Chapter 2
Classical Mechanics

2.1 Introduction

In this chapter we review classical mechanics, the theory that describes the behaviour of systems of classical non relativistic particles, as a necessary background for the discussion of quantum mechanics in the next chapters.

Quantum theory as physical theory is at present assumed to be universally valid. This means that it can, in principle, stand on its own. Neverthless, classical physics, from which it evolved in the twenties, is still very much present in its formulation, formulas and terminology. Moreover classical physics has not lost its value. Classical mechanics remains the appropriate vehicle for the study of most macroscopic situations. Studying, for example, the motion of billiard balls by quantum mechanics, is, of course, possible, but does not make sense. Approximating such a situation by classical mechanics is much simpler and the experimental results are in practice indistinguishable from those obtained by a quantum description. In Chap. 12, we compare classical physics and quantum physics, in particular classical mechanics and quantum mechanics, as 'algebraic dynamical systems', the first one with a commutative, the second with a noncommutative algebra of observables. Following this, we discuss in Chap. 13 quantum physics as a deformation of classical physics with Planck's constant $\hbar$ as a deformation parameter. Finally, even though the overall exposition of quantum theory in this book is 'axiomatic', we think that learning the theory should also include becoming familiar with the main lines of its historical development—as a matter of general education. For all these reasons the fairly extensive treatment of classical mechanics in this chapter precedes the chapters on quantum theory itself.

We start with a short historical overview of the subject in Sect. 2.2. For this Sect. 2.3 treats classical mechanics with Newton's equations as the basis of the subject; Sect. 2.4 gives the Lagrangian form of classical mechanics. Next, in Sect. 2.5, we review the Hamiltonian formalism, as it is obtained from Newton's equations via the Lagrange formalism. It is the proper vehicle for the transition to quantum mechanics. In Sect. 2.6 we discuss a more intrinsic, geometrical formulation of classical

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mechanics, with general dynamical systems as a notion defined on manifolds in Sect. 2.6.2, again Lagrangian and Hamiltonian systems in Sect. 2.6.3, and finally in Sect. 2.7 the algebraic version of the Hamiltonian formulation, which defines classical mechanics as an algebraic dynamical system and which will play a role in Chap. 12.

2.2 Historical Remarks

2.2.1 Aristotelian Physics

Classical mechanics describes the motion of bodies under the influence of forces. Superficial observation leads to the impression that physical objects are normally in rest and start moving only when forces are acting on them. Their velocities seem moreover to increase when the forces increase. Aristotle, the Greek philosopher who gave the first all-encompassing picture of the physical world based on empirical observation instead of pure speculation, followed this train of thought. One may—somewhat anachronistically—formulate his basic dynamical law of motion as

$$F = mv,$$

i.e. *the velocity of a moving body is proportional to the force acting on it*. One should add that to understand what happened when an object was thrown or was allowed to fall freely, it was necessary to devise special explanations, none of which now seem to us very convincing. The ideas of Aristotle dominated physics in the western world and in the world of Islam from classical antiquity until the end of the middle ages and the beginning of the renaissance.

2.2.2 Galileo and Newton

If one realizes the importance of friction in the motion of bodies and observes situations in which friction is negligible—think of a stone moving on a surface of ice, a different picture emerges: an object which moves with constant velocity will persist in this motion when left alone. The consequences of such observations were first clearly understood by Galileo. He was led to the general *principle of inertia*, which we may formulate as:

A physical body which is free, i.e. on which no forces act, is either in rest or moves in a straight line with constant velocity.

This principle has become the basis of what we at present call classical mechanics. Note that this is the modern form of the principle; Galileo thought of free motion as a motion in a great circle on the surface of the earth, still remaining in this way in the Aristotelian—Platonic view of circular motion as the ideal form of motion.

Galileo, who made many other important contributions to the new post-Aristotelian physics that arose in the sixteenth and seventeenth century, can be seen as the first representative of the method that led to the great successes of modern natural science: the combination of careful empirical observation with the use of precise mathematical models.

Another new and important insight that Galileo helped to establish was that the laws of physics are the same for events on earth and in the heavens. This made in a certain sense astronomy and in particular the study of planetary motion a part of mechanics, which greatly stimulated its further development. The heliocentric picture of the solar system had been put forward already by Copernicus. (Galileo was a strong defender of it, with very unpleasant consequences for him personally, as is well-known.)

Thinking—more or less—in terms of Copernicus' model and using the precise numerical data on planetary positions, collected by Tycho Brahe in years of observation, Johannes Kepler was able to establish that the planets move in ellipses with the sun in one of the focal points, thus finally breaking away from Plato's circles. This set the stage for the fundamental work of Newton. Starting from the principle of inertia he developed mechanics as a complete mathematical theory for the description of the motion of physical bodies under the influence of forces, with as central dynamical law the formula

$$F = ma,$$

stating that instead of the velocity the *acceleration*, i.e. the second derivative of the position with respect to time, should be proportional to the force, developing in the process differential and integral calculus as the appropriate mathematical tools for this. Introducing a universal gravitational force between two arbitrary massive bodies, proportional to the product of the masses of the two bodies and inversely proportional to the square of their distance, he was able to obtain—as a first application of his general ideas—a precise description of the motion of the moon around the earth, essentially in terms of Kepler's laws of planetary motion. All this he developed in his monumental "Philosophiae Naturalis Principia Mathematica" published in 1687, one the founding books of modern physics. See [1] for an English translation.

Newton's mechanics was further developed mathematically during the eighteenth and nineteenth century, by Laplace, Lagrange, Hamilton and Poincaré, however with no changes in its basic laws. It remains today a lively subject of mathematical research, particularly as celestial mechanics, with many interesting unsolved problems. Its modern formulation is geometrical, in terms of vector fields on differential manifolds, in particular so-called symplectic manifolds. Nevertheless, as a part of physics, 'classical mechanics' is essentially complete, a theory belonging to 19th century physics. The reason classical mechanics is discussed here in some detail is that it is necessary for the understanding of much of twentieth century physics, in particular quantum mechanics, the main topic of this book.

It is important to remark that classical mechanics, like much of the physics from the end of the nineteenth century, still describes many of the physical phenomena around us with very high precision. It fails however in situations where velocities comparable with the velocity of light are involved, or in situations in the submicroscopic world of atoms and molecules. In the first case Newton's theory has to be replaced by Einstein's theory of relativity and in the second case classical mechanics is superseded by quantum mechanics. This illustrates what was said about the domain of validity of physical theories in Sect. 1.4.3.

2.3 Newtonian Classical Mechanics

2.3.1 Newton's Equations for a System of Point Particles

Classical mechanics as it is taught nowadays to physics students is essentially Newton's mechanics, with some further developments that will be discussed in the next sections. Consider the typical situation of a system of N point particles with masses $m_1, \dots, m_N$, described by Cartesian coordinates $\mathbf{r}_1, \dots, \mathbf{r}_N$,

$$\mathbf{r}_j = (x_j, y_j, z_j), \quad j = 1, \dots, N.$$

We assume that there are forces acting on the particles, derived from a potential, i.e. the force $\mathbf{F}_j$ on the jth particle is equal to

$$\mathbf{F}_j(\mathbf{r}_1, \dots, \mathbf{r}_N) = -\frac{\partial}{\partial \mathbf{r}_j} V(\mathbf{r}_1, \dots, \mathbf{r}_N),$$

with $\frac{\partial}{\partial \mathbf{r}_j}$ denoting the triple of partial differentiations $(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial z_j})$, for $j = 1, \dots, N$, and $V(\mathbf{r}_1, \dots, \mathbf{r}_N)$ a given real function on $\mathbb{R}^{3N}$, the potential energy of the system. The time evolution of the system is described by the N vector-valued functions $\mathbf{r}_j(t)$, which are solutions of *Newton's equations*, in this case the system of coupled second order ordinary differential equations

$$m_j \frac{d^2 \mathbf{r}_{j(t)}}{dt^2} = -\frac{\partial}{\partial \mathbf{r}_j} V(\mathbf{r}_1(t), \dots, \mathbf{r}_N(t)),$$

for $j = 1, \dots, N$. Such a classical mechanical system is *deterministic*: if we mean by the state of the system at time $t = t_1$ the $2N$ positions and velocity vectors $\mathbf{r}_j(t_1)$ and $\mathbf{v}_j(t_1) = \frac{d}{dt}\mathbf{r}_j(t_1)$, for $j = 1, \dots, N$, then the state of the system at $t = t_1$ completely determines the state at a later time $t = t_2 > t_1$, because one can, for a sufficiently smooth potential function V, prove the existence of a unique solution for each given set of initial conditions $\mathbf{r}_j(t_1)$ and $\mathbf{v}_j(t_1)$. This does, of course, not mean that such a solution can always be found in an explicit form. For a system of two particles with masses m_1 and m_2, interacting through a potential

$$V(\mathbf{r}_1, \mathbf{r}_2) = -g \frac{m_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|},$$

with g a constant, Newton's equations can be solved in closed form. This is of course the problem of the sun and a planet attracting each other by gravitation; the periodic solutions are Kepler's elliptic planetary orbits. For a similar system consisting of three bodies Newton's equations cannot be solved exactly; the solutions can however be approximated to arbitrary precision.

2.3.2 Newton's Equations: A System of First Order Equations

By using the velocities $\mathbf{v}_j(t)$ as independent variables, Newton's equations can be written as a system of $2N$ vector-valued or $6N$ real-valued first order equations.

The position variables and the velocities can be written as $x_1, \ldots, x_n$ and $x_{n+1}, \ldots, x_{2n}$ instead of $\mathbf{r}_1, \ldots, \mathbf{r}_N$, and $\mathbf{v}_1, \ldots, \mathbf{v}_n$, with $n = 3N$. With these new variables Newton's equations take the form

$$\frac{d}{dt} x_s(t) = X_s(x_1(t), \ldots, x_{2n}(t)), \quad j = 1, \ldots, 2n,$$

with for $s = 1, \ldots, n$,

$$X_s(x_1(t), \ldots, x_{2n}(t)) = x_{s+n}(t), \quad s = 1, \ldots, n,$$

and for $s = n + 1, \ldots, 2n$,

$$X_j(x_1(t), \ldots, x_{2n}(t)) = -\frac{1}{m_{s-n}} \frac{\partial}{\partial x_{s-n}} V(x_1(t), \ldots x_n(t)),$$

with $m_1 = m_2 = m_3, m_4 = m_5 = m_6$, etc. This is the standard form for a general system of first order ordinary differential equations. Such a system is given by $2n$ functions X_s, defined on an open set $U \subset \mathbb{R}^{2n}$. It is called a *dynamical system*. With appropriate smoothness properties of the X_s there is a unique solution $x_s(t)$ on U with a prescribed value in $t = t_0$, on an open interval around t_0, for every point in U.

2.4 The Lagrangian Formulation of Classical Mechanics

2.4.1 Lagrangian Variational Problems

In the Lagrangian formulation Newtonian mechanics is treated as a particular example of a class of variational problems, i.e. problems in which a certain function, or a set of functions, is determined by finding the extremum of a given functional.

Let $\mathcal{U}$ be an open set in $\mathbb{R}^n$ and let L be a given real-valued function on $\mathcal{U} \times \mathbb{R}^n$. Consider curves in $\mathcal{U}$, i.e. functions γ from $\mathbb{R}$ into $\mathcal{U}$. To keep things simple all functions are supposed to be C^∞. We denote the n coordinates on $\mathcal{U}$ by $q^1, \ldots, q^n$ and those on $\mathcal{U} \times \mathbb{R}^n$ by $q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$. Consider a fixed finite interval $[t_1, t_2]$ on the real line. A curve γ, a set of functions $q^1(t), \ldots, q^n(t)$, determines an integral

$$I_{t_1,t_2}(\gamma) = \int_{t_1}^{t_2} L(q^1(t), \ldots, q^n(t), \frac{d}{dt}q^1(t), \ldots, \frac{d}{dt}q^n(t))\, dt.$$

This integral is a *functional* on the space of curves γ parametrized by t from the interval $[t_1, t_2]$.

The variational problem defined by this set-up is to find the curve or curves for which the integral is *extremal* with respect to variations which are arbitrary except that they leave the end points fixed. A curve γ is extremal in this sense if the integral is constant up to first order under each one parameter deformation $\gamma \mapsto \gamma_\varepsilon$ of the form

$$q^j(t) \mapsto q^j(t) + \varepsilon\, \eta^j(t),$$

for arbitrary (smooth) functions $\eta^j(t)$ with $\eta^j(t_1) = \eta^j(t_2) = 0$, for $j = 1, \ldots, n$, or

$$\frac{d}{d\varepsilon} I_{t_1,t_2}(\gamma_\varepsilon) = 0$$

in $\varepsilon = 0$ for such deformations.

(2.4.1,a) **Problem** Show that this condition leads to the following system of second order ordinary differential equations for the extremal functions $q^j(t)$, the *Euler-Lagrange equations*,

$$\frac{\partial L}{\partial q^j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) = 0, \quad j = 1, \ldots, n.$$

In physics and in some of the older mathematical literature on the variational calculus one employs a symbolic notation for ‘variations’. Written in this notation the statement above says that the requirement $\delta I_{t_1,t_2} = 0$ under variations δq^j with $\delta q^j(t_1) = \delta q^j(t_2) = 0$ implies the Euler-Lagrange equations.

2.4.2 *Newton's Equations as Variational Equations*

Consider a system of N point particles, as before. Let $\mathcal{U} = \mathbb{R}^n$, with $n = 3N$. The coordinates $q^1, \dots, q^n$ are the position variables $x_1, y_1, z_1, \dots, x_N, y_N, z_N$. Define the function L as

$$L(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n) = \sum_{j=1}^{n} \frac{1}{2} m_j (\dot{q}^j)^2 - V(q^1, \dots, q^n),$$

the *kinetic energy* minus the *potential energy* of the system, if we interprete the $\dot{q}^j$ as the components of the velocities of the particles. We have again $m_1 = m_2 = m_3, m_4 = m_5 = m_6$, etc. In this context the function L is called the *Lagrangian function*, or *Lagrangian* of the system and the integral $I_{t_1,t_2}(\gamma)$ is called the *action*.

(2.4.2,a) **Problem** Show that the Euler-Lagrange equations for this function L are just Newton's equations.

This result means that the time evolution of the mechanical system from t_1 to t_2 is described precisely by those curves γ for which the action is extremal, in fact *minimal* in this case.

Writing Newton's equations in Lagrangian form in this manner does of course not add anything to their contents, but has nevertheless great advantages:

a. There is no need to restrict oneself to Cartesian coordinates; the formulas hold for arbitrary curvilinear coordinates.
b. The Lagrangian formulation is very useful in situations where there are constraints on the system, for instance when the particles are restricted in their motion to a lower dimensional surface.
c. Symmetries and their implications such as conserved quantities can be easily read off from the Lagrangian.
d. The Lagrangian formulation of classical mechanics is the starting point for Feynman's path integral scheme, a semi-heuristic but very useful quantization scheme, which is discussed in Sect. 13.8.
e. The Lagrange formalism is convenient in relativistic field theory where space and time coordinates are treated on the same footing. Relativity theory will be discussed in Chap. 15 and relativistic field theory in Chap. 16.

2.5 The Hamiltonian Formulation of Classical Mechanics

In this section we finally obtain the formulation of classical mechanics which is the proper background for our presentation of quantum mechanics. Let us start from a Lagrangian variational system formulated in terms of local coordinate expressions, i.e. with coordinates $q^1, \dots, q^n$, not necessarily Cartesian, with the associated velocities $\dot{q}^1, \dots, \dot{q}^n$ and with a given Lagrange function

$L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$. Using L we introduce new variables, the *canonically conjugated momenta* $p_1, \ldots, p_n$, as

$$p_j = \frac{\partial}{\partial \dot{q}^j} L,$$

for $j = 1, \ldots, n$. We assume that this transformation, which is called the *Legendre transformation*, can be inverted, i.e. that the velocities $\dot{q}^j$ can be written as functions of $q^1, \ldots, q^n$ and the new momenta $p_1, \ldots, p_n$. If this is possible, the Lagrangian L is called *regular* or *nondegenerate*. If L is *singular* or *degenerate* the simple road to a Hamiltonian formalism breaks down at this point. More complicated procedures can be found to overcome the problem of degenerateness of L, but this will not be discussed here.

Define next the *Hamiltonian function*, or, for short, *Hamiltonian*, as

$$H = \sum_{j=1}^{n} p_j \dot{q}_j - L.$$

Note that this H should be seen as a function of $q^1, \ldots, q^n$ and $p_1, \ldots, p_n$. The time evolution, given in the Lagrangian formulation by functions $q^1(t), \ldots, q^n(t)$, $\dot{q}^1(t), \ldots, \dot{q}^n(t)$, satisfying the system of n Euler-Lagrange equations, is now given by functions $q^1(t), \ldots, q^n(t)$ and $p_1(t), \ldots, p_n(t)$, which are solutions of $2n$ first order equations involving H, namely

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q^j} \qquad \frac{dq^j}{dt} = \frac{\partial H}{\partial p_j},$$

for $j = 1, \ldots, n$.

(2.5,a) **Problem** Show that these equations—called, not surprisingly, *Hamilton's equations*—are equivalent to the Euler-Lagrange equations.

Hamilton's equations can be written in a more uniform manner as

$$\frac{dp_j}{dt} = \{H, p_j\} \quad \frac{dq^j}{dt} = \{H, q^j\},$$

for $j = 1, \ldots, n$, with the *Poisson bracket* $\{\cdot, \cdot\}$ defined for an arbitrary pair of functions f and g of the variables $p_1, \ldots, p_n, q^1, \ldots, q^n$ as

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial g}{\partial p_j} \frac{\partial f}{\partial q^j} \right).$$

See Supp. Sect. 20.2.7.2 for the most important properties of the Poisson bracket. This form of Hamilton's equations, in the more mathematically intrinsic form to be discussed further on, will be of great importance for an algebraic formulation of classical mechanics, and more generally for the common algebraic formulation of classical and quantum mechanics that will be formulated in Chap. 12.

The $2n$-dimensional space of the variables $p_1, \ldots, p_n, q^1, \ldots, q^n$ is called the *phase space* of the classical system. The time evolution, a *flow* in this space, has the property that it leaves the integration measure $dp_1 \ldots dp_n \, dq^1 \ldots dq^n$ invariant. This fact is known as *Liouville's theorem*. It is of basic importance in classical statistical mechanics, as will be shown in Chap. 10.

2.6 An Intrinsic Formulation

2.6.1 Introduction

So far we have restricted the discussion of classical mechanics to the case in which the phase space is just $\mathbb{R}^{2n}$. It is not hard to think of more general situations, for instance that of a particle moving in a circle or on the surface of a sphere. For a general discussion of classical mechanics in its various forms, as a general dynamical system, in Lagrangian and Hamiltonian form, we need as a mathematical background differential geometry, i.e. the general theory of C^∞-manifolds, and more in particular of symplectic manifolds. An extensive review of this is given in Supp. Chap. 20. A reader who is not familiar with the material appearing in this section should consult Supp. Chap. 20 and read it in parallel with this section.

The more intrinsic differential-geometric formulation has several advantages. It has, of course, great esthetic appeal. The formulation in terms of explicit formulas given so far is completely rigorous, but the differential geometric picture gives more structural insight. One important aspect of it is that this formulation, rather surprisingly, leads in a natural way to an algebraic framework, which can be used to describe both classical and quantum physics. This will be explained in Chap. 12.

2.6.2 General Dynamical Systems

A *dynamical system* is given by a pair $(\mathcal{M}, X)$, with $\mathcal{M}$ a smooth m-dimensional manifold and X a vector field on X. This X assigns to each point p of $\mathcal{M}$ in a smooth manner a tangent vector X_p and determines *integral curves*, i.e. curves described by smooth maps γ from an open interval in $\mathbb{R}^1$ into $\mathcal{M}$, such that they are tangent to the vector field in each point p of $\mathcal{M}$. The γ's form the solutions of a system of first order ordinary differential equations, given in local coordinates $\{x_s\}_s$, with X_j the component functions, as

$$\frac{dx^j(t)}{dt} = X^j(x^1(t), \ldots, x^m(t)), \quad j = 1, \ldots, m.$$

Newton's equations, as given in Sect. 2.3, form a particular example of such a dynamical system.

2.6.3 The Lagrange System

Starting from the position space, an n-dimensional manifold $\mathcal{Q}$, one constructs the space of positions and velocities, the tangent manifold $T(\mathcal{Q})$ of $\mathcal{Q}$. This is a vector bundle over $\mathcal{Q}$ with as fibres the tangent spaces at the points p of $\mathcal{Q}$. The dynamics of the system is given by the Lagrangian, a function L on $T(\mathcal{Q})$. Curves $\gamma(t)$ define in an obvious manner curves $\widehat{\gamma}(t)$ on the space of velocities. The action is the integral

$$I(\gamma) = \int_{t_1}^{t_2} L(\widehat{\gamma}(t))\, dt,$$

which is required to be extremal to give the evolution equations. Here we use the Lagrange formalism only as an intermediate step in the transition to the Hamilton formalism; so there is not much need to discuss it further here. It will however be used in our discussion of path-integral quantization in Sect. 13.8.

2.6.4 The Hamilton System

The tangent bundle $T(\mathcal{Q})$ has a dual bundle $T^*(\mathcal{Q})$, the *cotangent bundle*, dual in the sense of $C^\infty(\mathcal{Q})$-modules, constructed by smoothly welding together the real-linear duals $T_p^*(\mathcal{Q})$ of the tangent spaces $T_p(\mathcal{Q})$ at all points p of $\mathcal{Q}$. The cotangent bundle will be denoted as $\mathcal{M}$; it is a $2n$-dimensional manifold, the basic manifold in the Hamiltonian description of classical mechanics, the *phase space* of a classical mechanical system. As a cotangent bundle it has a natural closed nondegenerate 2-form ω, so it is a *symplectic manifold*. See Supp. Sect. 20.7.

In Sect. 2.5 we gave, in local coordinates, the road from the Lagrange to the Hamilton formalism by means of the Legendre transformation. In a more intrinsic picture it is a map s

$$s \,:\, T(\mathcal{Q}) \,\rightarrow\, T^*(\mathcal{Q}) = \mathcal{M},$$

a smooth welding of maps s_p

$$s_p \,:\, T_p(\mathcal{Q}) \rightarrow\, T_p^*(\mathcal{Q}) = \mathcal{M}_p, \quad \forall p \in \mathcal{M}.$$

An intrinsic formulation of this map can be given, but we refrain from doing this, as it involves the introduction of mathematical notions for which we have no further use. What is important is the result, a simple and transparent picture, the Hamilton formulation of classical mechanics, the standard point of departure for the transition to quantum mechanics.

The formalism for a Hamiltonian description of classical mechanics consists of the following elements:

- A symplectic manifold $(\mathcal{M}, \omega)$, the phase space.
- A Hamiltonian vector field X_H, with H the Hamiltonian function of the system, a function on $\mathcal{M}$ defined in Sect. 2.5. It determines a Hamiltonian dynamical system. The corresponding flow on $\mathcal{M}$ is the time evolution of the system.
- An important structural element is the Poisson bracket, defined in terms of the local coordinates $\{p_j\}_j$ and $\{q^k\}_k$ in Sect. 2.5, and in an intrinsic manner in Supp. Sect. 20.7.2.
- Symmetries of the system are described by canonical transformations of $\mathcal{M}$ and more particularly by Lie groups of such transformations. See Supp. Sect. 20.7.1. Their infinitesimal generators are Hamiltonian vector fields X_f, connected with a function f in the manner explained in Supp. Sect. 20.7.1. Such a function is in general an observable of the system, and in this particular situation a *constant of the motion* or a *conserved quantity*.

(2.6.4,a) **Problem** Show that for such a constant of the motion f the Poisson bracket of f with H vanishes.

Note that the Hamiltonian H itself, as a constant of the motion, is usually the energy of the system.

For an overview of the mathematical description of classical mechanics, see [2] and [3]. Two more elementary but valuable books on classical mechanics are [4] and [5].

2.7 An Algebraic Formulation

In Supp. Sect. 20.8 we give an algebraic formulation of differential geometry. Applying this to the geometric picture of classical mechanics, sketched in the preceding section, we obtain classical mechanics as an *algebraic dynamical system*.

This means a pair $(C^\infty(\mathcal{M}), X_H)$, consisting of a commutative algebra $C^\infty(\mathcal{M})$, with has an additional Poisson structure given by a Poisson bracket $(f, g) \mapsto \{f, g\}$, a derivation X_H of $C^\infty(\mathcal{M})$, which may be called a Hamiltonian derivation, as it is associated with an element H of $C^\infty(\mathcal{M})$. (Remember the equivalence between derivations of $C^\infty(\mathcal{M})$ and vector fields on $\mathcal{M}$. See Supp. Sect. 20.2.2.) It generates a one parameter group of time evolution automorphisms of $C^\infty(\mathcal{M})$, leaving the Poisson structure invariant. There may be additional groups of such automorphisms, representing symmetries, with as generators Hamiltonian derivations, associated with functions that are constants of the motion, i.e. have zero Poisson bracket with H.

This algebraic picture is particularly useful for comparing classical mechanics with quantum mechanics. It will be one of the main ingredients of Chaps. 12 and 13.

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