Chapter 2
Differential Forms

This chapter discusses integration on differentiable manifolds. Because there is no canonical choice of local coordinates, there is no natural notion of volume, and so only objects with appropriate transformation properties under coordinate changes can be integrated. These objects, called **differential forms**, were introduced by Élie Cartan in 1899; they come equipped with natural algebraic and differential operations, making them a fundamental tool of differential geometry.

Besides their role in integration, differential forms occur in many other places in differential geometry and physics: for instance, they can be used as a very efficient device for computing the curvature of Riemannian (Chap. 4) or Lorentzian (Chap. 6) manifolds; to formulate Hamiltonian mechanics (Chap. 5); or to write Maxwell's equations of electromagnetism in a compact and elegant form.

The algebraic structure of differential forms is set up in Sect. 2.1, which reviews the notions of **tensors** and **tensor product**, and introduces **alternating tensors** and their **exterior product**.

Tensor fields, which are natural generalizations of vector fields, are discussed in Sect. 2.2, where a new operation, the **pull-back** of a covariant tensor field by a smooth map, is defined. Differential forms are introduced in Sect. 2.3 as fields of alternating tensors, along with their **exterior derivative**. Important ideas which will not be central to the remainder of this book, such as the **Poincaré lemma**, **de Rham cohomology** or the **Lie derivative**, are discussed in the exercises.

The **integral** of a differential form on a smooth manifold in defined in Sect. 2.4. This makes use of another basic tool of differential geometry, namely the existence of **partitions of unity**.

The celebrated **Stokes theorem**, generalizing the fundamental theorems of vector calculus (Green's theorem, the divergence theorem and the classical Stokes theorem for vector fields) is proved in Sect. 2.5. Some of its consequences, such as **invariance by homotopy** of the integral of closed forms, or **Brouwer's fixed point theorem**, are explored in the exercises.

Finally, Sect. 2.6 studies the relation between orientability and the existence of special differential forms, called **volume forms**, which can be used to define a notion of volume on orientable manifolds.

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2.1 Tensors

Let V be an n-dimensional vector space. A k-**tensor** on V is a real multilinear function (meaning linear in each variable) defined on the product $V \times \cdots \times V$ of k copies of V. The set of all k-tensors is itself a vector space and is usually denoted by $\mathcal{T}^k(V^*)$.

Example 1.1

(1) The space of 1-tensors $\mathcal{T}^1(V^*)$ is equal to V^*, the **dual space** of V, that is, the space of real-valued linear functions on V.
(2) The usual inner product on $\mathbb{R}^n$ is an example of a 2-tensor.
(3) The determinant is an n-tensor on $\mathbb{R}^n$.

Given a k-tensor T and an m-tensor S, we define their **tensor product** as the $(k+m)$-tensor $T \otimes S$ given by

$$T \otimes S(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m}) := T(v_1, \ldots, v_k) \cdot S(v_{k+1}, \ldots, v_{k+m}).$$

This operation is bilinear and associative, but not commutative [cf. Exercise 1.15(1)].

Proposition 1.2 *If* $\{T_1, \ldots, T_n\}$ *is a basis for* $\mathcal{T}^1(V^*) = V^*$ *(the dual space of* V*), then the set* $\{T_{i_1} \otimes \cdots \otimes T_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$ *is a basis of* $\mathcal{T}^k(V^*)$*, and therefore* $\dim \mathcal{T}^k(V^*) = n^k$.

Proof We will first show that the elements of this set are linearly independent. If

$$T := \sum_{i_1,\ldots,i_k} a_{i_1 \ldots i_k} T_{i_1} \otimes \cdots \otimes T_{i_k} = 0,$$

then, taking the basis $\{v_1, \ldots, v_n\}$ of V dual to $\{T_1, \ldots, T_n\}$, meaning that $T_i(v_j) = \delta_{ij}$ (cf. Sect. 2.7.1), we have $T(v_{j_1}, \ldots, v_{j_k}) = a_{j_1 \ldots j_k} = 0$ for every $1 \leq j_1, \ldots, j_k \leq n$.

To show that $\{T_{i_1} \otimes \cdots \otimes T_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$ spans $\mathcal{T}^k(V^*)$, we take any element $T \in \mathcal{T}^k(V^*)$ and consider the k-tensor S defined by

$$S := \sum_{i_1,\ldots,i_k} T(v_{i_1}, \ldots, v_{i_k}) T_{i_1} \otimes \cdots \otimes T_{i_k}.$$

Clearly, $S(v_{i_1}, \ldots, v_{i_k}) = T(v_{i_1}, \ldots, v_{i_k})$ for every $1 \leq i_1, \ldots, i_k \leq n$, and so, by linearity, $S = T$. □

If we consider k-tensors on V^*, instead of V, we obtain the space $\mathcal{T}^k(V)$ (note that $(V^*)^* = V$, as shown in Sect. 2.7.1). These tensors are called **contravariant tensors** on V, while the elements of $\mathcal{T}^k(V^*)$ are called **covariant tensors** on V. Note that the contravariant tensors on V are the covariant tensors on V^*. The words covariant and contravariant are related to the transformation behavior of the tensor components under a change of basis in V, as explained in Sect. 2.7.1.

We can also consider **mixed** (k, m)-tensors on V, that is, multilinear functions defined on the product $V \times \cdots \times V \times V^* \times \cdots \times V^*$ of k copies of V and m copies of V^*. A (k, m)-tensor is then k times covariant and m times contravariant on V. The space of all (k, m)-tensors on V is denoted by $\mathcal{T}^{k,m}(V^*, V)$.

Remark 1.3

(1) We can identify the space $\mathcal{T}^{1,1}(V^*, V)$ with the space of linear maps from V to V. Indeed, for each element $T \in \mathcal{T}^{1,1}(V^*, V)$, we define the linear map from V to V, given by $v \mapsto T(v, \cdot)$. Note that $T(v, \cdot) : V^* \to \mathbb{R}$ is a linear function on V^*, that is, an element of $(V^*)^* = V$.
(2) Generalizing the above definition of tensor product to tensors defined on different vector spaces, we can define the spaces $\mathcal{T}^k(V^*) \otimes \mathcal{T}^m(W^*)$ generated by the tensor products of elements of $\mathcal{T}^k(V^*)$ by elements of $\mathcal{T}^m(W^*)$. Note that $\mathcal{T}^{k,m}(V^*, V) = \mathcal{T}^k(V^*) \otimes \mathcal{T}^m(V)$. We leave it as an exercise to find a basis for this space.

A tensor is called **alternating** if, like the determinant, it changes sign every time two of its variables are interchanged, that is, if

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

The space of all alternating k-tensors is a vector subspace $\Lambda^k(V^*)$ of $\mathcal{T}^k(V^*)$. Note that, for any alternating k-tensor T, we have $T(v_1, \ldots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$.

Example 1.4

(1) All 1-tensors are trivially alternating, that is, $\Lambda^1(V^*) = \mathcal{T}^1(V^*) = V^*$.
(2) The determinant is an alternating n-tensor on $\mathbb{R}^n$.

Consider now S_k, the group of all possible permutations of $\{1, \ldots, k\}$. If $\sigma \in S_k$, we set $\sigma(v_1, \ldots, v_k) = (v_{\sigma(1)}, \ldots, v_{\sigma(k)})$. Given a k-tensor $T \in \mathcal{T}^k(V^*)$ we can define a new alternating k-tensor, called $\operatorname{Alt}(T)$, in the following way:

$$\operatorname{Alt}(T) := \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, (T \circ \sigma),$$

where $\operatorname{sgn} \sigma$ is $+1$ or -1 according to whether σ is an even or an odd permutation. We leave it as an exercise to show that $\operatorname{Alt}(T)$ is in fact alternating.

Example 1.5 If $T \in \mathcal{T}^3(V^*)$,

$$\begin{aligned} \operatorname{Alt}(T)(v_1, v_2, v_3) = \tfrac{1}{6} \big(& T(v_1, v_2, v_3) + T(v_3, v_1, v_2) + T(v_2, v_3, v_1) \\ & - T(v_1, v_3, v_2) - T(v_2, v_1, v_3) - T(v_3, v_2, v_1) \big). \end{aligned}$$

We will now define the **wedge product** between alternating tensors: if $T \in \Lambda^k(V^*)$ and $S \in \Lambda^m(V^*)$, then $T \wedge S \in \Lambda^{k+m}(V^*)$ is given by

$$T \wedge S := \frac{(k+m)!}{k!\,m!} \operatorname{Alt}(T \otimes S).$$

Example 1.6 If $T, S \in \Lambda^1(V^*) = V^*$, then

$$T \wedge S = 2 \operatorname{Alt}(T \otimes S) = T \otimes S - S \otimes T,$$

implying that $T \wedge S = -S \wedge T$ and $T \wedge T = 0$.

It is easy to verify that this product is bilinear. To prove associativity we need the following proposition.

Proposition 1.7

(i) Let $T \in \mathcal{T}^k(V^*)$ and $S \in \mathcal{T}^m(V^*)$. If $\operatorname{Alt}(T) = 0$ *then*

$$\operatorname{Alt}(T \otimes S) = \operatorname{Alt}(S \otimes T) = 0;$$

(ii) $\operatorname{Alt}(\operatorname{Alt}(T \otimes S) \otimes R) = \operatorname{Alt}(T \otimes S \otimes R) = \operatorname{Alt}(T \otimes \operatorname{Alt}(S \otimes R))$.

Proof

(i) Let us consider

$$\begin{aligned}(k+m)!\ \operatorname{Alt}(T \otimes S)(v_1, \ldots, v_{k+m}) &= \\ = \sum_{\sigma \in S_{k+m}} (\operatorname{sgn} \sigma)\, T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) S(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+m)}).\end{aligned}$$

Taking the subgroup G of S_{k+m} formed by the permutations of $\{1, \ldots, k+m\}$ that leave $k+1, \ldots, k+m$ fixed, we have

$$\begin{aligned}&\sum_{\sigma \in G} (\operatorname{sgn} \sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) S(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+m)}) = \\ &= \left(\sum_{\sigma \in G} (\operatorname{sgn} \sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \right) S(v_{k+1}, \ldots, v_{k+m}) \\ &= k!\ (\operatorname{Alt}(T) \otimes S)\,(v_1, \ldots, v_{k+m}) = 0.\end{aligned}$$

Then, since G decomposes S_{k+m} into disjoint right cosets $G \cdot \widetilde{\sigma} := \{\sigma\widetilde{\sigma} \mid \sigma \in G\}$, and for each coset

$$\sum_{\sigma\in G\cdot\widetilde{\sigma}} (\operatorname{sgn}\sigma)(T\otimes S)(v_{\sigma(1)},\ldots,v_{\sigma(k+m)}) =$$
$$= (\operatorname{sgn}\widetilde{\sigma})\sum_{\sigma\in G}(\operatorname{sgn}\sigma)\,(T\otimes S)(v_{\sigma(\widetilde{\sigma}(1))},\ldots,v_{\sigma(\widetilde{\sigma}(k+m))})$$
$$= (\operatorname{sgn}\widetilde{\sigma})k!\ (\operatorname{Alt}(T)\otimes S)(v_{\widetilde{\sigma}(1)},\ldots,v_{\widetilde{\sigma}(k+m)}) = 0,$$

we have that $\operatorname{Alt}(T\otimes S)=0$. Similarly, we prove that $\operatorname{Alt}(S\otimes T)=0$.

(ii) By linearity of the operator Alt and the fact that $\operatorname{Alt}\circ\operatorname{Alt} = \operatorname{Alt}$ [cf. Exercise 1.15(3)], we have

$$\operatorname{Alt}(\operatorname{Alt}(S\otimes R)-S\otimes R)=0.$$

Hence, by (i),

$$\begin{aligned}0 &= \operatorname{Alt}(T\otimes(\operatorname{Alt}(S\otimes R)-S\otimes R))\\ &= \operatorname{Alt}(T\otimes\operatorname{Alt}(S\otimes R))-\operatorname{Alt}(T\otimes S\otimes R),\end{aligned}$$

and the result follows. □

Using these properties we can show the following.

Proposition 1.8 $(T\wedge S)\wedge R = T\wedge(S\wedge R)$.

Proof By Proposition 1.7, for $T\in\Lambda^k(V^*)$, $S\in\Lambda^m(V^*)$ and $R\in\Lambda^l(V^*)$, we have

$$\begin{aligned}(T\wedge S)\wedge R &= \frac{(k+m+l)!}{(k+m)!\,l!}\operatorname{Alt}((T\wedge S)\otimes R)\\ &= \frac{(k+m+l)!}{k!\,m!\,l!}\operatorname{Alt}(T\otimes S\otimes R)\end{aligned}$$

and

$$\begin{aligned}T\wedge(S\wedge R) &= \frac{(k+m+l)!}{k!\,(m+l)!}\operatorname{Alt}(T\otimes(S\wedge R))\\ &= \frac{(k+m+l)!}{k!\,m!\,l!}\operatorname{Alt}(T\otimes S\otimes R).\end{aligned}$$ □

We can now prove the following theorem.

Theorem 1.9 *If $\{T_1,\ldots,T_n\}$ is a basis for V^*, then the set*

$$\{T_{i_1}\wedge\cdots\wedge T_{i_k}\mid 1\le i_1<\ldots<i_k\le n\}$$

is a basis for $\Lambda^k(V^*)$*, and*

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof Let $T \in \Lambda^k(V^*) \subset \mathcal{T}^k(V^*)$. By Proposition 1.2,

$$T = \sum_{i_1,\ldots,i_k} a_{i_1\ldots i_k} T_{i_1} \otimes \cdots \otimes T_{i_k}$$

and, since T is alternating,

$$T = \mathrm{Alt}(T) = \sum_{i_1,\ldots,i_k} a_{i_1\cdots i_k} \mathrm{Alt}(T_{i_1} \otimes \cdots \otimes T_{i_k}).$$

We can show by induction that $\mathrm{Alt}(T_{i_1} \otimes \cdots \otimes T_{i_k}) = \frac{1}{k!} T_{i_1} \wedge T_{i_2} \wedge \cdots \wedge T_{i_k}$. Indeed, for $k = 1$, the result is trivially true, and, assuming it is true for k basis tensors, we have, by Proposition 1.7, that

$$\begin{aligned}
\mathrm{Alt}(T_{i_1} \otimes \cdots \otimes T_{i_{k+1}}) &= \mathrm{Alt}(\mathrm{Alt}(T_{i_1} \otimes \cdots \otimes T_{i_k}) \otimes T_{i_{k+1}}) \\
&= \frac{k!}{(k+1)!} \mathrm{Alt}(T_{i_1} \otimes \cdots \otimes T_{i_k}) \wedge T_{i_{k+1}} \\
&= \frac{1}{(k+1)!} T_{i_1} \wedge T_{i_2} \wedge \cdots \wedge T_{i_{k+1}}.
\end{aligned}$$

Hence,

$$T = \frac{1}{k!} \sum_{i_1,\ldots,i_k} a_{i_1\ldots i_k} T_{i_1} \wedge T_{i_2} \wedge \cdots \wedge T_{i_k}.$$

However, the tensors $T_{i_1} \wedge \cdots \wedge T_{i_k}$ are not linearly independent. Indeed, due to anticommutativity, if two sequences $(i_1, \ldots i_k)$ and $(j_1, \ldots j_k)$ differ only in their orderings, then $T_{i_1} \wedge \cdots \wedge T_{i_k} = \pm T_{j_1} \wedge \cdots \wedge T_{j_k}$. In addition, if any two of the indices are equal, then $T_{i_1} \wedge \cdots \wedge T_{i_k} = 0$. Hence, we can avoid repeating terms by considering only increasing index sequences:

$$T = \sum_{i_1 < \cdots < i_k} b_{i_1\ldots i_k} T_{i_1} \wedge \cdots \wedge T_{i_k}$$

and so the set $\{T_{i_1} \wedge \cdots \wedge T_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}$ spans $\Lambda^k(V^*)$. Moreover, the elements of this set are linearly independent. Indeed, if

$$0 = T = \sum_{i_1 < \cdots < i_k} b_{i_1\ldots i_k} T_{i_1} \wedge \cdots \wedge T_{i_k},$$

then, taking a basis $\{v_1, \dots, v_n\}$ of V dual to $\{T_1, \dots, T_n\}$ and an increasing index sequence $(j_1, \dots, j_k)$, we have

$$0 = T(v_{j_1}, \dots, v_{j_k}) = k! \sum_{i_1 < \dots < i_k} b_{i_1 \dots i_k} \operatorname{Alt}(T_{i_1} \otimes \dots \otimes T_{i_k})(v_{j_1}, \dots, v_{j_k})$$
$$= \sum_{i_1 < \dots < i_k} b_{i_1 \dots i_k} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, T_{i_1}(v_{j_{\sigma(1)}}) \cdots T_{i_k}(v_{j_{\sigma(k)}}).$$

Since $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ are both increasing, the only term of the second sum that may be different from zero is the one for which $\sigma = \mathrm{id}$. Consequently,

$$0 = T(v_{j_1}, \dots, v_{j_k}) = b_{j_1 \dots j_k}.$$

□

The following result is clear from the anticommutativity shown in Example 1.6.

Proposition 1.10 *If $T \in \Lambda^k(V^*)$ and $S \in \Lambda^m(V^*)$, then*

$$T \wedge S = (-1)^{km} S \wedge T.$$

Proof Exercise 1.15(4) □

Remark 1.11

(1) Another consequence of Theorem 1.9 is that $\dim(\Lambda^n(V^*)) = 1$. Hence, if $V = \mathbb{R}^n$, any alternating n-tensor in $\mathbb{R}^n$ is a multiple of the determinant.
(2) It is also clear that $\Lambda^k(V^*) = 0$ if $k > n$. Moreover, the set $\Lambda^0(V^*)$ is defined to be equal to $\mathbb{R}$ (identified with the set of constant functions on V).

A linear transformation $F : V \to W$ induces a linear transformation $F^* : \mathcal{T}^k(W^*) \to \mathcal{T}^k(V^*)$ defined by

$$(F^*T)(v_1, \dots, v_k) = T(F(v_1), \dots, F(v_k)).$$

This map has the following properties.

Proposition 1.12 *Let V, W, Z be vector spaces, let $F : V \to W$ and $H : W \to Z$ be linear maps, and let $T \in \mathcal{T}^k(W^*)$ and $S \in \mathcal{T}^m(W^*)$. We have:*

(1) $F^*(T \otimes S) = (F^*T) \otimes (F^*S)$;
(2) If T is alternating then so is F^*T;
(3) $F^*(T \wedge S) = (F^*T) \wedge (F^*S)$;
(4) $(F \circ H)^* = H^* \circ F^*$.

Proof Exercise 1.15(5) □

Another important fact about alternating tensors is the following.

Theorem 1.13 *Let* $F : V \to V$ *be a linear map and let* $T \in \Lambda^n(V^*)$. *Then* $F^*T = (\det A)T$, *where* A *is any matrix representing* F.

Proof As $\Lambda^n(V^*)$ is 1-dimensional and F^* is a linear map, F^* is just multiplication by some constant C. Let us consider an isomorphism H between V and $\mathbb{R}^n$. Then, $H^* \det$ is an alternating n-tensor in V, and so $F^*H^* \det = CH^* \det$. Hence

$$(H^{-1})^* F^* H^* \det = C \det \Leftrightarrow (H \circ F \circ H^{-1})^* \det = C \det \Leftrightarrow A^* \det = C \det,$$

where A is the matrix representation of F induced by H. Taking the standard basis in $\mathbb{R}^n$, $\{e_1, \ldots, e_n\}$, we have

$$A^* \det (e_1, \ldots, e_n) = C \det(e_1, \ldots, e_n) = C,$$

and so

$$\det (Ae_1, \ldots, Ae_n) = C,$$

implying that $C = \det A$. □

Remark 1.14 By the above theorem, if $T \in \Lambda^n(V^*)$ and $T \neq 0$, then two ordered basis $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are equivalently oriented if and only if $T(v_1, \ldots, v_n)$ and $T(w_1, \ldots, w_n)$ have the same sign.

Exercise 1.15

(1) Show that the tensor product is bilinear and associative but not commutative.
(2) Find a basis for the space $\mathcal{T}^{k,m}(V^*, V)$ of mixed (k, m)-tensors.
(3) If $T \in \mathcal{T}^k(V^*)$, show that

   (a) $\mathrm{Alt}(T)$ is an alternating tensor;
   (b) if T is alternating then $\mathrm{Alt}(T) = T$;
   (c) $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$.

(4) Prove Proposition 1.10.
(5) Prove Proposition 1.12.
(6) Let $T_1, \ldots, T_k \in V^*$. Show that

$$(T_1 \wedge \cdots \wedge T_k)(v_1, \ldots, v_k) = \det [T_i(v_j)].$$

(7) Show that Let $T_1, \ldots, T_k \in \Lambda^1(V^*) = V^*$ are linearly independent if and only if $T_1 \wedge \cdots \wedge T_k \neq 0$.

(8) Let $T \in \Lambda^k(V^*)$ and let $v \in V$. We define **contraction** of T by v, $\iota(v)T$, as the $(k-1)$-tensor given by

$$(\iota(v)T)(v_1, \ldots, v_{k-1}) = T(v, v_1, \ldots, v_{k-1}).$$

Show that:

(a) $\iota(v_1)(\iota(v_2)T) = -\iota(v_2)(\iota(v_1)T)$;
(b) if $T \in \Lambda^k(V^*)$ and $S \in \Lambda^m(V^*)$ then

$$\iota(v)(T \wedge S) = (\iota(v)T) \wedge S + (-1)^k T \wedge (\iota(v)S).$$

2.2 Tensor Fields

The definition of a vector field can be generalized to tensor fields of general type. For that, we denote by T_p^*M the dual of the tangent space T_pM at a point p in M (usually called the **cotangent space** to M at p).

Definition 2.1 A (k, m)-**tensor field** is a map that to each point $p \in M$ assigns a tensor $T \in \mathcal{T}^{k,m}(T_p^*M, T_pM)$.

Example 2.2 A vector field is a $(0, 1)$-tensor field (or a 1-contravariant tensor field), that is, a map that to each point $p \in M$ assigns the 1-contravariant tensor $X_p \in T_pM$.

Example 2.3 Let $f : M \to \mathbb{R}$ be a differentiable function. We can define a $(1, 0)$-tensor field df which carries each point $p \in M$ to $(df)_p$, where

$$(df)_p : T_pM \to \mathbb{R}$$

is the derivative of f at p. This tensor field is called the **differential** of f. For any $v \in T_pM$ we have $(df)_p(v) = v \cdot f$ (the directional derivative of f at p along the vector v). Considering a coordinate system $x : W \to \mathbb{R}^n$, we can write $v = \sum_{i=1}^n v^i \left(\frac{\partial}{\partial x^i}\right)_p$, and so

$$(df)_p(v) = \sum_i v^i \frac{\partial \hat{f}}{\partial x^i}(x(p)),$$

where $\hat{f} = f \circ x^{-1}$. Taking the coordinate functions $x^i : W \to \mathbb{R}$, we can obtain 1-forms dx^i defined on W. These satisfy

$$(dx^i)_p\left(\left(\frac{\partial}{\partial x^j}\right)_p\right) = \delta_{ij}$$

and so they form a basis of each cotangent space T_p^*M, dual to the coordinate basis $\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^n}\right)_p\right\}$ of T_pM. Hence, any $(1,0)$-tensor field on W can be written as $\omega = \sum_i \omega_i dx^i$, where $\omega_i : W \to \mathbb{R}$ is such that $\omega_i(p) = \omega_p\left(\left(\frac{\partial}{\partial x^i}\right)_p\right)$. In particular, df can be written in the usual way

$$(df)_p = \sum_{i=1}^{n} \frac{\partial \hat{f}}{\partial x^i}(x(p))(dx^i)_p.$$

Remark 2.4 Similarly to what was done for the tangent bundle, we can consider the disjoint union of all cotangent spaces and obtain the manifold

$$T^*M = \bigcup_{p \in M} T_p^*M$$

called the **cotangent bundle** of M. Note that a $(1,0)$-tensor field is just a map from M to T^*M defined by

$$p \mapsto \omega_p \in T_p^*M.$$

This construction can be easily generalized for arbitrary tensor fields.

The space of (k,m)-tensor fields is clearly a vector space, since linear combinations of (k,m)-tensors are still (k,m)-tensors. If W is a coordinate neighborhood of M, we know that $\left\{(dx^i)_p\right\}$ is a basis for T_p^*M and that $\left\{\left(\frac{\partial}{\partial x^i}\right)_p\right\}$ is a basis for T_pM. Hence, the value of a (k,m)-tensor field T at a point $p \in W$ can be written as the tensor

$$T_p = \sum a_{i_1 \cdots i_k}^{j_1 \cdots j_m}(p)(dx^{i_1})_p \otimes \cdots \otimes (dx^{i_k})_p \otimes \left(\frac{\partial}{\partial x^{j_1}}\right)_p \otimes \cdots \otimes \left(\frac{\partial}{\partial x^{j_m}}\right)_p$$

where the $a_{i_1 \ldots i_k}^{j_1 \ldots j_m} : W \to \mathbb{R}$ are functions which at each $p \in W$ give us the components of T_p relative to these bases of T_p^*M and T_pM. Just as we did with vector fields, we say that a tensor field is **differentiable** if all these functions are differentiable for all coordinate systems of the maximal atlas. Again, we only need to consider the coordinate systems of an atlas, since all overlap maps are differentiable [cf. Exercise 2.8(1)].

Example 2.5 The differential of a smooth function $f : M \to \mathbb{R}$ is clearly a differentiable $(1,0)$-tensor field, since its components $\frac{\partial \hat{f}}{\partial x^i} \circ x$ on a given coordinate system $x : W \to \mathbb{R}^n$ are smooth.

An important operation on covariant tensors is the **pull-back** by a smooth map.

Definition 2.6 Let $f : M \to N$ be a differentiable map between smooth manifolds. Then, each differentiable k-covariant tensor field T on N defines a k-covariant tensor field f^*T on M in the following way:

$$(f^*T)_p(v_1, \ldots, v_k) = T_{f(p)}((df)_p v_1, \ldots, (df)_p v_k),$$

for $v_1, \ldots, v_k \in T_pM$.

Remark 2.7 Notice that $(f^*T)_p$ is just the image of $T_{f(p)}$ by the linear map $(df)_p^* : \mathcal{T}^k(T_{f(p)}^*N) \to \mathcal{T}^k(T_p^*M)$ induced by $(df)_p : T_pM \to T_{f(p)}N$ (cf. Sect. 2.1). Therefore the properties $f^*(\alpha T + \beta S) = \alpha(f^*T) + \beta(f^*S)$ and $f^*(T \otimes S) = (f^*T) \otimes (f^*S)$ hold for all $\alpha, \beta \in \mathbb{R}$ and all appropriate covariant tensor fields T, S. We will see in Exercise 2.8(2) that the pull-back of a differentiable covariant tensor field is still a differentiable covariant tensor field.

Exercise 2.8

(1) Find the relation between coordinate functions of a tensor field in two overlapping coordinate systems.
(2) Show that the pull-back of a differentiable covariant tensor field is still a differentiable covariant tensor field.
(3) (*Lie derivative of a tensor field*) Given a vector field $X \in \mathfrak{X}(M)$, we define the **Lie derivative of a k-covariant tensor field** T along X as

$$L_X T := \frac{d}{dt}(\psi_t{}^*T)_{|_{t=0}},$$

where $\psi_t = F(\cdot, t)$ with F the local flow of X at p.

(a) Show that

$$L_X\left(T(Y_1, \ldots, Y_k)\right) = (L_X T)(Y_1, \ldots, Y_k) \\ + T(L_X Y_1, \ldots, Y_k) + \ldots + T(Y_1, \ldots, L_X Y_k),$$

i.e. show that

$$X \cdot (T(Y_1, \ldots, Y_k)) = (L_X T)(Y_1, \ldots, Y_k) \\ + T([X, Y_1], \ldots, Y_k) + \ldots + T(Y_1, \ldots, [X, Y_k]),$$

for all vector fields $Y_1, \ldots, Y_k$ [cf. Exercises 6.11(11) and 6.11(12) in Chap. 1].

(b) How would you define the Lie derivative of a (k, m)-tensor field?

2.3 Differential Forms

Fields of alternating tensors are very important objects called **forms**.

Definition 3.1 Let M be a smooth manifold. A **form of degree** k (or k**-form**) on M is a field of alternating k-tensors defined on M, that is, a map ω that, to each point $p \in M$, assigns an element $\omega_p \in \Lambda^k(T_p^*M)$.

The space of k-forms on M is clearly a vector space. By Theorem 1.9, given a coordinate system $x : W \to \mathbb{R}^n$, any k-form on W can be written as

$$\omega = \sum_I \omega_I dx^I$$

where $I = (i_1, \ldots, i_k)$ denotes any increasing index sequence of integers in $\{1, \ldots, n\}$, dx^I is the form $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, and the ω_I are functions defined on W. It is easy to check that the components of ω in the basis $\{dx^{i_1} \otimes \cdots \otimes dx^{i_k}\}$ are $\pm\omega_I$. Therefore ω is a differentiable $(k, 0)$-tensor (in which case it is called a **differential form**) if the functions ω_I are smooth for all coordinate systems of the maximal atlas. The set of differential k-forms on M is represented by $\Omega^k(M)$. From now on we will use the word "form" to mean a differential form.

Given a smooth map $f : M \to N$ between differentiable manifolds, we can induce forms on M from forms on N using the pull-back operation (cf. Definition 2.6), since the pull-back of a field of alternating tensors is still a field of alternating tensors.

Remark 3.2 If $g : N \to \mathbb{R}$ is a 0-form, that is, a function, the pull-back is defined as $f^*g = g \circ f$.

It is easy to verify that the pull-back of forms satisfies the following properties.

Proposition 3.3 *Let $f : M \to N$ be a differentiable map and α, β forms on N. Then,*

(i) $f^*(\alpha + \beta) = f^*\alpha + f^*\beta$;
(ii) $f^*(g\alpha) = (g \circ f) f^*\alpha = (f^*g)(f^*\alpha)$ *for any function* $g \in C^\infty(N)$;
(iii) $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$;
(iv) $g^*(f^*\alpha) = (f \circ g)^*\alpha$ *for any map* $g \in C^\infty(L, M)$*, where L is a differentiable manifold.*

Proof Exercise 3.8(1) □

Example 3.4 If $f : M \to N$ is differentiable and we consider coordinate systems $x : V \to \mathbb{R}^m$, $y : W \to \mathbb{R}^n$ respectively on M and N, we have $y^i = \hat{f}^i(x^1, \ldots, x^m)$ for $i = 1, \ldots, n$ and $\hat{f} = y \circ f \circ x^{-1}$ the local representation of f. If $\omega = \sum_I \omega_I dy^I$ is a k-form on W, then by Proposition 3.3,

$$f^*\omega = f^*\left(\sum_I \omega_I dy^I\right) = \sum_I (f^*\omega_I)(f^*dy^I) = \sum_I (\omega_I \circ f)(f^*dy^{i_1}) \wedge \cdots \wedge (f^*dy^{i_k}).$$

Moreover, for $v \in T_pM$,

$$(f^*(dy^i))_p(v) = (dy^i)_{f(p)}((df)_p v) = \Big(d(y^i \circ f)\Big)_p (v),$$

that is, $f^*(dy^i) = d(y^i \circ f)$. Hence,

$$\begin{aligned} f^*\omega &= \sum_I (\omega_I \circ f)\, d(y^{i_1} \circ f) \wedge \cdots \wedge d(y^{i_k} \circ f) \\ &= \sum_I (\omega_I \circ f)\, d(\hat{f}^{i_1} \circ x) \wedge \cdots \wedge d(\hat{f}^{i_k} \circ x). \end{aligned}$$

If $k = \dim M = \dim N = n$, then the pull-back $f^*\omega$ can easily be computed from Theorem 1.13, according to which

$$(f^*(dy^1 \wedge \cdots \wedge dy^n))_p = \det (d\hat{f})_{x(p)} (dx^1 \wedge \cdots \wedge dx^n)_p. \tag{2.1}$$

Given any form ω on M and a parameterization $\varphi : U \to M$, we can consider the pull-back of ω by φ and obtain a form defined on the open set U, called the **local representation** of ω on that parameterization.

Example 3.5 Let $x : W \to \mathbb{R}^n$ be a coordinate system on a smooth manifold M and consider the 1-form dx^i defined on W. The pull-back $\varphi^* dx^i$ by the corresponding parameterization $\varphi := x^{-1}$ is a 1-form on an open subset U of $\mathbb{R}^n$ satisfying

$$\begin{aligned} (\varphi^* dx^i)_x(v) &= (\varphi^* dx^i)_x \left(\sum_{j=1}^n v^j \left(\frac{\partial}{\partial x^j}\right)_x\right) = (dx^i)_p \left(\sum_{j=1}^n v^j (d\varphi)_x \left(\frac{\partial}{\partial x^j}\right)_x\right) \\ &= (dx^i)_p \left(\sum_{j=1}^n v^j \left(\frac{\partial}{\partial x^j}\right)_p\right) = v^i = (dx^i)_x(v), \end{aligned}$$

for $x \in U$, $p = \varphi(x)$ and $v = \sum_{j=1}^n v^j \left(\frac{\partial}{\partial x^j}\right)_x \in T_x U$. Hence, just as we had $\left(\frac{\partial}{\partial x^i}\right)_p = (d\varphi)_x \left(\frac{\partial}{\partial x^i}\right)_x$, we now have $(dx^i)_x = \varphi^*(dx^i)_p$, and so $(dx^i)_p$ is the 1-form in W whose local representation on U is $(dx^i)_x$.

If $\omega = \sum_I \omega_I dx^I$ is a k-form defined on an open subset of $\mathbb{R}^n$, we define a $(k+1)$-form called **exterior derivative** of ω as

$$d\omega := \sum_I d\omega_I \wedge dx^I.$$

Example 3.6 Consider the form $\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ defined on $\mathbb{R}^2\backslash\{0\}$. Then,

$$\begin{aligned} d\omega &= d\left(-\frac{y}{x^2+y^2}\right)\wedge dx + d\left(\frac{x}{x^2+y^2}\right)\wedge dy \\ &= \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\wedge dx + \frac{y^2-x^2}{(x^2+y^2)^2}\,dx\wedge dy = 0. \end{aligned}$$

The exterior derivative satisfies the following properties:

Proposition 3.7 *If $\alpha, \omega, \omega_1, \omega_2$ are forms on $\mathbb{R}^n$, then*

(i) $d(\omega_1+\omega_2) = d\omega_1 + d\omega_2$*;*
(ii) if ω is k-form, $d(\omega\wedge\alpha) = d\omega\wedge\alpha + (-1)^k\omega\wedge d\alpha$;
(iii) $d(d\omega) = 0$*;*
(iv) if $f:\mathbb{R}^m\to\mathbb{R}^n$ is smooth, $d(f^*\omega) = f^*(d\omega)$.

Proof Property (i) is obvious. Using (i), it is enough to prove (ii) for $\omega = a_I dx^I$ and $\alpha = b_J dx^J$:

$$\begin{aligned} d(\omega\wedge\alpha) &= d(a_I b_J\,dx^I\wedge dx^J) = d(a_I b_J)\wedge dx^I\wedge dx^J \\ &= (b_J\,da_I + a_I\,db_J)\wedge dx^I\wedge dx^J \\ &= b_J\,da_I\wedge dx^I\wedge dx^J + a_I\,db_J\wedge dx^I\wedge dx^J \\ &= d\omega\wedge\alpha + (-1)^k a_I dx^I\wedge db_J\wedge dx^J \\ &= d\omega\wedge\alpha + (-1)^k\omega\wedge d\alpha. \end{aligned}$$

Again, to prove (iii), it is enough to consider forms $\omega = a_I dx^I$. Since

$$d\omega = da_I\wedge dx^I = \sum_{i=1}^n \frac{\partial a_I}{\partial x^i}\,dx^i\wedge dx^I,$$

we have

$$\begin{aligned} d(d\omega) &= \sum_{j=1}^n\sum_{i=1}^n \frac{\partial^2 a_I}{\partial x^j\partial x^i}dx^j\wedge dx^i\wedge dx^I \\ &= \sum_{i=1}^n\sum_{j<i}\left(\frac{\partial^2 a_I}{\partial x^j\partial x^i} - \frac{\partial^2 a_I}{\partial x^i\partial x^j}\right)dx^j\wedge dx^i\wedge dx^I = 0. \end{aligned}$$

To prove (iv), we first consider a 0-form g:

$$f^*(dg) = f^*\left(\sum_{i=1}^n \frac{\partial g}{\partial x^i}dx^i\right) = \sum_{i=1}^n\left(\frac{\partial g}{\partial x^i}\circ f\right)df^i = \sum_{i,j=1}^n\left(\left(\frac{\partial g}{\partial x^i}\circ f\right)\frac{\partial f^i}{\partial x^j}\right)dx^j$$

$$= \sum_{j=1}^{n} \frac{\partial (g \circ f)}{\partial x^j} dx^j = d(g \circ f) = d(f^* g).$$

Then, if $\omega = a_I dx^I$, we have

$$\begin{aligned} d(f^*\omega) &= d((f^*a_I)df^I) = d(f^*a_I) \wedge df^I + (f^*a_I)d(df^I) = d(f^*a_I) \wedge df^I \\ &= \left(f^*da_I\right) \wedge (f^*dx^I) = f^*(da_I \wedge dx^I) = f^*(d\omega) \end{aligned}$$

(where df^I denotes the form $df^{i_1} \wedge \cdots \wedge df^{i_k}$), and the result follows. □

Suppose now that ω is a differential k-form on a smooth manifold M. We define the $(k+1)$-form $d\omega$ as the smooth form that is locally represented by $d\omega_\alpha$ for each parameterization $\varphi_\alpha : U_\alpha \to M$, where $\omega_\alpha := \varphi_\alpha^* \omega$ is the local representation of ω, that is, $d\omega = (\varphi_\alpha^{-1})^*(d\omega_\alpha)$ on $\varphi_\alpha(U)$. Given another parameterization $\varphi_\beta : U_\beta \to M$ such that $W := \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \varnothing$, it is easy to verify that

$$(\varphi_\alpha^{-1} \circ \varphi_\beta)^* \omega_\alpha = \omega_\beta.$$

Setting f equal to $\varphi_\alpha^{-1} \circ \varphi_\beta$, we have

$$f^*(d\omega_\alpha) = d(f^*\omega_\alpha) = d\omega_\beta.$$

Consequently,

$$\begin{aligned} (\varphi_\beta^{-1})^* d\omega_\beta &= (\varphi_\beta^{-1})^* f^*(d\omega_\alpha) \\ &= (f \circ \varphi_\beta^{-1})^*(d\omega_\alpha) \\ &= (\varphi_\alpha^{-1})^*(d\omega_\alpha), \end{aligned}$$

and so the two definitions agree on the overlapping set W. Therefore $d\omega$ is well defined. We leave it as an exercise to show that the exterior derivative defined for forms on smooth manifolds also satisfies the properties of Proposition 3.7.

Exercise 3.8

(1) Prove Proposition 3.3.
(2) (*Exterior derivative*) Let M be a smooth manifold. Given a k-form ω in M we can define its exterior derivative $d\omega$ without using local coordinates: given $k+1$ vector fields $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$,

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) := &\sum_{i=1}^{k+1} (-1)^{i-1} X_i \cdot \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}), \end{aligned}$$

where the hat indicates an omitted variable.

(a) Show that $d\omega$ defined above is in fact a $(k+1)$-form in M, that is,
   (i) $d\omega(X_1, \ldots, X_i + Y_i, \ldots, X_{k+1}) =$
   $d\omega(X_1, \ldots, X_i, \ldots, X_{k+1}) + d\omega(X_1, \ldots, Y_i, \ldots, X_{k+1})$;
   (ii) $d\omega(X_1, \ldots, fX_j, \ldots, X_{k+1}) = f d\omega(X_1, \ldots, X_{k+1})$ for any differentiable function f;
   (iii) $d\omega$ is alternating;
   (iv) $d\omega(X_1, \ldots, X_{k+1})(p)$ depends only on $(X_1)_p, \ldots, (X_{k+1})_p$.
(b) Let $x : W \to \mathbb{R}^n$ be a coordinate system of M and let $\omega = \sum_I a_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ be the expression of ω in these coordinates (where the a_I are smooth functions). Show that the local expression of $d\omega$ is the same as the one used in the local definition of exterior derivative, that is,

$$d\omega = \sum_I da_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

(3) Show that the exterior derivative defined for forms on smooth manifolds satisfies the properties of Proposition 3.7.
(4) Show that:

(a) if $\omega = f^1 dx + f^2 dy + f^3 dz$ is a 1-form on $\mathbb{R}^3$ then

$$d\omega = g^1 dy \wedge dz + g^2 dz \wedge dx + g^3 dx \wedge dy,$$

where $(g^1, g^2, g^3) = \mathrm{curl}(f^1, f^2, f^3)$;
(b) if $\omega = f^1 dy \wedge dz + f^2 dz \wedge dx + f^3 dx \wedge dy$ is a 2-form on $\mathbb{R}^3$, then

$$d\omega = \mathrm{div}(f^1, f^2, f^3)\, dx \wedge dy \wedge dz.$$

(5) (*De Rham cohomology*) A k-form ω is called **closed** if $d\omega = 0$. If it exists a $(k-1)$-form β such that $\omega = d\beta$ then ω is called **exact**. Note that every exact form is closed. Let Z^k be the set of all closed k-forms on M and define a relation between forms on Z^k as follows: $\alpha \sim \beta$ if and only if they differ by an exact form, that is, if $\beta - \alpha = d\theta$ for some $(k-1)$-form θ.

(a) Show that this relation is an equivalence relation.
(b) Let $H^k(M)$ be the corresponding set of equivalence classes (called the k-dimensional **de Rham cohomology space** of M). Show that addition and scalar multiplication of forms define indeed a vector space structure on $H^k(M)$.
(c) Let $f : M \to N$ be a smooth map. Show that:
   (i) the pull-back f^* carries closed forms to closed forms and exact forms to exact forms;
   (ii) if $\alpha \sim \beta$ on N then $f^*\alpha \sim f^*\beta$ on M;

(iii) f^* induces a linear map on cohomology $f^\sharp : H^k(N) \to H^k(M)$ naturally defined by $f^\sharp[\omega] = [f^*\omega]$;
(iv) if $g : L \to M$ is another smooth map, then $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$.

(d) Show that the dimension of $H^0(M)$ is equal to the number of connected components of M.
(e) Show that $H^k(M) = 0$ for every $k > \dim M$.

(6) Let M be a manifold of dimension n, let U be an open subset of $\mathbb{R}^n$ and let ω be a k-form on $\mathbb{R} \times U$. Writing ω as

$$\omega = dt \wedge \sum_I a_I dx^I + \sum_J b_J dx^J,$$

where $I = (i_1, \ldots, i_{k-1})$ and $J = (j_1, \ldots, j_k)$ are increasing index sequences, $(x^1, \ldots, x^n)$ are coordinates in U and t is the coordinate in $\mathbb{R}$, consider the operator $\mathcal{Q}$ defined by

$$\mathcal{Q}(\omega)_{(t,x)} = \sum_I \left(\int_{t_0}^t a_I ds \right) dx^I,$$

which transforms k-forms ω in $\mathbb{R} \times U$ into $(k-1)$-forms.

(a) Let $f : V \to U$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. Show that the induced diffeomorphism $\widetilde{f} := \mathrm{id} \times f : \mathbb{R} \times V \to \mathbb{R} \times U$ satisfies

$$\widetilde{f}^* \circ \mathcal{Q} = \mathcal{Q} \circ \widetilde{f}^*.$$

(b) Using (a), construct an operator $\mathcal{Q}$ which carries k-forms on $\mathbb{R} \times M$ into $(k-1)$-forms and, for any diffeomorphism $f : M \to N$, the induced diffeomorphism $\widetilde{f} := \mathrm{id} \times f : \mathbb{R} \times M \to \mathbb{R} \times N$ satisfies $\widetilde{f}^* \circ \mathcal{Q} = \mathcal{Q} \circ \widetilde{f}^*$. Show that this operator is linear.
(c) Considering the operator $\mathcal{Q}$ defined in (b) and the inclusion $i_{t_0} : M \to \mathbb{R} \times M$ of M at the "level" t_0, defined by $i_{t_0}(p) = (t_0, p)$, show that $\omega - \pi^* i_{t_0}^* \omega = d\mathcal{Q}\omega + \mathcal{Q}d\omega$, where $\pi : \mathbb{R} \times M \to M$ is the projection on M.
(d) Show that the maps $\pi^\sharp : H^k(M) \to H^k(\mathbb{R} \times M)$ and $i_{t_0}^\sharp : H^k(\mathbb{R} \times M) \to H(M)$ are inverses of each other (and so $H^k(M)$ is isomorphic to $H^k(\mathbb{R} \times M)$).
(e) Use (d) to show that, for $k > 0$ and $n > 0$, every closed k-form in $\mathbb{R}^n$ is exact, that is, $H^k(\mathbb{R}^n) = 0$ if $k > 0$.
(f) Use (d) to show that, if $f, g : M \to N$ are two **smoothly homotopic maps** between smooth manifolds (meaning that there exists a smooth map $H : \mathbb{R} \times M \to N$ such that $H(t_0, p) = f(p)$ and $H(t_1, p) = g(p)$ for some fixed $t_0, t_1 \in \mathbb{R}$), then $f^\sharp = g^\sharp$.
(g) We say that M is **contractible** if the identity map $\mathrm{id} : M \to M$ is smoothly homotopic to a constant map. Show that $\mathbb{R}^n$ is contractible.

(h) (*Poincaré lemma*) Let M be a contractible smooth manifold. Show that every closed form on M is exact, that is, $H^k(M) = 0$ for all $k > 0$.
(**Remark:** This exercise is based on an exercise in [GP73]).

(7) (*Lie derivative of a differential form*) Given a vector field $X \in \mathfrak{X}(M)$, we define the **Lie derivative of a form** ω along X as

$$L_X\omega := \frac{d}{dt}({\psi_t}^*\omega)_{|t=0},$$

where $\psi_t = F(\cdot, t)$ with F the local flow of X at p [cf. Exercise 2.8(3)]. Show that the Lie derivative satisfies the following properties:

(a) $L_X(\omega_1 \wedge \omega_2) = (L_X\omega_1) \wedge \omega_2 + \omega_1 \wedge (L_X\omega_2)$;
(b) $d(L_X\omega) = L_X(d\omega)$;
(c) **Cartan formula:** $L_X\omega = \iota(X)d\omega + d(\iota(X)\omega)$;
(d) $L_X(\iota(Y)\omega) = \iota(L_XY)\omega + \iota(Y)L_X\omega$
[cf. Exercise 6.11(12) on Chap. 1 and Exercise 1.15(8)].

2.4 Integration on Manifolds

Before we see how to integrate differential forms on manifolds, we will start by studying the $\mathbb{R}^n$ case. For that let us consider an n-form ω defined on an open subset U of $\mathbb{R}^n$. We already know that ω can be written as

$$\omega_x = a(x)\, dx^1 \wedge \cdots \wedge dx^n,$$

where $a : U \to \mathbb{R}$ is a smooth function. The **support** of ω is, by definition, the closure of the set where $\omega \neq 0$ that is,

$$\operatorname{supp}\omega = \overline{\{x \in \mathbb{R}^n \mid \omega_x \neq 0\}}.$$

We will assume that this set is compact (in which case ω is said to be **compactly supported**). We define

$$\int_U \omega = \int_U a(x)\, dx^1 \wedge \cdots \wedge dx^n := \int_U a(x)\, dx^1 \cdots dx^n,$$

where the integral on the right is a multiple integral on a subset of $\mathbb{R}^n$. This definition is almost well-behaved with respect to changes of variables in $\mathbb{R}^n$. Indeed, if $f : V \to U$ is a diffeomorphism of open sets of $\mathbb{R}^n$, we have from (2.1) that

$$f^*\omega = (a \circ f)(\det df)dy^1 \wedge \cdots \wedge dy^n,$$

and so
$$\int_V f^*\omega = \int_V (a \circ f)(\det df)dy^1 \cdots dy^n.$$

If f is orientation-preserving, then $\det(df) > 0$, and the integral on the right is, by the change of variables theorem for multiple integrals in $\mathbb{R}^n$ (cf. Sect. 2.7.2), equal to $\int_U \omega$. For this reason, we will only consider orientable manifolds when integrating forms on manifolds. Moreover, we will also assume that $\operatorname{supp}\omega$ is always compact to avoid convergence problems.

Let M be an oriented manifold, and let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ be an atlas whose parameterizations are orientation-preserving. Suppose that $\operatorname{supp}\omega$ is contained in some coordinate neighborhood $W_\alpha = \varphi_\alpha(U_\alpha)$. Then we define

$$\int_M \omega := \int_{U_\alpha} \varphi_\alpha^*\omega = \int_{U_\alpha} \omega_\alpha.$$

Note that this does not depend on the choice of coordinate neighborhood: if $\operatorname{supp}\omega$ is contained in some other coordinate neighborhood $W_\beta = \varphi_\beta(U_\beta)$, then $\omega_\beta = f^*\omega_\alpha$, where $f := \varphi_\alpha^{-1} \circ \varphi_\beta$ is orientation-preserving, and hence

$$\int_{U_\beta} \omega_\beta = \int_{U_\beta} f^*\omega_\alpha = \int_{U_\alpha} \omega_\alpha.$$

To define the integral in the general case we use a **partition of unity** (cf. Sect. 2.7.2) subordinate to the cover $\{W_\alpha\}$ of M, i.e. a family of differentiable functions on M, $\{\rho_i\}_{i\in I}$, such that:

(i) for every point $p \in M$, there exists a neighborhood V of p such that $V \cap \operatorname{supp}\rho_i = \varnothing$ except for a finite number of ρ_i;
(ii) for every point $p \in M$, $\sum_{i\in I} \rho_i(p) = 1$;
(iii) $0 \leq \rho_i \leq 1$ and $\operatorname{supp}\rho_i \subset W_{\alpha_i}$ for some element W_{α_i} of the cover.

Because of property (i), $\operatorname{supp}\omega$ (being compact) intersects the supports of only finitely many ρ_i. Hence we can assume that I is finite, and then

$$\omega = \left(\sum_{i\in I} \rho_i\right)\omega = \sum_{i\in I} \rho_i\omega = \sum_{i\in I} \omega_i$$

with $\omega_i := \rho_i\omega$ and $\operatorname{supp}\omega_i \subset W_{\alpha_i}$. Consequently we define:

$$\int_M \omega := \sum_{i\in I} \int_M \omega_i = \sum_{i\in I} \int_{U_{\alpha_i}} \varphi_{\alpha_i}^*\omega_i.$$

Remark 4.1

(1) When $\operatorname{supp}\omega$ is contained in one coordinate neighborhood W, the two definitions above agree. Indeed,

$$\int_M \omega = \int_W \omega = \int_W \sum_{i\in I} \omega_i = \int_U \varphi^*\left(\sum_{i\in I}\omega_i\right)$$
$$= \int_U \sum_{i\in I} \varphi^*\omega_i = \sum_{i\in I}\int_U \varphi^*\omega_i = \sum_{i\in I}\int_M \omega_i,$$

where we used the linearity of the pull-back and of integration on $\mathbb{R}^n$.

(2) The definition of integral is independent of the choice of partition of unity and the choice of cover. Indeed, if $\{\widetilde{\rho}_j\}_{j\in J}$ is another partition of unity subordinate to another cover $\{\widetilde{W}_\beta\}$ compatible with the same orientation, we have by (1)

$$\sum_{i\in I}\int_M \rho_i\omega = \sum_{i\in I}\sum_{j\in J}\int_M \widetilde{\rho}_j\rho_i\omega$$

and

$$\sum_{j\in J}\int_M \widetilde{\rho}_j\omega = \sum_{j\in J}\sum_{i\in I}\int_M \rho_i\widetilde{\rho}_j\omega.$$

(3) It is also easy to verify the linearity of the integral, that is,

$$\int_M a\omega_1 + b\omega_2 = a\int_M \omega_1 + b\int_M \omega_2.$$

for $a, b \in \mathbb{R}$ and ω_1, ω_2 two n-forms on M.

(4) The definition of integral can easily be extended to oriented manifolds with boundary.

Exercise 4.2

(1) Let M be an n-dimensional differentiable manifold. A subset $N \subset M$ is said to have **zero measure** if the sets $\varphi_\alpha^{-1}(N) \subset U_\alpha$ have zero measure for every parameterization $\varphi_\alpha : U_\alpha \to M$ in the maximal atlas.

(a) Prove that in order to show that $N \subset M$ has zero measure it suffices to check that the sets $\varphi_\alpha^{-1}(N) \subset U_\alpha$ have zero measure for the parameterizations in an arbitrary atlas.

(b) Suppose that M is oriented. Let $\omega \in \Omega^n(M)$ be compactly supported and let $W = \varphi(U)$ be a coordinate neighborhood such that $M\setminus W$ has zero measure. Show that

$$\int_M \omega = \int_U \varphi^*\omega,$$

where the integral on the right-hand side is defined as above and always exists.

(2) Let x, y, z be the restrictions of the Cartesian coordinate functions in $\mathbb{R}^3$ to S^2, oriented so that $\{(1,0,0); (0,1,0)\}$ is a positively oriented basis of $T_{(0,0,1)}S^2$, and consider the 2-form

$$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2).$$

Compute the integral

$$\int_{S^2} \omega$$

using the parameterizations corresponding to

(a) spherical coordinates;
(b) stereographic projection.

(3) Consider the manifolds

$$S^3 = \left\{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 2\right\};$$
$$T^2 = \left\{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = z^2 + w^2 = 1\right\}.$$

The submanifold $T^2 \subset S^3$ splits S^3 into two connected components. Let M be one of these components and let ω be the 3-form

$$\omega = zdx \wedge dy \wedge dw - xdy \wedge dz \wedge dw.$$

Compute the two possible values of $\int_M \omega$.

(4) Let M and N be n-dimensional manifolds, $f : M \to N$ an orientation-preserving diffeomorphism and $\omega \in \Omega^n(N)$ a compactly supported form. Prove that

$$\int_N \omega = \int_M f^*\omega.$$

2.5 Stokes Theorem

In this section we will prove a very important theorem.

Theorem 5.1 (Stokes) *Let M be an n-dimensional oriented smooth manifold with boundary, let ω be a $(n-1)$-differential form on M with compact support, and let $i : \partial M \to M$ be the inclusion of the boundary ∂M in M. Then*

$$\int_{\partial M} i^*\omega = \int_M d\omega,$$

where we consider ∂M with the induced orientation (cf. Sect. 9 in Chap. 1).

Proof Let us take a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to an open cover of M by coordinate neighborhoods compatible with the orientation. Then $\omega = \sum_{i\in I} \rho_i\omega$, where we can assume I to be finite (ω is compactly supported), and hence

$$d\omega = d\sum_{i\in I} \rho_i\omega = \sum_{i\in I} d(\rho_i\omega).$$

By linearity of the integral we then have,

$$\int_M d\omega = \sum_{i\in I}\int_M d(\rho_i\omega) \text{ and } \int_{\partial M} i^*\omega = \sum_{i\in I}\int_{\partial M} i^*(\rho_i\omega).$$

Hence, to prove this theorem, it is enough to consider the case where $\operatorname{supp}\omega$ is contained inside one coordinate neighborhood of the cover. Let us then consider an $(n-1)$-form ω with compact support contained in a coordinate neighborhood W. Let $\varphi : U \to W$ be the corresponding parameterization, where we can assume U to be bounded ($\operatorname{supp}(\varphi^*\omega)$ is compact). Then, the representation of ω on U can be written as

$$\varphi^*\omega = \sum_{j=1}^n a_j\, dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n,$$

(where each $a_j : U \to \mathbb{R}$ is a C^∞-function), and

$$\varphi^* d\omega = d\varphi^*\omega = \sum_{j=1}^n (-1)^{j-1} \frac{\partial a_j}{\partial x^j}\, dx^1 \wedge \cdots \wedge dx^n.$$

The functions a_j can be extended to C^∞-functions on $\mathbb{H}^n$ by letting

$$a_j(x^1, \cdots, x^n) = \begin{cases} a_j(x^1, \cdots, x^n) & \text{if } (x^1, \ldots, x^n) \in U \\ 0 & \text{if } (x^1, \ldots, x^n) \in \mathbb{H}^n \backslash U. \end{cases}$$

If $W \cap \partial M = \varnothing$, then $i^*\omega = 0$. Moreover, if we consider a rectangle I in $\mathbb{H}$ containing U defined by equations $b_j \le x^j \le c_j$ $(j = 1, \ldots, n)$, we have

$$\int_M d\omega = \int_U \left(\sum_{j=1}^n (-1)^{j-1} \frac{\partial a_j}{\partial x^j} \right) dx^1 \cdots dx^n = \sum_{j=1}^n (-1)^{j-1} \int_I \frac{\partial a_j}{\partial x^j}\, dx^1 \cdots dx^n$$

$$= \sum_{j=1}^{n}(-1)^{j-1}\int_{\mathbb{R}^{n-1}}\left(\int_{b_j}^{c_j}\frac{\partial a_j}{\partial x^j}dx^j\right)dx^1\cdots dx^{j-1}dx^{j+1}\cdots dx^n$$
$$= \sum_{j=1}^{n}(-1)^{j-1}\int_{\mathbb{R}^{n-1}}\Big(a_j(x^1,\ldots,x^{j-1},c_j,x^{j+1},\ldots,x^n)-$$
$$-a_j(x^1,\ldots,x^{j-1},b_j,x^{j+1},\ldots,x^n)\Big)\,dx^1\cdots dx^{j-1}dx^{j+1}\cdots dx^n = 0,$$

where we used the Fubini theorem (cf. Sect. 2.7.3), the fundamental theorem of Calculus and the fact that the a_j are zero outside U. We conclude that, in this case, $\int_{\partial M} i^*\omega = \int_M d\omega = 0$.

If, on the other hand, $W \cap \partial M \neq \varnothing$ we take a rectangle I containing U now defined by the equations $b_j \leq x^j \leq c_j$ for $j = 1,\ldots,n-1$, and $0 \leq x^n \leq c_n$. Then, as in the preceding case, we have

$$\int_M d\omega = \int_U\left(\sum_{j=1}^{n}(-1)^{j-1}\frac{\partial a_j}{\partial x^j}\right)dx^1\cdots dx^n = \sum_{j=1}^{n}(-1)^{j-1}\int_I\frac{\partial a_j}{\partial x^j}\,dx^1\cdots dx^n$$
$$= 0 + (-1)^{n-1}\int_{\mathbb{R}^{n-1}}\left(\int_0^{c_n}\frac{\partial a_n}{\partial x^n}dx^n\right)dx^1\cdots dx^{n-1}$$
$$= (-1)^{n-1}\int_{\mathbb{R}^{n-1}}\big(a_n(x^1,\ldots,x^{n-1},c_n) - a_n(x^1,\ldots,x^{n-1},0)\big)\,dx^1\cdots dx^{n-1}$$
$$= (-1)^n\int_{\mathbb{R}^{n-1}} a_n(x^1,\ldots,x^{n-1},0)\,dx^1\ldots dx^{n-1}.$$

To compute $\int_{\partial M} i^*\omega$ we need to consider a parameterization $\widetilde{\varphi}$ of ∂M defined on an open subset of $\mathbb{R}^{n-1}$ which preserves the standard orientation on $\mathbb{R}^{n-1}$ when we consider the induced orientation on ∂M. For that, we can for instance consider the set

$$\widetilde{U} = \{(x^1,\ldots,x^{n-1}) \in \mathbb{R}^{n-1} \mid ((-1)^n x^1, x^2,\ldots,x^{n-1},0) \in U\}$$

and the parameterization $\widetilde{\varphi} : \widetilde{U} :\to \partial M$ given by

$$\widetilde{\varphi}(x^1,\ldots,x^{n-1}) := \varphi\left((-1)^n x^1, x^2,\ldots,x^{n-1},0\right).$$

Recall that the orientation on ∂M obtained from φ by just dropping the last coordinate is $(-1)^n$ times the induced orientation on ∂M (cf. Sect. 9 in Chap. 1). Therefore $\widetilde{\varphi}$ gives the correct orientation. The local expression of $i : \partial M \to M$ on these coordinates ($\hat{i} : \widetilde{U} \to U$ such that $\hat{i} = \varphi^{-1} \circ i \circ \widetilde{\varphi}$) is given by

$$\hat{i}(x^1,\ldots,x^{n-1}) = \left((-1)^n x^1, x^2,\ldots,x^{n-1},0\right).$$

Hence,

$$\int_{\partial M} i^*\omega = \int_{\widetilde{U}} \widetilde{\varphi}^* i^*\omega = \int_{\widetilde{U}} (i \circ \widetilde{\varphi})^*\omega = \int_{\widetilde{U}} (\varphi \circ \hat{i})^*\omega = \int_{\widetilde{U}} \hat{i}^*\varphi^*\omega.$$

Moreover,

$$\begin{aligned}
\hat{i}^*\varphi^*\omega &= \hat{i}^* \sum_{j=1}^{n} a_j \, dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n \\
&= \sum_{j=1}^{n} (a_j \circ \hat{i}) \, d\hat{i}^1 \wedge \cdots \wedge d\hat{i}^{j-1} \wedge d\hat{i}^{j+1} \wedge \cdots \wedge d\hat{i}^n \\
&= (-1)^n (a_n \circ \hat{i}) \, dx^1 \wedge \cdots \wedge dx^{n-1},
\end{aligned}$$

since $d\hat{i}^1 = (-1)^n dx^1, d\hat{i}^n = 0$ and $d\hat{i}^j = dx^j$, for $j \neq 1$ and $j \neq n$. Consequently,

$$\begin{aligned}
\int_{\partial M} i^*\omega &= (-1)^n \int_{\widetilde{U}} (a_n \circ \hat{i}) \, dx^1 \cdots dx^{n-1} \\
&= (-1)^n \int_{\widetilde{U}} a_n \left((-1)^n x^1, x^2, \ldots, x^{n-1}, 0 \right) dx^1 \cdots dx^{n-1} \\
&= (-1)^n \int_{\mathbb{R}^{n-1}} a_n (x^1, x^2, \ldots, x^{n-1}, 0) \, dx^1 \cdots dx^{n-1} = \int_M d\omega
\end{aligned}$$

(where we have used the change of variables theorem). $\square$

Remark 5.2 If M is an oriented n-dimensional differentiable manifold (that is, a manifold with boundary $\partial M = \varnothing$), it is clear from the proof of the Stokes theorem that

$$\int_M d\omega = 0$$

for any $(n-1)$-differential form ω on M with compact support. This can be viewed as a particular case of the Stokes theorem if we define the integral over the empty set to be zero.

Exercise 5.3

(1) Use the Stokes theorem to confirm the result of Exercise 4.2(3).
(2) (*Homotopy invariance of the integral*) Recall that two maps $f_0, f_1 : M \to N$ are said to be smoothly homotopic if there exists a differentiable map $H : \mathbb{R} \times M \to N$ such that $H(0, p) = f_0(p)$ and $H(1, p) = f_1(p)$ [cf. Exercise 3.8(6)]. If M is a compact oriented manifold of dimension n and ω is a closed n-form on N, show that

$$\int_M f_0^*\omega = \int_M f_1^*\omega.$$

(3) (a) Let $X \in \mathfrak{X}(S^n)$ be a vector field with no zeros. Show that

$$H(t,p) = \cos(\pi t)p + \sin(\pi t)\frac{X_p}{\|X_p\|}$$

is a smooth homotopy between the identity map and the antipodal map, where we make use of the identification

$$X_p \in T_pS^n \subset T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}.$$

(b) Using the Stokes theorem, show that

$$\int_{S^n} \omega > 0,$$

where

$$\omega = \sum_{i=1}^{n+1} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}$$

and $S^n = \partial\{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ has the orientation induced by the standard orientation of $\mathbb{R}^{n+1}$.

(c) Show that if n is even then X cannot exist. What about when n is odd?

(4) (*Degree of a map*) Let M, N be compact, connected oriented manifolds of dimension n, and let $f : M \to N$ be a smooth map. It can be shown that there exists a real number $\deg(f)$ (called the **degree** of f) such that, for any n-form $\omega \in \Omega^n(N)$,

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

(a) Show that if f is not surjective then $\deg(f) = 0$.

(b) Show that if f is an orientation-preserving diffeomorphism then $\deg(f) = 1$, and that if f is an orientation-reversing diffeomorphism then $\deg(f) = -1$.

(c) Let $f : M \to N$ be surjective and let $q \in N$ be a regular value of f. Show that $f^{-1}(q)$ is a finite set and that there exists a neighborhood W of q in N such that $f^{-1}(W)$ is a disjoint union of opens sets V_i of M with $f|_{V_i} : V_i \to W$ a diffeomorphism.

(d) Admitting the existence of a regular value of f, show that $\deg(f)$ is an integer. (**Remark:** The **Sard theorem** guarantees that the set of critical values of a differentiable map f between manifolds with the same dimension has zero measure, which in turn guarantees the existence of a regular value of f).

(e) Given $n \in \mathbb{N}$, indicate a smooth map $f : S^1 \to S^1$ of degree n.

(f) Show that homotopic maps have the same degree.

(g) Let $f : S^n \to S^n$ be an orientation-preserving diffeomorphism if n is even, or an orientation-reversing diffeomorphism if n is odd. Prove that f has a fixed point, that is, a point $p \in S^n$ such that $f(p) = p$. (**Hint:** Show that if f had no fixed points then it would be possible to construct an homotopy between f and the antipodal map).

2.6 Orientation and Volume Forms

In this section we will study the relation between orientation and differential forms.

Definition 6.1 A **volume form** (or **volume element**) on a manifold M of dimension n is an n-form ω such that $\omega_p \neq 0$ for all $p \in M$.

The existence of a volume form is equivalent to M being orientable.

Proposition 6.2 *A manifold M of dimension n is orientable if and only if there exists a volume form on M.*

Proof Let ω be a volume form on M, and consider an atlas $\{(U_\alpha, \varphi_\alpha)\}$. We can assume without loss of generality that the open sets U_α are connected. We will construct a new atlas from this one whose overlap maps have derivatives with positive determinant. Indeed, considering the representation of ω on one of these open sets $U_\alpha \subset \mathbb{R}^n$, we have

$$\varphi_\alpha^* \omega = a_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n,$$

where the function a_α cannot vanish, and hence must have a fixed sign. If a_α is positive, we keep the corresponding parameterization. If not, we construct a new parameterization by composing φ_α with the map

$$(x^1, \ldots, x^n) \mapsto (-x^1, x^2, \ldots, x^n).$$

Clearly, in these new coordinates, the new function a_α is positive. Repeating this for all coordinate neighborhoods we obtain a new atlas for which all the functions a_α are positive, which we will also denote by $\{(U_\alpha, \varphi_\alpha)\}$. Moreover, whenever $W := \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \varnothing$, we have $\omega_\alpha = (\varphi_\beta^{-1} \circ \varphi_\alpha)^* \omega_\beta$. Hence,

$$\begin{aligned} a_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n &= (\varphi_\beta^{-1} \circ \varphi_\alpha)^* (a_\beta \, dx_\beta^1 \wedge \cdots \wedge dx_\beta^n) \\ &= (a_\beta \circ \varphi_\beta^{-1} \circ \varphi_\alpha)(\det(d(\varphi_\beta^{-1} \circ \varphi_\alpha))) \, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n \end{aligned}$$

and so $\det(d(\varphi_\beta^{-1} \circ \varphi_\alpha)) > 0$. We conclude that M is orientable.

Conversely, if M is orientable, we consider an atlas $\{(U_\alpha, \varphi_\alpha)\}$ for which the overlap maps $\varphi_\beta^{-1} \circ \varphi_\alpha$ are such that $\det d(\varphi_\beta^{-1} \circ \varphi_\alpha) > 0$. Taking a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to the cover of M by the corresponding coordinate neighborhoods, we may define the forms

$$\omega_i := \rho_i dx_i^1 \wedge \cdots \wedge dx_i^n$$

with $\operatorname{supp}\omega_i = \operatorname{supp}\rho_i \subset \varphi_{\alpha_i}(U_{\alpha_i})$. Extending these forms to M by making them zero outside $\operatorname{supp}\rho_i$, we may define the form $\omega := \sum_{i\in I}\omega_i$. Clearly ω is a well-defined n-form on M so we just need to show that $\omega_p \neq 0$ for all $p \in M$. Let p be a point in M. There is an $i \in I$ such that $\rho_i(p) > 0$, and so there exist linearly independent vectors $v_1, \ldots, v_n \in T_pM$ such that $(\omega_i)_p(v_1, \ldots, v_n) > 0$. Moreover, for all other $j \in I\backslash\{i\}$ we have $(\omega_j)_p(v_1, \ldots, v_n) \geq 0$. Indeed, if $p \notin \varphi_{\alpha_j}(U_{\alpha_j})$, then $(\omega_j)_p(v_1, \ldots, v_n) = 0$. On the other hand, if $p \in \varphi_{\alpha_j}(U_{\alpha_j})$, then by (2.1)

$$dx_j^1 \wedge \cdots \wedge dx_j^n = \det(d(\varphi_{\alpha_j}^{-1} \circ \varphi_{\alpha_i}))dx_i^1 \wedge \cdots \wedge dx_i^n$$

and hence

$$(\omega_j)_p(v_1, \ldots, v_n) = \frac{\rho_j(p)}{\rho_i(p)}(\det(d(\varphi_{\alpha_j}^{-1} \circ \varphi_{\alpha_i})))(\omega_i)_p(v_1, \ldots, v_n) \geq 0.$$

Consequently, $\omega_p(v_1, \ldots, v_n) > 0$, and so ω is a volume form. □

Remark 6.3 Sometimes we call a volume form an orientation. In this case the orientation on M is the one for which a basis $\{v_1, \ldots, v_n\}$ of T_pM is positive if and only if $\omega_p(v_1, \ldots, v_n) > 0$.

If we fix a volume form $\omega \in \Omega^n(M)$ on an orientable manifold M, we can define the **integral** of any compactly supported function $f \in C^\infty(M, \mathbb{R})$ as

$$\int_M f := \int_M f\omega$$

(where the orientation of M is determined by ω). If M is compact, we define its **volume** to be

$$\operatorname{vol}(M) := \int_M 1 = \int_M \omega.$$

Exercise 6.4

(1) Show that $M \times N$ is orientable if and only if both M and N are orientable.
(2) Let M be a compact oriented manifold with volume element $\omega \in \Omega^n(M)$. Prove that if $f > 0$ then $\int_M f\omega > 0$. (**Remark:** In particular, the volume of a compact manifold is always positive).
(3) Let M be a compact orientable manifold of dimension n, and let ω be an $(n-1)$-form in M.

    (a) Show that there exists a point $p \in M$ for which $(d\omega)_p = 0$.

(b) Prove that there exists no immersion $f : S^1 \to \mathbb{R}$ of the unit circle into $\mathbb{R}$.

(4) Let $f : S^n \to S^n$ be the antipodal map. Recall that the n-dimensional projective space is the differential manifold $\mathbb{R}P^n = S^n/\mathbb{Z}_2$, where the group $\mathbb{Z}_2 = \{1, -1\}$ acts on S^n through $1 \cdot x = x$ and $(-1) \cdot x = f(x)$. Let $\pi : S^n \to \mathbb{R}P^n$ be the natural projection.

(a) Prove that $\omega \in \Omega^k(S^n)$ is of the form $\omega = \pi^*\theta$ for some $\theta \in \Omega^k(\mathbb{R}P^n)$ if and only if $f^*\omega = \omega$.
(b) Show that $\mathbb{R}P^n$ is orientable if and only if n is odd, and that in this case,

$$\int_{S^n} \pi^*\theta = 2\int_{\mathbb{R}P^n} \theta.$$

(c) Show that for n even the sphere S^n is the orientable double covering of $\mathbb{R}P^n$ [cf. Exercise 8.6(9) in Chap. 1].

(5) Let M be a compact oriented manifold with boundary and $\omega \in \Omega^n(M)$ a volume element. The **divergence** of a vector field $X \in \mathfrak{X}(M)$ is the function $\operatorname{div}(X)$ such that

$$L_X\omega = (\operatorname{div}(X))\omega$$

[cf. Exercise 3.8(7)]. Show that

$$\int_M \operatorname{div}(X) = \int_{\partial M} \iota(X)\omega.$$

(6) (*Brouwer fixed point theorem*)

(a) Let M be an n-dimensional compact orientable manifold with boundary $\partial M \neq \varnothing$. Show that there exists no smooth map $f : M \to \partial M$ satisfying $f|_{\partial M} = \text{id}$.
(b) Prove the **Brouwer fixed point theorem**: Any smooth map $g : B \to B$ of the closed ball $B := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ to itself has a fixed point, that is, a point $p \in B$ such that $g(p) = p$. (**Hint:** For each point $x \in B$, consider the ray r_x starting at $g(x)$ and passing through x. There is only one point $f(x)$ different from $g(x)$ on $r_x \cap \partial B$. Consider the map $f : B \to \partial B$).

2.7 Notes

2.7.1 Section 2.1

(1) Given a finite dimensional vector space V we define its **dual space** as the space of linear functionals on V.

Proposition 7.1 If $\{v_1, \ldots, v_n\}$ is a basis for V then there is a unique basis $\{T_1, \ldots, T_n\}$ of V^* dual to $\{v_1, \ldots, v_n\}$, that is, such that $T_i(v_j) = \delta_{ij}$.

Proof By linearity, the equations $T_i(v_j) = \delta_{ij}$ define a unique set of functionals $T_i \in V^*$. Indeed, for any $v \in V$, we have $v = \sum_{j=1}^n a_j v_j$ and so

$$T_i(v) = \sum_{j=1}^n a_j T_i(v_j) = \sum_{j=1}^n a_j \delta_{ij} = a_i.$$

Moreover, these uniquely defined functionals are linearly independent. In fact, if

$$T := \sum_{i=1}^n b_i T_i = 0,$$

then, for each $j = 1, \ldots, n$, we have

$$0 = T(v_j) = \sum_{i=1}^n b_i T_i(v_j) = b_j.$$

To show that $\{T_1, \ldots, T_n\}$ generates V^*, we take any $S \in V^*$ and set $b_i := S(v_i)$. Then, defining $T := \sum_{i=1}^n b_i T_i$, we see that $S(v_j) = T(v_j)$ for all $j = 1, \ldots, n$. Since $\{v_1, \ldots, v_n\}$ is a basis for V, we have $S = T$. □

Moreover, if $\{v_1, \ldots, v_n\}$ is a basis for V and $\{T_1, \ldots, T_n\}$ is its dual basis, then, for any $v = \sum a_j v_j \in V$ and $T = \sum b_i T_i \in V^*$, we have

$$T(v) = \sum_{j=i}^n b_i T_i(v) = \sum_{i,j=1}^n a_j b_i T_i(v_j) = \sum_{i,j=1}^n a_j b_i \delta_{ij} = \sum_{i=1}^n a_i b_i.$$

If we now consider a linear functional F on V^*, that is, an element of $(V^*)^*$, we have $F(T) = T(v_0)$ for some fixed vector $v_0 \in V$. Indeed, let $\{v_1, \ldots, v_n\}$ be a basis for V and let $\{T_1, \ldots, T_n\}$ be its dual basis. Then if $T = \sum_{i=1}^n b_i T_i$, we have $F(T) = \sum_{i=1}^n b_i F(T_i)$. Denoting the values $F(T_i)$ by a_i, we get $F(T) = \sum_{i=1}^n a_i b_i = T(v_0)$ for $v_0 = \sum_{i=1}^n a_i v_i$. This establishes a one-to-one correspondence between $(V^*)^*$ and V, and allows us to view V as the space of linear functionals on V^*. For $v \in V$ and $T \in V^*$, we write $v(T) = T(v)$.

(2) Changing from a basis $\{v_1, \ldots, v_n\}$ to a new basis $\{v'_1, \ldots, v'_n\}$ in V, we obtain a **change of basis matrix** S, whose jth column is the vector of coordinates of the new basis vector v'_j in the old basis. We can then write the symbolic matrix equation

$$(v'_1, \ldots, v'_n) = (v_1, \ldots, v_n)S.$$

The coordinate (column) vectors a and b of a vector $v \in V$ (a contravariant 1-tensor on V) with respect to the old basis and to the new basis are related by

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = S^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = S^{-1}a,$$

since we must have $(v'_1, \ldots, v'_n)b = (v_1, \ldots, v_n)a = (v'_1, \ldots, v'_n)S^{-1}a$. On the other hand, if $\{T_1, \ldots, T_n\}$ and $\{T'_1, \ldots, T'_n\}$ are the dual bases of $\{v_1, \ldots, v_n\}$ and $\{v'_1, \ldots, v'_n\}$, we have

$$\begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} (v_1, \ldots, v_n) = \begin{pmatrix} T'_1 \\ \vdots \\ T'_n \end{pmatrix} \left(v'_1, \ldots, v'_n\right) = I$$

(where, in the symbolic matrix multiplication above, each coordinate is obtained by applying the covectors to the vectors). Hence,

$$\begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} \left(v'_1, \ldots, v'_n\right) S^{-1} = I \Leftrightarrow S^{-1} \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} \left(v'_1, \ldots, v'_n\right) = I,$$

implying that

$$\begin{pmatrix} T'_1 \\ \vdots \\ T'_n \end{pmatrix} = S^{-1} \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}.$$

The coordinate (row) vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of a 1-tensor $T \in V^*$ (a covariant 1-tensor on V) with respect to the old basis $\{T_1, \ldots, T_n\}$ and to the new basis $\{T'_1, \ldots, T'_n\}$ are related by

$$a \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} = b \begin{pmatrix} T'_1 \\ \vdots \\ T'_n \end{pmatrix} \Leftrightarrow aS \begin{pmatrix} T'_1 \\ \vdots \\ T'_n \end{pmatrix} = b \begin{pmatrix} T'_1 \\ \vdots \\ T'_n \end{pmatrix}$$

and so $b = aS$. Note that the coordinate vectors of the covariant 1-tensors on V transform like the basis vectors of V (that is, by means of the matrix S) whereas the coordinate vectors of the contravariant 1-tensors on V transform by means of the inverse of this matrix. This is the origin of the terms "covariant" and "contravariant".

2.7.2 Section 2.4

(1) *(Change of variables theorem)* Let $U, V \subset \mathbb{R}^n$ be open sets, let $g : U \to V$ be a diffeomorphism and let $f : V \to \mathbb{R}$ be an integrable function. Then

$$\int_V f = \int_U (f \circ g)|\det dg|.$$

(2) To define smooth objects on manifolds it is often useful to define them first on coordinate neighborhoods and then glue the pieces together by means of a **partition of unity**.

Theorem 7.1 *Let M be a smooth manifold and $\mathcal{V}$ an open cover of M. Then there is a family of differentiable functions on M, $\{\rho_i\}_{i\in I}$, such that:*

(i) for every point $p \in M$, there exists a neighborhood U of p such that $U \cap \operatorname{supp} \rho_i = \varnothing$ except for a finite number of ρ_i;
(ii) for every point $p \in M$, $\sum_{i\in I} \rho_i(p) = 1$;
(iii) $0 \le \rho_i \le 1$ and $\operatorname{supp} \rho_i \subset V$ for some element $V \in \mathcal{V}$.

Remark 7.2 This collection ρ_i of smooth functions is called partition of unity subordinate to the cover $\mathcal{V}$.

Proof Let us first assume that M is compact. For every point $p \in M$ we consider a coordinate neighborhood $W_p = \varphi_p(U_p)$ around p contained in an element V_p of $\mathcal{V}$, such that $\varphi_p(0) = p$ and $B_3(0) \subset U_p$ (where $B_3(0)$ denotes the ball of radius 3 around 0). Then we consider the C^∞-functions (cf. Fig. 2.1)

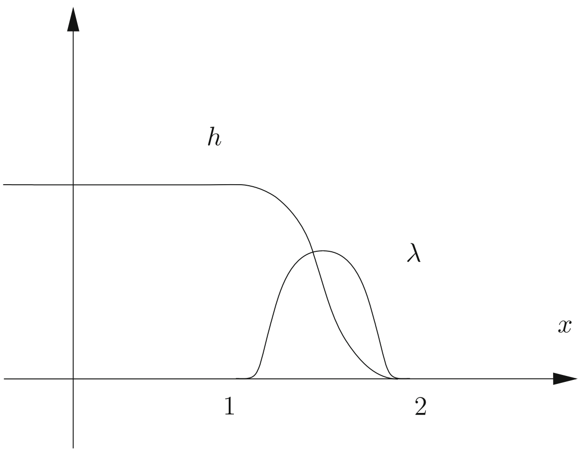

Fig. 2.1 Graphs of the functions λ and h

$$\begin{aligned}
\lambda : \mathbb{R} &\to \mathbb{R} \\
x &\mapsto \begin{cases} e^{\frac{1}{(x-1)(x-2)}} & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases} ,
\end{aligned}$$

$$\begin{aligned}
h : \mathbb{R} &\to \mathbb{R} \\
x &\mapsto \frac{\int_x^2 \lambda(t)\, dt}{\int_1^2 \lambda(t)\, dt} ,
\end{aligned}$$

$$\begin{aligned}
\beta : \mathbb{R}^n &\to \mathbb{R} \\
x &\mapsto h(\|x\|) .
\end{aligned}$$

Notice that h is a decreasing function with values $0 \leq h(x) \leq 1$, equal to zero for $x \geq 2$ and equal to 1 for $x \leq 1$. Hence, we can consider **bump functions** $\gamma_p : M \to [0, 1]$ defined by

$$\gamma_p(q) = \begin{cases} \beta(\varphi_p^{-1}(q)) & \text{if } q \in \varphi_p(U_p) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then $\operatorname{supp} \gamma_p = \overline{\varphi_p(B_2(0))} \subset \varphi_p(B_3(0)) \subset W_p$ is contained inside an element V_p of the cover. Moreover, $\{\varphi_p(B_1(0))\}_{p \in M}$ is an open cover of M and so we can consider a finite subcover $\{\varphi_{p_i}(B_1(0))\}_{i=1}^k$ such that $M = \cup_{i=1}^k \varphi_{p_i}(B_1(0))$. Finally we take the functions

$$\rho_i = \frac{\gamma_{p_i}}{\sum_{j=1}^k \gamma_{p_j}} .$$

Note that $\sum_{j=1}^k \gamma_{p_j}(q) \neq 0$ since q is necessarily contained inside some $\varphi_{p_i}(B_1(0))$ and so $\gamma_i(q) \neq 0$. Moreover, $0 \leq \rho_i \leq 1$, $\sum \rho_i = 1$ and $\operatorname{supp} \rho_i = \operatorname{supp} \gamma_{p_i} \subset V_{p_i}$.

If M is not compact we can use a **compact exhaustion**, that is, a sequence $\{K_i\}_{i \in \mathbb{N}}$ of compact subsets of M such that $K_i \subset \operatorname{int} K_{i+1}$ and $M = \cup_{i=1}^\infty K_i$. The partition of unity is then obtained as follows. The family $\{\varphi_p(B_1(0))\}_{p \in M}$ is a cover of K_1, so we can consider a finite subcover of K_1,

$$\left\{\varphi_{p_1}(B_1(0)), \ldots, \varphi_{p_{k_1}}(B_1(0))\right\} .$$

By induction, we obtain a finite number of points such that

$$\left\{\varphi_{p_1^i}(B_1(0)), \ldots, \varphi_{p_{k_i}^i}(B_1(0))\right\}$$

covers $K_i \setminus \operatorname{int} K_{i-1}$ (a compact set). Then, for each i, we consider the corresponding bump functions

$$\gamma_{p_1^i}, \dots, \gamma_{p_{k_i}^i} : M \to [0, 1].$$

Note that $\gamma_{p_1{}^i} + \cdots + \gamma_{p_{k_i}^i} > 0$ for every $q \in K_i \setminus \operatorname{int} K_{i-1}$ (as there is always one of these functions which is different from zero). As in the compact case, we can choose these bump functions so that $\operatorname{supp} \gamma_{p_j^i}$ is contained in some element of $\mathcal{V}$. We will also choose them so that $\operatorname{supp} \gamma_{p_j^i} \subset \int K_{i+1} \setminus K_{i-2}$ (an open set). Hence, $\{\gamma_{p_j^i}\}_{i \in \mathbb{N}, 1 \leq j \leq k_i}$ is locally finite, meaning that, given a point $p \in M$, there exists an open neighborhood V of p such that only a finite number of these functions is different from zero in V. Consequently, the sum $\sum_{i=1}^{\infty} \sum_{j=1}^{k_i} \gamma_{p_j^i}$ is a positive, differentiable function on M. Finally, making

$$\rho_j^i = \frac{\gamma_{p_j^i}}{\sum_{i=1}^{\infty} \sum_{j=1}^{k_i} \gamma_{p_j^i}},$$

we obtain the desired partition of unity (subordinate to $\mathcal{V}$). □

Remark 7.3 Compact exhaustions always exist on manifolds. In fact, if U is a bounded open set of $\mathbb{R}^n$, one can easily construct a compact exhaustion $\{K_i\}_{i \in \mathbb{N}}$ for U by setting

$$K_i = \left\{ x \in U \mid \operatorname{dist}(x, \partial U) \geq \frac{1}{n} \right\}.$$

If M is a differentiable manifold, one can always take a countable atlas $\mathcal{A} = \{(U_j, \varphi_j)\}_{j \in \mathbb{N}}$ such that each U_j is a bounded open set, thus admitting a compact exhaustion $\{K_i^j\}_{i \in \mathbb{N}}$. Therefore

$$\left\{ \bigcup_{i+j=l} \varphi_j \left(K_i^j \right) \right\}_{l \in \mathbb{N}}$$

is a compact exhaustion of M.

2.7.3 Section 2.5

(Fubini theorem) Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be compact intervals and let $f : A \times B \to \mathbb{R}$ be a continuous function. Then

$$\int_{A\times B} f = \int_A \left(\int_B f(x, y) dy^1 \cdots dy^m \right) dx^1 \cdots dx^n$$
$$= \int_B \left(\int_A f(x, y) dx^1 \cdots dx^n \right) dy^1 \cdots dy^m.$$

2.7.4 Bibliographical Notes

The material in this chapter can be found in most books on differential geometry (e.g. [Boo03, GHL04]). A text entirely dedicated to differential forms and their applications is [dC94]. The study of de Rham cohomology leads to a beautiful and powerful theory, whose details can be found for instance in [BT82].

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