Chapter 2
Simple Elections I

The simplest form of election is where there are a number of candidates for office, and one is to be elected. There is a well-defined set of voters—the *electorate*—and each voter casts one vote. The votes are then counted. We shall refer to such an election as a *simple election*.

In this chapter we shall look at some of the systems that have been suggested for simple ballots and are used in various places.

2.1 Elections with Two Candidates

Suppose two people are running for an office. After each elector makes his or her one vote, the person who gets more than half the votes wins. This is called the *majority* or *absolute majority* method. If there is an even number of electors then ties are possible—ties are possible in any electoral system—but apart from this the absolute majority method always produces a result.

2.2 Majority and Plurality

If there are three or more candidates, the majority method is not so good; there may quite easily be no winner. Several schemes have been devised that allow a candidate with an absolute majority to be elected, and try to find a good approximation when there is no "absolute" winner. These are called *majoritarian* or *plurality* systems.

W.D. Wallis, *The Mathematics of Elections and Voting*,
DOI 10.1007/978-3-319-09810-4_2

The first generalization is the *plurality* or *simple majority* method; it is often called "first-past-the-post" voting. Each voter makes one vote, and the person who receives the most votes wins. For example, if there were 3 candidates, A, B and C, and 70 voters, the absolute majority method requires 36 votes for a winner. If A received 30 votes and B and C each got 20, there would be no winner under that method. Under plurality, A would be elected.

The problem with the plurality method is that the winner might be very unpopular with a majority of voters. In our example, suppose all the supporters of B and C thought that both these candidates were better-qualified than A. Then the plurality method results in the election of the candidate that the majority thought was the worst choice. This problem is magnified if there are more candidates; even if there are only four or five candidates, people often think the plurality method elects the wrong person.

To overcome this difficulty in countries with only two major political parties, it is common for each party to endorse only one candidate. For example, in the United States, if there are two or more members of the Republican party who wish to run for some office, a preliminary election, called a *primary election*, is held, and party members vote on the proposed candidates; the one that receives the most votes is nominated by the party, and usually the others do not stand for election. This election would be called the *Republican primary*. There usually will also be a Democratic primary, and sometimes other parties run primaries. However, this method will not solve the problems if there are several major parties, or if the post for which the election is held is not a political one.

For further discussion of simple majority elections, see the paper of May [18].

2.3 Sequential Voting

Another technique used to avoid the problems of the plurality method is *sequential voting*. In this scheme a vote is taken, as a consequence a new set of candidates is selected; then a new vote is taken. The aim is to reduce the set of candidates to a manageable size—often to size two. The original election is again called a *primary*, but in this case all the candidates run in the primary election, not just those in one party.

For example, when the city of Carbondale, Illinois, votes for Mayor, there is a primary election for all the Mayoral candidates. This is run like an ordinary election. Later there is another election; the candidates in the final election are the two candidates who received the most votes in the primary. This second (*runoff*) election is decided by the majority method. A similar method is used in many other cities in the United States and in electing the Presidents of France and Chile and a few other countries.

We shall refer to this as the *runoff method* or *plurality runoff method.* The two top candidates are decided by plurality vote; all other candidates are eliminated; then a majority vote is taken. In the very unlikely case of a tie, some modification is necessary. For example if two candidates are tied for second place, those two and the vote leader would be considered in the runoff after the votes for any eliminated candidates have been distributed. In some cases the candidate with the smallest number of votes would be eliminated—essentially, it would be treated as a second runoff—while in others the race between those three candidates would be decided by plurality.

In electing state Governors in the United States and Presidents of a number of other countries (particularly in Europe), a modified runoff method is used. Separate primary elections are conducted by the different political parties, and each party nominates only one candidate in the final election. A more complicated system is used in electing the American President; for details, see Sect. 2.7.

In the real world there is usually a delay after the primary, and more campaigning takes place. As a result of this, individual voters' preferences may change. But for simplicity's sake we shall ignore this for the moment, and assume that every voter has a *preference list*, an order of preference between the candidates that remains fixed throughout the voting process; we assume that all candidates are included and there are no ties. We define the *preference profile* of an election to be the set of all the voters' preference lists. This can conveniently be written in a table. For example, say there are three candidates, A, B, C, and suppose:

5 voters like A best, then B, then C;
7 voters like B best, then A, then C;
4 voters like A best, then C, then B;
3 voters like C best, then B, then A;
no voters like B, then C, then A;
no voters like C, then B, then A.

We can represent this as

5	7	4	3
A	B	A	C
B	A	C	B
C	C	B	A

(We could also write

5	7	4	3	0	0
A	B	A	C	B	C
B	A	C	B	C	A
C	C	B	A	A	B

but we shall usually omit zero columns.)

In the primary election, each voter votes for the candidate he or she likes best. In the example, A would receive nine votes—five from those with preference list ABC, and four from those with list ACB. In general, a candidate receives the votes of those who put that candidate first in the preference list.

Sample Problem 2.1 *Suppose the preference profile of an election is*

5	7	4	3
A	B	A	C
B	A	C	B
C	C	B	A

What is the result of the election in the following cases?

(i) The majority method is used.

(ii) The plurality method is used.

(iii) The runoff method is used.

Solution. A receives nine votes, B receives seven, C receives three. So (i) there is no majority winner (as there are 19 voters, 10 votes would be needed), and (ii) A is the plurality winner. In a primary election, A and B are selected to contest the runoff. For the runoff, C is deleted, so the preference profile is

5	7	4	3
A	B	A	B
B	A	B	A

,

or (combining columns with the same preference list)

9	10
A	B
B	A

So B wins the runoff.

Practice Exercise. Repeat this question for an election with preference profile

7	5	8	3	4
A	A	B	B	C
B	C	A	C	B
C	B	C	A	A

2.4 The Hare Method

We introduced preference lists as a way of representing a voter's thoughts about the various candidates; they were not actual, physical lists. However, a number of methods have been devised that require a voter to present a preference list. These methods are known collectively as *preferential voting*. Some methods require the voter to list all candidates; others allow a partial list. Preferential voting is most useful in its more general form, for situations where several representatives are to be elected at once (see Chap. 5). In this section we look at the simpler version.

The *Hare method* or *alternative vote* system was invented by the English lawyer Sir Thomas Hare in 1859. In 1871, William Robert Ware proposed what is essentially the same idea, so the name "Ware's method" is sometimes used. Other names are "instant runoff voting" and "preferential voting" (although the name "preferential voting" is also used for other preferential systems).

The Hare method *requires* each voter to provide a preference list at the election. This list is called a *ballot*. All candidates are to be listed, and no ties are allowed. The candidate with the fewest first place votes is eliminated. Then the votes are tabulated again as if there were one fewer candidate, and again the one with the fewest first place votes in this new election is eliminated. When only two remain, the winner is decided by a majority vote. We shall assume that each preference list contains all candidates—what we shall call a *complete preference list*, but in some places the Hare system has been modified so that a voter lists only those candidates of whom she/he approves.

The Hare system can be used to give an ordered list of all candidates in terms of the electorate's preference. At any stage, when a candidate is deleted, that candidate goes below all candidates who have not yet been deleted; if two are deleted simultaneously they will have the same number of first-place votes at that time, and are considered tied.

Sample Problem 2.2 *Suppose there are four candidates for a position, and 24 voters whose preference profile is:*

5	7	4	3	3	2
A	B	A	C	D	D
C	D	D	D	C	C
D	A	C	B	A	B
B	C	B	A	B	A

Who would win using the following electoral systems?

(i) The plurality method.

(ii) The runoff method.

(iii) The Hare method.

Solution.

(i) The votes for A, B, C and D are 9, 7, 3 and 5 respectively, so A would win under plurality voting.

(ii) Under the runoff method there is a tie. A and B are retained, and the new preference profile is:

5	7	4	3	3	2
A	B	A	B	A	B
B	A	B	A	B	A

,

giving 12 votes to each candidate.

(iii) In the Hare method we first eliminate C, obtaining

5	7	4	3	3	2
A	B	A	D	D	D
B	D	D	B	A	B
D	A	B	A	B	A

Now we eliminate B:

5	7	4	3	3	2
A	D	A	D	D	D
D	A	D	A	A	A

So D wins 15 – 9.

Practice Exercise. Repeat the above question for the initial preference profile

6	7	7	7	2	7	5	2
A	B	A	C	D	D	B	D
C	D	D	D	C	C	D	B
D	A	C	A	A	B	C	C
B	C	B	B	B	A	A	A

As in all voting systems, it is possible that two candidates will receive the same number of first-place votes. If it happens that there is a tie between the two candidates with the smallest number of votes, both will be eliminated. This can lead to a candidate being elected without ever achieving more than the half the votes. For example, if the profile is

5	4	4
A	B	C
B	C	B
C	A	A

then both B and C will be eliminated, and A will win, although a majority prefer both B and C to A. However, such ties are extremely unlikely in political elections with typical numbers of voters.

As we said, one modification of the Hare method is to allow voters to cast votes only for the candidates of whom they approve. If there are five candidates, and you think A is best, B second, C third, but do not think either D or E is worthy of election, you simply vote A, B, C. If A, B and C are all eliminated, your vote is deleted, and the total number of votes cast is reduced accordingly. We shall refer to these generalized Hare systems as *instant runoff systems.*

In the above Sample Problem, suppose the seven voters represented by the second column all decided they did not wish to see A or D elected, while those represented by the fifth column did not like C. Then the preference profile is:

5	7	4	3	3	2
A	B	A	C	D	D
C	C	D	D	A	C
D		C	B	B	B
B		B	A		A

Again we first eliminate C,

5	7	4	3	3	2
A	B	A	D	D	D
B		D	B	A	B
D		B	A	B	A

Now we eliminate B:

5	4	3	3	2
A	A	D	D	D
D	D	A	A	A

and A wins 9–8.

Although nine votes is not a majority of the original 24 voters, it may be counted as a majority for this purpose. In some cases a *quota* is declared: a number is decided, and if neither of the last two candidates remaining achieves that many votes then the election is declared void. So, if there was a quota of 10 votes in the above example, the election would have to be held again.

A complication that can occur in real life is that a voter might not prefer one candidate over another: ties could occur in the voter's preferences. In practice, electoral systems that use preference lists do not allow ties, so that the voter must make a (possibly arbitrary) choice between the tied candidates. For simplicity, we shall assume that there are no ties in preference profiles.

Preferential voting is designed to avoid problems when there are three political parties all of which receive a substantial part of the vote. For example, suppose A and B are left-wing candidates and C is right-wing, and suppose the preference profile is

30	30	40
A	B	C
B	A	A
C	C	B

(where the numbers represent percentages). A majority, 60 %, of voters prefer the left-wing candidates, and would not wish to see C elected, but C would win under plurality. A would win under the preferential system.

The Hare system was introduced nationally in Australia in 1918. Up until then, the two major parties were the right-wing Liberal Party and the left-wing Labor Party. A new right-wing party, the Country Party, was formed to represent the interests of small farmers. The Country Party split the right-wing vote in some country areas, allowing the Labor Party to win elections where the majority of voters would have preferred either of the other two parties to Labor.

The Hare system meant that voters could vote for their preferred candidate, but give second preference to a candidate with similar views. If the first choice was unsuccessful, the vote would go to the second choice. This the problem of split votes, and meant that voters' views were represented more accurately. The Hare method or something very similar is still used in all Australian state elections, and in the federal elections.

2.5 The Coombs Rule

The Coombs rule was proposed by American psychologist Clyde Coombs ([12], 397–399) as "an alternative to the Hare system." Coombs was primarily concerned with the analysis of psychological data, preferences, and such; we shall look at his method as an electoral scheme, but it will be more relevant later in our chapter on group preferences and committees.

Essentially, Coombs proposed a method for situations where every voter produces a complete preference list. If no candidate receives an absolute majority, the candidate with the greatest number of *last*-place votes is eliminated, the preference profile is updated, and the method is applied to the new profile.

Sometimes the Hare method and the Coombs rule give the same result. For example, in Sample Problem 2.2, first B is eliminated, with 12 last-place votes. The revised preference profile is

5	7	4	3	3	2
A	D	A	C	D	D
C	A	D	D	C	C
D	C	C	A	A	A

C is next to be eliminated, with 11 last-place votes. The final profile is the same as it was for the Hare method, so again D wins. However, as the next example illustrates, the two methods do not always give the same result.

Sample Problem 2.3 *Suppose there are four candidates for a position, and there are 33 voters whose preference profile is:*

10	8	7	6
A	B	C	D
D	D	B	C
B	C	D	B
C	A	A	A

Who would win using the following electoral systems?

(i) The plurality method.

(ii) The runoff method.

(iii) The Hare method.

(iv) The Coombs rule.

Solution.

(i) A is the plurality winner with 10 votes.

(ii) Under the runoff method, the two candidates remaining are A and B, and B wins $21 - 10$.

(iii) Under the Hare method, D is eliminated, producing the profile

10	8	13
A	B	C
B	C	B
C	A	A

Now B is eliminated, and C wins $21 - 10$.

(iv) Under Coombs, A is the first candidate eliminated, leaving profile

10	8	7	6
D	B	C	D
B	D	B	C
C	C	D	B

Next C, with 18 last places, is eliminated. D beats B $16 - 15$.
So the four methods produce four different winners.

2.6 Point Methods

Pointscore methods have often been used in sporting contests. For example, they are commonly used in track meets and in motor racing. When the Olympic Games are being held, many newspapers publish informal medal tallies to rank the performance of the competing nations—the usual method is to allocate three points for a gold medal, two for a silver and one for a bronze, and then add.

In general, a fixed number of points are given for first, second, and so on. The points are totalled, and the candidate with the most points wins. If there are n competitors, a common scheme is to allocate n points to first, $n - 1$ to second, ..., or equivalently $n - 1$ to first, $n - 2$ to second, ... This case, where the points go in uniform steps, is called a *Borda count.*

One often sees scales like 5, 3, 2, 1, where the winner gets a bonus, or 3, 2, 1, 0, 0, ... (that is, all below a certain point are equal). Sometimes more complicated schemes are used; for example, in the Indy Racing League, the following system has been used:

1st gets	20	5th gets	10	9th gets	4
2nd gets	16	6th gets	8	10th gets	3
3rd gets	14	7th gets	6	11th gets	2
4th gets	12	8th gets	5	12th gets	1

Fastest qualifier gets one point.
Leader of most laps gets one point.

Pointscore methods are occasionally employed for elections, most often for small examples such as selection of the best applicant for a job.

Sometimes the result depends on the point scheme chosen.

Sample Problem 2.4 *What is the result of an election with preference table*

5	7	4	3
A	B	A	C
C	C	C	B
B	A	B	A

if a 3, 2, 1 count is used? What is the result if a 4, 2, 1 count is used?

Solution. With a 3, 2, 1 count the totals are $A : 37, B : 36, C : 41$, so C wins. With a 4, 2, 1 count the totals are $A : 46, B : 43, C : 44$, and A wins.

Practice Exercise. Repeat the question for the preference table

7	3	2	1
A	B	B	C
B	A	C	B
C	C	A	A

2.7 Electing the American President

The President of the United States is not elected by the people. Rather, he or she is elected by the members of the Electoral College (the "electors"), who in turn are appointed using the popular vote. Each state is apportioned the same number of electors as the number of members of Congress to which the state is entitled, and District of Columbia receives the same number of electors as the least populous state, currently three. For example, Illinois has 18 members in the House of Representatives and 2 senators, so it has 20 electors; Indiana has 11. In total, there are currently 538 electors, corresponding to the 435 members of the House of Representatives, 100 senators, and the 3 additional electors from the District of Columbia.

The electors pledge their support for a presidential nominee. In most states, all the electors are pledged to the presidential candidate who wins the most votes in the state; in 2012, all 20 Illinois electors were pledged to Obama, and all 11 Indiana electors were pledged to Romney. In Maine and Nebraska, an elector is selected for each congressional district, pledged to the candidate representing the party that received the most votes, and two more are allocated to the party that received the most votes in the state overall. Each elector then casts one vote for President and another vote for Vice President. While the electors are not required by federal law to honor their pledge, there have been very few occasions when an elector voted otherwise.

If no candidate receives a majority of the electoral college votes for President, then the House of Representatives selects the President, with each state having precisely one vote. If no candidate receives a majority for Vice President, then the Senate selects the Vice President, with each Senator having one vote.

Candidates for elector are nominated by their state political parties in the months prior to Election Day. In some states, the electors are nominated in primaries, the same way that other candidates are nominated. In other states, electors are nominated in party conventions. In Pennsylvania, the campaign committee of each presidential candidate names their candidates for elector.

Opinions on the Electoral College system vary. The system can be seen to favor smaller states, because those states receive more electors per capita than larger states. For example, California had a population of just over 38 million in June 2012, and with 55 electoral votes it has approximately one elector for every 720,000 citizens; Indiana has approximately one per 595,000; and Wyoming has one per 192,000. On the other hand, the United States is a union of states, so some argue that different states should have equally many electors. For more information on the Electoral College, and in particular on the various opinions, see [34].

In the 2000 election, Al Gore received over 500,000 more individual votes than George Bush, about 5 % of the total number of voters. But Bush received the majority in 29 states plus the District of Columbia, while Gore won 21 states. Bush won 271 electoral votes to Gore's 266. The election was only decided after the Supreme Court voted to block recounts in Florida. The final tally in Florida showed Bush winning that state by 537 votes, less than 0.01 % of the Florida votes, so Bush received all 25 Florida electors. If the Florida decision had been different, Gore would have won by 291 electoral college votes to Bush's 256.

Exercises 2

1. How many votes are needed for a majority winner if there are 55 voters?

2. In how many ways can a voter rank six candidates assuming ties are not allowed?

3. Twenty-eight electors vote between candidates A, B and C. Their votes are 4 for A, 15 for B and 9 for C. What is the result under the majority method? What is the result under the plurality method?

4. Eighty-five electors vote between candidates A, B and C. There were 30 votes for A and 33 vote for B. How many votes did C receive? What is the result under the majority method? What is the result under the plurality method?

5. Students in the University games club are voting for a Club president. There are three candidates, Smith, Jones, and Brown. The preference table is

25	27	14	22	35
S	S	J	J	B
J	B	S	B	J
B	J	B	S	S

(i) How many students voted?

(ii) How many first place votes did each candidate receive?

(iii) Who, if anybody, would win under the plurality method?

(iv) Who, if anybody, would win under the majority method?

6. At the Academy Awards there are three nominees for Best Actor: Arthur Andrews, Bob Brown and Clive Carter. The preference table is

123	101	442	212	310
A	A	B	C	C
B	C	A	A	B
C	B	C	B	A

(i) How many actors voted?

(ii) How many first place votes did each candidate receive?

(iii) Who, if anybody, would win under the plurality method?

(iv) Who, if anybody, would win under the majority method?

(v) Who would win under the runoff method?

7. In addition to plurality and the runoff method, two other techniques have been devised for cases when there is no majority winner. Both assume the full preference lists are known.

(a) The winner is the candidate with the fewest last-place votes.

(b) A runoff is held between the two candidates with the fewest last-place votes.

What are the results of using these two methods:

(i) Using the data of Exercise 5?

(ii) Using the data of Exercise 6?

8. Twenty-one electors must choose between five candidates: V, W, X, Y and Z. Their preference rankings are:

4	3	6	3	2	3
V	V	X	Y	Y	W
W	Y	Z	X	X	Z
X	W	Y	W	W	V
Y	X	V	Z	V	X
Z	Z	W	V	Z	Y

.

If there is no majority winner, all candidates with fewer than 20 % of the first-place votes (that is, those with fewer than 5) will be eliminated, and the preferences adjusted accordingly. If there is still no winner, a runoff is held. Which candidates are eliminated? What is the final result?

9. Given the following preference table, who would win under plurality voting? Who would win in a runoff?

6	3	4	2	1
A	C	C	B	E
E	B	D	A	A
B	E	A	C	B
C	D	E	D	C
D	A	B	E	D

10. A club with 36 members wishes to elect its president from four candidates, A, B, C and D. The preference profile is

16	10	8	2
A	B	C	B
B	A	B	A
C	D	A	C
D	C	D	D

.

(i) Who would be elected if the club used plurality voting?

(ii) Who would be elected if the club used the (3, 2, 1, 0) Borda count?

(iii) Who would be elected if the club used a modified Borda count with scores (5, 2, 1, 0)?

11. Eighteen delegates must elect one of four candidates, A, B, C and D. The preference profile is

8	6	4
A	B	C
B	D	D
C	A	A
D	C	B

.

(i) Who would be elected under the Hare method?

(ii) Who would be elected under the Coombs rule?

(iii) Who would be elected if the delegates used a modified Borda count with scores (2, 1, 0, 0)?

12. Fifty voters are to choose one of five candidates. Their preference profile is

20	10	14	6
A	B	B	C
C	A	A	D
E	C	D	B
B	D	C	A
D	E	E	E

.

What is the result under the following methods?

(i) Plurality.

(ii) Runoff.

(iii) The Hare method.

(iv) A modified Borda count with scores (5, 3, 2, 1, 0).

(v) The Coombs rule.

13. Fifteen committee members are to choose a new treasurer from four candidates, A, B, C and D. Their preference profile is

7	5	3
A	C	D
B	B	C
D	A	B
C	D	A

.

What is the result under:

(i) Plurality?

(ii) The Hare method?

(iii) The Coombs rule?

14. One hundred voters choose between four candidates, A, B, C, D. Their preference profile is

40	32	10	18
A	B	C	D
C	C	D	C
B	A	A	B
D	D	B	A

.

(i) What is the result if the Hare method is used?

(ii) What is the result if the Coombs rule is used?

15. Suppose there are three candidates in an election, where one candidate is to be elected. Is there any difference whether the runoff method or the Hare method is used?

16. Twenty-five electors vote for three candidates, resulting in the preference table

8	6	7	4
X	Y	Z	Y
Z	Z	X	X
Y	X	Y	Z

It is decided to use a scoring system where first place gets n points, second gets 2 and third gets 1, where n is some whole number greater than 2. For what ranges of values is X winner? What is the range for Y? For Z? Does it ever happen that there is no result?

17. Explain why majority rule is a reasonable electoral method in a country with only two political parties, but is not good in a country with four major political parties. (Why is the word "major" important here?)

18. When deciding an election, is it necessary to know the number of voters associated with each preference list, or is it sufficient to know the percentage of voters?

19. Construct a preference profile for four voters and four candidates, such that three voters prefer W to X, three prefer X to Y, three prefer Y to Z, and three prefer Z to W.

http://www.springer.com/978-3-319-09809-8

The Mathematics of Elections and Voting
Wallis, W.D.
2014, X, 96 p., Softcover
ISBN: 978-3-319-09809-8