Chapter 1
Introduction

Random sets originate in works published in the mid-sixties by well-known economists, Aumann [1] and Debreu [3] on the integration of set-valued functions. They have been given a full-fledged mathematical development by Kendall [10] and Matheron [16]. Random sets seem to have been originally used to handle uncertainty in spatial information, namely to tackle uncertainty in the definitions of geographical areas, in mathematical morphology, and in connection to geostatistics (to which Matheron is a major pioneering contributor, as seen by his work on kriging). Under this view, a precise realisation of a random set process is a precisely located set or region in an area of interest. This approach, especially applied to continuous spaces, raises subtle mathematical issues concerning the correct topology for handling set-valued realisations, that perhaps hide the intrinsic simplicity and intuitions behind random sets (e.g., casting them in a finite setting). The reason is that continuous random sets, like in geostatistics, were perhaps more easily found in applications than finite ones at that time. In any case this peculiarity has confined random sets to very specialised areas of mathematics.

Yet, as argued in this book, random sets, including and especially finite ones, can be useful in other areas, and especially information processing and knowledge representation. More precisely, the treatment of incomplete or imprecise statistical data can benefit from random sets. Indeed, sets can represent incomplete information about otherwise precise variables, that is, a random set can be a natural model for an ill-known random variable. A random set is then a natural representation of a set of possible random variables, one of which is the right one, what we can call the *epistemic* understanding of random sets. This kind of view is at work in the pioneering works of Dempster [4] who studied upper and lower probabilities induced by a multivalued mapping from a probability space to the range of an attribute of interest. That multivalued mapping is formally the same as a random set, but now, the set has a different meaning from the one in geostatistics. In the latter area, there is no such things as upper and lower probabilities. However, if the random variable is ill-known, the probabilities of events become ill-known as well. Dempster seems to have been interested in catching up with older debates on the meaning of probabilities and of the likelihood functions, in which the statistician Ronald Fisher was involved in the

I. Couso et al., *Random Sets and Random Fuzzy Sets as Ill-Perceived Random Variables*,
SpringerBriefs in Computational Intelligence, DOI: 10.1007/978-3-319-08611-8_1

first half of the twentieth century, and that were left unsolved.[1] In Dempster's view, if observable quantities can be related to the ill-known parameter characterising a probabilistic model via a function with a known probability distribution, observations generate a random set of possible parameter values for the model. On this ground, Shafer [19] developed his theory of evidence, breaking loose from the statistical setting of Dempster upper and lower probabilities, and considering them as a form of non-additive subjective probabilities that go back to some ill-studied part of Bernoulli's works. The mathematical building block of evidence theory is a probability distribution on the powerset of a finite set. The ensuing popularity of Shafer theory in artificial intelligence showed that there could be plenty of applications for finite random sets. Strangely enough there seems to have been very few cross-references from the Dempster–Shafer literature to the random set literature and conversely, if we except Nguyen's early 1978 paper pointing out the formal similarities between random sets and belief functions [17]. In this publication, the basic mathematics of random sets are discussed in detail in an elementary framework, with due credit to both traditions.

The emergence of fuzzy sets after Lotfi Zadeh's pioneering paper in 1965 [26][2] led to their hybridisation with random sets, first by the French mathematician Robert Féron in 1976, with his paper [9] in the Comptes Rendus de l'Académie des Sciences, written in the French language, and whose title strictly speaking means "random fuzzy sets". This paper and other more extensive ones (in a little known French mathematical economics outlet) appeared just after the introduction of random sets in geostatistics by another French pioneer, Matheron, and they are in the same spirit.

In 1978 and 1979, Kwakernaak [14, 15] proposed the notion of fuzzy random variable completely independently from the random set tradition, with a clear intention to represent an ill-known random variable (he explicitly mentions the fact that behind a fuzzy random variable there is an original one that is standard, but ill-perceived). Indeed, Kwakernaak does not refer to random sets at all, nor to Dempster's works. While Féron was fuzzifying random sets, Kwakernaak did randomise fuzzy intervals understood as incomplete information.

The epistemic stance of Kwakernaak approach is testified by the fact that his second paper [15] cites the seminal paper by Zadeh on possibility theory [27]. Zadeh's paper prompted Dubois and Prade to relate fuzzy sets, viewed as possibility distributions representing incomplete information, and Shafer belief functions. In their 1980 book [6], they point out that Zadeh's *possibility measures* are consonant plausibility functions, and introduce the name *necessity measures* for the conjugate functions of possibility measures, which are special case of belief functions. This book, along with some subsequent publications by Yager [23, 24] contributed to highlight possibility theory as an elementary basic building block of uncertainty representations. Consonant versions of Shafer's belief and plausibility functions

[1] In his paper, Dempster refers to the early works of Smith [21] on imprecise probabilities, but no reference to Aumann and Debreu is made (in fact these pioneers work more or less at the same time in different areas).

[2] Interestingly, the very same year as Auman and Debreu's papers.

actually date back to pioneering works from the late 1940s on by the economist Shackle [20], who should be considered as the forerunner of possibility theory. Possibility theory received an extensive treatment in [8], and has since then been acknowledged as one of the three major uncertainty theories along with Shafer's evidence theory and the one of imprecise probabilities [22].

The objectivist tradition was taken over in the 1980s, by Puri and Ralescu [18] who provide a rigorous mathematical foundation for fuzzy random sets, pursuing the line opened by Féron. They indeed seem to cast their work in the Auman-Debreu-Matheron tradition of random sets (even if citing many above-mentioned forerunners, with the notable exception of Dempster and Shafer and possibility theory). They started to extend standard probabilistic notions to this setting (expectation, limit theorems, normality...), while, much later on, Körner [11] construes the variance of a fuzzy random variable as a precise number. On the contrary, the book by Kruse and Meyer [13], whose works parallel the ones of Puri and Ralescu, catches up with the tradition, initiated by Kwakernaak, of viewing a fuzzy random variable as a tool for modeling incomplete fuzzy (they call it vague) information. As a consequence they consider the variance of a fuzzy random variable as a fuzzy set modeling what is known about the variance of the original random variable. Actually the difference between the two traditions cannot be observed by studying the expected value, which in the two traditions, is an interval or a fuzzy interval in the sense of Aumann integral. But the way the variance is defined[3] is a good indication of whether a random (fuzzy) set models incomplete information or not. Figure 1.1 provides an overview of the history of ideas and formal concepts outlined in this introduction.

Strangely enough, in the 1990s and later, many mathematical contributions to the theory of fuzzy random variables (see for instance [2]) followed the Puri and Ralescu tradition, not so much the Kwakernaak–Kruse one, even if, in these subsequent works, there was a deliberate intention of relating the mathematical developments and the ensuing statistical tools to ill-perceived outcomes to random experiments taking the form of linguistic variables in the sense of Zadeh, hence fuzzy numbers.[4] This is perhaps due to the lack of awareness of the existence of two distinct epistemological traditions, beyond the mathematical differences pertaining to the kind of metric spaces used, etc.

The aim of this book is to contribute to highlight the distinction between ontic and epistemic random sets, and presents the basics of the epistemic approach in connexion to imprecise probability theory, thus bridging the gaps between the works of the Kwakernaak–Kruse tradition, and Dempster's pioneering works in upper and lower probabilities. Our claim is that while the Puri–Ralescu "objectivist" tradition seems to be fit to the modeling of the variability of entities naturally taking the form of fuzzy sets, the question of handling epistemic uncertainty in statistical processes

---

[3] Compare the papers by Körner [11] and Kruse [12].

[4] In parallel, some works proposed a direct fuzzification of Shafer's theory of evidence (originating quite early in a paper by Zadeh [28]) such as the papers of Dubois and Prade [7] (also relying on Dempster's construction), Yen [25], or more recently Denœux [5]; these authors make no reference to fuzzy random variables.

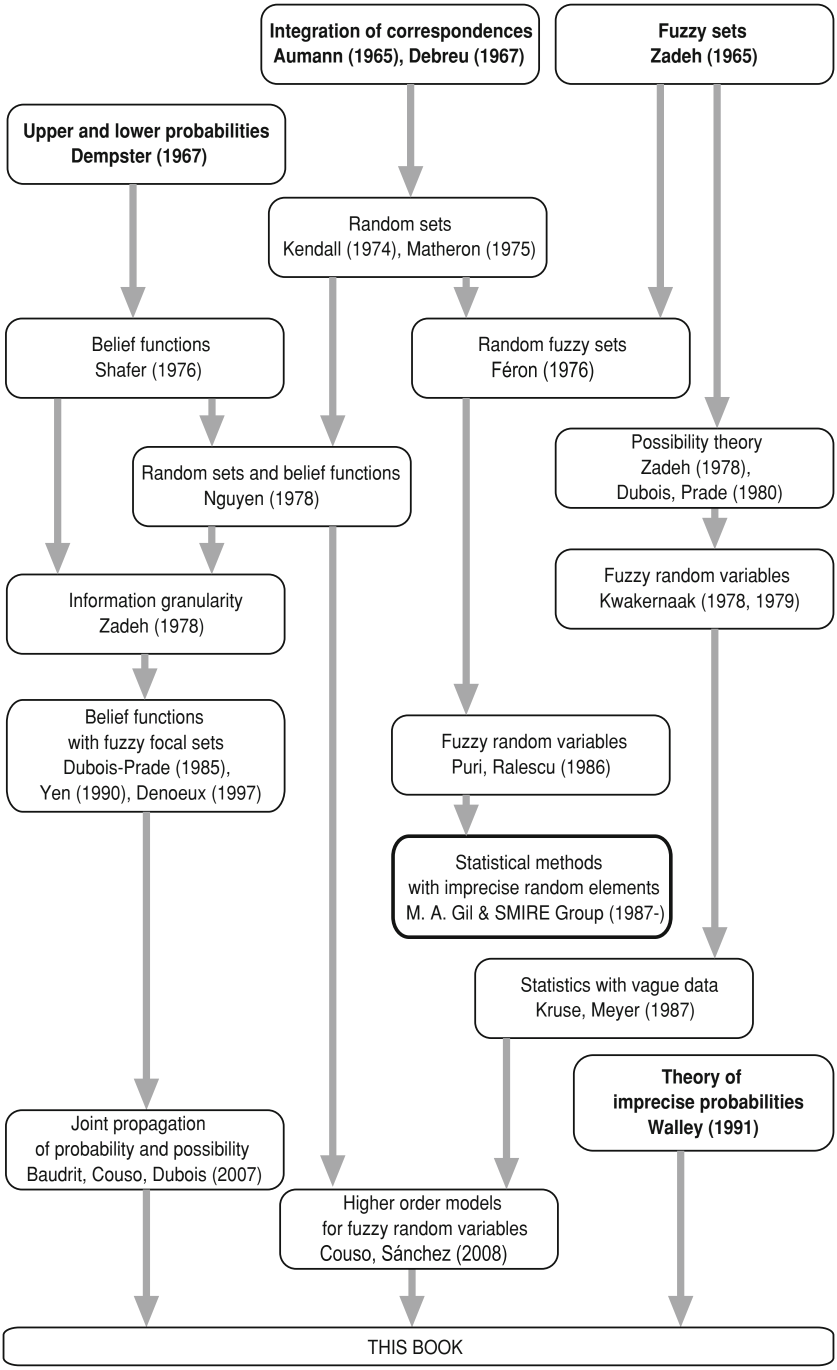

Fig. 1.1 History of random and fuzzy set representations of uncertainty

is better addressed following the Kwakernaak–Kruse tradition, which can as well be viewed as blending possibility and probability theories.

References

1. J. Aumann, Integral of set valued functions. J. Math. Anal. Appl. **12**, 1–12 (1965)
2. A. Colubi, R. Coppi, P. D'Urso, M.A. Gil. Statistics with fuzzy random variables. METRON-Int. J. Stat. **LXV**, 277–303 (2007)
3. G. Debreu, Integration of correspondences, in *Proceedings of the Fifth Berkeley Symposium of Mathematical Statistics and Probability* (Berkeley, USA, 1965), pp. 351–372
4. A.P. Dempster, Upper and lower probabilities induced by a multi-valued mapping. Ann. Math. Stat. **38**, 325–339 (1967)
5. T. Denœux, Modeling vague beliefs using fuzzy-valued belief structures. Fuzzy Sets Syst. **116**(2), 167–199 (2000)
6. D. Dubois, H. Prade, *Fuzzy Sets and Systems: Theory and Applications* (Academic Press, New York, 1980)
7. D. Dubois, H. Prade, Evidence measures based on fuzzy information. Automatica **21**(5), 547–562 (1985)
8. D. Dubois, H. Prade, *Possibility Theory* (Plenum Press, New York, 1988)
9. R. Féron, Ensembles aléatoires flous. C.R. Acad. Sci. Paris Ser. A **282**, 903–906 (1976)
10. D.G. Kendall, Foundations of a theory of random sets, in *Stochastic Geometry*, ed. by E.F. Harding, D.G. Kendall (Wiley, New York, 1974), pp. 322–376
11. R. Körner, On the variance of fuzzy random variables. Fuzzy Sets Syst. **92**, 83–93 (1997)
12. R. Kruse, On the variance of random sets. J. Math. Anal. Appl. **122**, 469–473 (1987)
13. R. Kruse, K.D. Meyer, *Statistics with Vague Data* (D. Reidel Publishing Company, Dordrecht, 1987)
14. H. Kwakernaak, Fuzzy random variables: definition and theorems. Inf. Sci. **15**, 1–29 (1978)
15. H. Kwakernaak, Fuzzy random variables: algorithms and examples in the discrete case. Inform. Sci. **17**, 253–278 (1979)
16. G. Matheron, *Random Sets and Integral Geometry* (Wiley, New York, 1975)
17. H.T. Nguyen, On random sets and belief functions. J. Math. Anal. Appl. **63**, 531–542 (1978)
18. M. Puri, D. Ralescu, Fuzzy random variables. J. Math. Anal. Appl. **114**, 409–422 (1986)
19. G. Shafer, *A Mathematical Theory of Evidence* (Princeton University Press, Princeton, 1976)
20. G.L.S. Shackle, *Decision, Order and Time in Human Affairs*, 2nd edn. (Cambridge University Press, UK, 1961)
21. C.A.B. Smith, Consistency in statistical inference and decision. J. R. Stat. Soc. B **23**, 1–37 (1961)
22. P. Walley, Measures of uncertainty in expert systems. Artif. Intell. **83**(1), 1–58 (1996)
23. R.R. Yager, A foundation for a theory of possibility. Cybern. Syst. **10**(1–3), 177–204 (1980)
24. R.R. Yager, An introduction to applications of possibility theory. Hum. Syst. Manag. **3**, 246–269 (1983)
25. R.R. Yager, An introduction to applications of possibility theory. Hum. Syst. Manag. **3**, 246–269 (1983)
26. L.A. Zadeh, Fuzzy sets. Inf. Control **8**, 338–353 (1965)
27. L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets Syst. **1**, 3–28 (1978)
28. L.A. Zadeh, Fuzzy sets and information granularity, in *Advances in Fuzzy Set Theory and Applications*, ed. by M.M. Gupta, R.K. Ragade, R.R. Yager. (North-Holland, Amsterdam, 1979), pp. 3–18

Springer

http://www.springer.com/978-3-319-08610-1

Random Sets and Random Fuzzy Sets as Ill-Perceived Random Variables
An Introduction for Ph.D. Students and Practitioners
Couso, I.; Dubois, D.; Sanchez, L.
2014, VIII, 97 p. 8 illus., Softcover
ISBN: 978-3-319-08610-1