Chapter 2
Representations of Quivers

Abstract In this chapter a new language is introduced to study the examples of matrix problems: that of representations of quivers. This approach leads naturally to a more sophisticated language known as categories and functors and large part of the chapter is devoted to the development of this new language. The benefit of it will be that the list of "normal forms" will be enhanced by some internal structure. At the end a the important example of a linear quiver is studied.

2.1 Quivers

Look again at the examples of Chap. 1. In the two subspace problem, we considered pairs of matrices (A, B) with the same number of rows under a certain equivalence relation. Identifying matrices with linear maps, we have thus considered diagrams

$$\begin{array}{ccc} K^{n'} & & K^{n''} \\ {\scriptstyle A}\searrow & & \swarrow{\scriptstyle B} \\ & K^{m} & \end{array} \tag{2.1}$$

and were trying to find "good bases" for the involved vector spaces. The **dual problem** (see Exercise 1.3.2) corresponds to diagrams of the form

$$\begin{array}{ccc} K^{n'} & & K^{n''} \\ {\scriptstyle A}\nwarrow & & \nearrow{\scriptstyle B} \\ & K^{m} & \end{array} .$$

In the Kronecker problem, we considered two matrices of the same size and in the three Kronecker problem three matrices of the same size under the equivalence

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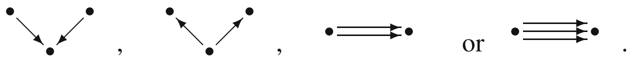
relation of simultaneous row transformations and simultaneous column transformations. Thus $[A|B]$ and $[A|B|C]$ corresponds to two and three "parallel" linear maps, respectively:

Again, the equivalence relation is given by arbitrary change of basis in the two vector spaces. Thus, these four matrix problems, are encoded by a simple diagram



(2.2)

The matrix problem may be recovered, by replacing the vertices by vector spaces with basis, the arrows by linear maps (between the corresponding vector spaces) and considering two corresponding tuples of matrices as equivalent if one can be obtained from the other by change of basis. The diagrams we consider are therefore oriented graphs, where loops and multiple arrows are explicitly allowed. We formalize this in the following.

A **quiver** is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are sets and s, t are two maps $Q_1 \to Q_0$. The elements of Q_0 are called **vertices**, the elements of Q_1 are called **arrows**. The vertices $s(\alpha)$ and $t(\alpha)$ are called the **starting vertex** respectively the **terminating vertex** of the arrow α. We also say that α **starts in** $s(\alpha)$ and **ends in** $t(\alpha)$. A quiver Q is **finite** if Q_0 and Q_1 are finite sets.

Examples 2.1 **(a)** The quiver on the left in (2.2) is called **two subspace quiver**.

(b) The third quiver from the left in (2.2) is called **Kronecker quiver**.

(c) The quiver on the right in (2.2) is called **three Kronecker quiver**. More generally, the n**-Kronecker quiver** consists of two vertices $1, 2$ and $n > 1$ arrows, which have 1 as starting vertex and 2 as terminating vertex. ◇

Usually, in examples, we will have $Q_0 = \{1, \ldots, n\}$. For arrows α with $s(\alpha) = i$ and $t(\alpha) = j$, we usually write. $\alpha: i \to j$. In general, we denote by $\mathbb{N}^{Q_0}$ the set of all functions $Q_0 \to \mathbb{N}$, thus, if $Q_0 = \{1, \ldots, n\}$ then $\mathbb{N}^{Q_0} = \mathbb{N}^n$.

To each quiver we can associate a matrix problem (the contrary is false, see Comment 2.6 (b) at the end of Sect. 2.2). Let Q be a finite quiver. Then the **matrix problem associated to** Q is the pair $(\mathscr{M}_Q, \sim_Q)$ where

$$\mathscr{M}_Q = \bigcup_{d \in \mathbb{N}^{Q_0}} \mathscr{M}_{Q,d}, \quad \mathscr{M}_{Q,d} = \{(M_\alpha)_{\alpha \in Q_1} \mid M_\alpha \in K^{d_{t(\alpha)} \times d_{s(\alpha)}}\}$$

and $(M_\alpha)_\alpha \sim_Q (N_\alpha)_\alpha$ if and only if there exists a family $(U_i)_{i\in Q_0}$ of invertible matrices such that

$$N_\alpha = U_{t(\alpha)} M_\alpha U_{s(\alpha)}^{-1} \tag{2.3}$$

for each arrow $\alpha \in Q_1$.

For each $M \in \mathscr{M}_{Q,d}$ the vector $d \in \mathbb{N}^{Q_0}$ is called the **dimension vector** of M and shall be denoted by $d = \underline{\dim}\, M$.

Exercises

2.1.1 Draw the diagram of the quiver Q given by the sets $Q_0 = \{1, 2, 3, 4, 5\}$, $Q_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, and the functions $s(\alpha_i) = i$ and $t(\alpha_i) = i + 1$ for $i = 1, \ldots, 4$.

2.1.2 Draw the quiver corresponding to the three subspace problem, described in Exercise 1.3.3. This quiver is called the **three subspace quiver**.

2.1.3 Let Q be the quiver $1 \overset{\alpha}{\to} 2$. Solve the corresponding matrix problem $(\mathscr{M}_Q, \sim_Q)$, that is, determine the indecomposables.

2.1.4 Determine the dimension vectors of the indecomposables in the Kronecker problem.

2.2 Representations

We will fix the ground field K and omit the dependence on K in our notation if no confusion can arise.

A **representation** of a quiver Q is a pair

$$V = \big((V_i)_{i\in Q_0}, (V_\alpha)_{\alpha\in Q_1}\big)$$

of two families: the first, indexed over the vertices of Q, is a family of finite-dimensional vector spaces and the second, indexed over the arrows of Q, consists of linear maps $V_\alpha \colon V_{s(\alpha)} \to V_{t(\alpha)}$.

The **zero representation**, denoted by 0, is the unique family with $V_i = 0$ (the zero vector space) for each $i \in Q_0$.

It is common to write a representation "graphically" by replacing each vertex i by the vectorspace V_i and each arrow $\alpha \colon i \to j$ by the linear map $V_\alpha \colon V_i \to V_j$. The **dimension vector** of a representation V is the vector $(\dim V_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$.

Example 2.2 The indecomposable elements of the Kronecker problem with square matrices are $[\mathbf{1}_m|J(m,\lambda)]$ for $\lambda \in K$ and $[J(m,0)|\mathbf{1}_m]$. They correspond to the following indecomposable representations of the Kronecker quiver

$$V_m \underset{X+\lambda}{\overset{1}{\rightrightarrows}} V_m \qquad \text{resp.} \qquad V_m \underset{1}{\overset{X\,,}{\rightrightarrows}} V_m$$

where $V_m = K[X]/(X^m)$. ◇

Let V and W be two representations of a finite quiver Q. A **morphism** from V to W is a family of linear maps $f = (f_i : V_i \to W_i)_{i \in Q_0}$ such that for each arrow $\alpha : i \to j$ we have

$$f_j V_\alpha = W_\alpha f_i. \tag{2.4}$$

We denote a morphism just like a function, that is, we write $f : V \to W$ to indicate that f is a morphism from V to W. Observe that Eq. (2.4) states that the following diagram commutes:

$$\begin{array}{ccc} V_i & \xrightarrow{f_i} & W_i \\ {\scriptstyle V_\alpha}\downarrow & & \downarrow{\scriptstyle W_\alpha} \\ V_j & \xrightarrow[f_j]{} & W_j \end{array} .$$

A morphism is an **isomorphism** if each f_i is invertible and we say that V and W are **isomorphic** representations if there exists an isomorphism from V to W.

A **basis** of a representation V is a family $(B_i)_{i \in Q_0}$ where B_i is a basis of the space V_i for each vertex $i \in Q_0$. Each such basis yields a family of matrices $V^B = (V_\alpha^B)_{\alpha \in Q_1}$ where V_α^B represents the linear map V_α in the bases $B_{s(\alpha)}$ and $B_{t(\alpha)}$.

Proposition 2.3 *Let V and W be two representations of a finite quiver Q. Then V is isomorphic to W if and only if for some (any) basis B of V and some (any) basis C of W we have that V^B and W^C are equivalent elements of the matrix problem associated to Q.*

Proof Notice that by chosing bases, we translate the linear invertible map f_i into an invertible matrix U_i. Condition (2.4) corresponds then to (2.3). □

Let V and W be two representations of a quiver Q. The direct sum $V \oplus W$ is then defined as the representation given by the spaces $(V \oplus W)_i = V_i \oplus W_i$ and the linear maps $(V \oplus W)_\alpha = V_\alpha \oplus W_\alpha$, which are defined componentwise. We denote $V \oplus V$ as a power by V^2 and inductively V^i for lager exponents i.

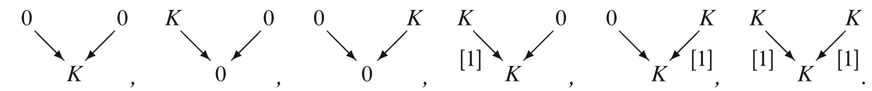
A representation V is **indecomposable** if and only if $V \neq 0$ and it is impossible to find an isomorphism $V \xrightarrow{\sim} V' \oplus V''$ for any non-zero representations V' and V''.

Proposition 2.4 *Let V be a representation of a finite quiver Q. Then V is indecomposable if and only if V^B is indecomposable for some (any) basis B of V.*

Proof This an immediate consequence of the definitions and Proposition 2.3. □

We thus achieved a perfect translation. Solving one of the matrix problems above corresponds to *classifying the indecomposable representations up to isomorphism.* For instance, we get the following result.

Proposition 2.5 *Each indecomposable representation of the quiver corresponding to the two subspace problem is isomorphic to precisely one representation of the following list.*

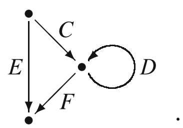


Comments 2.6 (a) Observe that the strange matrices of the two subspace problem occurring in (1.3) correspond to natural representations.

(b) Notice that not every matrix problem we considered admits such a straightforward translation. For instance, in the coupled four-block problem of Sect. 1.4 we looked at quadruples of matrices

$$\begin{bmatrix} C & D \\ E & F \end{bmatrix},$$

where the row and column transformations for D are coupled by conjugation. The corresponding quiver would look as follows (where we indicated the places of the matrices):



This quiver defines a wild case and does not correspond to our original matrix problem, since we have not expressed in our new language that we can add rows from the lower stripe to the upper stripe nor that we can add columns from the left to the right stripe.

Exercises

2.2.1 Write the matrices given in Proposition 1.8 as representations of the corresponding quiver.

2.2.2 Use the spaces V_n and V_{n+1} of Example 2.2 to write those indecomposable representations of the Kronecker quiver which were not already given in the Example.

2.2.3 Decompose the following representation into indecomposables

$$\begin{array}{ccccc} & K^2 & & K^2 & \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & & \searrow \quad \swarrow & & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ & & K^2 & & \end{array}$$

2.3 Categories and Functors

We shall briefly explain the language of categories and functors, since it provides a general language for the different concepts we shall encounter. If you are already familiar with categories and functors you can skip all of this section except the examples and read on in Sect. 2.4.

A **category** $\mathscr{C}$ is a class of **objects** (which we usually denote by the same letter as the whole category) together with a family of sets $\mathscr{C}(x, y)$ whose elements are called **morphisms** (one set for each pair of objects $x, y \in \mathscr{C}$) together with a family of **composition maps** $\mathscr{C}(y,z) \times \mathscr{C}(x, y) \to \mathscr{C}(x,z), (g, f) \mapsto g \circ f$ (one for each triple of objects $x, y, z \in \mathscr{C}$) such that for each object $x \in \mathscr{C}$ there exists an **identity morphism** $1_x \in \mathscr{C}(x, x)$, that is an element which satisfies $1_x \circ f = f$ and $g \circ 1_x = g$ for any $f \in \mathscr{C}(w, x)$, $g \in \mathscr{C}(x, y)$, any $w, y \in \mathscr{C}$ and such that the composition is associative, that is $(h \circ g) \circ f = h \circ (g \circ f)$ for any $f \in \mathscr{C}(w, x)$, $g \in \mathscr{C}(x, y)$, $h \in \mathscr{C}(y, z)$, any $w, x, y, z \in \mathscr{C}$.

This is a long definition! Intuitively, a category is something similar to what you obtain when you throw all sets and all maps between all these sets into one big bag called Set, the category of sets.

To summarize: There are objects, which form a class; between any two objects there is a set of morphisms (possibly the empty set); morphisms may be composed and the composition is associative; and there are identity morphisms. We usually write $f : x \to y$ for a morphism $f \in \mathscr{C}(x, y)$ to remind us of the similarity with maps. We also often omit the composition symbol and write gf instead of $g \circ f$.

Examples 2.7 **(a)** The category Set has as objects the class of all sets and as morphisms just all maps. The composition of morphisms in the category is just the composition of maps and the identity morphisms are the identity maps.

(b) The category Vec has as objects K-vector spaces with the linear maps as morphisms. The composition of morphisms and the identity morphism are again the obvious ones. The category vec has as objects the class of finite-dimensional K-vector spaces, again with the linear maps as morphisms.

(c) The category Top has the topological spaces as objects and continuous functions as morphisms. ◇

As you see you can take for the objects all representatives of a fixed algebraic structure like topological spaces, rings, groups, abelian groups, finitely generated abelian groups and so on and so on. For the morphisms you take the structure preserving maps between them, and for the composition just the composition of maps. You always get a category. Let us look now at some more and stranger categories.

Examples 2.8 **(a)** Let Q be a finite quiver. We will see that $\mathscr{M}_Q$ can be viewed as a category. The class of objects is by definition just the set $\mathscr{M}_Q$ itself. If $M, N \in \mathscr{M}_Q$ then let

$$\mathscr{M}_Q(M, N) = \{(U_i)_{i \in Q_0} \mid \forall \alpha \in Q_1, N_\alpha U_{s(\alpha)} = U_{t(\alpha)} M_\alpha\}.$$

Observe that the condition $N_\alpha U_{s(\alpha)} = U_{t(\alpha)} M_\alpha$ is the same as (2.3) except that we do not require the matrices U_i to be invertible.

It is easy to verify that $\mathscr{M}_Q$ is indeed a category if the composition is given by componentwise matrix multiplication and the identity morphisms are the tuples of identity matrices. However, morphisms are clearly not functions between two sets in this example.

(b) The category rep Q has as objects the representations of Q and as morphisms just the morphisms of representations. The composition is given by componentwise composition of linear functions and the identity morphisms are given by tuples of identity functions. Note, that as in the example before, morphisms are not given by a single function; in this case they consist of a family of functions satisfying some compatibility property. If V and W are representations of the quiver Q, we denote the morphism set rep $Q(V, W)$ also by $\mathrm{Hom}_Q(V, W)$. ◇

In a category $\mathscr{C}$ a morphism $f: x \to y$ is called an **isomorphism** if there exists a morphism $g: y \to x$ such that $f \circ g = 1_y$ and $g \circ f = 1_x$. Two objects are said to be **isomorphic in** $\mathscr{C}$ if there exists an isomorphism between them.

Example 2.9 In the category $\mathscr{M}_Q$ (see Example 2.8(a)) two objects are isomorphic precisely when they are equivalent. So, the categorical concept of isomorphism has just the right meaning we are interested in. ◇

The following concept will be used to relate different categories among each other.

If $\mathscr{C}$ and $\mathscr{D}$ are categories then a **covariant functor** $F: \mathscr{C} \to \mathscr{D}$ associates to each object $x \in \mathscr{C}$ an object of $Fx \in \mathscr{D}$ and to each morphism $g \in \mathscr{C}(x, y)$ a morphism $Fg \in \mathscr{D}(Fx, Fy)$ such that $F1_x = 1_{Fx}$ for each object $x \in \mathscr{C}$ and such that the composition is preserved, that is $F(h \circ g) = Fh \circ Fg$ for any $g \in \mathscr{C}(x, y)$, $h \in \mathscr{C}(y, z)$, any $x, y, z \in \mathscr{C}$. A **contravariant functor** $F: \mathscr{C} \to \mathscr{D}$ is very similar to a covariant functor except that it inverts the direction of the morphisms, that is $Fg \in \mathscr{D}(Fy, Fx)$ for any $g \in \mathscr{C}(x, y)$ and consequently $F(h \circ g) = Fg \circ Fh$ for any g and h.

We shall meet many functors during the course of this book and limit ourselves here to two simple examples of covariant functors.

Examples 2.10 **(a)** Let Q be a finite quiver. Define a functor $F: \operatorname{rep} Q \to \operatorname{vec}$ by $FV = \bigoplus_{i \in Q_0} V_i$ for any representation V of Q and $Fg = \bigoplus_{i \in Q_0} g_i$ for any morphism of representations g.

(b) Let Q be a finite quiver. Then define a functor $G: \mathscr{M}_Q \to \operatorname{rep} Q$ as follows: for an object M let $(GM)_i = K^{d_i}$, where $d = \underline{\dim}\, M$ is the dimension vector, and $(GM)_\alpha = M_\alpha$. Further, if $U: M \to N$ is a morphism in $\mathscr{M}_Q$, then define $GU = U$, with the abuse of notation that U denotes a matrix as well as the associated linear map in the canonical bases. ◇

If $F: \mathscr{C} \to \mathscr{D}$ and $G: \mathscr{D} \to \mathscr{E}$ are functors then we obtain a functor $GF: \mathscr{C} \to \mathscr{E}$ in the obvious way: $(GF)x = G(Fx)$ for each object x and $(GF)f = G(Ff)$ for each morphism f. The functor GF is called the **composition** of F with G.

If $\mathscr{C}$ is a category, then the functor $1_{\mathscr{C}}: \mathscr{C} \to \mathscr{C}$ defined by $1_{\mathscr{C}}x = x$ and $1_{\mathscr{C}}f = f$ for each object x and morphism f is called the **identity functor** of $\mathscr{C}$.

Two functors $F: \mathscr{C} \to \mathscr{D}$ and $G: \mathscr{D} \to \mathscr{C}$ are called **inverse to each other** if $FG = 1_{\mathscr{D}}$ and $GF = 1_{\mathscr{C}}$. Note that F and G necessarily must have the same variance, that is, they both must be covariant or both contravariant, see Exercise 2.3.4. If F and G are covariant, then they are called **isomorphisms** and the categories $\mathscr{C}$ and $\mathscr{D}$ are called **isomorphic**. If F and G are contravariant then they are called **dualizations** and the categories **dual**.

We will later see that it is rather seldom in practice that two categories are isomorphic, see Sect. 2.5, where we develop a weaker notion called **equivalence**. The last piece of categorical terminology relates two functors.

If $F: \mathscr{C} \to \mathscr{D}$ and $G: \mathscr{C} \to \mathscr{D}$ are two covariant functors, then a **morphism of functors** (or **natural transformation** or just **morphism**) $\varphi: F \to G$ is a family $(\varphi_x)_{x \in \mathscr{C}}$ of morphism $\varphi_x \in \mathscr{D}(Fx, Gx)$ such that $\varphi_y \circ Fh = Gh \circ \varphi_x$ for any morphism $h \in \mathscr{C}(x, y)$.

If $F: \mathscr{C} \to \mathscr{D}$ is a given covariant functor, then the morphism $F \to F$ given by the family $(1_{Fx})_{x\in\mathscr{C}}$ is called **identity morphism** and will be denoted by 1_F. A morphism $\varphi: F \to G$ of covariant functors is an **isomorphism** if there exists a morphism $\psi: G \to F$ such that $\psi\varphi = 1_F$ and $\varphi\psi = 1_G$.

Exercises

2.3.1 Prove that a morphism $f: V \to W$ of representations of a quiver is an isomorphism if and only if for each vertex i the linear map f_i is bijective. Show that in that case the family $(f_i^{-1})_{i\in Q_0}$ of inverse maps constitutes an isomorphism $W \to V$ of representations.

2.3.2 Prove a generalization of the previous exercise, namely, that a morphism $\varphi: F \to G$ of functors $F, G: \mathscr{C} \to \mathscr{D}$ is an isomorphism if and only if for each object $x \in \mathscr{C}$ the morphism $\varphi_x: Fx \to Gx$ is an isomorphism. Show that in that case the family $(\varphi_x^{-1})_{x\in\mathscr{C}}$ is an isomorphism of functors $G \to F$.

2.3.3 Verify carefully that all the properties stated in the definition of a category are satisfied in the two Examples 2.8.

2.3.4 Investigate when the functor GF is covariant and when it is contravariant, depending on the variance of F and G.

2.3.5 Let $F: \mathscr{C} \to \mathscr{D}$ and $G: \mathscr{C} \to \mathscr{D}$ be two contravariant functors. What is the appropriate condition for a family $(\varphi_x)_{x\in\mathscr{C}}$ of morphisms $\varphi_x \in \mathscr{D}(Fx, Gx)$ to be a *morphism of covariant functors*?

2.4 The Path Category

A representation looks very much like a functor $Q \to \text{vec}$ where Q is viewed "as a category" with vertices as objects and arrows as morphisms. But of course this is nonsense, since there are no identity morphisms and no composition of arrows in Q. In the following we will enhance the quiver Q to a proper category. Therefore we will need the concept of paths in Q.

Let Q be a quiver (possibly infinite). A **path of length** l is a $(l+2)$-tuple

$$w = (j|\alpha_l, \alpha_{l-1}, \ldots, \alpha_2, \alpha_1|i) \tag{2.5}$$

where $i, j \in Q_0$ and $\alpha_1, \ldots, \alpha_l \in Q_1$ such that $s(\alpha_1) = i$, $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, l-1$ and $t(\alpha_l) = j$.

We explicitly allow $l = 0$ but require then that $j = i$. The corresponding path $e_i := (i||i)$ is called the **identity path** or **trivial path** in i. The length l of a path w is denoted by $\text{len}(w)$.

We extend the functions s and t in the obvious way: $s(w) = i$ and $t(w) = j$ if w is the path (2.5). A path w of positive length $l > 0$ is called a **cycle** if $s(w) = t(w)$. Cycles are often also called **oriented cycles** in the literature. A cycle of length 1 is called **loop**.

The **composition** of two paths $v = (i|\alpha_l, \ldots, \alpha_1|h)$ and $w = (j|\beta_m, \ldots, \beta_1|i)$ is defined by

$$wv = (j|\beta_m, \ldots, \beta_1, \alpha_l, \ldots, \alpha_1|h).$$

Notice that we defined the composition of paths in the same order as functions, which is not at all standard in the literature, but rather up to the taste of the author. A path $(i|\alpha_l, \ldots, \alpha_1|h)$ will often be denoted by $\alpha_l \alpha_{l-1} \cdots \alpha_1$.

Let Q be a quiver (possibly infinite). The **path category** KQ of Q is the category whose objects are the vertices of Q and the morphisms from i to j form a vector space which has as basis the paths w with $s(w) = i$ and $t(w) = j$. The composition is extended bilinearly from the composition of paths.

At this point we should pause a little and look at the curious fact that we did not define the category of paths having as morphisms *just* the paths, as one might expect first. Indeed that would form a nice category also, but due to reasons which shall become clear in the next chapter, we "linearize" the paths such that we can take sums and multiples.

A category is a **K-category** if its morphism sets are endowed with a K-vector space structure such that the composition is K-bilinear.

A functor $F: \mathscr{C} \to \mathscr{D}$ between K-categories is **K-linear** if $\mathscr{C}(x, y) \to \mathscr{D}(Fx, Fy), h \mapsto Fh$ is K-linear for each pair of objects $x, y \in \mathscr{C}$. If $\mathscr{C}$ is a K-category then $\operatorname{mod}\mathscr{C}$ is the **category of K-linear functors** $\mathscr{C} \to \operatorname{vec}$, that is, the objects of $\operatorname{mod}\mathscr{C}$ are those functors and the morphisms are the morphisms of functors with the obvious composition. For two functors $F, G \in \operatorname{mod}\mathscr{C}$ we write $\operatorname{Hom}_{\mathscr{C}}(F, G)$ for the set of morphisms $(\operatorname{mod}\mathscr{C})(F, G)$.

Example 2.11 The path category KQ is a K-category. Moreover, each representation V of Q defines a K-linear (covariant) functor

$$\tilde{V}: KQ \to \operatorname{vec}.$$

Conversely, any such functor gives rise to a representation of Q. ◇

In a K-category $\mathscr{C}$ the **direct sum** of two objects x and y is defined as object z together with maps

$$x \underset{\iota_x}{\overset{\pi_x}{\leftrightarrows}} z \underset{\iota_y}{\overset{\pi_y}{\rightleftarrows}} y$$

such that $\pi_x \iota_x = \mathrm{id}_x$, $\pi_y \iota_y = \mathrm{id}_y$, $\iota_x \pi_x + \iota_y \pi_y = \mathrm{id}_z$, $\pi_y \iota_x = 0$ and $\pi_x \iota_y = 0$. So, formally a direct sum is a quintuple $(z, \pi_x, \pi_y, \iota_x, \iota_y)$. However, the object z is—up to isomorphism—uniquely determined by x and y, see Exercise 2.4.6. This justifies the common abuse of language to call z itself the direct sum of x and y and denote it as $x \oplus y$.

Example 2.12 If $\mathscr{C} = \operatorname{rep} Q$, the categorical direct sum corresponds to the direct sum defined for representations above. Also in case $\mathscr{C} = \mathscr{M}_Q$ the categorical direct sum corresponds to the direct sum in the language of matrix problems. ◇

Exercises

2.4.1 Verify that the morphisms of representations are precisely the morphisms between covariant K-linear functors $KQ \to \mathrm{vec}$. Show that the category $\mathrm{mod}(KQ)$ is isomorphic to the category $\operatorname{rep} Q$.

2.4.2 If Q denotes the Kronecker quiver:

$$1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \quad ,$$

then there are four morphism spaces in the category KQ. Determine the dimensions of these spaces. Determine $KQ(2, 1)$ as set. How many elements does it have?

2.4.3 Let Q be a finite quiver. Show that different objects of KQ are non-isomorphic.

2.4.4 For a finite quiver Q, prove that all morphism spaces in KQ are finite-dimensional if and only if there is no cycle in the quiver Q.

2.4.5 The **adjacency matrix** A_Q of a quiver Q with vertices $1, \dots, n$ is the matrix of size $n \times n$ whose entry $(A_Q)_{ij}$ is the number of arrows $\alpha \in Q_1$ with $s(\alpha) = j$ and $t(\alpha) = i$. Prove that A_Q is **nilpotent** (that is, there exists some positive integer t such that $A_Q^t = 0$) if and only if there is no cycle in Q. For this, show first, that for each t, the entry $(A_Q^t)_{ij}$ equals the number of paths w of length t with $s(w) = j$, $t(w) = i$.

Conclude from this that in case Q has no cycle then $A_Q^n = 0$, where n is the number of vertices. Furthermore show that the matrix $B = \mathbf{1}_n + A_Q + A_Q^2 + \dots + A_Q^{n-1}$ measures the dimension of the morphism spaces in KQ, namely $B_{ij} = \dim_K (KQ(j, i))$.

2.4.6 Let $\mathscr{C}$ be a K-category. Suppose that the quintuples $(z, \pi_x, \pi_y, \iota_x, \iota_y)$ and $(z', \pi'_x, \pi'_y, \iota'_x, \iota'_y)$ are two direct sums of the objects x and y in $\mathscr{C}$. Proof that z is isomorphic to z'.

2.5 Equivalence of Categories

As we have seen there is a very close relationship between the category of representations rep Q and the category of matrix problems $\mathscr{M}_Q$ associated to Q. In the following we would like to clarify this relationship completely.

We recall that two categories $\mathscr{C}$ and $\mathscr{D}$ are called **isomorphic** if there exist two functors $F: \mathscr{C} \to \mathscr{D}$ and $G: \mathscr{D} \to \mathscr{C}$ such that $GF = 1_{\mathscr{C}}$ and $FG = 1_{\mathscr{D}}$.

However, this notion is in most concrete cases far too restrictive. It is more convenient to look at some slight generalization: two categories $\mathscr{C}$ and $\mathscr{D}$ are called **equivalent** if there exist two functors $F: \mathscr{C} \to \mathscr{D}$ and $G: \mathscr{D} \to \mathscr{C}$ such that $GF \simeq 1_{\mathscr{C}}$ and $FG \simeq 1_{\mathscr{D}}$, that is, if there exists isomorphisms of functors $\varphi: GF \to 1_{\mathscr{C}}$ and $\psi: FG \to 1_{\mathscr{D}}$. In that case the functors F and G are called **equivalences** or **quasi-inverse to each other**.

Proposition 2.13 *Let Q be a finite quiver. Then the categories* rep Q *and $\mathscr{M}_Q$ are equivalent.*

Proof We already have constructed the functor

$$F: \mathscr{M}_Q \longrightarrow \operatorname{rep} Q,$$

as application on the objects, see Example 2.10(b). The definition on morphisms is straightforward.

To define a quasi-inverse G of F we choose a basis B^V for each representation V. Define $n_{V,i} = \dim V_i$ for each representation V and each vertex i. We recall that for each arrow $\alpha: i \to j$ we get a matrix $M_\alpha^V \in K^{n_{V,j} \times n_{V,i}}$ representing the linear map V_α in the bases B_i^V and B_j^V. Moreover, the tuple $G(V) = (M_\alpha^V)_{\alpha \in Q_1}$ defines an object of $\mathscr{M}_Q$. Furthermore, for each morphism $f: V \to W$ of representations of Q we define $G(f) = (U_i^f)_{i \in Q_0}$, where U_i^f is the matrix representing the linear map f_i in the bases B_i^V and B_i^W.

To see that

$$G: \operatorname{rep} Q \longrightarrow \mathscr{M}_Q$$

is a functor we have to verify that G preserves identity morphisms and the composition. Indeed $G(1_V) = 1_{G(V)}$ holds since the morphism 1_V is the family of identity maps $1_{V_i}: V_i \to V_i$ which are expressed as identity matrices, since for both spaces we choose the same basis B_i^V. For the composition, let U, V and W be representations of Q and $f: U \to V$ and $g: V \to W$ be morphisms of representations. Then, for a vertex i of the quiver, $G(f)_i$ is the matrix representing $f_i: U_i \to V_i$ in the basis B_i^U and B_i^V respectively. Similarly, $G(g)_i$ represents g_i in the Basis B_i^V and B_i^W respectively. Therefore $G(g \circ f)_i = G(g)_i G(f)_i$ holds for all i. This shows $G(g \circ f) = G(g)G(f)$ and finishes the proof that G is a functor.

It remains to see that F and G are quasi-inverse to each other. Furthermore, we denote by $\psi_{V,i}\colon K^{n_{V,i}} \to V_i$ the linear map representing the identity matrix in the canonical basis and B_i^V respectively. Note that $K^{n_{V,i}} = FG(V)_i$ holds by definition. To see that the family $\psi_V = (\psi_{V,i})_{i\in Q_0}$ defines a morphism $FG(V) \to V$, we have to verify that for each arrow $\alpha\colon i \to j$ the following diagram commutes:

$$\begin{array}{ccc} FG(V)_i & \xrightarrow{\psi_{V,i}} & V_i \\ {\scriptstyle FG(V)_\alpha}\downarrow & & \downarrow{\scriptstyle V_\alpha} \\ FG(V)_j & \xrightarrow[\psi_{V,j}]{} & V_j \end{array}.$$

We show this by looking at these maps in special bases: for V_i and V_j we chose the bases B_i^V and B_j^V respectively and for $FG(V)_i$ and $FG(V)_j$ we choose the canonical bases. The linear map V_α is then represented by the matrix $G(V)_\alpha = M_\alpha^V$, the maps $\psi_{V,i}$ and $\psi_{V,j}$ by identity matrices and $FG(V)_\alpha$ also by $G(V)_\alpha$. This shows that ψ_V is a morphism of representations. Since for each vertex $\psi_{V,i}$ is bijective, it is an isomorphism, see Exercise 2.3.1.

Thus we have now a family $(\psi_V)_{V\in \operatorname{rep} Q}$ of isomorphisms $\psi_V\colon FG(V) \to V$ of representations. To see that this family constitutes a morphism $FG \to 1_{\operatorname{rep} Q}$ of functors $\operatorname{rep} Q \to \operatorname{rep} Q$ it must be shown that for each morphism $f\colon V \to W$ of representations the following diagram on the left hand side commutes.

$$\begin{array}{ccc} FG(V) & \xrightarrow{\psi_V} & V \\ {\scriptstyle FG(f)}\downarrow & & \downarrow{\scriptstyle f} \\ FG(W) & \xrightarrow[\psi_W]{} & W \end{array} \qquad \begin{array}{ccc} FG(V)_i & \xrightarrow{\psi_{V,i}} & V_i \\ {\scriptstyle FG(f)_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ FG(W)_i & \xrightarrow[\psi_{W,i}]{} & W_i \end{array}$$

By definition, this means that for each vertex i the diagram on the right hand side commutes. Indeed, in the bases B_i^V and B_i^W for V_i and W_i respectively and the canonical bases for $FG(V)_i$ and $FG(W)_i$, the linear maps f_i and $FG(f)_i$ are represented by $G(f)_i$, whereas $\psi_{V,i}$ and $\psi_{W,i}$ are represented by identity matrices.

To see that ψ is an isomorphism of functors we have to give an inverse. For this we use Exercise 2.3.2 to see that for each representation V, the family $\psi_V^{-1} = (\psi_{V,i}^{-1})_{i\in Q_0}$ is an isomorphism of representations which is an inverse of ψ.

To get an isomorphism $GF \to 1_{\mathscr{M}_Q}$ we could proceed very similarly. But we will choose a much simpler way by restricting the choice of the bases for each vector space of the form K^t to be always the canonical basis. It then happens that $GF(M) = M$ for each $M \in \mathscr{M}_Q$. Thus $GF = 1_{\mathscr{M}_Q}$ and φ is the identity morphism.

□

We have intentionally written down all the details in the proof to show how each categorical definition, which is involved, can be brought down in our setting to statements about linear maps and matrices representing them.

Exercises

2.5.1 Let Q be the quiver which has a single vertex and no arrows. Show that vec and rep Q are isomorphic categories. In this sense the study of representations of quivers generalizes linear algebra of finite-dimensional vector spaces.

2.5.2 Let Mat be the category whose objects are the natural numbers (including 0) and whose morphism spaces $\mathrm{Mat}(n, m)$ are the sets $K^{m\times n}$ of matrices of size $m \times n$ and entries in the field K. The composition in Mat is given matrix multiplication. Show that vec and Mat are equivalent categories.

2.6 A New Example

We will consider a new class of problems starting from a family of quivers, which are called **linearly oriented**, and look as follows:

$$Q: \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-2}{\bullet} \xrightarrow{\alpha_{n-2}} \underset{n-1}{\bullet} \xrightarrow{\alpha_{n-1}} \underset{n}{\bullet}$$

We shall denote this quiver by $\overrightarrow{\mathbb{A}}_n$. The following result shows, that the classification problem can be solved completely for $\overrightarrow{\mathbb{A}}_n$.

Theorem 2.14 *Each indecomposable representation of $\overrightarrow{\mathbb{A}}_n$ is isomorphic to a representation*

$$[j, i]: \quad 0 \to \ldots \to 0 \to K \xrightarrow{[1]} \ldots \xrightarrow{[1]} K \to 0 \to \ldots \to 0,$$

where the first (that is, leftmost) occurrence of K happens in place j and the last (that is, rightmost) in place i for some $1 \le j \le i \le n$.

In particular, $\overrightarrow{\mathbb{A}}_n$ is of finite representation type and there are $\frac{n(n+1)}{2}$ indecomposables, up to isomorphism.

Proof Let V be an indecomposable representation of $\overrightarrow{\mathbb{A}}_n$. Let i be the minimal index such that V_{α_i} is not injective and set $i = n$ if no such index exists. Similarly, let j be the maximal index such that $V_{\alpha_{j-1}}$ is not surjective and set $j = 1$ if no such index exists. We shall show that V is isomorphic to $[j, i]$.

If $i < n$ then $V_{\alpha_1}, \dots, V_{\alpha_{i-1}}$ are all injective, but V_{α_i} is not. Then we let S_i be a complement of $L_i = \operatorname{Ker} V_{\alpha_i}$, and set inductively $S_h = V_{\alpha_h}^{-1}(S_{h+1})$, $L_h = V_{\alpha_h}^{-1}(L_{h+1})$ for $h = i-1, i-2, \dots, 1$. Note, that $S_h \oplus L_h = V_h$ for $h = 1, \dots i$. We thus see that V decomposes into

$$(L_1 \to \dots \to L_i \to 0 \to \dots \to 0) \oplus (S_1 \to \dots \to S_i \to V_{i+1} \to \dots \to V_n)$$

and since V is indecomposable and $L_i \neq 0$ the right summand must be zero.

Thus we have shown so far that $V_h = 0$ for $h > i$ and that all maps V_{α_h} are injective for $h < i$. This implies that $j \leq i$. We observe that if $i = n$ then all these statements are trivially true or void.

If $j > 1$ then $V_{\alpha_j}, V_{\alpha_{j+1}}, \dots, V_{\alpha_{i-1}}$ are surjective and hence bijective, but $V_{\alpha_{j-1}}$ is not. Let R_j be a complement of $M_j = V_{\alpha_{j-1}}(V_{j-1})$ and set inductively $M_h = V_{\alpha_{h-1}}(M_{h-1})$, $R_h = V_{\alpha_{h-1}}(R_{h-1})$, for $h = j+1, \dots i$. We therefore conclude that V decomposes into

$$\begin{aligned}(0 \to \dots \to 0 \to R_j \to \dots \to R_i \to 0 \to \dots \to 0) \oplus \\ (V_1 \to \dots \to V_{j-1} \to M_j \to \dots M_i \to 0 \to \dots \to 0).\end{aligned}$$

The indecomposability of V implies now that the latter one is zero, since $R_j \neq 0$.

This shows that $V_h = 0$ for $h < j$ and that V_{α_h} is bijective for $h = j, \dots, i-1$. We observe that in case $j = 1$ all these statements are trivially true or void. Thus V is isomorphic to

$$0 \to \dots \to 0 \to K^d \xrightarrow{\mathbf{1}_d} \dots \xrightarrow{\mathbf{1}_d} K^d \to 0 \to \dots \to 0,$$

where d denotes the dimension of the spaces $V_j, \dots, V_i$. But this representation is isomorphic to the direct sum of d copies of $[j, i]$. By the indecomposability of V it follows that $d = 1$ and that V is isomorphic to $[j, i]$. □

Thus, we have determined the objects of the category $\operatorname{rep} \overrightarrow{\mathbb{A}}_n$: they are, up to isomorphism, direct sums of the representations $[j, i]$. Now we turn our attention to morphisms between representations of $\overrightarrow{\mathbb{A}}_n$. If V and W are two representations, we first write them as direct sum of indecomposable representations, say $V \simeq \bigoplus_{a=1}^{s} V_a$ and $W \simeq \bigoplus_{b=1}^{t} W_b$, where each V_a and each W_b is of the form $[j, i]$ for some $1 \leq j \leq i \leq n$. A morphism $\varphi \colon V \to W$ is then given by a matrix of morphisms $\varphi_{ba} \colon V_a \to W_b$. Therefore, we are reduced to determine the morphisms between two indecomposable representations $[j, i]$ and $[j', i']$.

Lemma 2.15 *The morphism space* $\operatorname{Hom}([j, i], [j', i'])$ *is non-zero if and only if* $j' \leq j \leq i' \leq i$. *Moreover, in that case,* $\operatorname{Hom}([j, i], [j', i'])$ *is one-dimensional.*

Proof Suppose that $\psi\colon [j,i] \to [j',i']$ is a morphism. Clearly, there is always the **zero morphism** with $\psi_h = 0$ for all h. If we suppose that ψ is not identically zero then the two intervals $[j,i]$ and $[j',i']$ must have some intersection, that is, $m = \max(j,j') \le \min(i,i') = M$. For each h with $m \le h \le M$, the map ψ_h is just scalar multiplication with some factor λ_h. But if $m \le h, h+1 \le M$, then the commutative square

$$\begin{array}{ccc} K & \xrightarrow{[1]} & K \\ {\scriptstyle [\lambda_h]}\downarrow & & \downarrow{\scriptstyle [\lambda_{h+1}]} \\ K & \xrightarrow[{[1]}]{} & K \end{array}$$

shows that $\lambda_h = \lambda_{h+1}$. Hence, if there is a non-zero morphism then it is just a non-zero scalar multiple of the morphism $\iota = \iota_{j,i}^{j',i'}\colon [j,i] \to [j',i']$, where $\iota_h = [1]$ for each $h = \max(j,j'), \ldots, \min(i,i')$.

It remains to determine the condition, when a non-zero morphism ψ can exist. If $j < j'$ then we have a commuting square

$$\begin{array}{ccc} K & \xrightarrow{[1]} & K \\ {\scriptstyle \psi_{j'-1}}\downarrow & & \downarrow{\scriptstyle \psi_{j'}} \\ 0 & \xrightarrow[{[1]}]{} & K \end{array}$$

which shows that $\psi_{j'} = 0$ and consequently $\psi = 0$. Similarly, the case $i < i'$ is excluded. Since $m \le M$ we get $j \le i'$ and therefore $j' \le j \le i' \le i$ is a necessary condition for $\mathrm{Hom}([j,i],[j',i'])$ to be non-zero and in that case this space is one-dimensional. □

Hence we have

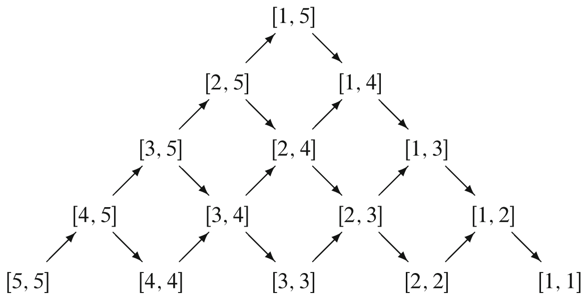
$$\mathrm{Hom}([j,i],[j',i']) = \begin{cases} K\,\iota_{j,i}^{j',i'}, & \text{if } j' \le j \le i' \text{ and } j \le i' \le i \\ 0, & \text{else.} \end{cases}$$

Notice that the maps $\iota_{j,i}^{j',i'}$ behave multiplicatively, that is $\iota_{j',i'}^{j'',i''} \circ \iota_{j,i}^{j',i'} = \iota_{j,i}^{j'',i''}$.

We call a non-zero morphism between two indecomposable representations of a quiver Q **irreducible** if it cannot be written as a sum of compositions, where each composition consists of two non-isomorphisms between indecomposables.

Hence from $[j,i]$ there are, up to scalar multiples, at most two irreducible morphisms starting, namely $\iota_{j,i}^{j-1,i}$ (if $1 < j$) and $\iota_{j,i}^{j,i-1}$ (if $j < i$).

Putting everything together, we have the following diagram of irreducible maps between indecomposable representations in case $n = 5$.



(2.6)

Notice that the whole diagram is commutative, i. e. each square in it is commutative. But there is still more structure inside of it which we shall discover in Sect. 6.3.

Exercises

2.6.1 Decompose the following representation V of $\overrightarrow{\mathbb{A}}_n$ into indecomposables: $V_i = K^2$ for all $i = 1, \ldots, n$ and $V_{\alpha_i} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for all $i = 1, \ldots, n-1$.

2.6.2 Show that if $f: V \to W$ is a morphism of representations of a quiver Q then $\operatorname{Im} f = (f_i(V_i))_{i \in Q_0}$ defines a **subrepresentation** of W, that is a family of subspaces $W_i' \subseteq W_i$ such that $W_\alpha(W_i') \subseteq W_j'$ for each arrow $\alpha: i \to j$. Explain how W_α defines $(\operatorname{Im} f)_\alpha$.

2.6.3 Use the previous exercise to prove that an irreducible morphism $f: V \to W$ between indecomposable representations satisfies that either all f_i are injective or all f_i are surjective.

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