Chapter 2
Stability

2.1 Introduction

It is well known that stability is an essential and important problem in control theory and dynamic system analysis. Under the assumption that the TRM of a MJS is exactly given in advance, many results on the stability of SMJSs have been achieved [1–5]. However, this assumption may not be satisfied in many practical applications. In fact, the corresponding TRM may be uncertain, partially unknown or designed without being given beforehand. Clearly, such general TRMs play important roles in the analysis of MJSs. When there are uncertainties in a TRM, it may lead to instability or destroy the system performance if the uncertainties are not considered. For normal state-space MJSs with uncertain TRM, some results were available in [6, 7]. Via considering the inherent probability constraints on rows of the TRM, improved results were proposed in [8, 9], while the LMI conditions were presented in [10]. When only a subset of the elements of an TRM are unknown, some results were developed in [11, 12] by separating known elements from the unknown ones. Improved results about the general case were further developed in [13, 14]. When the TRM is selected, [15] first considered the stabilization problem via designing the TRM and static output feedback gain simultaneously. By analysing the existing work for normal state-space MJSs, it is easy to see that most approaches employed to deal with such general cases cannot be extended to SMJSs.

In this chapter, the fundamental problem of stability for SMJSs with these general TRMs is considered. Since SMJSs contain both singular derivative matrix and Markov property, analysis on SMJSs becomes much different from normal state-space MJSs and it is usually more complicated. Therefore, it is necessary to develop conditions such that the considered system is not only stable but also regular and impulse-free over such general TRMs. This chapter will focus on the investigation on developing these conditions. Most results will be developed in terms of LMIs or LMIs with equation constraints. Moreover, the proposed approaches are to be extended to the problem of robust stability of Markovian jump singularly perturbed descriptor systems with uncertain switchings and nonlinear perturbations for any

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$\varepsilon \in (0, \bar{\varepsilon}]$. LMI conditions depending on $\bar{\varepsilon}$ instead of ε are obtained by exploiting an ε-dependent Lyapunov function such that the existence and uniqueness of a solution in addition to exponential stability in mean square can be guaranteed.

2.2 Stability with General TRMs

Consider a class of continuous-time SMJSs described as

$$E\dot{x}(t) = A(r_t)x(t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $E \in \mathbb{R}^{n\times n}$ may be singular with $\text{rank}(E) = r \le n$. The symbol $A(r_t)$ is a known matrix with compatible dimensions. Operation mode $\{(r_t), t \ge 0\}$ is a right-continuous Markov process taking values in a finite space $\mathbb{S} = \{1, 2, \ldots, N\}$ with TRM $\Pi = (\pi_{ij}) \in \mathbb{R}^{N\times N}$ given as

$$Pr\{r_{t+h} = j | r_t = i\} = \begin{cases} \pi_{ij}h + o(h) & i \ne j, \\ 1 + \pi_{ii}h + o(h) & i = j, \end{cases} \tag{2.2}$$

where $h > 0$, $\lim_{h\to 0^+}(o(h)/h) = 0$, and $\pi_{ij} \ge 0$, for $i \ne j$, are the transition rates from mode i at time t to mode j at time $t + h$, which satisfy

$$\pi_{ii} = -\sum_{j=1,\ j\ne i}^{N} \pi_{ij}. \tag{2.3}$$

For the simplification of notation in the subsequent analysis, for each possible $r_t = i \in \mathbb{S}$, the matrix $M(r_t)$ will be denoted by M_i and so on.

It is well known that singular systems including SMJSs usually have three types of modes: finite dynamic modes, infinite dynamic modes (impulsive modes) and non-dynamic modes. For any finite initial conditions, the time response of a singular systems may exhibit impulsive or non-causal behaviour. The undesired impulsive behaviour in a singular system results from the infinite dynamic modes. Sometimes, even if a singular system is impulse-free, there are still initial finite discontinuities because of the inconsistent initial conditions. Moreover, since both infinite dynamic modes and non dynamic modes are included in a singular system, the existence and uniqueness of a solution to a given singular system is not always guaranteed. Therefore, it is extremely important to develop conditions to guarantee that the considered singular system is not only stable but also regular and impulse-free.

For SMJS (2.1), the following definition is introduced:

Definition 2.1 [16, 17] System (2.1) or the matric pair (E, A_i), $\forall i \in \mathbb{S}$, is said to be

(1) regular if $det(sE - A_i)$ is not identically zero for every $i \in \mathbb{S}$;
(2) impulse-free if $deg(det(sE - A_i)) = rank(E)$ for every $i \in \mathbb{S}$;

(3) stochastically stable if

$$\mathscr{E}\left\{\int_0^\infty x^T(t)x(t)dt|x_0, r_0\right\} \leq M(x_0, r_0),$$

holds for any initial condition $x_0 \in \mathbb{R}^n$ and $\eta_0 \in \mathbb{S}$, where $M(\phi(t), \eta_0) > 0$ is a given constant;

(4) stochastically admissible if it is regular, impulse-free and stochastically stable.

Remark 2.1 It should be noted that the regularity of the pair (E, A_i) guarantees that system (2.1) has a unique solution for any specified initial conditions. Moreover, from Definition 2.1, it follows that the non-impulsiveness of the pair (E, A_i) implies regularity of the pair (E, A_i).

For system (2.1), several sets of necessary and sufficient conditions for stochastic admissibility are provided in the following.

Lemma 2.1 *[16, 17] System (2.1) is stochastically admissible if and only if there exists matrix P_i such that*

$$E^T P_i = P_i^T E \geq 0, \tag{2.4}$$

$$\left(A_i^T P_i\right)^\star + \sum_{j=1}^{N} \pi_{ij} E^T P_j < 0. \tag{2.5}$$

Lemma 2.2 *[18] System (2.1) is stochastically admissible if and only if there exist matrices $P_i > 0$ and Q_i such that*

$$\left(A_i^T P_i E + A_i^T U^T Q_i V^T\right)^\star + \sum_{j=1}^{N} \pi_{ij} E^T P_j E < 0, \tag{2.6}$$

where $U \in \mathbb{R}^{(n-r)\times n}$ is any matrix with full row rank satisfying $UE = 0$, and $V \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank satisfying $EV = 0$.

Now, two sets of necessary and sufficient conditions for stochastic admissibility of system (2.1), where system matrix A_i is separated from Lyapunov matrix P_i will be presented.

Theorem 2.1 *System (2.1) is stochastically admissible if and only if there exist matrices P_i, G_i and Z_i such that*

$$E^T P_i = P_i^T E \geq 0, \tag{2.7}$$

$$\begin{bmatrix} \left(A_i^T G_i\right)^\star + \sum_{j=1}^{N} \pi_{ij} E^T P_j & A_i^T Z_i + P_i^T - G_i^T \\ * & (-Z_i)^\star \end{bmatrix} < 0. \tag{2.8}$$

Proof In order to prove this theorem, the fact that (2.8) is equivalent to (2.5), is proved at first.
Sufficiency: Pre- and post-multiplying (2.8) by matrix

$$\left[\, I \;\; A_i^T \,\right],$$

and its transpose respectively, it is straightforward to see that (2.5) holds.
Necessity: Assume (2.5) holds. Then there is always a sufficient small scalar $\varepsilon_i > 0$ such that

$$\left(A_i^T P_i\right)^{\star} + \sum_{j=1}^{N} \pi_{ij} E^T P_j + A_i^T \frac{\varepsilon_i}{2} A_i < 0. \tag{2.9}$$

Let $\varepsilon_i I = Z_i$ and $P_i = G_i$. Using the Schur complements, (2.9) can be rewritten as (2.8). It is concluded that (2.7) and (2.8) are equivalent to (2.4) and (2.5) respectively.

From Lemma 2.1, it easy to see that system (2.1) is stochastically stable if and only if the conditions (2.7) and (2.8) hold. This completes the proof.

Theorem 2.2 *System (2.1) is stochastically admissible if and only if there exist matrices $P_i > 0$, Q_i, G_i and Z_i such that*

$$\begin{bmatrix} \left(A_i^T G_i\right)^{\star} + \sum_{j=1}^{N} \pi_{ij} E^T P_j E & A_i^T Z_i + \left(P_i E + U^T Q_i V^T\right)^T - G_i^T \\ * & (-Z_i)^{\star} \end{bmatrix} < 0. \tag{2.10}$$

where $U \in \mathbb{R}^{(n-r)\times n}$ and $V \in \mathbb{R}^{n\times(n-r)}$ are defined in Lemma 2.2.

Proof Similar to the proof of Theorem 2.1, it is concluded that (2.8) is equivalent to (2.6). By Lemma 2.2, it is easy to see that system (2.1) is stochastically admissible if and only if (2.9) holds for each $i \in \mathbb{S}$. This completes the proof.

Remark 2.2 Both Theorems 2.1 and 2.2 give necessary and sufficient conditions for system (2.1) being stochastically admissible. Especially, the conditions in Theorem 2.2 are strict LMIs, which can be solved directly. It should be noted that Theorems 2.1 and 2.2 are different from Lemmas 2.1 and 2.2 in which A_i and P_i are decoupled. Sometimes, this decoupling is very helpful to deal with many general cases such as a mode-independent case. When $E = I$, system (2.1) reduces to state-space MJSs, and similar results can be obtained directly.

By investigating conditions (2.5), (2.6), (2.8) and (2.10), it is observed that transition rate π_{ij} plays an important role in system analysis where the transition rate π_{ij} is exactly known and given beforehand. When it is not accessible, the aforementioned conditions should be reconsidered. In this sense, these results developed under the condition that the TRM is exact known, cannot be applied and thus their application is limited. Next, four cases of TRM Π including some general cases are described as follows:

Case 1 Π is assumed to be known exactly, which is described by (2.2);

Case 2 Π is obtained inexactly and has admissible uncertainty

$$\Pi = \tilde{\Pi} + \Delta\tilde{\Pi}, \tag{2.11}$$

in which $\tilde{\Pi} \triangleq (\tilde{\pi}_{ij})$ is an estimation of the known constant Π, and $\Delta\tilde{\Pi} \triangleq (\Delta\tilde{\pi}_{ij})$ with $\Delta\tilde{\pi}_{ij} \triangleq \pi_{ij} - \tilde{\pi}_{ij}$ denotes the estimated error, which satisfies (2.2). It is assumed that $\Delta\tilde{\pi}_{ij}$, $j \neq i$, takes any value in $[-\varepsilon_{ij}, \varepsilon_{ij}]$, and $\alpha_{ij} \triangleq \tilde{\pi}_{ij} - \varepsilon_{ij}$. Moreover, it is obtained that $|\Delta\tilde{\pi}_{ii}| \leq -\varepsilon_{ii}$, where $\varepsilon_{ii} \triangleq -\sum_{j=1, j\neq i}^{N} \varepsilon_{ij}$;

Case 3 Π with property (2.2) is partially known or accessible, in which some elements are unknown. For example, a partly unknown Π may be expressed as

$$\Pi = \begin{bmatrix} \pi_{11} & ? & \pi_{13} & ? \\ ? & ? & ? & \pi_{24} \\ \pi_{31} & ? & ? & \pi_{34} \\ ? & ? & \pi_{43} & \pi_{44} \end{bmatrix},$$

where '?' represents the unknown elements. Based on this, for any $i \in \mathbb{S}$, define $\mathbb{S}^i = \mathbb{S}_k^i + \mathbb{S}_{\bar{k}}^i$ where

$$\mathbb{S}_k^i = \{j : \pi_{ij} \text{ is known}\} \textit{ and } \mathbb{S}_{\bar{k}}^i = \{j : \pi_{ij} \text{ is unknown}\}, \tag{2.12}$$

They are further described, respectively, by

$$\mathbb{S}_k^i = \{k_1^i, \ldots, k_m^i\} \text{ and } \mathbb{S}_{\bar{k}}^i = \{\bar{k}_1^i, \ldots, \bar{k}_{N-m}^i\}, \tag{2.13}$$

where $k_m^i \in \mathbb{Z}^+$ is the column index of the mth known element in the ith row of Π, and the column index of the $(N-m)$th unknown element in the same row is denoted as $\bar{k}_{N-m}^i \in \mathbb{Z}^+$. In addition, $\tau = \min_{i \in \mathbb{S}_{\bar{k}}^i} \{\pi_{ii}\}$ is assumed to be known;

Case 4 Π in (2.2) is to be designed instead of being given beforehand.

When the admissible uncertainty described by (2.11) is added to an MJS, the system performance will be reduced and the system may be even unstable. For normal state-space MJSs, there have been many references [6–10] reporting the relevant study. But the referred approaches employed to deal with (2.11) cannot be applied to SMJSs. The main reason is that these methods require additional assumption on Lyapunouv matrix which should be satisfied firstly. It is not true for SMJSs because the underlying matrix is non-singular. Moreover, even if the above assumption holds, new problems such as the decoupling problem of system design emerge. Considering this, in the following, sufficient conditions for stochastic admissibility of SMJSs with uncertain TRM are established, some of which are within LMI framework.

Theorem 2.3 *System (2.1) is stochastically admissible under Case 2 if there exist matrices P_i, $W_i = W_i^T$ and $T_i > 0$, such that*

$$E^T P_i = P_i^T E \geq 0, \tag{2.14}$$

$$\begin{bmatrix} \Omega_i & W_i \\ * & -T_i \end{bmatrix} < 0, \tag{2.15}$$

$$E^T P_j - E^T P_i - W_i \leq 0,\ j \in \mathbb{S},\ j \neq i, \tag{2.16}$$

where

$$\Omega_i = \left(A_i^T P_i\right)^{\star} + 0.25\varepsilon_{ii}^2 T_i - \varepsilon_{ii} W_i + \sum_{j=1, j\neq i}^{N} \alpha_{ij} E^T (P_j - P_i).$$

Proof From Lemma 2.1 and (2.11), it follows that (2.5) is equivalent to

$$\begin{aligned} &\left(A_i^T P_i\right)^{\star} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T (P_j - P_i) - \Delta\tilde{\pi}_{ii} W_i - \varepsilon_{ii} W_i \\ &\quad + \sum_{j=1,\ j\neq i}^{N} (\Delta\tilde{\pi}_{ij} + \varepsilon_{ij}) \left(E^T P_j - E^T P_i - W_i\right) < 0, \end{aligned} \tag{2.17}$$

which is guaranteed by

$$\left(A_i^T P_i\right)^{\star} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T (P_j - P_i) - \Delta\tilde{\pi}_{ii} W_i - \varepsilon_{ii} W_i < 0, \tag{2.18}$$

$$\sum_{j=1,\ j\neq i}^{N} (\Delta\tilde{\pi}_{ij} + \varepsilon_{ij}) \left(E^T P_j - E^T P_i - W_i\right) \leq 0. \tag{2.19}$$

Furthermore, for any $T_i > 0$,

$$-\Delta\tilde{\pi}_{ii} W_i \leq 0.25(\Delta\tilde{\pi}_{ii})^2 T_i + W_i T_i^{-1} W_i \leq 0.25\varepsilon_{ii}^2 T_i + W_i T_i^{-1} W_i. \tag{2.20}$$

Taking into account (2.20), it is easy to see that condition (2.15) implies (2.18). On the other hand, from (2.11), it is seen that (2.16) implies (2.19). Thus, (2.4) and (2.5) are guaranteed by (2.14)–(2.16). This completes the proof.

Theorem 2.4 *System (2.1) is stochastically admissible under Case 2 if there exist matrices $P_i > 0$, Q_i, $W_i = W_i^T$ and $T_i > 0$, such that*

$$\left(A_i^T P_i E + A_i^T U^T Q_i V^T\right)^{\star} + 0.25\varepsilon_{ii}^2 T_i - \varepsilon_{ii} W_i + \sum_{j=1, j\neq i}^{N} \alpha_{ij} E^T (P_j - P_i) E < 0, \tag{2.21}$$

$$E^T P_j E - E^T P_i E - W_i \leq 0,\ j \in \mathbb{S},\ j \neq i. \tag{2.22}$$

Proof Similar to the proof of Theorem 2.3, it is obtained that (2.6) is equivalent to

$$\begin{aligned}\left(A_i^T P_i E + A_i^T U^T Q_i V^T\right)^{\star} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T (P_j - P_i) E - \Delta\tilde{\pi}_{ii} W_i - \varepsilon_{ii} W_i \\ + \sum_{j=1,\ j\neq i}^{N} (\Delta\tilde{\pi}_{ij} + \varepsilon_{ij}) \left(E^T P_j E - E^T P_i E - W_i\right) < 0,\end{aligned} \tag{2.23}$$

which is ensured by

$$\left(A_i^T P_i E + A_i^T U^T Q_i V^T\right)^{\star} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T (P_j - P_i) E - \Delta\tilde{\pi}_{ii} W_i - \varepsilon_{ii} W_i < 0, \tag{2.24}$$

$$\sum_{j=1,\ j\neq i}^{N} (\Delta\tilde{\pi}_{ij} + \varepsilon_{ij}) \left(E^T P_j E - E^T P_i E - W_i\right) \leq 0. \tag{2.25}$$

Based on (2.20), the condition (2.24) is guaranteed by (2.21). On the other hand, it is obvious that (2.22) implies (2.25). This completes the proof.

Based on Theorems 2.1 and 2.2 and by the method used for handling (2.11), the following theorems are ready to be presented:

Theorem 2.5 *System (2.1) is stochastically admissible under Case 2 if there exist matrices P_i, G_i, Z_i, $W_i = W_i^T$ and $T_i > 0$, such that*

$$E^T P_i = P_i^T E \geq 0, \tag{2.26}$$

$$\begin{bmatrix} \bar{\Omega}_i & A_i^T Z_i + P_i^T - G_i^T \\ * & (-Z_i)^{\star} \end{bmatrix} < 0, \tag{2.27}$$

$$E^T P_j - E^T P_i - W_i \leq 0,\ j \in \mathbb{S},\ j \neq i, \tag{2.28}$$

where

$$\bar{\Omega}_i = \left(A_i^T G_i\right)^{\star} + 0.25\varepsilon_{ii}^2 T_i - \varepsilon_{ii} W_i + \sum_{j=1, j\neq i}^{N} \alpha_{ij} E^T (P_j - P_i) < 0.$$

Proof From the proof of Theorem 2.1, it clear to see that the difference between Cases 1 and 2 only lies in π_{ij} which is related to $\sum_{j=1}^{N} \pi_{ij} E^T P_j$. Using the same method used for uncertain TRM, this theorem can be proved easily. This completes the proof.

Theorem 2.6 *System (2.1) is stochastically admissible under Case 2 if there exist matrices $P_i > 0$, Q_i, G_i, Z_i, $W_i = W_i^T$ and $T_i > 0$, such that*

$$\begin{bmatrix} \tilde{\Omega}_i & A_i^T Z_i + \left(P_i E + U^T Q_i V^T\right)^T - G_i^T \\ * & (-Z_i)^\star \end{bmatrix} < 0, \tag{2.29}$$

$$E^T P_j E - E^T P_i E - W_i \leq 0,\ j \in \mathbb{S},\ j \neq i, \tag{2.30}$$

where

$$\tilde{\Omega}_i = (A_i^T G_i)^\star + 0.25\varepsilon_{ii}^2 T_i - \varepsilon_{ii} W_i + \sum_{j=1, j\neq i}^{N} \alpha_{ij} E^T (P_j - P_i) E < 0.$$

Proof The proof can be obtained by Theorems 2.2 and 2.4, which is omitted here. This completes the proof.

Remark 2.3 Via using a slack variable method on TRM, several sets of conditions are established to ensure that system (2.1) with uncertain TRM is stochastic admissible, in which some results are in traditional LMI forms. Clearly, there is no additional restriction on system matrix P_i in these conditions. Moreover, such results are applicable to discuss system synthesis problems and the couplings among uncertain transition rates, singular matrix, system and Lyapunov matrices are decoupled and dealt with appropriately.

Next, the stochastic admissibility of system (2.1) under Case 3 will be considered.

Theorem 2.7 *System (2.1) is stochastically admissible under Case 3 if there exist matrices P_i and $W_i = W_i^T$ such that*

$$E^T P_i = P_i^T E \geq 0, \tag{2.31}$$

$$\left(A_i^T P_i\right)^\star + \sum_{j\in\mathbb{S}_k^i, j\neq i} \pi_{ij} E^T (P_j - P_i) + \sum_{j\in\mathbb{S}_k^i} \pi_{ij} W_i < 0,\ i \in \mathbb{S}_k^i, \tag{2.32}$$

$$\left(A_i^T P_i\right)^\star + \sum_{j\in\mathbb{S}_k^i, j\neq i} \pi_{ij} \left[E^T (P_j - P_i) + W_i\right] - \tau W_i < 0,\ i \in \bar{\mathbb{S}}_k^i, \tag{2.33}$$

$$E^T P_j - E^T P_i - W_i \leq 0,\ i \in \mathbb{S},\ j \in \bar{\mathbb{S}}_k^i,\ j \neq i. \tag{2.34}$$

Proof For any $W_i = W_i^T$, it is known that

$$\begin{aligned}\left(A_i^T P_i\right)^\star + \sum_{j=1}^{N} \pi_{ij} E^T P_j &= \left(A_i^T P_i\right)^\star + \sum_{j\in\mathbb{S}, j\neq i} \pi_{ij} E^T (P_j - P_i) - \sum_{j=1}^{N} \pi_{ij} W_i \\ &= \left(A_i^T P_i\right)^\star + \sum_{j\in\mathbb{S}_k^i, j\neq i} \pi_{ij} \left[E^T (P_j - P_i) - W_i\right] \\ &\quad + \sum_{j\in\bar{\mathbb{S}}_k^i, j\neq i} \left[E^T (P_j - P_i) - W_i\right] - \pi_{ii} W_i < 0,\end{aligned} \tag{2.35}$$

which is guaranteed by

$$\left(A_i^T P_i\right)^\star + \sum_{j\in\mathbb{S}_k^i, j\neq i} \pi_{ij} \left[E^T (P_j - P_i) - W_i\right] - \pi_{ii} W_i < 0, i \in \mathbb{S}, \tag{2.36}$$

and (2.34). When $i \in \mathbb{S}_k^i$, (2.36) is rewritten to (2.32). To the contrary, if $i \in \bar{\mathbb{S}}_k^i$, it is obtained that (2.33) implies (2.36). This completes the proof.

Similarly, the following theorems can be obtained directly.

Theorem 2.8 *System (2.1) is stochastically admissible under Case 3 if there exist matrices $P_i > 0$, Q_i and $W_i = W_i^T$ such that*

$$\left(A_i^T P_i E + A_i^T U^T Q_i V^T\right)^\star + \sum_{j\in\mathbb{S}_k^i, j\neq i} \pi_{ij} E^T (P_j - P_i) E + \sum_{j\in\mathbb{S}_k^i} \pi_{ij} W_i < 0, i \in \mathbb{S}_k^i, \tag{2.37}$$

$$\left(A_i^T P_i E + A_i^T U^T Q_i V^T\right)^\star + \sum_{j\in\mathbb{S}_k^i, j\neq i} \pi_{ij} \left[E^T (P_j - P_i) E + W_i\right] - \tau W_i < 0, i \in \bar{\mathbb{S}}_k^i, \tag{2.38}$$

$$E^T P_j E - E^T P_i E - W_i \leq 0, i \in \mathbb{S}, j \in \bar{\mathbb{S}}_k^i, j \neq i. \tag{2.39}$$

Theorem 2.9 *System (2.1) is stochastically admissible under Case 3 if there exist matrices P_i, G_i, Z_i and $W_i = W_i^T$, such that*

$$E^T P_i = P_i^T E \geq 0, \tag{2.40}$$

$$\begin{bmatrix} \hat{\Omega}_i & A_i^T Z_i + P_i^T - G_i^T \\ * & (-Z_i)^\star \end{bmatrix} < 0, i \in \mathbb{S}_k^i, \tag{2.41}$$

$$\begin{bmatrix} \check{\Omega}_i & A_i^T Z_i + P_i^T - G_i^T \\ * & (-Z_i)^\star \end{bmatrix} < 0, i \in \bar{\mathbb{S}}_k^i, \tag{2.42}$$

$$E^T P_j - E^T P_i - W_i \le 0, i \in \mathbb{S}, j \in \bar{\mathbb{S}}_k^i, j \ne i. \tag{2.43}$$

where

$$\hat{\Omega}_i = \left(A_i^T G_i\right)^\star + \sum_{j \in \mathbb{S}_k^i, j \ne i} \pi_{ij} E^T (P_j - P_i) + \sum_{j \in \mathbb{S}_k^i} \pi_{ij} W_i,$$
$$\check{\Omega}_i = \left(A_i^T G_i\right)^\star + \sum_{j \in \mathbb{S}_k^i, j \ne i} \pi_{ij} \left[E^T (P_j - P_i) + W_i\right] - \tau W_i.$$

Theorem 2.10 *System (2.1) is stochastically admissible under Case 3 if there exist matrices $P_i > 0$, Q_i, G_i, Z_i and $W_i = W_i^T$, such that*

$$\begin{bmatrix} \Theta_i & A_i^T Z_i + \left(P_i E + U^T Q_i V^T\right)^T - G_i^T \\ * & (-Z_i)^\star \end{bmatrix} < 0, i \in \mathbb{S}_k^i, \tag{2.44}$$

$$\begin{bmatrix} \bar{\Theta}_i & A_i^T Z_i + \left(P_i E + U^T Q_i V^T\right)^T - G_i^T \\ * & (-Z_i)^\star \end{bmatrix} < 0, i \in \bar{\mathbb{S}}_k^i, \tag{2.45}$$

$$E^T P_j E - E^T P_i E - W_i \le 0, i \in \mathbb{S}, j \in \bar{\mathbb{S}}_k^i, j \ne i, \tag{2.46}$$

where

$$\left(A_i^T G_i\right)^\star + \sum_{j \in \mathbb{S}_k^i, j \ne i} \pi_{ij} E^T (P_j - P_i) E + \sum_{j \in \mathbb{S}_k^i} \pi_{ij} W_i,$$
$$\left(A_i^T G_i\right)^\star + \sum_{j \in \mathbb{S}_k^i, j \ne i} \pi_{ij} \left[E^T (P_j - P_i) E + W_i\right] - \tau W_i < 0.$$

Finally, consider the stochastic admissibility of system (2.1) under Case 4. From the criteria given above, it is seen that although the TRM may be exact known, uncertain or partially unknown, the proposed methods are all based on a precondition that all or some elements of an TRM are given beforehand. In some cases, an appropriate TRM may be selected for MJSs. From the results presented in this chapter, it is seen that for a given TRM, inequalities such as (2.5) are linear to matrix P_i. However, if TRM is unknown, characterization (2.5) turns out to be bilinear due to the product terms of non-singular matrix P_i and elements in Π. When similar problem is discussed in [15], the positive-definite property of P_i for normal state-space MJSs plays important roles in system analysis and synthesis. Due to the singular derivative matrix and the only non-singular Lyapunov matrix, the property of P_i is not true for SMJSs. Thus, such problems should be reconsidered for SMJSs.

Theorem 2.11 *There exists an TRM such that system (2.1) is stochastically admissible, if there exist matrices P_i, $\hat{\pi}_{ij} \ge 0$, $i \ne j$, $W_i > 0$ and $Z_i > 0$, such that*

$$E^T P_i = P_i^T E \ge 0, \tag{2.47}$$

$$\begin{bmatrix} \left(A_i^T P_i\right)^\star & \Omega_{i2} \\ * & \Omega_{i3} \end{bmatrix} < 0, \tag{2.48}$$

$$E^T P_j - E^T P_i - W_i \leq 0,\ j \in \mathbb{S},\ j \neq i, \tag{2.49}$$

$$W_i Z_i = I, \tag{2.50}$$

where

$$\begin{aligned} \Omega_{i2} &= \left[\, \hat{\pi}_{i1} I \ \ldots\ \hat{\pi}_{i(i-1)} I \ \hat{\pi}_{i(i+1)} I \ \ldots\ \hat{\pi}_{iN} I \,\right], \\ \Omega_{i3} &= -\mathrm{diag}\{Z_i, \ldots, Z_i\}. \end{aligned}$$

In this case, a stabilizing TRM is given as

$$\pi_{ij} = \hat{\pi}_{ij}^2,\ \pi_{ii} = -\sum_{i \neq j} \pi_{ij}. \tag{2.51}$$

Proof By Theorem 2.1 and conditions in Theorem 2.10, it is seen that only (2.5) needs to be proved. Similarly, it is equivalent to

$$\left(A_i^T P_i\right)^\star + \sum_{j=1,\ j\neq i}^{N} \pi_{ij} W_i + \sum_{j=1,\ j\neq i}^{N} \pi_{ij} \left(E^T P_j - E^T P_i - W_i\right) < 0. \tag{2.52}$$

which is guaranteed by

$$\left(A_i^T P_i\right)^\star + \sum_{j=1,\ j\neq i}^{N} \pi_{ij} W_i < 0, \tag{2.53}$$

$$\sum_{j=1,\ j\neq i}^{N} \pi_{ij} \left(E^T P_j - E^T P_i - W_i\right) \leq 0. \tag{2.54}$$

Based on (2.51), it is concluded that (2.48)–(2.50) imply (2.53) and (2.54). This completes the proof.

Remark 2.4 Theorem 2.11 gives an approach of designing a stabilizing TRM, in which the corresponding matrix P_i is not necessary positive-definite. In addition, this approach can be extended to the other system analysis and synthesis problems easily. In the case when $E = I$, Theorem 2.11 is used to deal with normal state-space MJSs with TRM designed. In this sense, this theorem can be considered as an extension of normal state-space MJSs to SMJSs.

Theorem 2.12 *There exists a TRM such that system (2.1) is stochastically admissible, if there exist matrices $P_i > 0$, Q_i, $\hat{\pi}_{ij} \geq 0$, $i \neq j$, $W_i > 0$ and $Z_i > 0$, such that*

$$\begin{bmatrix} \left(A_i^T\left(P_iE + U^TQ_iV^T\right)\right)^{\star} & \Omega_{i2} \\ * & \Omega_{i3} \end{bmatrix} < 0, \tag{2.55}$$

$$E^TP_jE - E^TP_iE - W_i \leq 0,\ j \in \mathbb{S},\ j \neq i, \tag{2.56}$$

$$W_iZ_i = I. \tag{2.57}$$

In addition, the stabilizing TRM can be calculated by (2.51).

Theorem 2.13 *There exists a TRM such that system (2.1) is stochastically admissible, if there exist matrices P_i, $\hat{\pi}_{ij} \geq 0$, $i \neq j$, $W_i > 0$ and $Z_i > 0$, such that*

$$E^TP_i = P_i^TE \geq 0, \tag{2.58}$$

$$\begin{bmatrix} \left(A_i^TG_i\right)^{\star} & A_i^TZ_i + P_i^T - G_i^T & \Omega_{i2} \\ * & (-Z_i)^{\star} & 0 \\ * & * & \Omega_{i3} \end{bmatrix} < 0, \tag{2.59}$$

$$E^TP_j - E^TP_i - W_i \leq 0,\ j \in \mathbb{S},\ j \neq i, \tag{2.60}$$

$$W_iZ_i = I. \tag{2.61}$$

Then, a stabilizing SPRM can be solved by (2.51).

Theorem 2.14 *There exists a TRM such that system (2.1) is stochastically admissible, if there exist matrices $P_i > 0$, Q_i, $\hat{\pi}_{ij} \geq 0$, $i \neq j$, $W_i > 0$ and $Z_i > 0$, such that*

$$\begin{bmatrix} \left(A_i^TG_i\right)^{\star} & A_i^TZ_i + \left(P_iE + U^TQ_iV^T\right)^T - G_i^T & \Omega_{i2} \\ * & (-Z_i)^{\star} & 0 \\ * & * & \Omega_{i3} \end{bmatrix} < 0, \tag{2.62}$$

$$E^TP_jE - E^TP_iE - W_i \leq 0,\ j \in \mathbb{S},\ j \neq i, \tag{2.63}$$

$$W_iZ_i = I. \tag{2.64}$$

Then, (2.51) is used to compute a stabilizing SPRM.

Remark 2.5 It can be seen that both Theorems 2.11 and 2.14 are proposed as a set of LMIs with equation constraints such as (2.50) and cannot be solved directly because of such non-convex conditions. However, there are many existing numerical approaches to deal with this problem. Among those approaches, LMI-based approaches are favourable and promising. Both cone complementarity linearization (CCL) algorithm [19] and sequential linear programming matrix (SLPM) algorithm [20] can be easily to solve the inversion constraints.

In order to utilize the CCL algorithm to solve the proposed problem, we first define a convex set of all the feasible solutions of LMIs (2.47)–(2.49) as follows:

$$\mathscr{S} \triangleq \{\mathscr{X} \mid \mathscr{X} \ \textit{satisfies LMIs}\ (2.47)\text{–}(2.49)\}, \tag{2.65}$$

where

$$\begin{aligned}\mathscr{X} \triangleq \{&\hat{P}_i = \hat{P}_i^T, W_i > 0, Z_i > 0,\\ &\hat{\pi}_{ij} \geq 0, \forall i, j \in \mathbb{S}, j \neq i\}.\end{aligned} \tag{2.66}$$

It is known that for any matrices $W_i > 0$ and $Z_i > 0$, $i \in \mathbb{S}$, if LMI

$$\begin{bmatrix} W_i & I \\ I & Z_i \end{bmatrix} \geq 0, \tag{2.67}$$

is feasible, then $\text{Trace}(W_i Z_i) \geq n$, and $\text{Trace}(W_i Z_i) = n$ if and only if $W_i Z_i = I$. Define a set as

$$\mathscr{T} \triangleq \left\{ \begin{bmatrix} W_i & I \\ I & Z_i \end{bmatrix} \geq 0, \ for\ all\ i \in \mathbb{S} \right\}. \tag{2.68}$$

By the CCL approach, the above non-convex problem of (2.50) is equivalent to the following minimization problem:

$$\min_{\mathscr{X} \in \mathscr{S} \cap \mathscr{T}} \text{Trace} \sum_{i=1}^{N} W_i Z_i. \tag{2.69}$$

It is seen that the optimal solution to problem (2.50) is $N\tilde{n}$ satisfying

$$\text{Trace}(W_i Z_i) = \tilde{n}, \forall i \in \mathbb{S}. \tag{2.70}$$

Based on the analysis, a computational algorithm to solve this problem can be proposed.

Algorithm 2.1 is described as follows:

Step 1: Given system (2.1) with given γ and error accuracy δ;
Step 2: Find any initial solution $\mathscr{X}_0 \in \mathscr{S}$, and set $k = 0$;
Step 3: Define function

$$f_k(\mathscr{X}) = \text{Trace}\left(\Sigma_{i=1}^{N} (W_i Z_{ik} + Z_i W_{ik})\right). \tag{2.71}$$

Find $\mathscr{X}$ via solving the following convex programming:

$$\min_{\mathscr{X} \in \mathscr{S}} \{f_k(\mathscr{X}) | \begin{bmatrix} W_i & I \\ I & Z_i \end{bmatrix} \geq 0, \forall i \in \mathbb{S}\}; \tag{2.72}$$

Step 4: If $|f_k(\mathscr{X}) - 2N\tilde{n}| \leq \delta$, a stabilizing TRM Π can be got by (2.51), and then exit; otherwise, go to step 5;

Step 5: Let $W_{i(k+1)} = W_{ik}$, $Z_{i(k+1)} = Z_{ik}$ and $k = k + 1$. If $k < k_{max}$, then go to step 3, else exit.

In this section, the stability problem of SMJS (2.1) with TRM satisfying Cases 1–4 is considered. Different from the similar results in [16], the presented results here have the following properties: (1) Instead of assuming that the TRM of an SMJS is known exactly, the corresponding TRM of the results proposed in this section may be uncertain, partially known and designed; (2) Several sets of necessary and sufficient conditions for stochastic admissibility are established, where system matrix A_i and Lyapunov matrix P_i are decoupled successfully. This property is very suitable to system synthesis, such as stabilization via mode-independent controllers, partially mode-dependent H_∞ filtering and so on; (3) For Case 1, using the methods proposed in this section, the terms $E_i^T P_i$ and $\sum_{j=1}^N \pi_{ij} E_j^T P_j$ will not be enlarged by introducing additional variables and inequalities; (4) Without coupled LMIs, in this section, all the results are linear LMIs and can be solved easily; (5) Without transforming $E_i^T P_i = P_i^T E_i \geq 0$ into additional LMIs by minimizing a common scalar, another approach in this section is proposed to deal with such constraints and makes the conditions solved directly. In this section, different methods are developed to discuss stability problem under Case 1, and new problems in terms of TRM of SMJSs satisfying some general cases are studied by new techniques. It is obvious that the proposed results are more general and therefore can be considered as necessary supplementary to the existing results such as the ones developed in [16].

2.3 Robust Stability

Consider a general class of Markovian jump singularly perturbed descriptor systems described as

$$\begin{cases} E\dot{x}(t) = A_1(r_t)x(t) + A_2(r_t)z(t) + B_1(r_t)f_1(t, r_t, x, z), \\ \varepsilon\dot{z}(t) = A_3(r_t)x(t) + A_4(r_t)z(t) + B_2(r_t)f_2(t, r_t, x, z), \\ x(0) = x_0, \\ z(0) = z_0, \end{cases} \tag{2.73}$$

where $x(t) \in \mathbb{R}^n$ and $z(t) \in \mathbb{R}^m$ are the state vectors of slow and fast dynamics. Matrix $E \in \mathbb{R}^{n\times n}$ may be singular, which is assumed to be $rank(E) = r \leq n$. $A_1(r_t)$, $A_2(r_t)$, $A_3(r_t)$, $A_4(r_t)$, $B_1(r_t)$ and $B_2(r_t)$ are known matrices of compatible dimensions. Parameter $\{r_t, t \geq 0\}$ is defined by (2.2) and (2.3), whose TRM Π is imprecise and described in (2.11). For any $r_t = i \in \mathbb{S}$, $f_k(t, i, x, z)$, $k = 1, 2$, is a time-varying nonlinear perturbation with $f_k(t, i, 0, 0) = 0$ for all $t \geq 0$, which satisfies the following Lipschitz condition for all (t, x, z), $(t, \tilde{x}, \tilde{z}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$:

$$\|f_k(t,i,x,z) - f_k(t,i,\tilde{x},\tilde{z})\| \le \gamma_i \|F_{ki}(x-\tilde{x}) + G_{ki}(z-\tilde{z})\|, \ k = 1,2, \quad (2.74)$$

where $\gamma_i > 0$, F_{ki} and G_{ki} are constant matrices with appropriate dimensions. Moreover, from (2.74), it follows that

$$\|f_k(t,i,x,z)\| \le \gamma_i \|F_{ki}x + G_{ki}z\|, \quad k = 1,2, \quad (2.75)$$

In order to simplify notation, system (2.73) is rewritten as

$$\begin{cases} E_\varepsilon \dot{\xi}(t) = A(r_t)\xi(t) + B(r_t)f(t,r_t,\xi(t)), \\ \quad \xi(0) = \xi_0, \end{cases} \quad (2.76)$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \ f(t,r_t,\xi(t)) = \begin{bmatrix} f_1(t,r_t,x(t),z(t)) \\ f_2(t,r_t,x(t),z(t)) \end{bmatrix}, \ E_\varepsilon = \begin{bmatrix} E & 0 \\ 0 & \varepsilon I \end{bmatrix},$$
$$A(r_t) = \begin{bmatrix} A_1(r_t) & A_2(r_t) \\ A_3(r_t) & A_4(r_t) \end{bmatrix}, \ B(r_t) = \begin{bmatrix} B_1(r_t) & 0 \\ 0 & B_2(r_t) \end{bmatrix},$$

and $f(t,r_t,\xi)$ satisfies

$$f^T(t,r_t,\xi)f(t,r_t,\xi) \le \gamma(r_t)\xi^T F^T(r_t)F(r_t)\xi, \quad (2.77)$$

with

$$F(r_t) = \begin{bmatrix} F_1(r_t) & G_1(r_t) \\ F_2(r_t) & G_2(r_t) \end{bmatrix}.$$

Remark 2.6 Due to the presence of small parameter ε, system (2.73) will lead to ill-conditioned problem in system analysis and synthesis when ε tends to be zero. Description (2.73) is more general in terms of containing some special cases, for example, when $E = I$, system (2.73) without nonlinear perturbation becomes Markovian jump singularly perturbed systems with or without time delay [21–23]; when $\varepsilon = 0$ and there is no nonlinear perturbation, it will become singular Markovian jump systems [16–18, 24, 25]; when there is no jumping, system (2.73) belongs to singularly perturbed descriptor systems whose robust stability was considered in [26].

Consider system (2.73). The following definitions are introduced:

Definition 2.2 For any given $\varepsilon > 0$, the pair $(E_\varepsilon, A(r_t))$ is said to be:

(1) regular if $\det(sE_\varepsilon - A(r_t))$ is not identically zero for every $r_t \in \mathbb{S}$;
(2) impulse-free if $\deg(\det(sE_\varepsilon - A(r_t))) = \operatorname{rank}(E_\varepsilon)$ for every $r_t \in \mathbb{S}$.

Definition 2.3 System (2.73) with (2.11) and (2.74) is said to be exponentially mean-square stable, if there exist scalars $a > 0$ and $b > 0$ such that

$$\mathscr{E}\{\|\xi(t)\|^2|\xi_0, r_0\} \le ae^{-bt}\|\xi_0\|^2,$$

for any initial conditions $\xi_0 \in \mathbb{R}^{n+m}$ and $r_0 \in \mathbb{S}$.

From [17], it is seen that for any given $\varepsilon > 0$, there always exist two non-singular matrices $M(\varepsilon)$ and $N(\varepsilon)$ such that

$$M_\varepsilon E_\varepsilon N_\varepsilon = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; M_\varepsilon A_\varepsilon(r_t) N_\varepsilon = \begin{bmatrix} A_\varepsilon^1(r_t) & A_\varepsilon^2(r_t) \\ A_\varepsilon^3(r_t) & A_\varepsilon^4(r_t) \end{bmatrix}.$$

Then, the pair $(E_\varepsilon, A_\varepsilon(r_t))$ is impulse-free if and only if $A_\varepsilon^4(r_t)$ is non-singular for every $r_t \in \mathbb{S}$.

Remark 2.7 From Definition 2.2, it is concluded that, for any given $\varepsilon > 0$, impulse free implies regular. In addition, it is easy to verify that for any given $\varepsilon > 0$, the pair $(E_\varepsilon, A_\varepsilon(r_t))$ is regular and impulse-free if and only if the $(E, A_1(r_t))$ is regular and impulse-free.

Lemma 2.3 *(S-procedure lemma)[27] Let $\Omega_0(z)$ and $\Omega_1(z)$ be two arbitrary quadratic forms over $\mathbb{R}^s$. Then $\Omega_0(z) < 0$ for all $z \in \mathbb{R}^s - \{0\}$ satisfying $\Omega_1(z) \le 0$ if and only if there exists a scalar $\tau \ge 0$ such that*

$$\Omega_0(z) - \tau\Omega_1(z) < 0, \quad \forall z \in \mathbb{R}^s - \{0\}.$$

Lemma 2.4 *For any given positive scalar $\varepsilon \in (0, \bar{\varepsilon}]$, if*

$$\Omega_1 \ge 0, \tag{2.78}$$

$$\Omega_1 + \bar{\varepsilon}\Omega_2 \ge 0, \tag{2.79}$$

$$\Omega_1 + \bar{\varepsilon}\Omega_2 + \bar{\varepsilon}^2\Omega_3 > 0, \tag{2.80}$$

where Ω_1, Ω_2 and Ω_3 are symmetric matrices with appropriate dimensions, then

$$\Omega_1 + \varepsilon\Omega_2 + \varepsilon^2\Omega_3 > 0, \; \forall \varepsilon \in (0, \bar{\varepsilon}]. \tag{2.81}$$

Proof Since $\varepsilon \in (0, \bar{\varepsilon}]$, it is rewritten as $\varepsilon = \lambda\bar{\varepsilon}$ with $\lambda \in (0, 1]$. From (2.78) to (2.80), it follows that

$$(1-\lambda)\Omega_1 \ge 0, \tag{2.82}$$

$$\lambda(1-\lambda)\Omega_1 + (1-\lambda)\varepsilon\Omega_2 \ge 0, \tag{2.83}$$

$$\lambda^2\Omega_1 + \lambda\varepsilon\Omega_2 + \varepsilon^2\Omega_3 > 0, \tag{2.84}$$

which imply (2.81). This completes the proof.

Now, consider the regularity of system (2.73) in addition to free-impulse. Then, the uniqueness of the solution will be guaranteed.

Theorem 2.15 *If there exist matrices P_{i1}, P_{i3}, $U_i > 0$, $S_i > 0$, and scalar $\tau > 0$ such that*

$$E^T P_{i1} = P_{i1}^T E \geq 0, \tag{2.85}$$

$$E^T P_{j1} - E^T P_{i1} - U_i \leq 0, \tag{2.86}$$

$$\begin{bmatrix} \Theta_i & P_{i1}^T B_{i1} & U_i \\ * & -\tau I & 0 \\ * & * & -S_i \end{bmatrix} < 0, \tag{2.87}$$

where

$$\Theta_i = \left(A_{i1}^T P_{i1}\right)^{\star} + \left(A_{i3}^T P_{i3} E\right)^{\star} + 0.25\delta_{ii}^2 S_i - \delta_{ii} U_i + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T \left(P_{j1} - P_{i1}\right) + \tau\gamma_i^2 F_{i1}^T F_{i1}.$$

Then for any $\varepsilon > 0$, the pair $(E_\varepsilon, A(r_t))$ is regular and impulse-free for every $r_t \in \mathbb{S}$. Moreover, equation (2.73) or (2.76) with (2.11) and (2.74) has a unique solution on $[0, \infty)$.

Proof First of all, it is required to show that (2.17) and (2.87) imply inequality

$$\begin{bmatrix} \tilde{\Theta}_i & P_{i1}^T B_{i1} \\ * & -\tau I \end{bmatrix} < 0, \tag{2.88}$$

where

$$\tilde{\Theta}_i = \left(A_{i1}^T P_{i1}\right)^{\star} + \left(A_{i3}^T P_{i3} E\right)^{\star} + \sum_{j=1}^{N} \pi_{ij} E^T P_{j1} + \tau\gamma_i^2 F_{i1}^T F_{i1}.$$

It is easy to see that (2.88) is transformed into

$$\tilde{\Theta}_i + P_{i1}^T B_{i1} \tau^{-1} B_{i1}^T P_{i1} < 0. \tag{2.89}$$

which, under condition (2.11), is equivalent to

$$\begin{aligned} &\left(A_{i1}^T P_{i1}\right)^{\star} + \left(A_{i3}^T P_{i3} E\right)^{\star} + \sum_{j=1, j\neq i}^{N} \alpha_{ij} E^T (P_{j1} - P_{i1}) - \varepsilon_{ii} U_i + \tau\gamma_i^2 F_{i1}^T F_{i1} \\ &\quad + P_{i1}^T B_{i1} \tau^{-1} B_{i1}^T P_{i1} - \Delta\tilde{\pi}_{ii} U_i + \sum_{j=1, j\neq i}^{N} (\Delta\tilde{\pi}_{ij} + \varepsilon_{ij}) \left(E^T P_{j1} - E^T P_{i1} - U_i\right) < 0. \end{aligned} \tag{2.90}$$

For any $S_i > 0$, it is obtained that

$$\Delta\tilde{\pi}_{ii} U_i \leq 0.25(\Delta\tilde{\pi}_{ii})^2 S_i + U_i S_i^{-1} U_i \leq 0.25\varepsilon_{ii}^2 S_i + U_i S_i^{-1} U_i. \tag{2.91}$$

Substituting it into (2.90), it is verified that (2.86) and (2.87) imply (2.90) which is equivalent to (2.88).

Next, the objective to prove that the uniqueness of the solution to equation (2.73) with (2.11) and (2.74) on $[0, \infty)$ is guaranteed by (2.85) and (2.88). Let $t_0 = 0$ and define a sequence of stopping time

$$t_{k+1} = \inf\{t > t_k : r_t \neq r_{t_k}\},$$

for all $k \geq 0$. It is concluded that for any $k \geq 0$, $r_t = r_{t_k}$ is constant for all $t \in [t_k, t_{k+1})$ and $t_k \to \infty$ as $k \to \infty$. First, we show there is a unique solution to equation (2.73) with $r_t = i$ on interval $[t_0, t_1)$. Since $rank(E) = r \leq n$, there are two non-singular matrices $M = \begin{bmatrix} M_1^T & M_2^T \end{bmatrix}^T$ and $N = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$ such that

$$\begin{aligned} MEN &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad MA_{i1}N = \begin{bmatrix} A_{i1}^1 & A_{i1}^2 \\ A_{i1}^3 & A_{i1}^4 \end{bmatrix}, \\ M^{-T}P_{i1}N &= \begin{bmatrix} P_{i1}^1 & P_{i1}^2 \\ P_{i1}^3 & P_{i1}^4 \end{bmatrix}, \quad P_{i3}M^{-1} = \begin{bmatrix} P_{i3}^1 & P_{i3}^2 \end{bmatrix}. \end{aligned} \tag{2.92}$$

Pre- and post-multiplying (2.85) by N^T and its transpose, respectively, it follows that $N^T E^T M^T M^{-T} P_{i1} N = N^T P_{i1}^T M^{-1} MEN$ which implies $P_{i1}^2 = 0$. Similarly, pre- and post-multiplying $\tilde{\Theta}_i < 0$ by N^T and N, respectively,

$$\left(\left(P_{i1}^4\right)^T A_{i1}^4\right)^\star + \tau\gamma_i^2 N_2^T F_{i1}^T F_{i1} N_2 < 0, \tag{2.93}$$

which implies A_{i1}^4 is non-singular. Then, the pair $(E, A_1(r_t))$ is regular and impulse-free, and there are two non-singular matrices $\tilde{M} = \begin{bmatrix} \tilde{M}_1^T & \tilde{M}_2^T \end{bmatrix}^T$ and $\tilde{N} = \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \end{bmatrix}$ such that

$$\begin{aligned} \tilde{M}E\tilde{N} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{M}A_{i1}\tilde{N} = \begin{bmatrix} \tilde{A}_{i1} & 0 \\ 0 & I \end{bmatrix}, \\ \tilde{M}^{-T}P_{i1}\tilde{N} &= \begin{bmatrix} \tilde{P}_{i1}^1 & 0 \\ \tilde{P}_{i1}^3 & \tilde{P}_{i1}^4 \end{bmatrix}, \quad \tilde{M}B_{i1} = \begin{bmatrix} \tilde{B}_{i1}^1 \\ \tilde{B}_{i1}^2 \end{bmatrix}. \end{aligned} \tag{2.94}$$

Similarly, by pre- and post-multiplying (2.88) by $diag\{\tilde{N}^T, I\}$ and its transpose, respectively,

$$\left(\tilde{P}_{i1}^{4}\right)^{\star} + \tau\gamma_i^2 \tilde{N}_2^T F_{i1}^T F_{i1} \tilde{N}_2 + \tau^{-1} \left(\tilde{P}_{i1}^{4}\right)^T \tilde{B}_{i1}^2 \left(\tilde{B}_{i1}^2\right)^T \tilde{P}_{i1}^4 < 0. \tag{2.95}$$

Thus for every $i \in \mathbb{S}$, there is always a small scalar $\mu > 0$ such that

$$\left(\tilde{P}_{i1}^{4}\right)^{\star} + \tau\gamma_i^2 \tilde{N}_2^T F_{i1}^T F_{i1} \tilde{N}_2 + \tau^{-1} \left(\tilde{P}_{i1}^{4}\right)^T \tilde{B}_{i1}^2 \left(\tilde{B}_{i1}^2\right)^T \tilde{P}_{i1}^4 + \tau^{-1} \left(\tilde{P}_{i1}^{4}\right)^T \mu I \tilde{P}_{i1}^4 < 0. \tag{2.96}$$

By rewriting (2.96),

$$\begin{bmatrix} \tau\gamma_i^2 \tilde{N}_2^T F_{i1}^T F_{i1} \tilde{N}_2 - \tau \left(\tilde{B}_{i1}^2 \left(\tilde{B}_{i1}^2\right)^T + \mu I\right)^{-1} & \tau \left[\left(\tilde{B}_{i1}^2 \left(\tilde{B}_{i1}^2\right)^T + \mu I\right) \tilde{P}_{i1}^4 + I\right]^T \\ * & -\tau \left(\tilde{B}_{i1}^2 \left(\tilde{B}_{i1}^2\right)^T + \mu I\right) \end{bmatrix} < 0, \tag{2.97}$$

which implies

$$\gamma_i^2 \tilde{N}_2^T F_{i1}^T F_{i1} \tilde{N}_2 < \left(\tilde{B}_{i1}^2 \left(\tilde{B}_{i1}^2\right)^T + \mu I\right)^{-1}. \tag{2.98}$$

Since $\tilde{B}_{i1}^2 (\tilde{B}_{i1}^2)^T + \mu I > 0$,

$$\| F_{i1} \hat{N}_{i2} \| < \frac{1}{\gamma_i}, \tag{2.99}$$

where $\hat{N}_{i2} = \tilde{N}_2 (\tilde{B}_{i1}^2 (\tilde{B}_{i1}^2)^T + \mu I)^{\frac{1}{2}}$. Then for any $i \in \mathbb{S}$, there exists a sufficient small $\kappa > 0$ such that

$$\| F_{i1} \hat{N}_{i2} \| < \frac{1}{\gamma_i (1 + \kappa)}. \tag{2.100}$$

Let

$$\hat{M}_i = \left[\, \tilde{M}_1^T \;\; \hat{M}_{i2}^T \,\right]^T, \; \hat{N}_i = \left[\, \tilde{N}_1 \;\; \hat{N}_{i2} \,\right],$$

with $\hat{M}_{i2} = (\tilde{B}_{i1}^2 (\tilde{B}_{i1}^2)^T + \mu I)^{-\frac{1}{2}} \tilde{M}_2$. It is easy to verify that $\hat{M}_i$ and $\hat{N}_i$ are nonsingular and

$$\begin{aligned} &\hat{M}_i E \hat{N}_i = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \hat{M}_i A_{i1} \hat{N}_i = \begin{bmatrix} \tilde{A}_{i1} & 0 \\ 0 & I \end{bmatrix}, \\ &\hat{M}_i A_{i2} = \begin{bmatrix} \hat{A}_{i2}^1 \\ \hat{A}_{i2}^2 \end{bmatrix}, A_{i3} \hat{N}_i = \left[\, \hat{A}_{i3}^1 \;\; \hat{A}_{i3}^2 \,\right], \hat{M}_i B_{i1} = \begin{bmatrix} \tilde{B}_{i1}^1 \\ \hat{B}_{i1}^2 \end{bmatrix}, \end{aligned} \tag{2.101}$$

where $\hat{B}_{i1}^2 (\hat{B}_{i1}^2)^T = (\tilde{B}_{i1}^2 (\tilde{B}_{i1}^2)^T + \mu I)^{-\frac{1}{2}} \tilde{B}_{i1}^2 (\tilde{B}_{i1}^2)^T (\tilde{B}_{i1}^2 (\tilde{B}_{i1}^2)^T + \mu I)^{-\frac{1}{2}} < I$. Define $\hat{N}_i^{-1} x = \left[\, \hat{x}_1^T \;\; \hat{x}_2^T \,\right]^T$ and taking into account (2.101). System (2.73) is rewritten as

$$\begin{cases} \dot{\hat{x}}_1(t) = \tilde{A}_{i1}\hat{x}_1(t) + \hat{A}^1_{i2}z(t) + \tilde{B}^1_{i1} f_{i1}\left(t, \tilde{N}_1\hat{x}_1 + \hat{N}_{i2}\hat{x}_2, z\right), \\ \quad 0 = \hat{x}_2(t) + \hat{A}^2_{i2}z(t) + \hat{B}^2_{i1} f_{i1}\left(t, \tilde{N}_1\hat{x}_1 + \hat{N}_{i2}\hat{x}_2, z\right), \\ \varepsilon\dot{z}(t) = \hat{A}^1_{i3}\hat{x}_1(t) + \hat{A}^2_{i3}\hat{x}_2(t) + A_{i4}z(t) + B_{i2} f_{i2}\left(t, \tilde{N}_1\hat{x}_1 + \hat{N}_{i2}\hat{x}_2, z\right). \end{cases} \tag{2.102}$$

Based on [27], there is a unique solution to equation (2.73) with any compatible initial condition on $[t_0, t_1)$. Similarly, it can be also shown that there is a unique solution on $[t_1, t_2)$ for any given admissible condition $\xi(t_1)$, and so on. So it is obtained that Eq. (2.73) with (2.11) and (2.74) has a unique solution on $[0, \infty)$. This completes the proof.

Theorem 2.16 *Give a scalar $\bar{\varepsilon} > 0$, if there exist matrices $P_{i1} > 0$, P_{i2}, P_{i3}, $P_{i4} = P_{i4}^T$, $P_{i5} = P_{i5}^T$, $P_{i6} = P_{i6}^T$, $U_{i1} > 0$, U_{i2}, $U_{i3} > 0$, $S_{i1} > 0$, S_{i2}, $S_{i3} > 0$ and scalar $\tau > 0$ such that the following LMIs hold for all $i, j \in \mathbb{S}$, $j \neq i$:*

$$\Phi_{i1} + \bar{\varepsilon}\Phi_{i2} \geq 0, \tag{2.103}$$

$$\Phi_{i1} + \bar{\varepsilon}\Phi_{i2} + \bar{\varepsilon}^2\Phi_{i3} > 0, \tag{2.104}$$

$$\Psi^j_{i1} \leq 0, \tag{2.105}$$

$$\Psi^j_{i1} + \bar{\varepsilon}\Psi^j_{i2} \leq 0, \tag{2.106}$$

$$\Psi^j_{i1} + \bar{\varepsilon}\Psi^j_{i2} + \bar{\varepsilon}^2\Psi^j_{i3} \leq 0, \tag{2.107}$$

$$\Omega_{i1} < 0, \tag{2.108}$$

$$\Omega_{i1} + \bar{\varepsilon}\Omega_{i2} \leq 0, \tag{2.109}$$

$$\Omega_{i1} + \bar{\varepsilon}\Omega_{i2} + \bar{\varepsilon}^2\Omega_{i3} < 0, \tag{2.110}$$

$$\begin{bmatrix} U_{i1} & U_{i2} \\ * & U_{i3} \end{bmatrix} > 0, \tag{2.111}$$

$$\begin{bmatrix} S_{i1} & S_{i2} \\ * & S_{i3} \end{bmatrix} > 0, \tag{2.112}$$

where

$$\Phi_{i1} = \begin{bmatrix} P_{i1} & 0 \\ * & 0 \end{bmatrix},\ \Phi_{i2} = \begin{bmatrix} P_{i5} & P_{i3}^T \\ * & P_{i4} \end{bmatrix},\ \Phi_{i3} = \begin{bmatrix} 0 & 0 \\ * & P_{i6} \end{bmatrix},$$

$$\Psi_{i1}^j = \begin{bmatrix} E^T(P_{j1} - P_{i1})E - U_{i1} & -U_{i2} \\ * & -U_{i3} \end{bmatrix},$$

$$\Psi_{i2}^j = \begin{bmatrix} E^T(P_{j5} - P_{i5})E & E^T(P_{j3}^T - P_{i3}^T) \\ * & P_{j4} - P_{i4} \end{bmatrix},$$

$$\Psi_{i3}^j = \begin{bmatrix} 0 & 0 \\ * & P_{j6} - P_{i6} \end{bmatrix},$$

$$\Omega_{i1} = \begin{bmatrix} \Omega_{i1}^{11} & \Omega_{i1}^{12} & \Omega_{i1}^{13} & \Omega_{i1}^{14} & U_{i1} & U_{i2} \\ * & \Omega_{i1}^{22} & 0 & \Omega_{i1}^{23} & U_{i2}^T & U_{i3} \\ * & * & -\tau I & 0 & 0 & 0 \\ * & * & * & -\tau I & 0 & 0 \\ * & * & * & * & -S_{i1} & -S_{i2} \\ * & * & * & * & * & -S_{i3} \end{bmatrix},$$

$$\Omega_{i2} = \begin{bmatrix} \Omega_{i2}^{11} & \Omega_{i2}^{12} & \Omega_{i2}^{13} & 0 & 0 & 0 \\ * & \Omega_{i2}^{22} & \Omega_{i2}^{23} & \Omega_{i2}^{24} & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Omega_{i3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Omega_{i3}^{1} & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\begin{aligned}
\Omega_{i1}^{11} =& \left(A_{i1}^T P_{i1} E\right)^\star + \left(A_{i1}^T V P_{i2}\right)^\star + \left(A_{i3}^T P_{i3} E\right)^\star + 0.25\varepsilon_{ii}^2 S_{i1} \\
& - \varepsilon_{ii} U_{i1} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T (P_{j1} - P_{i1}) E + \tau\gamma_i^2 \left(F_{i1}^T F_{i1} + F_{i2}^T F_{i2}\right), \\
\Omega_{i1}^{12} =& A_{i3}^T P_{i4} + E^T P_{i1} A_{i2} + P_{i2}^T V^T A_{i2} + E^T P_{i3}^T A_{i4} \\
& + 0.25\varepsilon_{ii}^2 S_{i2} - \varepsilon_{ii} U_{i2} + \tau\gamma_i^2 \left(F_{i1}^T G_{i1} + F_{i2}^T G_{i2}\right), \\
\Omega_{i1}^{13} =& E^T P_{i1} B_{i1} + P_{i2}^T V^T B_{i1},\ \Omega_{i1}^{14} = E^T P_{i3}^T B_{i2},
\end{aligned}$$

$$\Omega_{i1}^{22} = \left(A_{i4}^T P_{i4}\right)^\star + 0.25\varepsilon_{ii}^2 S_{i3} - \varepsilon_{ii} U_{i3} + \tau\gamma_i^2 \left(G_{i1}^T G_{i1} + G_{i2}^T G_{i2}\right), \ \Omega_{i1}^{23} = P_{i4} B_{i2},$$

$$\Omega_{i2}^{11} = \left(A_{i1}^T P_{i5} E\right)^\star + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T (P_{j5} - P_{i5}) E,$$

$$\Omega_{i2}^{12} = A_{i1}^T P_{i3}^T + A_{i3}^T P_{i6} + E^T P_{i5} A_{i2} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T \left(P_{j3}^T - P_{i3}^T\right),$$

$$\Omega_{i2}^{13} = E^T P_{i5} B_{i1}, \ \Omega_{i2}^{22} = \left(A_{i4}^T P_{i6}\right)^\star + (P_{i3} A_{i2})^\star + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} (P_{j4} - P_{i4}),$$

$$\Omega_{i2}^{23} = P_{i3} B_{i1}, \ \Omega_{i2}^{24} = P_{i6} B_{i2}, \ \Omega_{i3}^{1} = \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} (P_{j6} - P_{i6}).$$

where V is any appropriate matrix with full column rank satisfying $V^T E = 0$. Then for any $\varepsilon \in (0, \bar{\varepsilon}]$, Eq. (2.73) with (2.11) and (2.74) has a unique solution on $[0, \infty)$ and is exponentially mean-square stable over all the admissible uncertainty.

Proof First, it needs to be proved that if conditions in Theorem 2.16 hold, equation (2.73) has a unique solution on $[0, \infty)$. Define $\tilde{P}_{i1} = P_{i1} E + V P_{i2}$, it is concluded that (2.85) holds, and (2.105) implies (2.86). That is,

$$E^T \left(\tilde{P}_{j1} - \tilde{P}_{i1}\right) - U_{i1} < 0. \tag{2.113}$$

From (2.108) and taking into account (2.113) and expression (2.76), it is obtained that

$$\begin{bmatrix} \hat{\Theta}_i & \tilde{P}_{i1}^T B_{i1} & U_{i1} \\ * & -\tau I & 0 \\ * & * & -S_{i1} \end{bmatrix} < 0, \tag{2.114}$$

where

$$\hat{\Theta}_i = \left(A_{i1}^T \tilde{P}_{i1}\right)^\star + \left(A_{i3}^T P_{i3} E\right)^\star + 0.25\varepsilon_{ii}^2 S_{i1} - \varepsilon_{ii} U_{i1} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} E^T \left(\tilde{P}_{j1} - \tilde{P}_{i1}\right) + \tau\gamma_i^2 F_{i1}^T F_{i1}.$$

By Theorem 2.15, one gets that there is a unique solution to system (2.73) with (2.11) and (2.74) on $[0, \infty)$ for any given $\varepsilon > 0$.

Next, it is to prove that system (2.73) is exponentially mean-square stable. For any $r_t = i \in \mathbb{S}$, define

$$P_{i\varepsilon} = \begin{bmatrix} (P_{i1} + \varepsilon P_{i5}) E + V P_{i2} & \varepsilon P_{i3}^T \\ P_{i3} E & P_{i4} + \varepsilon P_{i6} \end{bmatrix}.$$

For any $\varepsilon \in (0, \bar{\varepsilon}]$, it is obtained that

$$E_{\varepsilon}^{T} P_{i\varepsilon} = P_{i\varepsilon}^{T} E_{\varepsilon} = \tilde{E}^{T} \left(\Phi_{i1} + \varepsilon \Phi_{i2} + \varepsilon^{2} \Phi_{i3} \right) \tilde{E} \geq 0, \tag{2.115}$$

where

$$\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}.$$

Since $P_{i1} > 0$, it is concluded that

$$\Phi_{i1} \geq 0. \tag{2.116}$$

Taking into account (2.103), (2.104) and (2.116), and by Lemma 2.4, (2.115) holds. For any $\varepsilon \in (0, \bar{\varepsilon}]$, choose an ε-dependent Lyapunov function for system (2.73) such that

$$V(\xi(t)) = \xi^{T}(t) E_{\varepsilon}^{T} P_{i\varepsilon} \xi(t), \ \ \varepsilon \in (0, \bar{\varepsilon}]. \tag{2.117}$$

Let $\mathscr{L}$ be the weak infinitesimal generator of random process $\{\xi(t), r_t\}$, for each $r_t = i \in \mathbb{S}$, which is defined as

$$\mathscr{L}[V(\xi(t), r_t = i)] = \lim_{h \to 0^{+}} \frac{1}{h} [\mathscr{E}(V(\xi(t+h), r_{t+h}) | \xi(t), r_t = i) - V(\xi(t), i)]. \tag{2.118}$$

Then,

$$\mathscr{L}[V(\xi(t), r_t)] = \left[(A_i \xi(t) + B_i f_i(t))^{T} P_{i\varepsilon} \right]^{\star} + \xi^{T}(t) \sum_{j=1}^{N} \tilde{\pi}_{ij} E_{j\varepsilon}^{T} P_{j\varepsilon} \xi(t) < 0. \tag{2.119}$$

By the S-procedure Lemma, the inequality $\mathscr{L}[V(x_t, r_t)] < 0$ is equivalent to that there is a $\tau > 0$ such that

$$\mathscr{L}[V(\xi(t), r_t)] - \tau \left(f_i^{T}(t) f_i(t) - \gamma_i^{2} \xi^{T}(t) F_i^{T} F_i \xi(t) \right) < 0, \tag{2.120}$$

which is equivalent to

$$\begin{aligned} &\left[(A_i \xi(t) + B_i f_i(t))^{T} P_{i\varepsilon} \right]^{\star} - \tau \left(f_i^{T}(t) f_i(t) - \gamma_i^{2} \xi^{T}(t) F_i^{T} F_i \xi(t) \right) \\ &\qquad + \xi^{T}(t) [\sum_{j=1, j \neq i}^{N} \alpha_{ij} E_{\varepsilon}^{T} (P_{j\varepsilon} - P_{i\varepsilon}) - \varepsilon_{ii} U_i - \Delta \tilde{\pi}_{ii} U_i \\ &\qquad + \sum_{j=1, j \neq i}^{N} (\Delta \tilde{\pi}_{ij} + \varepsilon_{ij}) \left(E_{\varepsilon}^{T} P_{j\varepsilon} - E_{\varepsilon}^{T} P_{i\varepsilon} - U_i \right)] \xi(t) < 0. \end{aligned} \tag{2.121}$$

Similar to (2.90), it is easy to see that (2.121) is guaranteed by

$$E_\varepsilon^T P_{j\varepsilon} - E_\varepsilon^T P_{i\varepsilon} - U_i \le 0, \tag{2.122}$$

$$\begin{aligned}&\left[(A_i\xi(t) + B_i f_i(t))^T P_{i\varepsilon}\right]^\star - \tau\left(f_i^T(t) f_i(t) - \gamma_i^2 \xi^T(t) F_i^T F_i \xi(t)\right) + \xi^T(t)\\ &\quad \times \Big[\sum_{j=1, j\neq i}^{N} \alpha_{ij} E_\varepsilon^T (P_{j\varepsilon} - P_{i\varepsilon}) - \varepsilon_{ii} U_i + 0.25\varepsilon_{ii}^2 S_i + U_i S_i^{-1} U_i\Big]\xi(t) < 0.\end{aligned} \tag{2.123}$$

Substituting (2.111) and (2.115) into (2.122) and by Lemma 2.4, it is obtained that (2.105)–(2.107) implies (2.122). Moreover, (2.123) is rewritten as

$$\begin{bmatrix} \xi(t) \\ f_i(t) \end{bmatrix}^T \begin{bmatrix} \Lambda_{i\varepsilon} & P_{i\varepsilon}^T B_i \\ * & -\tau I \end{bmatrix} \begin{bmatrix} \xi(t) \\ f_i(t) \end{bmatrix} < 0, \tag{2.124}$$

where

$$\Lambda_{i\varepsilon} = \left(A_i^T P_{i\varepsilon}\right)^\star + \gamma_i^2 \tau F_i^T F_i + \sum_{j=1, j\neq i}^{N} \alpha_{ij} E_\varepsilon^T (P_{j\varepsilon} - P_{i\varepsilon}) - \varepsilon_{ii} U_i + 0.25\varepsilon_{ii}^2 S_i + U_i S_i^{-1} U_i.$$

It is concluded that (2.124) is guaranteed by

$$\begin{bmatrix} \bar{\Lambda}_{i\varepsilon} & P_{i\varepsilon}^T B_i & U_i \\ * & -\tau I & 0 \\ * & * & -S_i \end{bmatrix} < 0, \tag{2.125}$$

where

$$\bar{\Lambda}_{i\varepsilon} = \Lambda_{i\varepsilon} - U_i S_i^{-1} U_i.$$

By substituting the parameters of (2.76), (2.111), (2.112) and $P_{i\varepsilon}$ into (2.125),

$$\Omega_{i1} + \varepsilon \Omega_{i2} + \varepsilon^2 \Omega_{i3} < 0, \ \forall \varepsilon \in (0, \bar{\varepsilon}], \tag{2.126}$$

which is obtained by (2.108)–(2.110). Thus, (2.120) hods for all $i \in \mathbb{S}$, and there is a constant $\theta_\varepsilon > 0$ such that

$$\mathscr{L}[V(\xi(t), i)] \le -\theta_\varepsilon ||\xi(t)||^2. \tag{2.127}$$

On the other hand, for any $i \in \mathbb{S}$, let $\hat{z}(t) = z(t)$ and define

$$\xi(t) = \left[\begin{array}{c|c} \hat{N}_i & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hline \hat{z}(t) \end{array}\right] = \left[\begin{array}{c|c} \hat{N}_i & 0 \\ \hline 0 & I \end{array}\right] \hat{\xi}(t).$$

(2.117) becomes

$$\begin{aligned} V\left(\hat{\xi}, i\right) &= \hat{\xi}^T(t) \begin{bmatrix} \hat{N}_i^T & 0 \\ 0 & I \end{bmatrix} E_\varepsilon^T P_{i\varepsilon} \begin{bmatrix} \hat{N}_i & 0 \\ 0 & I \end{bmatrix} \hat{\xi}(t) \\ &= \hat{\xi}^T(t) \begin{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \Xi_{i\varepsilon} \begin{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \hat{\xi}(t), \end{aligned} \tag{2.128}$$

where

$$\Xi_{i\varepsilon} = \begin{bmatrix} \hat{M}_i^{-T} & 0 \\ 0 & I \end{bmatrix} \left(\Phi_{i1} + \varepsilon \Phi_{i2} + \varepsilon^2 \Phi_{i3}\right) \begin{bmatrix} \hat{M}_i^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

From (2.103), (2.104) and (2.116), there is a scalar $\bar{\theta}_\varepsilon = \min_{i \in \mathbb{S}} \lambda_{\min}(\Xi_{i\varepsilon}) > 0$ such that

$$V\left(\hat{\xi}, i\right) \geq \bar{\theta}_\varepsilon \left(\|\hat{x}_1\|^2 + \|\hat{z}\|^2\right). \tag{2.129}$$

From (2.127),

$$\mathscr{L}[V(\xi(t), i)] \leq -\theta_\varepsilon b_1 \left(\|\hat{x}_1\|^2 + \|\hat{z}\|^2\right), \tag{2.130}$$

where

$$b_1 = \min_{i \in \mathbb{S}} \sigma_{\min}^2 \left(\begin{bmatrix} \hat{N}_i & 0 \\ 0 & I \end{bmatrix}\right).$$

Taking into account (2.129) and (2.130) and by Dynkin's formula,

$$\bar{\theta}_\varepsilon \mathscr{E}\left\{\|\hat{x}_1\|^2 + \|\hat{z}\|^2\right\} \leq \mathscr{E}\{V(\xi_0, r_0)\} - \theta_\varepsilon b_1 \int_0^t \mathscr{E}\left\{\|\hat{x}_1\|^2 + \|\hat{z}\|^2\right\} ds. \tag{2.131}$$

Applying Gronwall-Bellman lemma to (2.131), it is easy to see that there is a scalar $a_\varepsilon > 0$ such that

$$\mathscr{E}\left\{\|\hat{x}_1\|^2 + \|\hat{z}\|^2\right\} \leq a_\varepsilon \|\xi_0\|^2 e^{-b_\varepsilon t}, \tag{2.132}$$

where $b_\varepsilon = \theta_\varepsilon b_1 \bar{\theta}_\varepsilon^{-1}$. Then $(\hat{x}_1, \hat{z})$ is exponentially mean-square stable. From (2.75) and (2.102), we obtain that

$$\begin{aligned} \|\hat{x}_2\| &\leq \|\hat{A}_{i2}^2\| \|z\| + \gamma_i \|\hat{B}_{i1}^2\| \|F_{i1} \tilde{N}_1 \hat{x}_1 + F_{i1} \hat{N}_{i2} \hat{x}_2 + G_{i1} \hat{z}\| \\ &\leq \left(\|\hat{A}_{i2}^2\| + \gamma_i \|G_{i1}\|\right) \|z\| + \gamma_i \|F_{i1} \tilde{N}_1\| \|\hat{x}_1\| + \gamma_i \|F_{i1} \hat{N}_{i2}\| \|\hat{x}_2\|. \end{aligned} \tag{2.133}$$

Under condition (2.100) and taking into account (2.132), it is obtained that

$$\mathscr{E}\{\|\hat{x}_2\|\} \leq \hat{a}\|\xi_0\|e^{-\frac{b_\varepsilon}{2}t}, \tag{2.134}$$

where

$$\hat{a} = \frac{2(1+\kappa)}{\kappa}\sqrt{a_\varepsilon}\max_{i\in\mathbb{S}}\left(\|\hat{A}_{i2}^2\| + \gamma_i\|G_{i1}\|, \gamma_i\|F_{i1}\tilde{N}_1\|\right).$$

Therefore, system (2.73) is globally exponentially mean-square stable. This completes the proof.

Remark 2.8 In Theorem 2.16, not only is the existence condition of stability bound $\bar{\varepsilon}$ of system (2.73) with uncertain TRM presented, but also an estimation of $\bar{\varepsilon}$ is given. In addition, the ε-dependent Lyapunov function given by (2.117) is more general, in which more slack matrices are introduced. That is, when $P_{i5} = 0$ and $P_{i6} = 0$, one can have the corresponding ones in [22, 23, 26, 28]. It should be emphasized that the stability bound $\bar{\varepsilon}$ was estimated directly without introducing additional inequalities.

When $E = I$, system (2.73) reduces to the following system:

$$\begin{cases} \dot{x}(t) = A_1(r_t)x(t) + A_2(r_t)z(t) + B_1(r_t)f_1(t, r_t, x, z), \\ \varepsilon\dot{z}(t) = A_3(r_t)x(t) + A_4(r_t)z(t) + B_2(r_t)f_2(t, r_t, x, z), \\ x(0) = x_0, \\ z(0) = z_0, \end{cases} \tag{2.135}$$

with constraints (2.11) and (2.74). Although system (2.135) is a normal singular perturbed system with Markovian switching, it is also dealt with by a descriptor approach which could reduce the conservativeness. It is easy to see that system (2.135) is equivalent to

$$\begin{cases} \bar{E}\dot{\bar{x}}(t) = \bar{A}_1(r_t)\bar{x}(t) + \bar{A}_2(r_t)z(t) + \bar{B}_1(r_t)\bar{f}_1(t, r_t, \bar{x}, z), \\ \varepsilon\dot{z}(t) = \bar{A}_3(r_t)\bar{x}(t) + A_4(r_t)z(t) + B_2(r_t)\bar{f}_2(t, r_t, \bar{x}, z), \\ x(0) = x_0, \\ z(0) = z_0, \end{cases} \tag{2.136}$$

where

$$\begin{aligned} &\bar{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \ \bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{A}_1(r_t) = \begin{bmatrix} 0 & I \\ A_1(r_t) & -I \end{bmatrix}, \\ &\bar{A}_2(r_t) = \begin{bmatrix} 0 \\ A_2(r_t) \end{bmatrix}, \ \bar{B}_1(r_t) = \begin{bmatrix} 0 \\ B_1(r_t) \end{bmatrix}, \ \bar{f}_1(t, r_t, \bar{x}, z) = f_1(t, r_t, x, z), \\ &\bar{A}_3(r_t) = \begin{bmatrix} A_3(r_t) & 0 \end{bmatrix}, \ \bar{f}_2(t, r_t, \bar{x}, z) = f_2(t, r_t, x, z), \\ &\bar{F}_1(r_t) = \begin{bmatrix} F_1(r_t) & 0 \end{bmatrix}, \ \bar{F}_2(r_t) = \begin{bmatrix} F_2(r_t) & 0 \end{bmatrix}. \end{aligned}$$

Using the similar method as in Theorem 2.16, the following corollary is ready to be presented:

Corollary 2.1 *Give a scalar* $\bar{\varepsilon} > 0$*, if there exist matrices* $P_{i1} > 0$*,* P_{i2}*,* P_{i3}*,* $P_{i4} = P_{i4}^T$*,* $P_{i5} = P_{i5}^T$*,* $P_{i6} = P_{i6}^T$*,* $U_{i1} > 0$*,* U_{i2}*,* $U_{i3} > 0$*,* $S_{i1} > 0$*,* S_{i2}*,* $S_{i3} > 0$ *and scalar* $\tau > 0$ *such that LMIs (2.103), (2.104), (2.111), (2.112) and*

$$\bar{\Psi}_{i1}^{j} \leq 0, \tag{2.137}$$

$$\bar{\Psi}_{i1}^{j} + \bar{\varepsilon}\bar{\Psi}_{i2}^{j} \leq 0, \tag{2.138}$$

$$\bar{\Psi}_{i1}^{j} + \bar{\varepsilon}\bar{\Psi}_{i2}^{j} + \bar{\varepsilon}^2\Psi_{i3}^{j} \leq 0, \tag{2.139}$$

$$\bar{\Omega}_{i1} < 0, \tag{2.140}$$

$$\bar{\Omega}_{i1} + \bar{\varepsilon}\bar{\Omega}_{i2} \leq 0, \tag{2.141}$$

$$\bar{\Omega}_{i1} + \bar{\varepsilon}\bar{\Omega}_{i2} + \bar{\varepsilon}^2\Omega_{i3} < 0, \tag{2.142}$$

where

$$\bar{\Psi}_{i1}^{j} = \begin{bmatrix} \bar{E}^T(P_{j1} - P_{i1})\bar{E} - U_{i1} & -U_{i2} \\ * & -U_{i3} \end{bmatrix},$$

$$\bar{\Psi}_{i2}^{j} = \begin{bmatrix} \bar{E}^T(P_{j5} - P_{i5})\bar{E} & \bar{E}^T(P_{j3}^T - P_{i3}^T) \\ * & P_{j4} - P_{i4} \end{bmatrix},$$

$$\bar{\Omega}_{i1} = \begin{bmatrix} \bar{\Omega}_{i1}^{11} & \bar{\Omega}_{i1}^{12} & \bar{\Omega}_{i1}^{13} & \bar{\Omega}_{i1}^{14} & U_{i1} & U_{i2} \\ * & \Omega_{i1}^{22} & 0 & \Omega_{i1}^{23} & U_{i2}^T & U_{i3} \\ * & * & -\tau I & 0 & 0 & 0 \\ * & * & * & -\tau I & 0 & 0 \\ * & * & * & * & -S_{i1} & -S_{i2} \\ * & * & * & * & * & -S_{i3} \end{bmatrix},$$

$$\bar{\Omega}_{i2} = \begin{bmatrix} \bar{\Omega}_{i2}^{11} & \bar{\Omega}_{i2}^{12} & \bar{\Omega}_{i2}^{13} & 0 & 0 & 0 \\ * & \bar{\Omega}_{i2}^{22} & \bar{\Omega}_{i2}^{23} & \Omega_{i2}^{24} & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\begin{aligned}
\bar{\Omega}_{i1}^{11} =& \left(\bar{A}_{i1}^{T} P_{i1} E\right)^{\star} + \left(\bar{A}_{i1}^{T} V P_{i2}\right)^{\star} + \left(\bar{A}_{i3}^{T} P_{i3} \bar{E}\right)^{\star} + 0.25\varepsilon_{ii}^{2} S_{i1} \\
& - \varepsilon_{ii} U_{i1} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} \bar{E}^{T} (P_{j1} - P_{i1}) \bar{E} + \tau \gamma_i^2 \left(\bar{F}_{i1}^{T} \bar{F}_{i1} + \bar{F}_{i2}^{T} \bar{F}_{i2}\right), \\
\bar{\Omega}_{i1}^{12} =& \bar{A}_{i3}^{T} P_{i4} + \bar{E}^{T} P_{i1} \bar{A}_{i2} + P_{i2}^{T} V^{T} \bar{A}_{i2} + \bar{E}^{T} P_{i3}^{T} A_{i4} \\
& + 0.25\varepsilon_{ii}^{2} S_{i2} - \varepsilon_{ii} U_{i2} + \tau \gamma_i^2 \left(\bar{F}_{i1}^{T} G_{i1} + \bar{F}_{i2}^{T} G_{i2}\right), \\
\bar{\Omega}_{i1}^{13} =& \bar{E}^{T} P_{i1} \bar{B}_{i1} + P_{i2}^{T} V^{T} \bar{B}_{i1},\ \bar{\Omega}_{i1}^{14} = \bar{E}^{T} P_{i3}^{T} B_{i2}, \\
\bar{\Omega}_{i2}^{11} =& \left(\bar{A}_{i1}^{T} P_{i5} \bar{E}\right)^{\star} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} \bar{E}^{T} (P_{j5} - P_{i5}) \bar{E}, \\
\bar{\Omega}_{i2}^{12} =& \bar{A}_{i1}^{T} P_{i3}^{T} + \bar{A}_{i3}^{T} P_{i6} + \bar{E}^{T} P_{i5} \bar{A}_{i2} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} \bar{E}^{T} \left(P_{j3}^{T} - P_{i3}^{T}\right), \\
\bar{\Omega}_{i2}^{13} =& \bar{E}^{T} P_{i5} \bar{B}_{i1},\ \bar{\Omega}_{i2}^{22} = \left(A_{i4}^{T} P_{i6}\right)^{\star} + \left(P_{i3} \bar{A}_{i2}\right)^{\star} + \sum_{j=1,\ j\neq i}^{N} \alpha_{ij} (P_{j4} - P_{i4}), \\
\bar{\Omega}_{i2}^{23} =& P_{i3} \bar{B}_{i1},
\end{aligned}$$

hold for all $i, j \in \mathbb{S}$, $j \neq i$. *Then for any* $\varepsilon \in (0, \bar{\varepsilon}]$, *equation (2.135) with (2.11) and (2.74) has a unique solution on* $[0, \infty)$ *and is exponentially mean-square stable over all the admissible uncertainties.*

When π_{ij} is accessible accurately, the corresponding result can be obtained directly.

Corollary 2.2 *Give a scalar* $\bar{\varepsilon} > 0$, *if there exist matrices* $P_{i1} > 0$, P_{i2}, P_{i3}, $P_{i4} = P_{i4}^{T}$, $P_{i5} = P_{i5}^{T}$, $P_{i6} = P_{i6}^{T}$ *and scalar* $\tau > 0$ *such that LMIs (2.103), (2.104) and*

$$\tilde{\Omega}_{i1} < 0, \tag{2.143}$$

$$\tilde{\Omega}_{i1} + \bar{\varepsilon}\tilde{\Omega}_{i2} \leq 0, \tag{2.144}$$

$$\tilde{\Omega}_{i1} + \bar{\varepsilon}\tilde{\Omega}_{i2} + \bar{\varepsilon}^2\tilde{\Omega}_{i3} < 0, \tag{2.145}$$

where

$$\tilde{\Omega}_{i1} = \begin{bmatrix} \tilde{\Omega}_{i1}^{11} & \tilde{\Omega}_{i1}^{12} & \Omega_{i1}^{13} & \Omega_{i1}^{14} \\ * & \tilde{\Omega}_{i1}^{22} & 0 & \tilde{\Omega}_{i1}^{23} \\ * & * & -\tau I & 0 \\ * & * & * & -\tau I \end{bmatrix},$$

$$\tilde{\Omega}_{i2} = \begin{bmatrix} \tilde{\Omega}_{i2}^{11} & \tilde{\Omega}_{i2}^{12} & \tilde{\Omega}_{i2}^{13} & 0 \\ * & \tilde{\Omega}_{i2}^{22} & \Omega_{i2}^{23} & \Omega_{i2}^{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix},$$

$$\tilde{\Omega}_{i3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & \tilde{\Omega}_{i3}^{1} & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix},$$

$$\begin{aligned}
\tilde{\Omega}_{i1}^{11} =& \left(A_{i1}^T P_{i1} E\right)^\star + \left(A_{i1}^T V P_{i2}\right)^\star + \left(A_{i3}^T P_{i3} E\right)^\star \\
&+ \sum_{j=1}^{N} \pi_{ij} E^T P_{j1} E + \tau \gamma_i^2 \left(F_{i1}^T F_{i1} + F_{i2}^T F_{i2}\right), \\
\tilde{\Omega}_{i1}^{12} =& A_{i3}^T P_{i4} + E^T P_{i1} A_{i2} + P_{i2}^T V^T A_{i2} \\
&+ E^T P_{i3}^T A_{i4} + \tau \gamma_i^2 \left(F_{i1}^T G_{i1} + F_{i2}^T G_{i2}\right), \\
\tilde{\Omega}_{i1}^{22} =& \left(A_{i4}^T P_{i4}\right)^\star + \tau \gamma_i^2 \left(G_{i1}^T G_{i1} + G_{i2}^T G_{i2}\right), \\
\tilde{\Omega}_{i2}^{11} =& \left(A_{i1}^T P_{i5} E\right)^\star + \sum_{j=1}^{N} \pi_{ij} E^T P_{j5} E, \\
\tilde{\Omega}_{i2}^{12} =& A_{i1}^T P_{i3}^T + A_{i3}^T P_{i6} + E^T P_{i5} A_{i2} + \sum_{j=1}^{N} \pi_{ij} E^T P_{j3}^T, \\
\tilde{\Omega}_{i2}^{22} =& \left(A_{i4}^T P_{i6}\right)^\star + (P_{i3} A_{i2})^\star + \sum_{j=1}^{N} \pi_{ij} P_{j4}, \\
\tilde{\Omega}_{i3}^{1} =& \sum_{j=1}^{N} \pi_{ij} P_{j6},
\end{aligned}$$

hold for all $i \in \mathbb{S}$. Then for any $\varepsilon \in (0, \bar{\varepsilon}]$, Eq. (2.73) with (2.2) and (2.3) has a unique solution on $[0, \infty)$ and is exponentially mean-square stable.

When there is no Markovian switching, system (2.73) becomes

$$\begin{cases} E\dot{x}(t) = A_1x(t) + A_2z(t) + B_1 f_1(t, x, z), \\ \varepsilon\dot{z}(t) = A_3x(t) + A_4z(t) + B_2 f_2(t, x, z), \\ x(0) = x_0, \\ z(0) = z_0, \end{cases} \tag{2.146}$$

where $f_k(t, x, z)$, $k = 1, 2$, satisfies

$$\| f_k(t, x, z) - f_k(t, \tilde{x}, \tilde{z})\| \leq \gamma \| F_k(x - \tilde{x}) + G_k(z - \tilde{z})\|, \quad k = 1, 2, \tag{2.147}$$

and

$$\| f_k(t, x, z)\| \leq \gamma \| F_k x + G_k z\|, \quad k = 1, 2, \tag{2.148}$$

where $\gamma > 0$, F_k and G_k are constant matrices with appropriate dimensions.

Corollary 2.3 *Give a scalar $\bar{\varepsilon} > 0$, if there exist matrices $P_1 > 0$, P_2, P_3, $P_4 = P_4^T$, $P_5 = P_5^T$, $P_6 = P_6^T$ and scalar $\tau > 0$ such that*

$$\hat{\Phi}_1 + \bar{\varepsilon}\hat{\Phi}_2 \geq 0, \tag{2.149}$$

$$\hat{\Phi}_1 + \bar{\varepsilon}\hat{\Phi}_2 + \bar{\varepsilon}^2\hat{\Phi}_3 > 0, \tag{2.150}$$

$$\hat{\Omega}_1 < 0, \tag{2.151}$$

$$\hat{\Omega}_1 + \bar{\varepsilon}\hat{\Omega}_2 < 0, \tag{2.152}$$

where

$$\hat{\Phi}_1 = \begin{bmatrix} P_1 & 0 \\ * & 0 \end{bmatrix}, \ \hat{\Phi}_2 = \begin{bmatrix} P_5 & P_3^T \\ * & P_4 \end{bmatrix}, \ \hat{\Phi}_3 = \begin{bmatrix} 0 & 0 \\ * & P_6 \end{bmatrix},$$

$$\hat{\Omega}_1 = \begin{bmatrix} \hat{\Omega}_1^{11} & \hat{\Omega}_1^{12} & \hat{\Omega}_1^{13} & \hat{\Omega}_1^{14} \\ * & \hat{\Omega}_1^{22} & 0 & \hat{\Omega}_1^{23} \\ * & * & -\tau I & 0 \\ * & * & * & -\tau I \end{bmatrix},$$

$$\hat{\Omega}_2 = \begin{bmatrix} \hat{\Omega}_2^{11} & \hat{\Omega}_2^{12} & \hat{\Omega}_2^{13} & 0 \\ * & \hat{\Omega}_2^{22} & \hat{\Omega}_2^{23} & \hat{\Omega}_2^{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix},$$

$$\hat{\Omega}_1^{11} = \left(A_1^T P_1 E\right)^\star + \left(A_1^T V P_2\right)^\star + \left(A_3^T P_3 E\right)^\star + \tau\gamma^2 \left(F_1^T F_1 + F_2^T F_2\right),$$
$$\hat{\Omega}_1^{12} = A_3^T P_4 + E^T P_1 A_2 + P_2^T V^T A_2 + E^T P_3^T A_4 + \tau\gamma^2 \left(F_1^T G_1 + F_2^T G_2\right),$$
$$\hat{\Omega}_1^{13} = E^T P_1 B_1 + P_2^T V^T B_1,\ \hat{\Omega}_1^{14} = E^T P_3^T B_2,$$
$$\hat{\Omega}_1^{22} = \left(A_4^T P_4\right)^\star + \tau\gamma^2 \left(G_1^T G_1 + G_2^T G_2\right),\ \hat{\Omega}_1^{23} = P_4 B_2,$$
$$\hat{\Omega}_2^{11} = \left(A_1^T P_5 E\right)^\star,\ \hat{\Omega}_2^{12} = A_1^T P_3^T + A_3^T P_6 + E^T P_5 A_2,\ \hat{\Omega}_2^{13} = E^T P_5 B_1,$$
$$\hat{\Omega}_2^{22} = \left(A_4^T P_6\right)^\star + (P_3 A_2)^\star,\ \hat{\Omega}_2^{23} = P_3 B_1,\ \hat{\Omega}_2^{24} = P_6 B_2,$$

hold. Then for any $\varepsilon \in (0, \bar{\varepsilon}]$, *Eq.(2.146) with (2.147) and (2.148) has a unique solution on* $[0, \infty)$ *and is exponentially stable.*

To illustrate the results developed above, some numerical examples are presented as follows:

Example 2.1 Consider the following singularly perturbed system from [29]:

$$\begin{cases} \dot{x} = x - z + \dfrac{|x|z}{1 + 4z^2}, \\ \varepsilon\dot{z} = 2x - z + \dfrac{|x|z}{1 + 4x^2}. \end{cases} \tag{2.153}$$

For system (2.153), it is concluded that $f_1(x, z) = |x|z/(1 + 4z^2)$ and $f_2(x, z) = |x|z/(1 + 4x^2)$ satisfy (2.74) with $F_1 = G_2 = 0.25$, $G_1 = F_2 = 0$ and $\gamma = 1$ respectively. By method [29], it is obtained that the stability bound is $\bar{\varepsilon} = 9.5 \times 10^{-3}$, while the stability bound computed by method [26] is $\bar{\varepsilon} = 0.3395$. From Corollary 2.3, it is obtained that the stability bound $\bar{\varepsilon} = 0.4528$, which is large, and thus the result given in Corollary 2.3 is less conservative.

Example 2.2 Consider the following Markovian jump singularly perturbed descriptor system with two modes such as

Mode 1:

$$\begin{cases} \dot{x}_1 = x_2 + 0.2 f_{11}^1(t, x_1, x_2, z), \\ 0 = 0.4x_1 - x_2 - z - 0.5 f_{11}^2(t, x_1, x_2, z), \\ \varepsilon\dot{z} = x_1 + x_2 - z + f_{12}(t, x_1, x_2, z), \end{cases} \tag{2.154}$$

where

$$f_{11}^1(t, x_1, x_2, z) = \frac{|x_1|z}{1 + 16z^2,}$$
$$f_{11}^2(t, x_1, x_2, z) = \frac{|x_1|x_2}{1 + 16x_2^2},$$
$$f_{12}(t, x_1, x_2, z) = 0.25z \sin(x_2 + z).$$

Mode 2:

$$\begin{cases} \dot{x}_1 = 0.3x_1 + 0.4x_2 - 0.9z + 0.3f_{21}^1(t, x_1, x_2, z), \\ 0 = -0.9x_1 + x_2 + 0.2z + 0.2f_{21}^2(t, x_1, x_2, z), \\ \varepsilon\dot{z} = 0.2x_1 - 0.5z + f_{22}(t, x_1, x_2, z), \end{cases} \tag{2.155}$$

where

$$f_{21}^1(t, x_1, x_2, z) = \frac{|x_1|z}{16 + z^2},$$
$$f_{21}^2(t, x_1, x_2, z) = \frac{|x_1|x_2}{16 + x_2^2},$$
$$f_{22}(t, x_1, x_2, z) = 0.4z\cos(x_1 - 2x_2).$$

It is concluded that the nonlinear perturbations of systems (2.154) and (2.155) satisfy (2.74) and (2.75) with $\gamma_i = 1, i = 1, 2$. First, it is assumed that TRM is given exactly, that is,

$$\Pi = \begin{bmatrix} -1.2 & 1.2 \\ 0.4 & -0.4 \end{bmatrix}$$

By methods in [21–23, 30], there is no information on stability bound $\bar{\varepsilon}$. But from Corollary 2.2, the above system has a unique solution and is exponentially mean-square stable for $\forall\varepsilon \in (0, \bar{\varepsilon}]$ with a stability bound $\bar{\varepsilon} = 0.5229$. If TRM Π is not obtained exactly, only the estimated transition rates are got as $\tilde{\pi}_{11} = -1.2$ and $\tilde{\pi}_{22} = -0.4$, where uncertainty $\Delta\tilde{\Pi}$ satisfies $|\Delta\tilde{\pi}_{12}| \le \varepsilon_{12} \triangleq 0.5\tilde{\pi}_{12}$ and $|\Delta\tilde{\pi}_{21}| \le \varepsilon_{21} \triangleq 0.5\tilde{\pi}_{21}$ respectively. From Theorem 2.16, an estimation of stability bound $\bar{\varepsilon} = 0.3219$ which guarantees that the aforementioned system is exponentially mean-square stable for any $\varepsilon \in (0, \bar{\varepsilon}]$.

Example 2.3 Consider the following singularly perturbed system controlled by a DC motor, which is illustrated in Fig. 2.1. It is described as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \dfrac{g}{l}\sin x_1(t) + \dfrac{NK_m}{ml^2}z(t), \\ \dot{z}(t) = \dfrac{K_bN}{L_a}x_2(t) - \dfrac{R(r_t)}{L_a}z(t) + \dfrac{1}{L_a}u(t), \end{cases} \tag{2.156}$$

where $x_1(t) = \theta_p(t)$, $x_2(t) = \dot{\theta}_p(t)$ and $z(t) = I_a(t)$ are system states, $u(t)$ is the control input, K_m is the motor torque constant, K_b is the back emf constant, N is the gear ratio, and $R(r_t)$ is defined as

$$R(r_t) = \begin{cases} R_a, & \text{if } r_t = 1, \\ R_b, & \text{otherwise } r_t = 2, \end{cases}$$

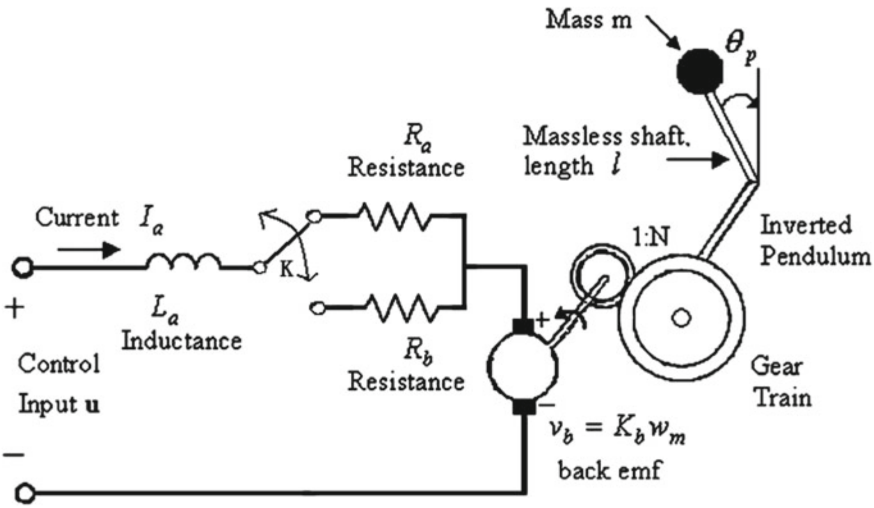

Fig. 2.1 DC motor controlling an inverted pendulum

where $\{r_t, t \geq 0\}$ is a Markov process taking values in a finite set $\mathbb{S} = \{1, 2\}$.

Let $L_a = \varepsilon H$, system (2.156) becomes a normal SPS with Markovian switching, which is described as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \dfrac{g}{l} \sin x_1(t) + \dfrac{NK_m}{ml^2} z(t), \\ \varepsilon \dot{z}(t) = -K_b N x_2(t) - R(r_t) z(t) + u(t). \end{cases} \tag{2.157}$$

The parameters of this system are given as $g = 9.8\,\mathrm{m/s^2}$, $l = 1\,\mathrm{m}$, $m = 1\,\mathrm{kg}$, $N = 10$, $l = 1\,\mathrm{m}$, $K_m = 0.1\,\mathrm{Nm/A}$, $K_b = 0.1\,\mathrm{Vs/rad}$, $R_a = 1\Omega$ and $R_b = 2\Omega$. Substituting the parameters into (2.157) and letting $u(t) = -20x_1 - 2x_2$, one has

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = z(t) + 9.8 \sin x_1(t), \\ \varepsilon \dot{z}(t) = -20x_1 - 3x_2 - R(r_t) z(t), \end{cases} \tag{2.158}$$

where TRM is first assumed to be given exactly, that is,

$$\Pi = \begin{bmatrix} -1.5 & 1.5 \\ 0.7 & -0.7 \end{bmatrix}.$$

For this case, it is also seen that the methods in [21–23, 30] fail in giving an estimation of stability bound $\bar{\varepsilon}$. By Corollary 2.2, it is concluded that the corresponding closed-loop system (2.158) has a unique solution and is exponentially mean-square stable

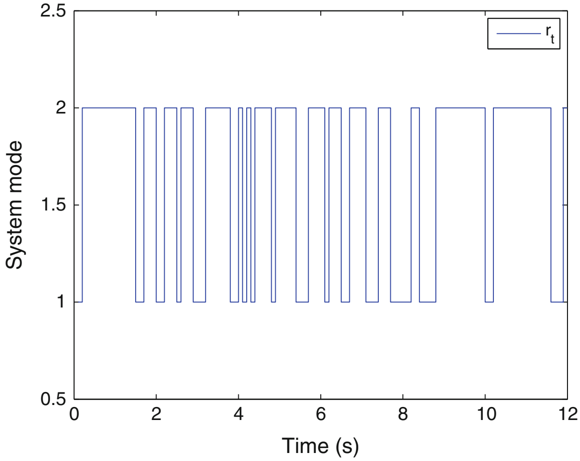

Fig. 2.2 The mode of the closed-loop system with $\varepsilon = 0.01$

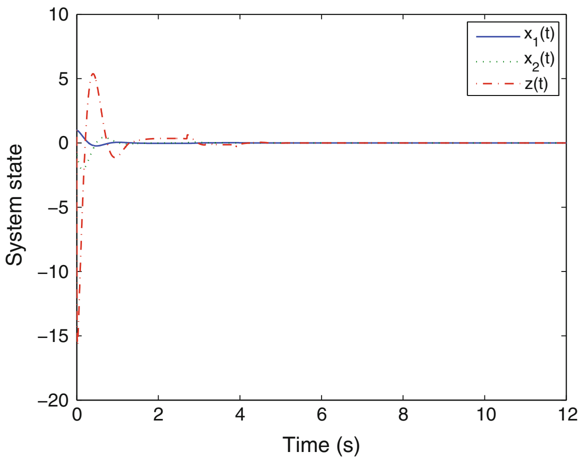

Fig. 2.3 The states of the closed-loop system with $\varepsilon = 0.01$

for any $\varepsilon \in (0, 0.1022]$. Let initial conditions $x_1(0) = 1$, $x_2(0) = -1$, $z(0) = 1$ and $r_0 = 1$. The simulation of system mode $r(t)$ is shown in Fig. 2.2, and the evolution of system state is given in Fig. 2.3.

Moreover, if TRM Π is an estimation $\tilde{\Pi}$, and the uncertainties satisfy $|\Delta\tilde{\pi}_{12}| \le \varepsilon_{12} \triangleq 0.5\tilde{\pi}_{12}$ and $|\Delta\tilde{\pi}_{21}| \le \varepsilon_{21} \triangleq 0.5\tilde{\pi}_{21}$ respectively, then from Theorem 2.16, it can be obtained that the corresponding estimation of stability bound is $\bar{\varepsilon} = 0.0685$. Especially, when there is no jumping parameter in system (2.158), that is $R_a = R_b = 1\Omega$, it becomes a deterministic singularly perturbed system. Then, it is obtained that there is no solution if the approach proposed in [28] is employed. However, the stability bound can be got as $\bar{\varepsilon} = 0.0388$ by Corollary 2.3. This example, again, shows that our result is less conservative.

2.4 Conclusion

This chapter has addressed the stability of SMJSs with general TRMs, whose TRMs may be exactly known, uncertain, partially unknown and designed. The conditions guaranteeing a given SMJS stochastically admissible are expressed in terms of LMIs or LMIs with equation constraints, which can be efficiently solved by using standard numerical algorithms. Especially, when TRM is given exactly, necessary and sufficient conditions with different forms are developed. Then, the robust stability of Markovian jump singularly perturbed systems with uncertain switchings and nonlinear perturbations for any perturbation parameter $\varepsilon \in (0, \bar{\varepsilon}]$ are solved by an LMI approach. Instead of containing ε, such conditions guaranteeing the existence and uniqueness of a solution as well as stochastic admissibility, are established by choosing an ε-dependent Lyapunov function and only depend on stability bound $\bar{\varepsilon}$. It is worth mentioning that the stability results proposed in this chapter will play important roles in dealing with other problems. Part of the results presented in this chapter are available in [31, 32].

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