Chapter 2
Sequences and Series

This chapter deals with sequences, series, and products of real numbers, and the fundamental concept of convergence of these entities. We shall treat, too, approximation of real numbers by rational numbers, and we shall introduce the Euler number e.

2.1 Approximation by Rational Numbers

A good deal of the computational work with real numbers is being done on subsets of rational numbers (there is no way to store in a computer an infinite sequence of digits, as the decimal—or the binary—expansion of an irrational number). Fortunately, as we showed in Proposition 85, the set of rational numbers, as well as the set of dyadic numbers, are dense in the set of real numbers. This allows, then, to approximate any real number by rational numbers or by dyadic numbers as wished. A set of techniques to properly manage approximation is, certainly, needed. These, as part of what is known as "approximation theory", have been well developed since, and the reader certainly realizes their importance for dealing with the world around us.

> *Although this may seem a paradox, all exact science is dominated by the idea of approximation.*
> Bertrand Russell

> *Le Calcul infinitésimal, [...], est l'apprentissage du maniement des inégalités bien plus que des égalités, et on pourrait le résumer en trois mots: MAJORER, MINORER, APPROCHER.*

> (*The infinitesimal calculus, [...], consists of learning the use of inequalities, rather than equalities themselves, and may be summarize in three actions: to SEARCH FOR UPPER BOUNDS, to SEARCH FOR LOWER BOUNDS, to APPROXIMATE.*)
> Jean Dieudonné

Suppose we want to store the number $\frac{1}{3}$ in a computer that is able to manage only dyadic numbers from the list $\{\frac{k}{2^n} : k = 0, 1, 2, \ldots, 2^n, n = 0, 1, 2, 3\}$. We find that the closest dyadic number to $\frac{1}{3}$ in our list is the number $\frac{3}{8}$ ($= 0.375$ in base 10). This may not be very satisfactory. Suppose now that we can store *all* dyadic numbers (something impossible) in a computer. We still would not be able to store the number $\frac{1}{3}$, since $\frac{1}{3}$ is not a dyadic number (see Exercise 13.20).

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Proposition 114 *Let* $\mathfrak{b} \geq 2$ *be a natural number, and let*

$$\mathfrak{b} = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s} \tag{2.1}$$

be the (unique up to reordering) prime-number factorization of $\mathfrak{b}$ (see Proposition 8). *Then a proper fraction* α *has a finite expansion in the base* $\mathfrak{b}$ *if, and only if,* $\alpha = p/q$, *where* $p, q \in \mathbb{Z}$, $q \neq 0$, *and the prime-number factorization of* q *uses only prime numbers in (2.1).*

Proof Assume that α has a finite expansion in the base $\mathfrak{b}$. Then, for some $m \in \mathbb{N}$ and integers $n_1, \ldots, n_m$,

$$\begin{aligned}\alpha &= \frac{n_1}{\mathfrak{b}} + \frac{n_2}{\mathfrak{b}^2} + \cdots + \frac{n_m}{\mathfrak{b}^m} \\ &= \frac{n_1 \mathfrak{b}^{m-1} + n_2 \mathfrak{b}^{m-2} + \cdots + n_m}{\mathfrak{b}^m} = \frac{L}{p_1^{mk_1} \cdots p_s^{mk_s}},\end{aligned}$$

where $L \in \mathbb{Z}$.
Conversely, assume that, for some $p \in \mathbb{Z}$, and $l_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \ldots, s$, we have

$$\alpha = \frac{p}{p_1^{l_1} \cdots p_s^{l_s}}.$$

Choose $\{n_1, \cdots, n_s\}$ so that

$$\left(p_1^{l_1} \cdots p_s^{l_s}\right)\left(p_1^{n_1} \cdots p_s^{n_s}\right) = \mathfrak{b}^N$$

for some natural number N. Clearly we have

$$\alpha = \frac{p\left(p_1^{n_1} \cdots p_s^{n_s}\right)}{\mathfrak{b}^N},$$

so the number α has a finite expansion in the base $\mathfrak{b}$. □

As a particular case of Proposition 114, observe that fractions (between 0 and 1) that have finite decimal expansions are, precisely, those of the form $\frac{p}{q}$, for $q = 2^l 5^k$, where $k, l \in \mathbb{N} \cup \{0\}$.

Corollary 115 *Consider a base* $\mathfrak{b}$, *where* $\mathfrak{b}$ *is a prime number. Then a fraction* α *has a finite expansion in the base* $\mathfrak{b}$ *if, and only if,*

$$\alpha = \frac{p}{\mathfrak{b}^l}, \text{ where } p \in \mathbb{Z} \text{ and } l \in \mathbb{N} \cup \{0\}.$$

Imagine the ideal—impossible—situation where we can store all fractions in a computer. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every irrational number $x \in \mathbb{R}$ there exists a fraction $\frac{p}{q}$ so that $|x - \frac{p}{q}|$ can be made arbitrarily small. There is a price we have to pay: the more accurate the approximation, the larger the denominator of the fraction.

Increasing the size of the denominators of our fractions can be computationally costly; we then need to know, how close we can get to an irrational number with a fixed size of the denominator. The two following results are due to the German mathematician J. P. G. Lejeune Dirichlet.

Lemma 116 (Dirichlet) *Let* $\theta \in \mathbb{R}$ *and* $t \in \mathbb{N}$. *Then there exist integers* p *and* q *so that* $0 < q \leq t$ *and*

$$|q\theta - p| < \frac{1}{t}.$$

Proof For every $k \in \{0, \ldots, t\}$ choose an integer n_k so that

$$0 \leq k\theta - n_k < 1$$

and set $x_k = k\theta - n_k$. Split the interval $[0, 1)$ into t intervals $I_1, I_2, \ldots, I_t$, each of them having length $\frac{1}{t}$:

$$I_1 = \left[0, \frac{1}{t}\right), \; I_2 = \left[\frac{1}{t}, \frac{2}{t}\right), \ldots, \; I_t = \left[\frac{t-1}{t}, 1\right).$$

Having t intervals and $t + 1$ numbers $\{x_k\}$, $k \in \{0, \ldots, t\}$, we conclude that at least one interval I_n contains at least two numbers x_k and x_l for $k \neq l$. Set $q = k - l$ and $p = n_k - n_l$ and observe

$$|q\theta - p| = |(k - l)\theta - (n_k - n_l)| = |x_k - x_l| < \frac{1}{t}.$$

□

In the next result, fractions are not necessarily written in an irreducible form (i.e., numerator and denominator may have common prime factors). However, see Remark 118 in this connection.

Theorem 117 (Dirichlet) *Let* $\theta \in \mathbb{R}$ *and* $Q \in \mathbb{N}$. *Then there exist* p *and* q *so that*

$$\boxed{\left|\theta - \frac{p}{q}\right| < \frac{1}{Qq} \leq \frac{1}{q^2}, \quad p, q \text{ integers}, \; 0 < q \leq Q.} \tag{2.2}$$

Proof From Lemma 116 it follows that, for arbitrary $Q \in \mathbb{N}$, there exist integers p and q so that $0 < q \leq Q$ and

$$|q\theta - p| < \frac{1}{Q}.$$

Therefore

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{Qq} \leq \frac{1}{q^2}.$$

□

As an example, consider the number $x = \sqrt{2}$. Set the upper bound on the denominators of the approximating fractions to be $Q := 6$. The condition $|x - \frac{p}{q}| \leq \frac{1}{qQ}$

(with $q \le Q$) is met just by one fraction, precisely $7/5$, as the reader may check easily. Note that $|\sqrt{2} - 7/5| = 0,01421356\ldots < 1/30 = 0,0\overline{3}$.

Remark 118

1. It is a trivial observation—it will be needed later, see Remark 110— that we can always take, in Theorem 117, an irreducible fraction p/q (i.e., p and q without common prime factors) for the approximation in (2.2). Indeed, if $|\theta - p/q| < 1/(Qq)$, then when we pass to the corresponding irreducible fraction p'/q' we have $0 < q' \le q$, so

$$\left|\theta - \frac{p'}{q'}\right| = \left|\theta - \frac{p}{q}\right| < \frac{1}{Qq} \le \frac{1}{Qq'}, \text{ and } 0 < q' \le q \le Q.$$

2. If θ in Theorem 117 is rational, then we may prove that the number of irreducible expressions p/q that satisfy (2.2), *for Q running through all elements in* $\mathbb{N}$, is finite. Since this fact will be not needed in this text, we omit the proof. However, we shall need later (see Remark 110 and Example 380.2) that, on the contrary, *if θ is irrational, the number of expressions p/q that satisfy (2.2) for Q running through all elements in* $\mathbb{N}$ *is infinite*. This can be proved in the following way (we follow [HaWr75, 11.3]): assume that the number of expressions p/q that satisfy (2.2) with arbitrary Q is finite, say

$$\left\{\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}\right\}. \tag{2.3}$$

Since θ is irrational, there exists $Q \in \mathbb{N}$ such that

$$\left|\theta - \frac{p_k}{q_k}\right| > \frac{1}{Q}, \quad k = 1, 2, \ldots, n. \tag{2.4}$$

For this Q we may find, due to Theorem 117, an expression p/q that satisfies (2.2), i.e.,

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{qQ} \quad \left(\le \frac{1}{Q}\right),$$

so p/q is not in the list (2.3), a contradiction. □

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In Definition 39 we introduced the floor and ceiling functions, and denoted by $\mathrm{fr}\,(x)$ the fractional part of x, i.e., $\mathrm{fr}\,(x) := x - \lfloor x \rfloor$, for all $x \in \mathbb{R}$. Observe that $\mathrm{fr}\,(x) \in [0, 1)$ for all $x \in \mathbb{R}$.

Theorem 119 *Let $\theta \in [0, 1]$. Then the set*

$$A = \{\mathrm{fr}\,(n\theta)\}_{n=1}^{\infty}$$

is dense in $[0, 1]$ *if, and only if, θ is irrational.*

Proof Let $\theta \in [0, 1]$ be a given irrational number. Let $x \in [0, 1]$ be arbitrary and let $\varepsilon \in (0, 1)$. Choose $t \in \mathbb{N}$ large enough so that $\frac{1}{t} < \varepsilon$ (Proposition 24). According to Lemma 116, there exist integers p and q so that

$$\beta := |q\theta - p| < \varepsilon.$$

Since θ is not a rational number we have $\beta \neq 0$. Note that we either have $q\theta - p = \beta$ or $q\theta - p = -\beta$.

Assume first that $q\theta - p = \beta$. Then $\mathrm{fr}\,(q\theta) = \beta$. Choose $n \in \mathbb{N}$ so that $n\beta < 1$ and $(n+1)\beta > 1$ (recall that β is irrational). Note that $\mathrm{fr}\,(kq\theta) = k\beta$ for all $k \in \{1, \dots, n\}$ and thus

$$|\mathrm{fr}\,(k_0 q\theta) - x| \leq \beta$$

for some $k_0 \in \{1, \dots, n\}$. The result then follows.

Assume now that $q\theta - p = -\beta$. Then $\mathrm{fr}\,(q\theta) = 1 - \beta$. Choose $n \in \mathbb{N}$ so that $n\beta < 1$ and $(n+1)\beta > 1$. We have $\mathrm{fr}\,(kq\theta) = 1 - k\beta$ for all $k \in \{1, \dots, n\}$ and thus

$$|\mathrm{fr}\,(k_0 q\theta) - x| \leq \beta$$

for some $k_0 \in \{1, \dots, n\}$. This finishes the proof of this implication.

If, on the contrary, θ is a fraction, then the set A can not be dense. This follows from the fact that if $\theta = p/q$ for some $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $q \neq 0$, then $q\theta \in \mathbb{Z}$, so

$$\mathrm{fr}\,((q + n)\theta) \in \{\mathrm{fr}\,(\theta), \mathrm{fr}\,(2\theta), \mathrm{fr}\,(3\theta), \dots, \mathrm{fr}\,(q\theta)\}, \text{ for all } n \in \mathbb{N}.$$

□

Taking for granted the continuity of the trigonometric functions $\sin x$ and $\cos x$ (see Sect. 5.2.5), Theorem 119 implies the density in the unit circle of the set $\{(\cos n\theta, \sin n\theta) : n \in \mathbb{N}\}$ whenever θ is an irrational multiple of π. See Exercise 13.86.

2.2 Sequences

2.2.1 Basics on Sequences

A *sequence* of real numbers is a mapping s from $\mathbb{N}$ into $\mathbb{R}$. It is customary, instead of writing $s(n)$ for the image of the element $n \in \mathbb{N}$, to use just s_n. Then we represent a sequence as a *list* in the form $\{s_1, s_2, s_3, \dots\}$, also denoted just by $\{s_n\}_{n=1}^{\infty}$. If there is no risk of misunderstanding, we will write $\{s_n\}$ instead[1].

[1] Note that we consider only infinite sequences, if nothing is said on the contrary. This does not mean that the *range* of the sequence should be necessarily an infinite set. For example, a *constant* sequence $\{1, 1, 1, \dots\}$ is a perfectly acceptable sequence.

Definition 120 We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is *bounded above (bounded below) (bounded)* if the set $\{x_n : n \in \mathbb{N}\} \subset \mathbb{R}$ is bounded above (respectively, bounded below) (respectively, bounded).

The following are examples of sequences of real numbers:

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \left(= \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}\right),\ \{\sqrt[n]{n}\}_{n=1}^{\infty}\ (= \{1, \sqrt{2}, \sqrt[3]{3}, \dots\}), \{1, 0, 1, 0, 1, \dots\}.$$

Definition 121 below is central in Analysis. It is a typical ε-versus-n_0 game. Let us first explain the main idea in the mechanism behind. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. We want to convey the idea that *it approaches some point* $x \in \mathbb{R}$, that the approximation turns out to be *as good as we wish if we let the sequence "run"*, and that after some moment, if the approximation is good, *it will be good later on as well*. All this may be expressed in the following quantitative way, that has a "two player game flavor":

Player 1 plays positive real numbers ε, and player 2 plays in response a natural number n having a previously fixed property. Player 1 wins (and then the sequence $\{x_n\}_{n=1}^{\infty}$ does not converges to x) if he/she can produce $\varepsilon > 0$ such that no answer n from player 2 fits. Player 2 wins (and then the sequence converges to x) if he/she has a strategy that produces n *for any* $\varepsilon > 0$ (the natural number n depends on ε).

More precisely: Player 1 starts by playing some positive number ε. Player 2, in response, plays $n_\varepsilon \in \mathbb{N}$ (we use this notation to stress that n depends on ε) in such a way that for all $n \geq n_\varepsilon$, $|x_n - x| < \varepsilon$. Player 1 plays another ε (he/she tries to beat player 2—i.e., so that player 2 cannot find the corresponding number n_ε—so he/she is interested in playing a small positive number). No way. The second player produces another $n_\varepsilon \in \mathbb{N}$ (probably much bigger that the first one) having the same property: for $n \geq n_\varepsilon$, x_n is ε-close to x. If the game continues forever, in the sense that no $\varepsilon > 0$ played by the first player can make the second player to surrender (in other words, if the second player can *always* provide the $n_\varepsilon \in \mathbb{N}$ needed—the second player has a "winning strategy"), we say that *the sequence* $\{x_n\}_{n=1}^{\infty}$ *converges to* x.

Let us formalize precisely this as a definition, a most fundamental and truly ingenious one, essentially due to the Czech mathematician B. Bolzano and the French mathematician A. L. Cauchy.

Definition 121 A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is said to *converge to a real number* x if

for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\lvert x_n - x\rvert < \varepsilon$ for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

If this is the case, we write

$$\boxed{\lim_{n\to\infty} x_n = x,}$$

in short,

$$\boxed{\lim_n x_n = x, \quad \text{or} \quad \lim x_n = x, \quad \text{or even} \quad x_n \to x,}$$

and we say that x *is the limit of the sequence* $\{x_n\}_{n=1}^{\infty}$—equivalently, that the sequence $\{x_n\}_{n=1}^{\infty}$ *tends to* x. A sequence $\{x_n\}_{n=1}^{\infty}$ that has a limit in $\mathbb{R}$ is said to be *convergent*. Otherwise, the sequence is said to be *divergent*.

Example 122 As a first example, let us define a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way: Put $x_n = 10^{-3}$ if n is odd, otherwise $x_n = 0$. The question is whether $\{x_n\}_{n=1}^{\infty}$ approaches 0. Player 1 plays $\varepsilon = 1/2$; player 2 replies by *any* n_ε. Player 1 plays $1/3$. Player 2 again may play *any* n_ε. The game continues until player 1 plays $\varepsilon = 10^{-3}$. Then the game ends, since there is no choice of $n_\varepsilon \in \mathbb{N}$ that will satisfy $|x_n - 0| < 10^{-3}$ for all $n \geq n_\varepsilon$. This shows that the sequence $\{x_n\}_{n=1}^{\infty}$ *does not* converge to 0. ◇

Remark 123

1. Note that the statement $|x_n - x| < \varepsilon$ in Definition 121 is equivalent to $x_n \in (x - \varepsilon, x + \varepsilon)$.
2. A terminological remark: given a sequence $\{x_n\}_{n=1}^{\infty}$, we say that a property of the terms x_n occurs *eventually* whenever there exists $N \in \mathbb{N}$ such that the property holds for all $n \geq N$. We say that the property occurs *frequently* in case that, for every $N \in \mathbb{N}$, there exists $n \geq N$ such that the property holds for x_n.

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It is important to realize that the concept of limit of a sequence is not ambiguous. Precisely, we have the following result.

Proposition 124 *If a sequence in* $\mathbb{R}$ *is convergent, its limit is unique.*

Proof Assume that a sequence $\{x_n\}_{n=1}^{\infty}$ has two limits, say x and y. Then, given $\varepsilon > 0$, we can find n_0 and n_1 such that, for $n \geq n_0$, $|x - x_n| < \varepsilon/2$, and for $n \geq n_1$, $|y - x_n| < \varepsilon/2$. Then, for $n = \max\{n_0, n_1\}$, we have $|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $x = y$. □

Remark 125 The reader will find that computations concerning limits in many of the subsequent arguments give that a certain quantity is less than $K\varepsilon$, where K is a given fixed constant, and ε is an arbitrary number. Thanks to the arbitrariness of ε and the fact that K is a constant, fixed along the entire argument, the reader can safely substitute $K\varepsilon$ by ε and derive the conclusion. Thus, for example, $\lim x_n = 0$ if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_n| < 10\varepsilon$ for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and $\lim x_n = 0$ in turns implies that for every $\varepsilon > 0$ there exists $n'_\varepsilon > 0$ such that $|x_n| < \varepsilon/6$ for all $n \in \mathbb{N}$, $n \geq n'_\varepsilon$. ®

Proposition 126 *We have*

$$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0.}$$

Proof Let $\varepsilon > 0$. Choose $n_\varepsilon \in \mathbb{N}$ so that $n_\varepsilon > 1/\varepsilon$ (this follows from Proposition 24), hence $1/n_\varepsilon < \varepsilon$. If $n \geq n_\varepsilon$ then $\frac{1}{n} \leq \frac{1}{n_\varepsilon}$, thus

$$\left|\frac{1}{n} - 0\right| \leq \left|\frac{1}{n_\varepsilon} - 0\right| < \varepsilon, \text{ if } n \geq n_\varepsilon.$$

According to the definition, this means that $1/n \to 0$. □

Example 127 The sequence $\{x_n\}_{n=1}^{\infty}$, where $x_n = (-1)^n$ for $n \in \mathbb{N}$, is not convergent. Indeed, assume that it converges to some $x \in \mathbb{R}$. Put $\varepsilon = \frac{1}{2}$. Let n_ε be such that $|x_n - x| < \varepsilon = \frac{1}{2}$ for every $n \geq n_\varepsilon$. If $n \geq n_\varepsilon$, then $|x_n - x_{n+1}| \leq |x_n - x| + |x - x_{n+1}| \leq 2.\left(\frac{1}{2}\right) = 1$, which is not true since for every $n \in \mathbb{N}$ we have $|x_n - x_{n+1}| = 2$. ◇

It is important to note that all topological concepts on $\mathbb{R}$ —i.e., concepts like open or closed sets, neighborhoods, etc.— can be described by using sequences. This is a consequence of the fact that the closed subsets of $\mathbb{R}$ (and then their complements, the open subsets of $\mathbb{R}$) can be characterized by sequences in the following way.

Proposition 128 *A subset F of $\mathbb{R}$ is closed if, and only if, whenever a sequence $\{x_n\}_{n=1}^{\infty}$ in F converges to $x \in \mathbb{R}$, then necessarily $x \in F$.*

Proof If F is closed, then F^c is open. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence if F that converges to some $x \in \mathbb{R}$. Assume that $x \in F^c$. There exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset F^c$. The sequence $\{x_n\}_{n=1}^{\infty}$ is in $(x - \varepsilon, x + \varepsilon)$ $(\subset F^c)$ for n big enough, a contradiction.

Assume now that F is not closed. Then F^c is not open. Thus, there exists $x \in F^c$ such that $(x - 1/n, x + 1/n) \cap F \neq \emptyset$ for every $n \in \mathbb{N}$. We can choose then $x_n \in (x - 1/n, x + 1/n) \cap F$ for $n \in \mathbb{N}$. The sequence $\{x_n\}_{n=1}^{\infty}$ is in F and converges to x ($\notin F$). Indeed, fix $\varepsilon > 0$. Find $n_\varepsilon \in \mathbb{N}$ such that $1/n_\varepsilon < \varepsilon$ (this is possible thanks to Proposition 126, see also Proposition 24). For $n \geq n_\varepsilon$ we get $1/n \leq 1/n_\varepsilon < \varepsilon$, hence $|x - x_n| < \varepsilon$ for $n \geq n_\varepsilon$ (see Remark 123.1). This shows that $x_n \to x$. □

Proposition 129 *Every convergent sequence in $\mathbb{R}$ is bounded.*

Proof Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ that converges to x. Fix $\varepsilon = 1$ and find $n_1 \in \mathbb{N}$ such that, if $n \geq n_1$, then $|x_n - x| < 1$. This gives $|x_n| \leq |x| + 1$ for $n \geq n_1$. Thus $|x_n| \leq \max\{|x_1|, \ldots, |x_{n_1}|, |x| + 1\}$ for all $n \in \mathbb{N}$. □

Remark 130 There is a notational device that may help to shorten some expressions. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Assume that given any natural number N we can find $n_N \in \mathbb{N}$ such that $x_n \geq N$ for all $n \geq n_N$. In this case, we write $\lim_{n\to\infty} x_n = +\infty$. Observe that, according to Definition 121, the sequence $\{x_n\}_{n=1}^{\infty}$ *does not converge* (see also Proposition 129). In a similar way, if given $N \in \mathbb{N}$ there exists $n_N \in \mathbb{N}$ such that $x_n \leq -N$ for all $n \geq n_N$, we write $\lim_{n\to\infty} x_n = -\infty$, despite that the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge. ®

In order to prove Corollary 132 below, we will need a simple yet useful inequality, named after the Swiss mathematician D. Bernoulli.

Lemma 131 **(Bernoulli's inequality)** *Let $x \in \mathbb{R}$ and $n \geq 2$ be a positive integer. If $x > -1$ and $x \neq 0$ we have*

$$\boxed{(1 + x)^n > 1 + nx.} \tag{2.5}$$

Proof Given $x \in \mathbb{R}$, $x > -1$, and $x \neq 0$, we proceed by induction on $n \in \mathbb{N}$, starting with $n = 2$. In this case we get

$$(1 + x)^2 = x^2 + 2x + 1 > 1 + 2x,$$

so (2.5) holds for $n = 2$. Assume now that the inequality holds for some $n \geq 2$. Since $(1 + x) > 0$ and $x \neq 0$, we have

$$(1+x)^{n+1} = (1+x)(1+x)^n > (1+x)(1+nx) = 1+(n+1)x+nx^2 > 1+(n+1)x$$

and the inequality holds for $n + 1$. By the finite induction principle, we get that the inequality holds for every $n \in \mathbb{N}$, $n \geq 2$. □

Corollary 132 *For $x \in \mathbb{R}$, the sequence $\{x^n\}_{n=1}^{\infty}$ converges if, and only if, $x \in (-1, 1]$. If $|x| < 1$, then $x^n \to 0$.*

Proof If $x = 1$, the sequence $\{x^n\}_{n=1}^{\infty}$ obviously converges (to 1).

If $x = -1$, the sequence does not converge (see Example 127). For another argument, see the example after Proposition 140.

If $x = 0$, again the sequence is obviously convergent (to 0). Assume that $0 < |x| < 1$. Then $y := 1/|x| > 1$. Write $y = (1+\varepsilon)$, where $\varepsilon > 0$. Then, by Lemma 131, we have $y^n = (1 + \varepsilon)^n > 1 + n\varepsilon$ for all $n \in \mathbb{N}$, $n \geq 2$. This shows that given $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $y^n > k$ for every $n \geq n_0$. Then $|x^n| = 1/y^n < 1/k$ for every $n \geq n_0$. Since, by Proposition 126, $\{1/k\}_{k=1}^{\infty} \to 0$, we obtain $|x^n| \to 0$, hence $x^n \to 0$. Finally, if $|x| > 1$, we have $|x| = 1 + \varepsilon$ for some $\varepsilon > 0$. Again by Lemma 131, $|x^n| = |x|^n = (1 + \varepsilon)^n > 1 + n\varepsilon$, hence $\{x^n\}_{n=1}^{\infty}$ is unbounded, and so, by Proposition 129, $\{x_n\}_{n=1}^{\infty}$ does not converge. □

Proposition 133 *Consider two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ that converge to limits A and B in $\mathbb{R}$, respectively. Then we have*

(i) $\lim_{n\to\infty} (x_n + y_n) = A + B$.
(ii) $\lim_{n\to\infty} x_n y_n = AB$.
(iii) $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{A}$, *provided $x_n \neq 0$ for all $n \in \mathbb{N}$, and $A \neq 0$.*

Proof Let $\varepsilon > 0$. Let $n_\varepsilon \in \mathbb{N}$ be large enough so that, if $n \geq n_\varepsilon$, then

$$|x_n - A| < \varepsilon \text{ and } |y_n - B| < \varepsilon.$$

It follows that

$$|(x_n + y_n) - (A + B)| \leq |x_n - A| + |y_n - B| < \varepsilon + \varepsilon = 2\varepsilon, \text{for } n \geq n_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(x_n + y_n) \to A + B$, and this proves (i).

Recall now that every convergent sequence in $\mathbb{R}$ is bounded (see Proposition 129). In particular, there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Then, for $n \geq n_\varepsilon$,

$$\begin{aligned}|x_n y_n - AB| &= |x_n y_n - x_n B + x_n B - AB| \\ &\leq |x_n(y_n - B)| + |B(x_n - A)| \leq M\varepsilon + |B|\varepsilon = \varepsilon(M + |B|).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude (see Remark 125) then that $x_n y_n \to AB$, and this proves (ii).

Let $\varepsilon > 0$ be such that $\varepsilon < |A|/2$. Observe that, for $n \geq n_\varepsilon$,

$$|x_n| = |A - (A - x_n)| \geq |A| - |A - x_n| \geq |A| - |A|/2 = |A|/2.$$

Then, for $n \geq n_\varepsilon$,

$$\left|\frac{1}{x_n} - \frac{1}{A}\right| = \frac{|x_n - A|}{|x_n A|} \leq \frac{2|x_n - A|}{|A|^2} < \frac{2\varepsilon}{|A|^2}.$$

Since $0 < \varepsilon < |A|/2$ is, otherwise, arbitrary, we conclude (see again Remark 125) that $1/x_n \to 1/A$. □

Definition 134 We say a sequence $\{x_n\}_{n=1}^\infty$ is *increasing (decreasing)* if $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$ (respectively, $x_n \geq x_{n+1}$ for every $n \in \mathbb{N}$). We say that $\{x_n\}_{n=1}^\infty$ is *strictly increasing (strictly decreasing)* if $x_n < x_{n+1}$ for every $n \in \mathbb{N}$ (respectively $x_n > x_{n+1}$ for every $n \in \mathbb{N}$).

Theorem 135 *Every increasing (decreasing) and bounded above (respectively, bounded below) sequence in $\mathbb{R}$ is convergent (to the supremum (respectively, infimum) of its values). If a sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{R}$ is increasing (decreasing) and unbounded, then we have $\lim_{n\to\infty} x_n = +\infty$ (respectively, $\lim_{n\to\infty} x_n = -\infty$), see Remark 130.*

Proof Consider a sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{R}$ that is increasing and bounded above. The set $\{x_n : n \in \mathbb{N}\}$ is bounded above. Let $x := \sup\{x_n : n \in \mathbb{N}\}$. It exists thanks to Theorem 45. We shall show that

$$\lim_{n\to\infty} x_n = x.$$

To this end, let $\varepsilon > 0$ be an arbitrary positive number. By the definition of the supremum, there exists n_ε so that

$$|x - x_{n_\varepsilon}| < \varepsilon.$$

Since the sequence is increasing we have $x_{n_\varepsilon} \leq x_n \leq x$ for all $n \geq n_\varepsilon$, hence

$$|x - x_n| = x - x_n \leq x - x_{n_\varepsilon} = |x - x_{n_\varepsilon}| < \varepsilon \text{ for all } n \geq n_\varepsilon,$$

and thus $\lim_{n\to\infty} x_n = x$. The situation in which the sequence is decreasing is similar.

If the sequence is increasing and unbounded, given $r \in \mathbb{R}$ there exists $n_r \in \mathbb{N}$ such that $x_{n_r} > r$. Since the sequence is increasing, $r < x_{n_r} \leq x_n$ for all $n \geq n_r$, and this shows that $x_n \to +\infty$. The remaining case is treated similarly. □

If a sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{R}$ is increasing and converges to some $x \in \mathbb{R}$, we shall write $x_n \uparrow x$. Analogously, we shall write $x_n \downarrow x$ for a sequence $\{x_n\}_{n=1}^\infty$ that decreases and converges to x.

Remark 136 The property exhibited in Theorem 135 is equivalent to the completeness of $\mathbb{R}$ (see Theorem 1074). ®

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