, y_k)$ be an optimal burning sequence for G_1. Without loss of generality, by Corollary 1, and the structure of G_1, we can assume that $\{y_1, y_2, \ldots, y_{k-1}\} \subseteq G_0$. Then, we have two possibilities; either $y_k = x$ or $y_k = x' \in V(G_1) \setminus V(G_0)$, for some $x \in V(G_0)$. If the former holds, then to burn x' by the end of the k-th step, one of the nodes in the neighbourhood of x must be burned in an earlier stage, which is a contradiction. Since in this case $(y_1, y_2, \ldots, y_k)$ forms a burning sequence for G_0. If the latter holds, that is, $y_k = x' \in V(G_1) \setminus V(G_0)$, for some $x \in V(G_0)$, then, we must have $x = y_{k-1}$ (Note that all the neighbours of x must be burned either in the $(k-1)$-th step or the k-th step; Otherwise, y_k is burned before the k-th step, which is a contradiction). Otherwise, if $x \neq y_{k-1}$, to burn x by the k-th step, one of the neighbours of x must be burned in an earlier stage. But then in this case, $(y_1, \ldots, y_{k-1}, x)$ forms an optimal burning sequence for G_0 such that one of the neighbours of x is burned in the $(k-1)$-th step which is a contradiction with the assumption. Thus, $x = y_{k-1}$.

If all the neighbours of x, including y, are burned in the $(k-1)$-th step, then $(y_1, \ldots, y_{k-2}, y, x)$ forms an optimal burning sequence for G_0. But this is a contradiction with the assumption. If at least one of the neighbours of x like y is burned at the k-th step, then $(y_1, \ldots, y_{k-2}, x, y)$ forms an optimal burning sequence for G_0, which is again a contradiction with the assumption. Therefore, in this case, $b(G_1) = b(G_0)$ is impossible, and hence, $b(G_1) = b(G_0) + 1$.

Conversely, suppose that $b(G_1) = b(G_0) + 1$, and $(x_1, \ldots, x_k)$ is an optimal burning sequence for G_0. If x_k has a neighbour that is burned at stage $k-1$, then x'_k is also burned at stage k. Therefore, $(x_1, \ldots, x_k)$ is a burning sequence for G_1, and we have that $b(G_1) = b(G_0)$, which is a contradiction. Thus, $b(G_1) = b(G_0)+1$, if and only if for every optimal burning sequence of G_0, say $(x_1, \ldots, x_k)$, all the neighbours of x_k are burned in stage k. By induction, we can conclude that $b(G_t) = b(G_0) + 1$ if and only if for every optimal burning sequence of G_0, say $(x_1, \ldots, x_k)$, all the neighbours of x_k are burned in stage k. Since starting from any graph G_0, for any $t \geq 1$, $b(G_t) = b(G_0)$, or $b(G_t) = b(G_0) + 1$, we conclude that $b(G_t) = b(G_0)$ if and only if for every optimal burning sequence of G_0, say $(x_1, \ldots, x_k)$ one of the neighbours of x_k is burned in stage $k-1$. □

4 Cartesian Grids

The *Cartesian product* of graphs G and H, written $G \square H$, has nodes $V(G) \times V(H)$ with (u, v) adjacent to (x, y) if $u = x$ and $vy \in E(H)$ or $v = y$ and $ux \in E(G)$. The *Cartesian $m \times n$ grid* is $P_m \square P_n$. We prove the following theorem.

Theorem 9. *If G is a Cartesian $m \times n$ grid with $1 \leq m \leq n$, then we have that*

$$b(G) = \begin{cases} \Theta(n^{1/2}) & \text{if } m = O(n^{1/2}) \\ \Theta((mn)^{1/3}) & \text{if } m = \Omega(n^{1/2}). \end{cases}$$

Proof. First, we find a general upper bound by applying the covering idea in Theorem 4 as follows. Using a layout as shown in Figure 3 we may provide a covering of G by a collection of t closed neighbourhoods of radius r. Note that the r-th neighbourhood of a vertex in a grid is a subset of a "diamond" with diameter $2r+1$ in the Cartesian grid plane. Thus, by a simple counting argument we have that

$$\begin{aligned} t &= \left\lceil \frac{m}{2r+1} \right\rceil \left\lceil \frac{n}{2r+1} \right\rceil + \left(\left\lceil \frac{m}{2r+1} \right\rceil + 1 \right) \left(\left\lceil \frac{n}{2r+1} \right\rceil + 1 \right) \\ &\leq 2 \left(\left\lceil \frac{m}{2r+1} \right\rceil + 1 \right) \left(\left\lceil \frac{n}{2r+1} \right\rceil + 1 \right). \end{aligned}$$

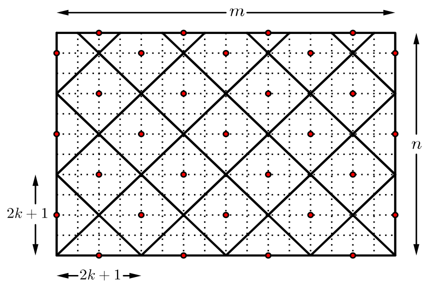

Fig. 3. A covering of the Cartesian grid

Therefore, $t = O(\frac{mn}{r^2} + \frac{m}{r} + \frac{n}{r})$, and consequently, by Theorem 4,

$$b(G) = O\left(r + \frac{mn}{r^2} + \frac{m}{r} + \frac{n}{r}\right). \tag{2}$$

First, we consider the case that $m = O(n^{1/2})$: Since P_n is an isometric subgraph of G, then by Theorem 3, we have that $b(G) = \Omega(n^{1/2})$. Moreover, by taking $r = n^{1/2}$, we derive that $\frac{mn}{r^2} = m = O(n^{1/2})$, and $\frac{m}{r} + \frac{n}{r} \le 2\frac{n}{r} = O(n^{1/2})$. Thus, by equation (2), $b(G) = O(n^{1/2})$, and we conclude that in this case, $b(G) = \Theta(n^{1/2})$.

Now, suppose $m = \Omega(n^{1/2})$. Let $S = (x_1, x_2, \ldots, x_k)$ be a burning sequence for G. Thus, every node in G must be in the $(k-i)$-th neighborhood of a node x_i, for some $1 \le i \le k$. By direct checking, the number of nodes in the r-th closed neighborhood of a node x in G equals

$$\begin{aligned} |N_r[x]| = |\{y \in G : d(x,y) \le r\}| &= 1 + 4 + \cdots + 4r \\ &= 1 + 2r(r+1). \end{aligned}$$

Therefore, by double counting the nodes of G and by (1), we have that

$$\begin{aligned} mn = |G| &\le |N_{k-1}[x_1]| + |N_{k-2}[x_2]| + \cdots + |N_0[x_k]| \\ &= k + \sum_{i=1}^{k-1} 2i(i+1) = \frac{2k^3 + k}{3}. \end{aligned}$$

Since the above inequality holds for all burning sequences, we conclude that $b(G) = \Omega((mn)^{1/3})$. On the other hand, by taking $r = (mn)^{1/3}$ in equation (2), we derive that $b(G) = O((mn)^{1/3})$. Hence, the proof follows. □

5 Conclusions and Future Work

We introduced a new graph parameter, the burning number of a graph, written $b(G)$. The burning number measures how rapidly social contagion spreads in a given graph. We gave a characterization of the burning number in terms of decompositions into trees, and gave bounds on the burning number which allow us to compute it for a variety of graphs. We determined the asymptotic order of the burning number of grids, and determined the burning number in the Iterated Local Transitive model for social networks.

Several problems remain on the burning number. We conjecture that for a connected graph G of order n, $b(G) \le \lceil n^{1/2} \rceil$. Determining the burning number remains open for many classes of graphs, including trees and disconnected graphs. It remains open to consider the burning number in real-world social networks such as Facebook or LinkedIn. As Theorem 8 suggests, the burning number of on-line social networks is likely of constant order as the network grows over time. We remark that burning number generalizes naturally to directed graphs; one interesting direction is to determine the burning number on Kleinberg's small world model [15], which adds random directed edges to the Cartesian grid.

A simple variation which leads to complex dynamics is to change the rules for nodes to burn. As in graph bootstrap percolation [2], the rules could be varied so nodes burn only if they are adjacent to at least r burned neighbors, where $r > 1$. We plan on studying this variation in future work.

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