Chapter 2
Topologies for Spaces of Vector Fields

In this chapter we review the definitions of the topologies we use for spaces of Lipschitz, finitely differentiable, smooth, and real analytic vector fields. We comment that all topologies we define are locally convex topologies, of which the normed topologies are a special case. However, few of the topologies we define, and none of the interesting ones, are normable. We, therefore, begin with a very rapid review of locally convex topologies, and why they are inevitable in work such as we undertake here.

2.1 An Overview of Locally Convex Topologies for Vector Spaces

In this section we provide a "chatty" overview of locally convex topologies, since this work relies on these in an essential way. The presentation here should be regarded as that of a bare bones introduction, and a reader wishing to understand the subject deeply will wish to refer to references such as [1, 3, 5, 7, 10, 12]. We particularly suggest [10] as a good place to start learning the theory.

2.1.1 Motivation

As mentioned above, few of the topologies we introduce below arise from a norm, and the most interesting ones, e.g., the topologies for spaces of smooth and real analytic vector fields, are decidedly not norm topologies. Let us reflect on why locally convex topologies, such as we use in this work, are natural. Consider first the task of putting a norm on the space $C^0(\mathbb{R})$ of continuous $\mathbb{R}$-valued functions on $\mathbb{R}$. Spaces of continuous functions are in the domain of classical analysis, and so are well-known to the readership of this monograph, e.g., [4, Theorem 7.9]. This is often considered

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for continuous functions defined on compact spaces, e.g., compact intervals, where the sup-norm suffices to describe the topology in an adequate manner. For continuous functions on noncompact spaces, the sup-norm obviously no longer applies. In such cases, it is common to consider instead functions that "die off" at infinity, as the sup-norm again functions perfectly well for these classes. For the entire space of continuous functions, say $\mathrm{C}^0(\mathbb{R})$, the sup-norm is no longer a viable candidate for defining a topology. Instead one can use a family of natural seminorms, one for each compact set $K \subseteq \mathbb{R}$. To be precise, we define

$$p_K(f) = \sup\{|f(x)| \mid x \in K\}.$$

The collection p_K, $K \subseteq \mathbb{R}$ compact, of such seminorms can then be used to define a topology (in a manner that we make precise in Definition 2.2). If one wishes to apply the same reasoning to functions of class C^m, $m \in \mathbb{Z}_{>0}$, we can use the seminorms

$$p_K^m(f) = \sup\{|\boldsymbol{D}^j f(x)| \mid x \in K,\ j \in \{0, 1, \dots, m\}\}, \qquad K \subseteq \mathbb{R} \text{ compact},$$

on $\mathrm{C}^m(\mathbb{R})$, and it is not hard to imagine that this can be used to describe a suitable topology; we define these sorts of topologies precisely below.

By being slightly more clever, one can imagine adapting the above procedure for topologising $\mathrm{C}^m(\mathbb{R})$, $m \in \mathbb{Z}_{\geq 0}$, to topologising the space $\mathrm{C}^m(\mathsf{M})$ of functions on a smooth manifold M of class C^m. If M is compact, such a space is actually a normed space, since supremums can be taken over the compact set M. If one wishes to topologise the space $\mathrm{C}^\infty(\mathsf{M})$ of smooth functions on a smooth manifold, one must account for all derivatives. Let us indicate how to do this for $\mathrm{C}^\infty(\mathbb{R})$; we handle the general case in Sect. 2.2.2. For $\mathrm{C}^\infty(\mathbb{R})$ we define the seminorms

$$p_{K,m}^\infty(f) = \sup\{|\boldsymbol{D}^j f(x)| \mid x \in K,\ j \in \{0, 1, \dots, m\}\}, \qquad K \subseteq \mathbb{R} \text{ compact},\ m \in \mathbb{Z}_{\geq 0}.$$

Note that the appropriate adaptation of these seminorms to manifolds will never yield a normed topology, since there will always be infinitely many derivatives to account for.

The point of the preceding motivation is this: topologies defined by families of seminorms arise in natural ways when topologising spaces of functions in differential geometry.

2.1.2 Families of Seminorms and Topologies Defined by These

With the preceding remarks as motivation, let us provide a few precise definitions and state a few facts (without proof) arising from these definitions.

We begin with the notion of a seminorm.

Definition 2.1 (Seminorm) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let V be an $\mathbb{F}$-vector space. A ***seminorm*** for V is a function $p \colon \mathsf{V} \to \mathbb{R}_{\geq 0}$ such that

(i) $p(av) = |a| p(v)$ for $a \in \mathbb{F}$ and $v \in \mathsf{V}$;
(ii) $p(v_1 + v_2) \leq p(v_1) + p(v_2)$. ○

The reader will note that the missing norm element is the positive definiteness. A moments reflection on the examples above indicates why this omission is necessary. Nonetheless, one can use families of seminorms to define a topology.

Definition 2.2 (The topology defined by a family of seminorms) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let V be an $\mathbb{F}$-vector space, and let $\mathscr{P}$ be a family of seminorms for V. The ***topology*** defined by $\mathscr{P}$ is that topology for which the sets

$$\{v \in \mathsf{V} \mid\ p(v) < r\}, \qquad p \in \mathscr{P},\ r \in \mathbb{R}_{>0},$$

are a subbasis, i.e., open sets in the topology are unions of finite intersections of these sets. The resulting topology is called a ***locally convex*** topology, and an $\mathbb{F}$-vector space with a locally convex topology is called an $\mathbb{F}$***-locally convex topological vector space***, or simply a ***locally convex space***. ○

Now let us simply list some attributes of these topologies, referring to the references for details. In the following list, we let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let U and V be $\mathbb{F}$-locally convex spaces defined by families $\mathscr{Q}$ and $\mathscr{P}$, respectively, of seminorms.

1. The locally convex topology on V is Hausdorff if and only if, for each $v \in \mathsf{V}$, there exists $p \in \mathscr{P}$ such that $p(v) \neq 0$ [10, Theorem 1.37]. Locally convex spaces are often assumed to be Hausdorff, and we shall suppose this to be true for our statements below.
2. Locally convex topologies are translation-invariant, i.e., a neighbourhood basis at 0 translates (by adding v) to a neighbourhood basis at $v \in \mathsf{V}$ [10, Theorem 1.37].
3. We say that a subset $\mathcal{B}$ is ***von Neumann bounded*** if, for any neighbourhood $\mathcal{N}$ of 0, there exists $\lambda \in \mathbb{R}_{>0}$ such that $\mathcal{B} \subseteq \lambda\mathcal{N}$. A subset is von Neumann bounded if and only if $p|\mathcal{B}$ is bounded for every $p \in \mathscr{P}$ [10, Theorem 1.37(b)].
4. A locally convex topology is ***normable*** if it can be defined by a single seminorm which is a norm. A locally convex space is normable if and only if there exists a convex bounded neighbourhood of 0 [10, Theorem 1.39].
5. Compact subsets of locally convex spaces are closed and bounded. However, closed and bounded subsets are not necessarily compact, e.g., closed balls in infinite-dimensional Banach spaces are not compact.
6. Unlike the situation for Banach spaces, there *are* infinite-dimensional locally convex spaces for which closed and bounded sets are compact. An important class of such spaces are the so-called nuclear spaces [9, Proposition 4.47]. A normed space is nuclear if and only if it is finite-dimensional. In this work, many of the spaces we deal with are nuclear.
7. A locally convex space is ***metrisable*** if its topology can be defined by a translation-invariant metric. A locally convex space is metrisable if and only if it can be defined by a countable family of seminorms [10, Remark 1.38(c)].

8. Metrisable topologies are characterised by their convergent sequences. This is a general assertion, following from the fact that metric spaces are first-countable [13, Corollary 10.5]. However, we will encounter locally convex spaces that are not metrisable, and so convergence in such spaces is determined by using nets rather than sequences. Recall that a ***net*** in a set is indexed by points in a directed set, i.e., a partially ordered set $(I, \preceq)$ with the attribute that, given $i_1, i_2 \in I$, there exists $i \in I$ such that $i_1, i_2 \preceq i$. A net $(x_i)_{i \in I}$ in a topological space ***converges*** to x_0 if, for every neighbourhood $\mathcal{O}$ of x_0, there exists $i_0 \in I$ such that $x_i \in \mathcal{O}$ for all $i_0 \preceq i$.
9. A net $(v_i)_{i \in I}$ in V converges to v_0 if and only if, for each $p \in \mathscr{P}$ and each $\varepsilon \in \mathbb{R}_{>0}$, there exists $i_0 \in I$ such that $p(v_i - v_0) < \varepsilon$ for $i_0 \preceq i$.
10. A ***Cauchy net*** in V is a net $(v_i)_{i \in I}$ such that, for each $p \in \mathscr{P}$ and each $\varepsilon \in \mathbb{R}_{>0}$, there exists $i_0 \in I$ such that, if $i_0 \preceq i_1, i_2$, then $p(v_{i_1} - v_{i_2}) < \varepsilon$. A locally convex space is ***complete*** if every Cauchy net converges.
11. A linear map $L : \mathsf{U} \to \mathsf{V}$ is continuous if and only if, for each $p \in \mathscr{P}$, there exist $q_1, \dots, q_k \in \mathscr{Q}$ and $C_1, \dots, C_k \in \mathbb{R}_{>0}$ such that

$$p\,(L(u)) \leq C_1 q_1(u) + \cdots + C_k q_k(u),$$

cf. the discussion in [12, §III.1.1]. We denote by $\mathrm{L}(\mathsf{U}; \mathsf{V})$ the set of continuous linear maps from U to V.

The preceding is all that we shall make direct reference to in this monograph. We mention, however, that our work here relies on the recent work of Jafarpour and Lewis [6], and in this work, especially the development of the topology for spaces of real analytic vector fields, many deep properties of locally convex topologies are used. We shall skirt around these issues, for the most part, in the present monograph.

2.2 Seminorms for Locally Convex Spaces of Vector Fields

We now describe in a little detail the seminorms we use for spaces of vector fields with various regularity, Lipschitz, finitely differentiable, smooth, and real analytic. We also characterise spaces of holomorphic vector fields, because these can often be useful in understanding real analytic vector fields.

While our interest is primarily in spaces of vector fields, it is actually less confusing notationally and conceptually to work instead with spaces of sections of a vector bundle. Thus, throughout this section we will work with a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ that is either smooth, real analytic, or holomorphic, depending on our needs.

2.2.1 Fibre Norms for Jet Bundles

The classes of sections we consider are all characterised by their derivatives in some manner. The appropriate device for considering derivatives of sections is the theory of jet bundles, for which we refer to [11] and [8, §12]. By $\mathsf{J}^m\mathsf{E}$ we denote the vector bundle of m-jets of sections of a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, with $\pi_m\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$ denoting the projection. If ξ is a smooth section of E, we denote by $j_m\xi$ the corresponding smooth section of $\mathsf{J}^m\mathsf{E}$.

Sections of $\mathsf{J}^m\mathsf{E}$ should be thought of as sections of E along with their first m derivatives. In a local trivialisation of E, one has the local representatives of the derivatives, order-by-order. Such an order-by-order decomposition of derivatives is not possible globally, however. Nonetheless, following [6, §2.1], we shall mimic this order-by-order decomposition globally using a linear connection ∇^0 on E and an affine connection ∇ on M. First note that ∇ defines a connection on $\mathsf{T}^*\mathsf{M}$ by duality. Also, ∇ and ∇^0 together define a connection ∇^m on $\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ by asking that the Leibniz Rule be satisfied for the tensor product. Then, for a smooth section ξ of E, we denote

$$\nabla^{(m)}\xi = \nabla^m \cdots \nabla^1\nabla^0\xi,$$

which is a smooth section of $\mathrm{T}^{m+1}(\mathsf{T}^*\mathsf{M} \otimes \mathsf{E})$. By convention we take $\nabla^{(-1)}\xi = \xi$.

We then have a map

$$\begin{aligned} S^m_{\nabla,\nabla^0}\colon \mathsf{J}^m\mathsf{E} &\to \oplus_{j=0}^m(\mathrm{S}^j\mathsf{T}^*\mathsf{M} \otimes \mathsf{E}) \\ j_m\xi(x) &\mapsto (\xi(x), \mathrm{Sym}_1 \otimes \mathrm{id}_{\mathsf{E}}(\nabla^0\xi)(x), \dots, \mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{E}}(\nabla^{(m-1)}\xi)(x)), \end{aligned} \tag{2.1}$$

which can be verified to be an isomorphism of vector bundles [6, Lemma 2.1]. Here $\mathrm{Sym}_m\colon \mathrm{T}^m(\mathsf{V}) \to \mathrm{S}^m(\mathsf{V})$ is defined by

$$\mathrm{Sym}_m(v_1 \otimes \cdots \otimes v_m) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

Now we note that inner products on the components of a tensor product induce in a natural way inner products on the tensor product [6, Lemma 2.2]. Thus, if we suppose that we have a fibre metric $\mathbb{G}_0$ on E and a Riemannian metric $\mathbb{G}$ on M, there is induced a natural fibre metric $\mathbb{G}_m$ on $\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ for each $m \in \mathbb{Z}_{\geq 0}$. We then define a fibre metric $\overline{\mathbb{G}}_m$ on $\mathsf{J}^m\mathsf{E}$ by

$$\begin{aligned} &\overline{\mathbb{G}}_m(j_m\xi(x), j_m\eta(x)) \\ &\qquad = \sum_{j=0}^m \mathbb{G}_j\Big(\frac{1}{j!}\mathrm{Sym}_j \otimes \mathrm{id}_{\mathsf{E}}(\nabla^{(j-1)}\xi)(x), \frac{1}{j!}\mathrm{Sym}_j \otimes \mathrm{id}_{\mathsf{E}}(\nabla^{(j-1)}\eta)(x)\Big). \end{aligned}$$

(The factorials are required to make things work out with the real analytic topology.) The corresponding fibre norm we denote by $\|\cdot\|_{\overline{\mathbb{G}}_m}$.

2.2.2 Seminorms for Spaces of Smooth Vector Fields

Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle. Using the fibre norms from the preceding section, it is a straightforward matter to define appropriate seminorms that prescribe the locally convex topology for $\Gamma^\infty(\mathsf{E})$. For $K \subseteq \mathsf{M}$ compact and for $m \in \mathbb{Z}_{\geq 0}^m$, define a seminorm $p^\infty_{K,m}$ on $\Gamma^\infty(\mathsf{E})$ by

$$p^\infty_{K,m}(\xi) = \sup\{\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K\}.$$

The family of seminorms $p^\infty_{K,m}$, $K \subseteq \mathsf{M}$ compact, $m \in \mathbb{Z}_{\geq 0}$, defines a locally convex topology, called the C^∞***-topology***,[1] with the following properties:

1. it is Hausdorff, metrisable, and complete, i.e., it is a Fréchet topology;
2. it is separable;
3. it is nuclear;
4. it is characterised by the sequences converging to zero, which are the sequences $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ such that, for each $K \subseteq \mathsf{M}$ and $m \in \mathbb{Z}_{\geq 0}$, the sequence $(j_m\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero.

In this paper we shall not make reference to other properties of the C^∞-topology, but we mention that there are other properties that play an important rôle in the results in Chap. 3. For these details, and for references where the above properties are proved, we refer to [6, §3.2].

2.2.3 Seminorms for Spaces of Finitely Differentiable Vector Fields

We again take $\pi\colon \mathsf{E} \to \mathsf{M}$ to be a smooth vector bundle, and we fix $m \in \mathbb{Z}_{\geq 0}$. For the space $\Gamma^m(\mathsf{E})$ of m-times continuously differentiable sections, we define seminorms p^m_K, $K \subseteq \mathsf{M}$ compact, for $\Gamma^m(\mathsf{E})$ by

$$p^m_K(\xi) = \sup\{\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K\}.$$

The locally convex topology defined by the family of seminorms p^m_K, $K \subseteq \mathsf{M}$ compact, we call the $\mathbf{C}^m$***-topology***, and it has the following properties:

1. it is Hausdorff, metrisable, and complete, i.e., it is a Fréchet topology;
2. it is separable;

[1] This is actually not a very good name. A better name, and the name used by Jafarpour and Lewis [6], would be the "smooth compact-open topology". However, we wish to keep things simple here, and also use notation that is common between regularity classes.

3. it is characterised by the sequences converging to zero, which are the sequences $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ such that, for each $K \subseteq \mathsf{M}$, the sequence $(j_m\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero;
4. if M is compact, then p^m_{M} is a norm that gives the C^m-topology.

As with the C^∞-topology, we refer to [6, §3.4] for details.

2.2.4 Seminorms for Spaces of Lipschitz Vector Fields

In this section we again work with a smooth vector bundle $\pi : \mathsf{E} \to \mathsf{M}$. In defining the fibre metrics from Sect. 2.2.1, for the Lipschitz topologies the affine connection ∇ is required to be the Levi-Civita connection for the Riemannian metric $\mathbb{G}$ and the linear connection ∇^0 is required to be $\mathbb{G}_0$-orthogonal. While Lipschitz vector fields are often used, spaces of Lipschitz vector fields are not. Nonetheless, one may define seminorms for spaces of Lipschitz vector fields rather analogous to those defined above in the smooth and finitely differentiable cases. Let $m \in \mathbb{Z}_{\geq 0}$. By $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ we denote the space of sections of E that are m-times continuously differentiable and whose m-jet is locally Lipschitz. (One can think of this in coordinates, but Jafarpour and Lewis [6] provide geometric definitions, if the reader is interested.) If a section ξ is of class $\mathrm{C}^{m+\mathrm{lip}}$, then, by Rademacher's Theorem [2, Theorem 3.1.6], its $(m+1)$st derivative exists almost everywhere. Thus we define

$$\begin{aligned}\mathrm{dil}\, j_m\xi(x) = \inf\{\sup\{\|\nabla^{[m]}_{v_y} j_m\xi\|_{\overline{\mathbb{G}}_m} \mid\ y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}} = 1,\\ j_m\xi \text{ differentiable at } y\} \mid\ \mathcal{U} \text{ is a relatively compact neighbourhood of } x\},\end{aligned}$$

which is the ***local sectional dilatation*** of ξ. Here $\nabla^{[m]}$ is the connection in $\mathsf{J}^m\mathsf{E}$ defined by the decomposition (2.1). Let $K \subseteq \mathsf{M}$ be compact and define

$$\lambda^m_K(\xi) = \sup\{\mathrm{dil}\, j_m\xi(x) \mid\ x \in K\}$$

for $\xi \in \Gamma^{m+\mathrm{lip}}(\mathsf{E})$. We can then define a seminorm $p^{m+\mathrm{lip}}_K$ on $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ by

$$p^{m+\mathrm{lip}}_K(\xi) = \max\{\lambda^m_K(\xi),\, p^m_K(\xi)\}.$$

The family of seminorms $p^{m+\mathrm{lip}}_K$, $K \subseteq \mathsf{M}$ compact, defines a locally convex topology for $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$, which we call the **$\mathrm{C}^{m+\mathrm{lip}}$*-topology***, having the following attributes:

1. it is Hausdorff, metrisable, and complete, i.e., it is a Fréchet topology;
2. it is separable;
3. it is characterised by the sequences converging to zero, which are the sequences $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ such that, for each $K \subseteq \mathsf{M}$, the sequence $(j_m\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero in both seminorms λ^m_K and p^m_K;
4. if M is compact, then $p^{m+\mathrm{lip}}_{\mathsf{M}}$ is a norm that gives the $\mathrm{C}^{m+\mathrm{lip}}$-topology.

We refer to [6, §3.5] for details.

2.2.5 Seminorms for Spaces of Holomorphic Vector Fields

Now we consider an holomorphic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ and denote by $\Gamma^{\mathrm{hol}}(\mathsf{E})$ the space of holomorphic sections of E. We let $\mathbb{G}$ be an Hermitian metric on the vector bundle and denote by $\|\cdot\|_{\mathbb{G}}$ the associated fibre norm. For $K \subseteq \mathsf{M}$ compact, denote by p_K^{hol} the seminorm

$$p_K^{\mathrm{hol}}(\xi) = \sup\{\|\xi(z)\|_{\mathbb{G}} \mid z \in K\}$$

on $\Gamma^{\mathrm{hol}}(\mathsf{E})$. The family of seminorms p_K^{hol}, $K \subseteq \mathsf{M}$ compact, defines a locally convex topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ that we call the **C$^{\mathbf{hol}}$*-topology***. This topology has the following properties:

1. it is Hausdorff, metrisable, and complete, i.e., it is a Fréchet topology;
2. it is separable;
3. it is nuclear;
4. it is characterised by the sequences converging to zero, which are the sequences $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ such that, for each $K \subseteq \mathsf{M}$, the sequence $(\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero;
5. if M is compact, then $p_{\mathsf{M}}^{\mathrm{hol}}$ is a norm that gives the C$^{\mathrm{hol}}$-topology.

We refer to [6, §4.2] and the references therein for details about the C$^{\mathrm{hol}}$-topology.

2.2.6 Seminorms for Spaces of Real Analytic Vector Fields

The topologies described above for spaces of smooth, finitely differentiable, Lipschitz, and holomorphic sections of a vector bundle are quite simple to understand in terms of their converging sequences. The topology one considers for real analytic sections does not have this attribute. There is a bit of a history to the characterisation of real analytic topologies, and we refer to [6, §5] for *four* equivalent characterisations of the real analytic topology for the space of real analytic sections of a vector bundle. Here we will give the most elementary of these definitions to state, although it is probably not the most practical definition. In practice, it is probably best to somehow complexify and use the holomorphic topology; we give instances of this in Theorems 3.9 and 3.17 below.

In this section we let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\Gamma^{\omega}(\mathsf{E})$ be the space of real analytic sections. One can show that there exist a real analytic linear connection ∇^0 on E, a real analytic affine connection ∇ on M, a real analytic fibre metric on E, and a real analytic Riemannian metric on M [6, Lemma 2.3]. Thus we can define real analytic fibre metrics $\overline{\mathbb{G}}_m$ on the jet bundles $\mathsf{J}^m\mathsf{E}$ as in Sect. 2.2.1.

To define seminorms for $\Gamma^{\omega}(\mathsf{E})$, let $c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ denote the space of sequences in $\mathbb{R}_{>0}$, indexed by $\mathbb{Z}_{\geq 0}$, and converging to zero. We shall denote a typical element of $c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ by $\mathbf{a} = (a_j)_{j \in \mathbb{Z}_{\geq 0}}$. Now, for $K \subseteq \mathsf{M}$ and $\mathbf{a} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, we define a seminorm $p^{\omega}_{K,\mathbf{a}}$ for $\Gamma^{\omega}(\mathsf{E})$ by

$$p^{\omega}_{K,\mathbf{a}}(\xi) = \sup\{a_0 a_1 \cdots a_m \| j_m \xi(x) \|_{\overline{\mathbb{G}}_m} \mid x \in K,\ m \in \mathbb{Z}_{\geq 0}\}.$$

The family of seminorms $p^{\omega}_{K,\mathbf{a}}$, $K \subseteq \mathsf{M}$ compact, $\mathbf{a} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, defines a locally convex topology on $\Gamma^{\omega}(\mathsf{E})$ that we call the **C$^{\omega}$*-topology***. This topology has the following attributes:

1. it is Hausdorff and complete;
2. it is not metrisable (and so it not a Fréchet topology);
3. it is separable;
4. it is nuclear.

We shall generally avoid dealing with the rather complicated structure of this topology, and shall be able to do what we need by just working with the seminorms. That this is possible is one of the main contributions of the work [6].

2.2.7 Summary and Notation

In the real case, the degrees of regularity are ordered according to

$$\mathrm{C}^0 \supset \mathrm{C}^{\mathrm{lip}} \supset \mathrm{C}^1 \supset \cdots \supset \mathrm{C}^m \supseteq \mathrm{C}^{m+\mathrm{lip}} \supset \mathrm{C}^{m+1} \supset \cdots \supset \mathrm{C}^{\infty} \supset \mathrm{C}^{\omega},$$

and in the complex case the ordering is the same, of course, but with an extra $\mathrm{C}^{\mathrm{hol}}$ on the right. Sometimes it will be convenient to write $\nu + \mathrm{lip}$ for $\nu \in \{\mathbb{Z}_{\geq 0}, \infty, \omega\}$, and in doing this we adopt the obvious convention that $\infty + \mathrm{lip} = \infty$ and $\omega + \mathrm{lip} = \omega$.

Where possible, we will state definitions and results for all regularity classes at once. To do this, we will let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, and consider the regularity classes $\nu \in \{m + m', \infty, \omega\}$. In such cases we shall require that the underlying manifold be of class "C^r, $r \in \{\infty, \omega\}$, as required". This has the obvious meaning, namely that we consider class C^{ω} if $\nu = \omega$ and class C^{∞} otherwise. Proofs will typically break into the four cases $\nu = \infty$, $\nu = m$, $\nu = m + \mathrm{lip}$, and $\nu = \omega$. In some cases there is a structural similarity in the way arguments are carried out, so we will sometimes do all cases at once. In doing this, we will, for $K \subseteq \mathsf{M}$ be compact, for $k \in \mathbb{Z}_{\geq 0}$, and for $\mathbf{a} \in c_0(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, denote

$$p_K = \begin{cases} p^{\infty}_{K,k}, & \nu = \infty, \\ p^{m}_{K}, & \nu = m, \\ p^{m+\mathrm{lip}}_{K}, & \nu = m + \mathrm{lip}, \\ p^{\omega}_{K,\mathbf{a}}, & \nu = \omega. \end{cases} \tag{2.2}$$

The convenience and brevity more than make up for the slight loss of preciseness in this approach.

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