

Construction of Conservation Laws Using Symmetries

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Abstract The concept of *nonlinear self-adjointness* of differential equations, introduced by the author in 2010, is discussed in detail. All linear equations and systems are nonlinearly self-adjoint. Moreover, the class of nonlinearly self-adjoint equations includes all nonlinear equations and systems having at least one *local* conservation law. It follows, in particular, that the *integrable systems* possessing infinite set of *Lie-Bäcklund* symmetries (*higher-order tangent transformations*) are nonlinearly self-adjoint. An explicit formula for conserved vectors associated with symmetries is provided for all nonlinearly self-adjoint differential equations and systems. The number of equations contained in the systems under consideration can be different from the number of dependent variables. A utilization of conservation laws for constructing exact solutions is discussed and illustrated by computing non-invariant solutions of the Chaplygin equations in gas dynamics.

1 Nonlinear Self-Adjointness

The concept of self-adjointness of nonlinear equations was introduced [1, 2] for constructing conservation laws associated with symmetries of differential equations. To extend the possibilities of the new method for constructing conservation laws the notion of quasi self-adjointness was suggested in [3]. I introduce here the general concept of *nonlinear self-adjointness*. It embraces the previous notions of self-adjoint

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and quasi self-adjoint equations and includes the linear self-adjointness as a particular case. But the set of nonlinearly self-adjoint equations is essentially wider and contains, in particular, all *linear equations* and *nonlinear equations and systems having at least one local conservation law*, including the so-called integrable systems. The construction of conservation laws demonstrates a practical significance of the nonlinear self-adjointness. Namely, *conservation laws can be associated with symmetries for all nonlinearly self-adjoint differential equations and systems*.

1.1 Preliminaries

1.1.1 Notation

We will use the following notation. The independent variables are denoted by

$$x = (x^1, \dots, x^n).$$

The dependent variables are

$$u = (u^1, \dots, u^m).$$

They are used together with their first-order partial derivatives $u_{(1)}$

$$u_{(1)} = \{u_i^\alpha\}, \quad u_i^\alpha = D_i(u^\alpha)$$

and higher-order derivatives $u_{(2)}, \dots, u_{(s)}, \dots$, where

$$\begin{aligned} u_{(2)} &= \{u_{ij}^\alpha\}, & u_{ij}^\alpha &= D_i D_j(u^\alpha), \dots \\ u_{(s)} &= \{u_{i_1 \dots i_s}^\alpha\}, & u_{i_1 \dots i_s}^\alpha &= D_{i_1} \dots D_{i_s}(u^\alpha). \end{aligned}$$

Here D_i is the total differentiation with respect to x^i

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots \quad (1)$$

A locally analytic function $f(x, u, u_{(1)}, \dots, u_{(k)})$ of any finite number of the variables $x, u, u_{(1)}, u_{(2)}, \dots$ is called a *differential function*. The set of all differential functions is denoted by \mathcal{A} . For more details see [4, Chap. 8].

1.1.2 Linear Self-Adjointness

Recall that the adjoint operator F^* to a linear operator F in a Hilbert space H with a scalar product (u, v) is defined by

$$(Fu, v) = (u, F^*v), \quad u, v \in H. \quad (2)$$

Let us consider, for the sake of simplicity, the case of one dependent variable u and denote by H the Hilbert space of real valued functions $u(x)$ such that $u^2(x)$ is integrable. The scalar product is given by

$$(u, v) = \int_{\mathbb{R}^n} u(x)v(x)dx.$$

Let F be a linear differential operator in H . Its action on the dependent variable u is denoted by $F[u]$. The Definition (2) of the adjoint operator F^* to F

$$(F[u], v) = (u, F^*[v])$$

can be written, using the divergence theorem, in the simple form

$$vF[u] - uF^*[v] = D_i(p^i) \quad (3)$$

where v is a new dependent variable, and p^i are any functions of $x, u, v, u_{(1)}, v_{(1)}, \dots$. It is manifest from Eq. (3) that the operators F and F^* are mutually adjoint

$$(F^*)^* = F. \quad (4)$$

In other words, the adjointness of linear operators is a *symmetric relation*.

The linear operator F is said to be self-adjoint if $F^* = F$. In this case we say that the equation $F[u] = 0$ is self-adjoint. Thus, the self-adjointness of a linear equation $F[u] = 0$ can be expressed by the equation

$$F^*[v] \Big|_{v=u} = F[u]. \quad (5)$$

1.1.3 Adjoint Equations to Nonlinear Differential Equations

Let us consider a system of m differential equations (linear or nonlinear)

$$F_\alpha(x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad \alpha = 1, \dots, m \quad (6)$$

with m dependent variables $u = (u^1, \dots, u^m)$. Equation (6) involve the partial derivatives $u_{(1)}, \dots, u_{(s)}$ up to order s .

Definition 1 The *adjoint equation* to equation (6) are given by

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) = 0, \quad \alpha = 1, \dots, m \quad (7)$$

with

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) = \frac{\delta \mathcal{L}}{\delta u^\alpha} \quad (8)$$

where \mathcal{L} is the *formal Lagrangian* for Eq.(6) defined by ¹

$$\mathcal{L} = v^\beta F_\beta \equiv \sum_{\beta=1}^m v^\beta F_\beta. \quad (9)$$

Here $v = (v^1, \dots, v^m)$ are new dependent variables, $v_{(1)}, \dots, v_{(s)}$ are their derivatives, e.g., $v_{(1)} = \{v_i^\alpha\}$, $v_i^\alpha = D_i(v^\alpha)$. We use $\delta/\delta u^\alpha$ for the Euler-Lagrange operator

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m$$

so that

$$\frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = \frac{\partial(v^\beta F_\beta)}{\partial u^\alpha} - D_i \left(\frac{\partial(v^\beta F_\beta)}{\partial u_i^\alpha} \right) + D_i D_k \left(\frac{\partial(v^\beta F_\beta)}{\partial u_{ik}^\alpha} \right) - \dots$$

The total differentiation (1) is extended to the new dependent variables

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + v_i^\alpha \frac{\partial}{\partial v^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + v_{ij}^\alpha \frac{\partial}{\partial v_j^\alpha} + \dots \quad (10)$$

The adjointness of *nonlinear* equations is not a symmetric relation. In other words, nonlinear equations, unlike the linear ones, do not obey the condition (4) of mutual adjointness. Instead, the following equation holds

$$(F^*)^* = \hat{F} \quad (11)$$

where \hat{F} is the *linear approximation* to F defined as follows. We use the temporary notation $F[u]$ for the left-hand side of Eq.(6) and consider $F[u + w]$ by letting $w \ll 1$. Then neglecting the nonlinear terms in w we define \hat{F} by the equation

¹ See [2]. An approach in terms of variational principles is developed in [5].

$$F[u + w] \approx F[u] + \hat{F}[w]. \quad (12)$$

For linear equations we have $\hat{F} = F$, and hence Eq.(11) is identical with Eq.(4). Let us illustrate Eq.(11) by the equation

$$F \equiv u_{xy} - \sin u = 0. \quad (13)$$

Equation (8) yields

$$F^* \equiv \frac{\delta}{\delta u} [v(u_{xy} - \sin u)] = v_{xy} - v \cos u \quad (14)$$

and

$$(F^*)^* \equiv \frac{\delta}{\delta v} [w(v_{xy} - v \cos u)] = w_{xy} - w \cos u. \quad (15)$$

Let us find \hat{F} by using Eq.(12). Since $\sin w \approx w$, $\cos w \approx 1$ when $w \ll 1$, we have

$$\begin{aligned} F[u + w] &\equiv (u + w)_{xy} - \sin(u + w) \\ &= u_{xy} + w_{xy} - \sin u \cos w - \sin w \cos u \\ &\approx u_{xy} - \sin u + w_{xy} - w \cos u \\ &= F[u] + w_{xy} - w \cos u. \end{aligned}$$

Hence, by (12) and(15), we have

$$\hat{F}[w] = w_{xy} - w \cos u = (F^*)^* \quad (16)$$

in accordance with Eq.(11).

1.1.4 The Case of One Dependent Variable

Let us consider the differential equation

$$F(x, u, u_{(1)}, \dots, u_{(s)}) = 0 \quad (17)$$

with one dependent variable u and any number of independent variables. In this case Definition 1 of the adjoint equation is written

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) = 0 \quad (18)$$

where

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) = \frac{\delta(vF)}{\delta u}. \quad (19)$$

1.1.5 Construction of Adjoint Equations to Linear Equations

The following statement has been formulated in [1, 2].

Proposition 1 *In the case of linear differential equations and systems, the adjoint equations determined by Eq. (8) and by Eq. (3) coincide.*

Proof The proof is based on the statement (see Proposition 8 in Sect. 2.1.2) that a function $Q(u, v)$ is a divergence, i.e., $Q = D_i(h^i)$, if and only if

$$\frac{\delta Q}{\delta u^\alpha} = 0, \quad \frac{\delta Q}{\delta v^\alpha} = 0, \quad \alpha = 1, \dots, m. \quad (20)$$

Let the adjoint operator F^* be constructed according to Eq. (3). Let us consider the case of many dependent variables and write Eq. (3) as follows

$$v^\beta F_\beta[u] = u^\beta F_\beta^*[v] + D_i(p^i). \quad (21)$$

Applying to (21) the variational differentiations and using Eq. (20) we obtain

$$\frac{\delta(v^\beta F_\beta[u])}{\delta u^\alpha} = \delta_\alpha^\beta F_\beta^*[v] \equiv F_\alpha^*[v].$$

Hence, (8) coincides with $F_\alpha^*[v]$ given by (3).

Conversely, let $F^*[v]$ be given by (8), i.e.,

$$F_\beta^*[v] = \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta}.$$

Consider the expression Q defined by

$$Q = v^\beta F_\beta[u] - u^\beta F_\beta^*[v] \equiv v^\beta F_\beta[u] - u^\beta \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta}.$$

Applying to the first expression for Q the variational differentiations $\delta/\delta u^\alpha$ we obtain

$$\frac{\delta Q}{\delta u^\alpha} = \frac{\delta(v^\beta F_\beta[u])}{\delta u^\alpha} - \delta_\alpha^\beta F_\beta^*[v] \equiv F_\alpha^*[v] - \delta_\alpha^\beta F_\beta^*[v] = 0.$$

Applying $\delta/\delta v^\alpha$ to the second expression for Q we obtain

$$\frac{\delta Q}{\delta v^\alpha} = \delta_\alpha^\beta F_\beta[u] - \frac{\delta}{\delta v^\alpha} \left[u^\beta \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta} \right] \equiv F_\alpha[u] - \frac{\delta}{\delta v^\alpha} \left[u^\beta \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta} \right].$$

The reckoning shows that

$$\frac{\delta}{\delta v^\alpha} \left[u^\beta \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta} \right] = F_\alpha[u]. \quad (22)$$

Thus Q solves Eq. (20) and hence Eq. (21) is satisfied. This completes the proof.

Remark 1 Let us discuss the proof of Eq. (22) in the case of a second-order linear operator for one dependent variable

$$F[u] = a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u.$$

Then we have

$$u \frac{\delta(vF[u])}{\delta u} = u \left[cv - vD_i(b^i) + vD_iD_j(a^{ij}) - b^i v_i + 2v_i D_j(a^{ij}) + a^{ij} v_{ij} \right].$$

Whence, after simple calculations we obtain

$$\begin{aligned} \frac{\delta}{\delta v} \left[u \frac{\delta(vF[u])}{\delta u} \right] &= \left[cu + b^i u_i + a^{ij} u_{ij} \right] \\ &\quad + \left\{ D_i D_j (a^{ij} u) - D_i (a^{ij} u_j) - D_i [u D_j (a^{ij})] \right\} \end{aligned}$$

and, noting that the expression in the braces vanishes, arrive at Eq. (22).

Let us illustrate Proposition 1 by the following simple example.

Example 1 Consider the heat equation

$$F[u] \equiv u_t - u_{xx} = 0 \quad (23)$$

and construct the adjoint operator to the linear operator

$$F = D_t - D_x^2 \quad (24)$$

by using Eq. (3). Noting that due to

$$\begin{aligned} v u_t &= D_t(uv) - u v_t \\ v u_{xx} &= D_x(vu_x) - v_x u_x = D_x(vu_x - uv_x) + uv_{xx} \end{aligned}$$

we have

$$vF[u] \equiv v(u_t - u_{xx}) = u(-v_t - v_{xx}) + D_t(uv) + D_x(uv_x - vu_x).$$

Hence

$$vF[u] - u(-v_t - v_{xx}) = D_t(uv) + D_x(uv_x - vu_x).$$

Therefore, denoting $t = x^1$, $x = x^2$, we obtain Eq. (3) with $F^*[v] = -v_t - v_{xx}$ and $p^1 = uv$, $p^2 = uv_x - vu_x$. Thus, the adjoint operator to the linear operator (24) is

$$F^* = -D_t - D_x^2 \quad (25)$$

and the adjoint equation to the heat Eq. (23) is written $-v_t - v_{xx} = 0$, or

$$v_t + v_{xx} = 0. \quad (26)$$

The derivation of the adjoint Eq. (26) and the adjoint operator (25) by the definition (19) is much simpler. Indeed, we have

$$F^* = \frac{\delta(vu_t - vu_{xx})}{\delta u} = -D_t(v) - D_x^2(v) = -(v_t + v_{xx}).$$

1.1.6 Self-Adjointness and Quasi Self-Adjointness

Recall that a linear differential operator F is called a *self-adjoint operator* if it is identical with its adjoint operator, $F = F^*$. Then the equation $F[u] = 0$ is also said to be self-adjoint. Thus, the self-adjointness of a *linear differential equation* $F[u] = 0$ means that the adjoint equation $F^*[v] = 0$ coincides with $F[u] = 0$ upon the substitution $v = u$. This property has been extended to nonlinear equations in [2]. It will be called here the *strict self-adjointness* and defined as follows.

Definition 2 We say that the differential equation (17) is strictly self-adjoint if the adjoint Eq. (18) becomes equivalent to the original Eq. (17) upon the substitution

$$v = u. \quad (27)$$

It means that the equation

$$F^*(x, u, u, \dots, u_{(s)}, u_{(s)}) = \lambda F(x, u, \dots, u_{(s)}) \quad (28)$$

holds with a certain (in general, variable) coefficient λ .

Example 2 The Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + uu_x$$

is strictly self-adjoint [1]. Indeed, its adjoint Eq. (18) has the form

$$v_t = v_{xxx} + uv_x$$

and coincides with the KdV equation upon the substitution (27).

In the case of linear equations the strict self-adjointness is identical with the usual self-adjointness of linear equations.

Example 3 Consider the linear equation

$$u_{tt} + a(x)u_{xx} + b(x)u_x + c(x)u = 0. \quad (29)$$

According to Eqs. (18) and (19), the adjoint equation to equation (29) is

$$\frac{\delta}{\delta u} \{v[u_{tt} + a(x)u_{xx} + b(x)u_x + c(x)u]\} \equiv D_t^2(v) + D_x^2(av) - D_x(bv) + cv = 0.$$

Upon substituting $v = u$ and performing the differentiations it becomes

$$u_{tt} + au_{xx} + (2a' - b)u_x + (a'' - b' + c)u = 0. \quad (30)$$

According to Definition 2, Eq. (29) is strictly self-adjoint if Eq. (30) coincides with Eq. (29). This is possible if

$$b(x) = a'(x). \quad (31)$$

Definition 2 is too restrictive. Moreover, it is inconvenient in the case of systems with several dependent variables $u = (u^1, \dots, u^m)$ because in this case Eq. (27) is not uniquely determined as it is clear from the following example.

Example 4 Let us consider the system of two equations

$$u_y^1 + u^2 u_x^2 - u_t^2 = 0, \quad u_y^2 - u_x^1 = 0 \quad (32)$$

with two dependent variables, $u = (u^1, u^2)$, and three independent variables t, x, y . Using the formal Lagrangian (9)

$$\mathcal{L} = v^1(u_y^1 + u^2 u_x^2 - u_t^2) + v^2(u_y^2 - u_x^1)$$

and Eq. (8) we write the adjoint Eq. (7), changing their sign, in the form

$$v_y^2 + u^2 v_x^1 - v_t^1 = 0, \quad v_y^1 - v_x^2 = 0. \quad (33)$$

If we use here the substitution (27), $v = u$ with $v = (v^1, v^2)$, i.e., let

$$v^1 = u^1, \quad v^2 = u^2$$

then the adjoint system (33) becomes

$$u_y^2 + u^2 u_x^1 - u_t^1 = 0, \quad u_y^1 - u_x^2 = 0$$

which is not connected with the system (32) by the equivalence relation (28). But if we set

$$v^1 = u^2, \quad v^2 = u^1$$

the adjoint system (33) coincides with the original system (32).

The concept of quasi self-adjointness generalizes Definition 2 and is more convenient for dealing with systems (6). This concept was formulated in [3] as follows. The system (6) is *quasi self-adjoint* if the adjoint system (7) becomes equivalent to the original system (6) upon a substitution

$$v = \varphi(u) \tag{34}$$

such that its derivative does not vanish in a certain domain of u

$$\varphi'(u) \neq 0, \quad \text{where} \quad \varphi'(u) = \left\| \frac{\partial \varphi^\alpha(u)}{\partial u^\beta} \right\|. \tag{35}$$

Remark 2 The substitution (34) defines a mapping

$$v^\alpha = \varphi^\alpha(u), \quad \alpha = 1, \dots, m$$

from the m -dimensional space of variables $u = (u^1, \dots, u^m)$ into the m -dimensional space of variables $v = (v^1, \dots, v^m)$. It is assumed that this mapping is continuously differentiable. The condition (35) guarantees that it is invertible, and hence Eqs. (7) and (6) are equivalent. The equivalence means that the following equations hold with certain coefficients λ_α^β

$$F_\alpha^*(x, u, \varphi, \dots, u_{(s)}, \varphi_{(s)}) = \lambda_\alpha^\beta F_\beta(x, u, \dots, u_{(s)}), \quad \alpha = 1, \dots, m \tag{36}$$

where

$$\varphi = \{\varphi^\alpha(u)\}, \quad \varphi_{(\sigma)} = \{D_{i_1} \cdots D_{i_\sigma}(\varphi^\alpha(u))\}, \quad \sigma = 1, \dots, s. \tag{37}$$

It can be shown that the matrix $\|\lambda_\alpha^\beta\|$ is invertible due to the condition (35).

Example 5 The quasi self-adjointness of nonlinear wave equations of the form

$$u_{tt} - u_{xx} = f(t, x, u, u_t, u_x)$$

is investigated in [6]. The results of the paper [6] show that, e.g., the equation

$$u_{tt} - u_{xx} + u_t^2 - u_x^2 = 0 \tag{38}$$

is quasi self-adjoint and that in this case the substitution (34) has the form

$$v = e^u. \quad (39)$$

Indeed, the adjoint equation to equation (38) is written

$$v_{tt} - v_{xx} - 2vu_{tt} - 2u_t v_t + 2vu_{xx} + 2u_x v_x = 0. \quad (40)$$

After the substitution (39) the left-hand side of Eq.(40) takes the form(36)

$$v_{tt} - v_{xx} - 2vu_{tt} - 2u_t v_t + 2vu_{xx} + 2u_x v_x = -e^u [u_{tt} - u_{xx} + u_t^2 - u_x^2]. \quad (41)$$

It is manifest from Eq.(41) that v given by (39) solves the adjoint Eq.(40) if one replaces u by any solution of Eq.(38).

In constructing conservation laws one can relax the condition (35). Therefore I generalize the previous definition of quasi self-adjointness as follows.

Definition 3 The system (6) is said to be quasi self-adjoint if the adjoint Eq.(7) are satisfied for all solutions u of the original system (6) upon a substitution

$$v^\alpha = \varphi^\alpha(u), \quad \alpha = 1, \dots, m \quad (42)$$

such that

$$\varphi(u) \neq 0. \quad (43)$$

In other words, the Eq.(36) hold after the substitution (42), where not all $\varphi^\alpha(u)$ vanish simultaneously.

Remark 3 The condition (43), unlike (35), does not guarantee the equivalence of Eqs.(7) and (6) because the matrix $\|\lambda_\alpha^\beta\|$ may be singular.

Example 6 It is well known that the linear heat Eq. (23) is not self-adjoint (not strictly self-adjoint in the sense of Definition 2). It is clear from Eqs.(23) and (26). Let us test Eq.(23) for quasi self-adjointness. Letting $v = \varphi(u)$, we obtain

$$v_t = \varphi' u_t, \quad v_x = \varphi' u_x, \quad v_{xx} = \varphi' u_{xx} + \varphi'' u_x^2$$

and the condition (36) is written

$$\varphi'(u)[u_t + u_{xx}] + \varphi''(u)u_x^2 = \lambda[u_t - u_{xx}].$$

Whence, comparing the coefficients of u_t in both sides, we obtain $\lambda = \varphi'(u)$. Then the above equation becomes

$$\varphi'(u)[u_t + u_{xx}] + \varphi''(u)u_x^2 = \varphi'(u)[u_t - u_{xx}].$$

This equation yields that $\varphi'(u) = 0$. Hence, Eq. (23) is quasi self-adjoint with the substitution $v = C$, where C is any non-vanishing constant. This substitution does not satisfy the condition (35).

Example 7 Let us consider the Fornberg-Whitham equation [7]

$$u_t - u_{txx} - uu_{xxx} - 3u_x u_{xx} + uu_x + u_x = 0. \quad (44)$$

Equations (18) and (19) give the following adjoint equation

$$F^* \equiv -v_t + v_{txx} + uv_{xxx} - uv_x - v_x = 0. \quad (45)$$

It is manifest from the Eqs. (44) and (45) that the Fornberg-Whitham equation is not strictly self-adjoint. Let us test it for quasi self-adjointness. Inserting in (45) the substitution $v = \varphi(u)$ and its derivatives

$$v_t = \varphi' u_t, \quad v_x = \varphi' u_x, \quad v_{xx} = \varphi' u_{xx} + \varphi'' u_x^2, \quad v_{tx} = \varphi' u_{tx} + \varphi'' u_t u_x, \dots$$

then writing the condition (36) and comparing the coefficients for u_t , u_{tx} , u_{xx} , ... one can verify that $\varphi'(u) = 0$. Hence, Eq. (44) is quasi self-adjoint but does not satisfy the condition (35).

1.2 Strict Self-Adjointness Via Multipliers

It is commonly known that numerous linear equations used in practice, e.g., linear evolution equations, are not self-adjoint in the classical meaning of the self-adjointness. Likewise, useful nonlinear equations such as the nonlinear heat equation, the Burgers equation, etc. are not strictly self-adjoint. We will see here that these and many other equations can be rewritten in a strictly self-adjoint equivalent form by using multipliers. The general discussion of this approach will be given in Sect. 1.3.7.

1.2.1 Motivating Examples

Example 8 Let us consider the following second-order nonlinear equation

$$u_{xx} + f(u)u_x - u_t = 0. \quad (46)$$

Its adjoint Eq. (18) is written

$$v_{xx} - f(u)v_x + v_t = 0. \quad (47)$$

It is manifest that the substitution $v = u$ does not map Eq. (47) into Eq. (46). Hence Eq. (46) is not strictly self-adjoint.

Let us clarify if Eq. (46) can be written in an equivalent form

$$\mu(u)[u_{xx} + f(u)u_x - u_t] = 0 \quad (48)$$

with a certain multiplier $\mu(u) \neq 0$ so that Eq. (48) is strictly self-adjoint. The formal Lagrangian for Eq. (48) is

$$\mathcal{L} = v\mu(u)[u_{xx} + f(u)u_x - u_t].$$

We have

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} &= D_x^2[\mu(u)v] - D_x[\mu(u)f(u)v] + D_t[\mu(u)v] \\ &\quad + \mu'(u)v[u_{xx} + f(u)u_x - u_t] + \mu(u)f'(u)vu_x \end{aligned}$$

whence, upon performing the differentiations

$$\frac{\delta \mathcal{L}}{\delta u} = \mu v_{xx} + 2\mu'vu_{xx} + 2\mu'u_xv_x + \mu''vu_x^2 - \mu f v_x + \mu v_t.$$

The strict self-adjointness requires that

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=u} = \lambda[u_{xx} + f(u)u_x - u_t].$$

This provides the following equation for the unknown multiplier $\mu(u)$

$$(\mu + 2u\mu')u_{xx} + (2\mu' + u\mu'')u_x^2 - \mu f u_x + \mu u_t = \lambda[u_{xx} + f(u)u_x - u_t]. \quad (49)$$

Since the right side of Eq. (49) does not contain u_x^2 we should have $2\mu' + u\mu'' = 0$, whence $\mu = C_1u^{-1} + C_2$. Furthermore, comparing the coefficients of u_t in both sides of Eq. (49) we obtain $\lambda = -\mu$. Now Eq. (49) takes the form

$$(C_2 - C_1u^{-1})u_{xx} - (C_1u^{-1} + C_2)fu_x = -(C_1u^{-1} + C_2)[u_{xx} + f(u)u_x]$$

and yields $C_2 = 0$. Thus, $\mu = C_1u^{-1}$. We can let $C_1 = -1$ and formulate the result.

Proposition 2 Equation (46) becomes strictly self-adjoint if we rewrite it in the form

$$\frac{1}{u} [u_t - u_{xx} - f(u)u_x] = 0. \quad (50)$$

Example 9 One can verify that the n -th order nonlinear evolution equation

$$\frac{\partial u}{\partial t} - f(u) \frac{\partial^n u}{\partial x^n} = 0, \quad f(u) \neq 0 \quad (51)$$

with one spatial variable x is not strictly self-adjoint. The following statement shows that it becomes strictly self-adjoint after using an appropriate multiplier.

Proposition 3 *Equation (51) becomes strictly self-adjoint upon rewriting it in the following equivalent form*

$$\frac{1}{uf(u)} \left[\frac{\partial u}{\partial t} - f(u) \frac{\partial^n u}{\partial x^n} \right] = 0. \quad (52)$$

Proof Multiplying Eq. (51) by $\mu(u)$ and taking the formal Lagrangian

$$\mathcal{L} = v\mu(u)[u_t - f(u)u_n]$$

where $u_n = D_x^n(u)$, we have

$$\frac{\delta \mathcal{L}}{\delta u} = -D_t[\mu(u)v] - D_x^n[\mu(u)f(u)v] + v\mu'(u)u_t - v[\mu(u)f(u)]'u_n.$$

Noting that $-D_t[\mu(u)v] + v\mu'(u)u_t = -\mu(u)v_t$ and letting $v = u$ we obtain

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=u} = -\mu(u)u_t - D_x^n[\mu(u)f(u)u] - [\mu(u)f(u)]'uu_n.$$

If we take $\mu(u) = [uf(u)]^{-1}$, then $\mu(u)f(u)u = 1$, $\mu(u)f(u) = u^{-1}$, and hence

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=u} = -\frac{1}{uf(u)} [u_t - f(u)u_n].$$

Thus, Eq. (52) satisfies the strict self-adjointness condition (28) with $\lambda = -1$. \square

1.2.2 Linear Heat Equation

Taking in (50) $f(u) = 0$, we rewrite the classical linear heat equation $u_t = u_{xx}$ in the following strictly self-adjoint form

$$\frac{1}{u} [u_t - u_{xx}] = 0. \quad (53)$$

This result can be extended to the heat equation

$$u_t - \Delta u = 0 \quad (54)$$

where Δu is the Laplacian with n variables $x = (x^1, \dots, x^n)$. Namely, the strictly self-adjoint form of Eq. (54) is

$$\frac{1}{u} [u_t - \Delta u] = 0. \quad (55)$$

Indeed, the formal Lagrangian (9) for Eq. (55) has the form

$$\mathcal{L} = \frac{v}{u} [u_t - \Delta u].$$

Substituting it in (19) we obtain

$$F^* = -D_t \left(\frac{v}{u} \right) - \Delta \left(\frac{v}{u} \right) - \frac{v}{u^2} [u_t - \Delta u].$$

Upon letting $v = u$ it becomes

$$F^* = -\frac{1}{u} [u_t - \Delta u].$$

Hence, Eq. (55) satisfies the condition (28) with $\lambda = -1$.

1.2.3 Nonlinear Heat Equation

Consider the nonlinear heat equation $u_t - D_x (k(u)u_x) = 0$, or

$$u_t - k(u)u_{xx} - k'(u)u_x^2 = 0. \quad (56)$$

Its adjoint equation has the form

$$v_t + k(u)v_{xx} = 0.$$

Therefore it is obvious that (56) does not satisfy Definition 2. But it becomes strictly self-adjoint if we rewrite it in the form

$$\frac{1}{u} [u_t - k(u)u_{xx} - k'(u)u_x^2] = 0. \quad (57)$$

Indeed, the formal Lagrangian (9) for Eq. (57) is written

$$\mathcal{L} = \frac{v}{u} [u_t - k(u)u_{xx} - k'(u)u_x^2].$$

Substituting it in (19) we obtain

$$F^* = -D_t \left(\frac{v}{u} \right) - D_x^2 \left(\frac{v}{u} k(u) \right) + 2D_x \left(\frac{v}{u} k'(u) u_x \right) \\ - \frac{v}{u} k'(u) u_{xx} - \frac{v}{u} k''(u) u_x^2 - \frac{v}{u^2} \left[u_t - k(u) u_{xx} - k'(u) u_x^2 \right].$$

Letting here $v = u$ we have

$$F^* = -\frac{1}{u} \left[u_t - k(u) u_{xx} - k'(u) u_x^2 \right].$$

Hence, Eq. (55) satisfies the strict self-adjointness condition (28) with $\lambda = -1$.

1.2.4 The Burgers Equation

Taking in (50) $f(u) = u$ we obtain the strictly self-adjoint form

$$\frac{1}{u} [u_t - u_{xx}] - u_x = 0 \quad (58)$$

of the Burgers equation $u_t = u_{xx} + uu_x$.

1.2.5 Heat Conduction in Solid Hydrogen

According to [8], the heat conduction in solid crystalline molecular hydrogen at low pressures is governed by the nonlinear equation (up-to positive constant coefficient)

$$u_t = u^2 \Delta u. \quad (59)$$

It is derived from the Fourier equation

$$\rho c_* \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T)$$

using the empirical information that the density ρ at low pressures has a constant value, whereas the specific heat c_* and the thermal conductivity k have the estimations

$$c_* \cong T^3, \quad k \cong T^3 (1 + T^4)^{-2}.$$

It is also shown in [8] that the one-dimensional Eq. (59)

$$u_t = u^2 u_{xx} \quad (60)$$

is related to the linear heat equation by a non-point transformation (Eq. (5) in [8]). A similar relation was found in [9] for another representation of Eq. (60). The non-point transformation of Eq. (60) to the linear heat equation

$$w_s = w_{\xi\xi} \quad (61)$$

is written in [10] as the differential substitution

$$t = s, \quad x = w, \quad u = w_\xi. \quad (62)$$

It is also demonstrated in [10], Sect. 20, that Eq. (60) is the unique equation with nontrivial Lie-Bäcklund symmetries among the equations of the form

$$u_t = f(u) + h(u, u_x), \quad f'(u) \neq 0.$$

The connection between Eq. (60) and the heat equation is treated in [11] as a reciprocal transformation [11]. It is shown in [12] that this connection, together with its extensions, allows to solve certain moving boundary problems in nonlinear heat conduction.

Our Example 9 from Sect. 1.2.1 reveals one more remarkable property of Eq. (60). Namely, taking $n = 2$ and $f(u) = u^2$ in Eq. (52) we see that Eq. (60) *becomes strictly self-adjoint if we rewrite it in the form*

$$\frac{u_t}{u^3} = \frac{u_{xx}}{u}. \quad (63)$$

1.2.6 Harry Dym Equation

Taking in Example 9 from Sect. 1.2.1 $n = 3$ and $f(u) = u^3$ we see that the Harry Dym equation

$$u_t - u^3 u_{xxx} = 0 \quad (64)$$

becomes strictly self-adjoint upon rewriting it in the form

$$\frac{u_t}{u^4} - \frac{u_{xxx}}{u} = 0.$$

1.2.7 Kompaneets Equation

The equations considered in Sects. 1.2.1–1.2.6 are quasi self-adjoint. For example, for Eq. (51) we have

$$F^* = -v_t - D_x^n(f(u)v) - vf'(u)u_n$$

whence making the substitution

$$v = \frac{1}{f(u)}$$

we obtain

$$F^* = \frac{f'}{f^2}u_t - \frac{f'}{f}u_n = \frac{f'}{f^2}[u_t - f(u)u_n].$$

Hence, Eq. (51) is quasi self-adjoint.

Example 10 The Kompaneets equation

$$u_t = \frac{1}{x^2}D_x[x^4(u_x + u + u^2)] \quad (65)$$

provides an example of an equation that is not quasi self-adjoint. Indeed, Eq. (65) has the formal Lagrangian

$$\mathcal{L} = v[-u_t + x^2u_{xx} + (x^2 + 4x + 2x^2u)u_x + 4x(u + u^2)].$$

The calculation yields the following adjoint equation to (65)

$$\frac{\delta \mathcal{L}}{\delta u} \equiv v_t + x^2v_{xx} - x^2(1 + 2u)v_x + 2(x + 2xu - 1)v = 0. \quad (66)$$

Letting $v = \varphi(u)$ one obtains

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} \Big|_{v=\varphi(u)} &= \varphi'(u)u_t + x^2u_{xx} - x^2(1 + 2u)u_x \\ &\quad + \varphi''(u)x^2u_x^2 + 2(x + 2xu - 1)\varphi(u). \end{aligned}$$

Writing the quasi self-adjointness condition (36) in the form

$$\frac{\delta \mathcal{L}}{\delta u} \Big|_{v=\varphi(u)} = \lambda[-u_t + x^2u_{xx} + (x^2 + 4x + 2x^2u)u_x + 4x(u + u^2)]$$

and comparing the coefficients for u_t in both sides one obtains $\lambda = -\varphi'(u)$, so that the quasi self-adjointness condition takes the form

$$\begin{aligned} & \varphi'(u)[u_t + x^2 u_{xx} - x^2(1 + 2u)u_x] + \varphi''(u)x^2 u_x^2 + 2(x + 2xu - 1)\varphi(u) \\ &= \varphi'(u)[u_t - x^2 u_{xx} - (x^2 + 4x + 2x^2 u)u_x - 4x(u + u^2)]. \end{aligned}$$

Comparing the coefficients for u_{xx} in both sides we obtain $\varphi'(u) = 0$. Then the above equation becomes $(x + 2xu - 1)\varphi(u) = 0$ and yields $\varphi(u) = 0$. Hence the Kompaneets equation is not quasi self-adjoint because the condition (43) is not satisfied.

But we can rewrite Eq. (65) in the strictly self-adjoint form by using a more general multiplier than above, namely, the multiplier

$$\mu = \frac{x^2}{u}. \quad (67)$$

Indeed, upon multiplying by this μ Eq. (65) is written

$$\frac{x^2}{u} u_t = \frac{1}{u} D_x \left[x^4 (u_x + u + u^2) \right].$$

Its formal Lagrangian

$$\mathcal{L} = \frac{v}{u} \left\{ -x^2 u_t + D_x \left[x^4 (u_x + u + u^2) \right] \right\}$$

satisfies the strict self-adjointness condition (28) with $\lambda = -1$

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=u} = -\frac{1}{u} \left\{ -x^2 u_t + D_x \left[x^4 (u_x + u + u^2) \right] \right\}.$$

Remark 4 Note that $v = x^2$ solves Eq. (66) for any u . The connection of this solution with the multiplier (67) is discussed in Sect. 1.3.7. See also Sect. 1.4.

1.3 General Concept of Nonlinear Self-Adjointness

Motivated by the examples discussed in Sects. 1.1 and 1.2 as well as other similar examples, I have suggested in [13] the general concept of *nonlinear self-adjointness* of systems consisting of any number of equations with m dependent variables. This concept encapsulates Definition 2 of strict self-adjointness and Definition 3 of quasi self-adjointness. The new concept has two different features. They are expressed below by two different but equivalent definitions.

1.3.1 Two Definitions and Their Equivalence

Definition 4 The system of \bar{m} differential equations (compare with Eq. 6)

$$F_{\bar{\alpha}}(x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad \bar{\alpha} = 1, \dots, \bar{m} \quad (68)$$

with m dependent variables $u = (u^1, \dots, u^m)$ is said to be *nonlinearly self-adjoint* if the *adjoint equations*

$$F_{\alpha}^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(v^{\bar{\beta}} F_{\bar{\beta}})}{\delta u^{\alpha}} = 0, \quad \alpha = 1, \dots, m \quad (69)$$

are satisfied for all solutions u of the original system (68) upon a substitution

$$v^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x, u), \quad \bar{\alpha} = 1, \dots, \bar{m} \quad (70)$$

such that

$$\varphi(x, u) \neq 0. \quad (71)$$

In other words, the following equations hold

$$F_{\alpha}^*(x, u, \varphi(x, u), \dots, u_{(s)}, \varphi_{(s)}) = \lambda_{\alpha}^{\bar{\beta}} F_{\bar{\beta}}(x, u, \dots, u_{(s)}), \quad \alpha = 1, \dots, m \quad (72)$$

where $\lambda_{\alpha}^{\bar{\beta}}$ are undetermined coefficients, and $\varphi_{(\sigma)}$ are derivatives of (70)

$$\varphi_{(\sigma)} = \{D_{i_1} \cdots D_{i_{\sigma}}(\varphi^{\bar{\alpha}}(x, u))\}, \quad \sigma = 1, \dots, s.$$

Here v and φ are the \bar{m} -dimensional vectors

$$v = (v^1, \dots, v^{\bar{m}}), \quad \varphi = (\varphi^1, \dots, \varphi^{\bar{m}})$$

and Eq. (71) means that not all components $\varphi^{\bar{\alpha}}(x, u)$ of φ vanish simultaneously.

Remark 5 If the system (68) is over-determined, i.e., $\bar{m} > m$, then the adjoint system (69) is sub-definite since it contains $m < \bar{m}$ equations for \bar{m} new dependent variables v . Vice versa, if $\bar{m} < m$, then the system (68) is sub-definite and the adjoint system (69) is over-determined.

Remark 6 The adjoint system (69), upon substituting there any solution $u(x)$ of Eq. (68), becomes a linear homogeneous system for the new dependent variables $v^{\bar{\alpha}}$. The essence of Eq. (72) is that for the self-adjoint system (68) there exist functions (70) that provide a non-trivial (not identically zero) solution to the adjoint system (69) for all solutions of the original system (68). This property can be taken as the following alternative definition of the nonlinear self-adjointness.

Definition 5 The system (68) is nonlinearly self-adjoint if there exist functions $v^{\bar{\alpha}}$ given by (70) that solve the adjoint system (69) for all solutions $u(x)$ of Eq. (68) and satisfy the condition (71).

Proposition 4 *The above two definitions are equivalent.*

Proof Let the system (68) be nonlinearly self-adjoint by Definition 4. Then, according to Remark 6, the system (68) satisfies the condition of Definition 5.

Conversely, let the system (68) be nonlinearly self-adjoint by Definition 5. Namely, let the functions $v^{\bar{\alpha}}$ given by (70) and satisfying the condition (71) solve the adjoint system (69) for all solutions $u(x)$ of Eq. (68). This is possible if and only if Eq. (72) hold. Then the system (68) is nonlinearly self-adjoint by Definition 4. \square

Example 11 It has been mentioned in Example 2 that the KdV equation

$$u_t = u_{xxx} + uu_x \tag{73}$$

is strictly self-adjoint. In terms of Definition 5 it means that $v = u$ solves the adjoint equation

$$v_t = v_{xxx} + uv_x \tag{74}$$

for all solutions of the KdV Eq. (73). One can verify that the general substitution of the form (70), $v = \varphi(t, x, u)$, satisfying Eq. (72) is given by

$$v = A_1 + A_2u + A_3(x + tu) \tag{75}$$

where A_1, A_2, A_3 are arbitrary constants. One can also check that v given by Eq. (75) solves the adjoint Eq. (74) for all solutions u of the KdV equation. The solution $v = x + tu$ is an invariant of the Galilean transformation of the KdV equation and appears in different approaches (see [10, Sect. 22.5] and [14]). Thus, the KdV equation is nonlinearly self-adjoint with the substitution (75).

Proposition 5 *Any linear equation is nonlinearly self-adjoint.*

Proof This property is the direct consequence of Definition 5 because the adjoint equation $F^*[v] = 0$ to a linear equation $F[u] = 0$ does not involve the variable u . \square

1.3.2 Remark on Differential Substitutions

One can further extend the concept of self-adjointness by replacing the *point-wise* substitution (70) with *differential* substitutions of the form

$$v^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x, u, u_{(1)}, \dots, u_{(r)}), \quad \bar{\alpha} = 1, \dots, \bar{m}. \tag{76}$$

Then Eq. (72) will be written, e.g., in the case $r = 1$, as follows

$$F_{\alpha}^{*}(x, u, \varphi, \dots, u_{(s)}, \varphi_{(s)}) = \lambda_{\alpha}^{\bar{\beta}} F_{\bar{\beta}} + \lambda_{\alpha}^{j\bar{\beta}} D_j(F_{\bar{\beta}}). \quad (77)$$

Example 12 The reckoning shows that the equation

$$u_{xy} = \sin u \quad (78)$$

is not self-adjoint via a point-wise substitution $v = \varphi(x, y, u)$, but it is self-adjoint in the sense of Definition 4 with the following differential substitution

$$v = \varphi(x, y, u_x, u_y) \equiv A_1[xu_x - yu_y] + A_2u_x + A_3u_y \quad (79)$$

where A_1, A_2, A_3 are arbitrary constants. The adjoint equation to equation (78) is

$$v_{xy} - v \cos u = 0$$

and the self-adjointness condition (77) with the function φ given by (79) is satisfied in the form

$$\begin{aligned} \varphi_{xy} - \varphi \cos u &= (A_1x + A_2)D_x(u_{xy} - \sin u) \\ &+ (A_3 - A_1y)D_y(u_{xy} - \sin u). \end{aligned} \quad (80)$$

1.3.3 Nonlinear Heat Equation

One-Dimensional Case

Let us apply the new viewpoint to the nonlinear heat Eq.(56), $u_t = (k(u)u_x)_x$, discussed in Sect. 1.2.3. We will take it in the expanded form

$$u_t - k(u)u_{xx} - k'(u)u_x^2 = 0, \quad k(u) \neq 0. \quad (81)$$

The adjoint Eqs. (18) to (81) is

$$v_t + k(u)v_{xx} = 0. \quad (82)$$

We take the substitution (70) written together with the necessary derivatives

$$\begin{aligned} v &= \varphi(t, x, u) \\ v_t &= \varphi_u u_t + \varphi_t, \quad v_x = \varphi_u u_x + \varphi_x \\ v_{xx} &= \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx} \end{aligned} \quad (83)$$

and arrive at the following self-adjointness condition (72)

$$\begin{aligned} \varphi_u u_t + \varphi_t + k(u)[\varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}] \\ = \lambda[u_t - k(u)u_{xx} - k'(u)u_x^2]. \end{aligned} \quad (84)$$

The comparison of the coefficients of u_t in both sides of Eq. (84) yields $\lambda = \varphi_u$. Then, comparing the terms with u_{xx} we see that $\varphi_u = 0$. Hence Eq. (84) reduces to

$$\varphi_t + k(u)\varphi_{xx} = 0 \quad (85)$$

and yields $\varphi_t = 0$, $\varphi_{xx} = 0$, whence $\varphi = C_1 x + C_2$, where $C_1, C_2 = \text{const}$. We have demonstrated that Eq. (81) is nonlinearly self-adjoint by Definition 4 and that the substitution (70) has the form

$$v = C_1 x + C_2. \quad (86)$$

The same result can be easily obtained by using Definition 5. We look for the solution of the adjoint Eq. (82) in the form $v = \varphi(t, x)$. Then Eq. (82) has the form (85). Since it should be satisfied for all solutions u of Eq. (81), we obtain $\varphi_t = 0$, $\varphi_{xx} = 0$, and hence Eq. (86).

Multi-dimensional Case

The similar analysis can be applied to the nonlinear heat equation with several variables $x = (x^1, \dots, x^n)$

$$u_t = \nabla \cdot (k(u)\nabla u) \quad (87)$$

or

$$u_t - k(u) \Delta u - k'(u)|\nabla u|^2 = 0. \quad (88)$$

The reckoning shows that the adjoint Eqs. (18) to (88) is written

$$v_t + k(u) \Delta v = 0. \quad (89)$$

It is easy to verify the nonlinear self-adjointness by Definition 5. Namely, searching for the solution of the adjoint Eq. (89) in the form $v = \varphi(t, x^1, \dots, x^n)$, one obtains

$$\varphi_t + k(u) \Delta \varphi = 0$$

whence

$$\varphi_t = 0, \quad \Delta \varphi = 0.$$

We conclude that Eq. (88) is self-adjoint and that the substitution (70) is given by

$$v = \varphi(x^1, \dots, x^n) \quad (90)$$

where $\varphi(x^1, \dots, x^n)$ is any solution of the n -dimensional Laplace equation $\Delta \varphi = 0$.

1.3.4 Anisotropic Nonlinear Diffusion Equation

Two-Dimensional Case

Consider the diffusion equation

$$u_t = (f(u)u_x)_x + (g(u)u_y)_y \quad (91)$$

in an anisotropic two-dimensional medium (see [15–17], vol. 1, Sect. 10.8) with arbitrary functions $f(u)$ and $g(u)$. The adjoint equation is

$$v_t + f(u)v_{xx} + g(u)v_{yy} = 0. \quad (92)$$

Using Definition 5 we obtain the following equations for nonlinear self-adjointness of Eq. (91)

$$\varphi_t = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0. \quad (93)$$

Integrating equations (93) we obtain the following substitution (70)

$$v = C_1 xy + C_2 x + C_3 y + C_4. \quad (94)$$

Three-Dimensional Case

The three-dimensional anisotropic nonlinear diffusion equation has the following form (see [15–17], vol. 1, Sect. 10.9)

$$u_t = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z. \quad (95)$$

Its adjoint equation is

$$v_t + f(u)v_{xx} + g(u)v_{yy} + h(u)v_{zz} = 0. \quad (96)$$

Equation (95) is nonlinearly self-adjoint. In this case the substitution (94) is replaced by

$$v = C_1 xyz + C_2 xy + C_3 xz + C_4 yz + C_5 x + C_6 y + C_7 z + C_8. \quad (97)$$

1.3.5 Nonlinear Wave Equations

One-Dimensional Case

Consider the following one-dimensional nonlinear wave equation

$$u_{tt} = (k(u)u_x)_x, \quad k(u) \neq 0 \quad (98)$$

or in the expanded form

$$u_{tt} - k(u)u_{xx} - k'(u)u_x^2 = 0. \quad (99)$$

The adjoint Eqs. (18) to (98) is written

$$v_{tt} - k(u)v_{xx} = 0. \quad (100)$$

Proceeding as in One-Dimensional Case or applying Definition 5 to Eqs. (99) and (100) by letting $v = \varphi(t, x)$, we obtain the following equations that guarantee the nonlinear self-adjointness of Eq. (98)

$$\varphi_{tt} = 0, \quad \varphi_{xx} = 0. \quad (101)$$

Integrating equation (101) we obtain the following substitution

$$v = C_1 tx + C_2 t + C_3 x + C_4. \quad (102)$$

Multi-Dimensional Case

The multi-dimensional version of Eq. (98) with $x = (x^1, \dots, x^\nu)$ is written

$$u_{tt} = \nabla \cdot (k(u)\nabla u) \quad (103)$$

or

$$u_{tt} - k(u)\Delta u - k'(u)|\nabla u|^2 = 0. \quad (104)$$

The adjoint equation is

$$v_{tt} - k(u)\Delta v = 0. \quad (105)$$

Using Definition 5 and searching the solution of the adjoint Eq. (105) in the form $v = \varphi(t, x^1, \dots, x^\nu)$, we obtain the equations

$$\varphi_{tt} = 0, \quad \Delta \varphi = 0.$$

Solving them we arrive at the following substitution (70)

$$v = a(x)t + b(x) \quad (106)$$

where $a(x)$ and $b(x)$ solve the ν -dimensional Laplace equation

$$\Delta a(x^1, \dots, x^\nu) = 0, \quad \Delta b(x^1, \dots, x^\nu) = 0.$$

Hence Eq. (103) is nonlinearly self-adjoint.

Nonlinear Vibration of Membranes

Vibrations of a uniform membrane whose tension varies during deformations are described by the following Lagrangian

$$L = \frac{1}{2} \left[u_t^2 - k(u)(u_x^2 + u_y^2) \right], \quad k'(u) \neq 0. \quad (107)$$

The corresponding Euler-Lagrange equation

$$\frac{\partial L}{\partial u} - D_t \left(\frac{\partial L}{\partial u_t} \right) - D_x \left(\frac{\partial L}{\partial u_x} \right) - D_y \left(\frac{\partial L}{\partial u_y} \right) = 0$$

provides the nonlinear wave equation

$$u_{tt} = k(u)(u_{xx} + u_{yy}) + \frac{1}{2} k'(u)(u_x^2 + u_y^2). \quad (108)$$

Note that Eq. (108) differs from the two-dimensional nonlinear wave Eq. (104) by the coefficient $1/2$. Let us find out if this difference affects self-adjointness.

By applying (69) to the formal Lagrangian of Eq. (108) we obtain

$$\begin{aligned} F^* &= v_{tt} - k(u)(v_{xx} + v_{yy}) - k'(u)(u_x v_x + u_y v_y + v u_{xx} + v u_{yy}) \\ &\quad - \frac{v}{2} k''(u)(u_x^2 + u_y^2). \end{aligned}$$

We take the substitution (70) together with the necessary derivatives (see Eq. 83)

$$\begin{aligned} v &= \varphi(t, x, y, u), & v_t &= \varphi_u u_t + \varphi_t \\ v_x &= \varphi_u u_x + \varphi_x, & v_y &= \varphi_u u_y + \varphi_y \\ v_{xx} &= \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx} \\ v_{yy} &= \varphi_u u_{yy} + \varphi_{uu} u_y^2 + 2\varphi_{yu} u_y + \varphi_{yy} \\ v_{tt} &= \varphi_u u_{tt} + \varphi_{uu} u_t^2 + 2\varphi_{tu} u_t + \varphi_{tt} \end{aligned} \quad (109)$$

and substitute the expressions (109) in the self-adjointness condition (72)

$$F^*|_{v=\varphi} = \lambda[u_{tt} - k(u)(u_{xx} + u_{yy}) - \frac{1}{2}k'(u)(u_x^2 + u_y^2)].$$

Comparing the coefficients of u_{tt} we obtain $\lambda = \varphi_u$. Then we compare the coefficients of u_{xx} and obtain $\varphi k'(u) = 0$. This equation yields $\varphi = 0$ because $k'(u) \neq 0$. Thus, the condition (71) is not satisfied for the point-wise substitution (70). Further investigation of Eq. (108) for the nonlinear self-adjointness requires differential substitutions.

1.3.6 Anisotropic Nonlinear Wave Equation

Two-Dimensional Case

The two-dimensional anisotropic nonlinear wave equation is (see [15–17, vol. 1, Sect. 12.6])

$$u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y. \quad (110)$$

Its adjoint equation has the form

$$v_{tt} - f(u)v_{xx} - g(u)v_{yy} = 0. \quad (111)$$

Proceeding as in Sect. 1.3.4 we obtain the following equations that guarantee the self-adjointness of Eq. (110)

$$\varphi_{tt} = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0. \quad (112)$$

Integrating equation (112) we obtain the following substitution (70)

$$v = C_1 txy + C_2 tx + C_3 ty + C_4 xy + C_5 t + C_6 x + C_7 y + C_8. \quad (113)$$

Remark 7 I provide here detailed calculations in integrating Eq. (112). The general solution to the linear second-order equation $\varphi_{tt} = 0$ is given by

$$\varphi = A(x, y)t + B(x, y) \quad (114)$$

with arbitrary functions $A(x, y)$ and $B(x, y)$. Substituting this expression for φ in the second and third Eq. (112) and splitting with respect to t we obtain the following equations for $A(x, y)$ and $B(x, y)$

$$A_{xx} = 0, \quad A_{yy} = 0, \quad B_{xx} = 0, \quad B_{yy} = 0.$$

Substituting the general solution

$$A = a_1(y)x + a_2(y)$$

of the equation $A_{xx} = 0$ in the equation $A_{yy} = 0$ and splitting with respect to x , we obtain $a_1'' = 0$, $a_2'' = 0$, whence

$$a_1 = c_{11}y + c_{12}, \quad a_2 = c_{21}y + c_{22}$$

where c_{11}, \dots, c_{22} are arbitrary constants. Substituting these in the above expression for A we obtain

$$A = c_{11}xy + c_{12}x + c_{21}y + c_{22}.$$

Proceeding likewise with the equations for $B(x, y)$, we have

$$B = d_{11}xy + d_{12}x + d_{21}y + d_{22}$$

with arbitrary constant coefficients d_{11}, \dots, d_{22} . Finally, we substitute the resulting A and B in the expression (114) for φ and, changing the notation, arrive at (113).

Three-Dimensional Case

The three-dimensional anisotropic nonlinear wave equation

$$u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z \quad (115)$$

has the following adjoint equation

$$v_{tt} - f(u)v_{xx} - g(u)v_{yy} - h(u)v_{zz} = 0. \quad (116)$$

In this case Eq. (112) are replaced by

$$\varphi_{tt} = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy}, \quad \varphi_{zz} = 0$$

and yield the following substitution (70)

$$\begin{aligned} v = & C_1txyz + C_2txy + C_3txz + C_4tyz + C_5tx + C_6ty + C_7tz \\ & + C_8xy + C_9xz + C_{10}yz + C_{11}t + C_{12}x + C_{13}y + C_{14}z + C_{15}. \end{aligned} \quad (117)$$

1.3.7 From Nonlinear to Strict Self-Adjointness

The approach of this section is not used for constructing conservation laws. But it may be useful for other applications of the nonlinear self-adjointness.

Theorem 1 *The differential equation (17)*

$$F(x, u, u_{(1)}, \dots, u_{(s)}) = 0 \quad (118)$$

is nonlinearly self-adjoint (Definition 4) if and only if it becomes strictly self-adjoint (Definition 2) upon rewriting in the equivalent form

$$\mu(x, u)F(x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad \mu(x, u) \neq 0 \quad (119)$$

with an appropriate multiplier $\mu(x, u)$.

Proof We will write the condition (72) for nonlinear self-adjointness of Eq. (118) in the form

$$\left. \frac{\delta(vF)}{\delta u} \right|_{v=\varphi(x,u)} = \lambda(x, u)F(x, u, u_{(1)}, \dots, u_{(s)}). \quad (120)$$

Furthermore, invoking that the Eqs. (119) and (118) are equivalent, we will write the condition (28) for strict self-adjointness of Eq. (119) in the form

$$\left. \frac{\delta(w\mu F)}{\delta u} \right|_{w=u} = \tilde{\lambda}(x, u)F(x, u, u_{(1)}, \dots, u_{(s)}). \quad (121)$$

Since w is a dependent variable and $\mu = \mu(x, u)$ is a certain function of x, u , the variational derivative in the left-hand side of (121) can be written as follows

$$\begin{aligned} \frac{\delta(w\mu F)}{\delta u} &= w \frac{\partial \mu}{\partial u} F + \mu w \frac{\partial F}{\partial u} - D_i \left(\mu w \frac{\partial F}{\partial u_i} \right) + D_i D_j \left(\mu w \frac{\partial F}{\partial u_{ij}} \right) - \dots \\ &= w \frac{\partial \mu}{\partial u} F + \frac{\delta(vF)}{\delta u} \end{aligned}$$

where v is the new dependent variable instead of w defined by the formula

$$v = \mu(x, u)w. \quad (122)$$

Now the left side of Eq. (121) is written

$$\left. \frac{\delta(w\mu F)}{\delta u} \right|_{w=u} = u \frac{\partial \mu}{\partial u} F + \left. \frac{\delta(vF)}{\delta u} \right|_{v=u\mu(x,u)}. \quad (123)$$

Let us assume that Eq. (118) is nonlinearly self-adjoint. Then Eq. (120) holds with a certain given function $\varphi(x, u)$. Therefore, we take the multiplier

$$\mu(x, u) = \frac{\varphi(x, u)}{u} \quad (124)$$

and reduce Eq. (123) to the following form

$$\frac{\delta(w\mu F)}{\delta u} \Big|_{w=u} = \left(\lambda + \frac{\partial \varphi}{\partial u} - \frac{\varphi}{u} \right) F.$$

This proves that Eq. (121) holds with

$$\tilde{\lambda} = \frac{\partial \varphi}{\partial u} - \frac{\varphi}{u} + \lambda.$$

Hence, Eq. (119) with the multiplier μ given by (124) is strictly self-adjoint.

Let us assume now that Eq. (119) with a certain multiplier $\mu(x, u)$ is strictly self-adjoint. Then Eq. (121) holds. Therefore, if we take the function φ defined by (see 124)

$$\varphi(x, u) = u\mu(x, u) \tag{125}$$

Equation (123) yields

$$\frac{\delta(vF)}{\delta u} \Big|_{v=\varphi(x,u)} = \left(\tilde{\lambda} - u \frac{\partial \mu}{\partial u} \right) F.$$

It follows that Eq. (120) holds with

$$\lambda = \tilde{\lambda} - u \frac{\partial \mu}{\partial u}.$$

We conclude that Eq. (118) is nonlinearly self-adjoint, thus completing the proof. \square

Example 13 The multiplier (67) used in Example 10 and the function $\varphi = x^2$ that provides a solution of the adjoint Eq. (66) to the Kompaneets equation are related by Eq. (125).

Example 14 Let us consider the one-dimensional nonlinear wave Eq. (99),

$$u_{tt} - k(u)u_{xx} - k'(u)u_x^2 = 0.$$

If we substitute in (124) the function φ given by the right-hand side of (102) we will obtain the multiplier that maps Eq. (99) into the strictly self-adjoint equivalent form. For example, taking (102) with $C_1 = C_3 = C_4 = 0, C_2 = 1$ we obtain the multiplier

$$\mu = \frac{t}{u}.$$

The corresponding equivalent equation to equation (99) has the formal Lagrangian

$$\mathcal{L} = \frac{tv}{u} [u_{tt} - k(u)u_{xx} - k'(u)u_x^2].$$

We have

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} = & D_t^2 \left(\frac{tv}{u} \right) - \frac{tv}{u^2} u_{tt} - D_x^2 \left(\frac{tv}{u} k(u) \right) - \frac{tv}{u} k'(u) u_{xx} + \frac{tv}{u^2} k(u) u_{xx} \\ & + 2D_x \left(\frac{tv}{u} k'(u) u_x \right) - \frac{tv}{u} k''(u) u_x^2 + \frac{tv}{u^2} k'(u) u_x^2. \end{aligned}$$

Letting here $v = u$ we see that the strict self-adjointness condition is satisfied in the following form

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=u} = -\frac{t}{u} [u_{tt} - k(u) u_{xx} - k'(u) u_x^2].$$

1.4 Generalized Kompaneets Equation

1.4.1 Introduction

The equation

$$\frac{\partial n}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n + n^2 \right) \right] \quad (126)$$

known as the Kompaneets equation or the *photon diffusion equation*, was derived independently by Kompaneets² [18] and Weymann [19]. They take as a starting point the kinetic equations for the distribution function of a photon gas³ and arrive, at certain idealized conditions, at Eq. (126). This equation provides a mathematical model for describing the time development of the energy spectrum of a low energy homogeneous photon gas interacting with a rarefied electron gas via the Compton scattering. Here n is the density of the photon gas (photon number density), t is time and x is connected with the photon frequency ν by the formula

$$x = \frac{h\nu}{kT_e} \quad (127)$$

where h is Planck's constant and kT_e is the *electron temperature* with the standard notation k for Boltzmann's constant. According to this notation, $h\nu$ has the meaning of the *photon energy*. The nonrelativistic approximation is used, i.e., it is assumed that the electron temperatures satisfy the condition $kT_e \ll mc^2$, where m is the

² He mentions in his paper that the work has been done in 1950 and published in *Report # 336* of the Institute of Chemical Physics of the USSR Academy of Sciences.

³ Weymann uses Dreicer's kinetic equation [20] for a photon gas interacting with a plasma which is slightly different from the equation used by Kompaneets.

electron mass and c is the light velocity. The term *low energy photon gas* means that $h\nu \ll mc^2$.

The question arises if the idealized conditions assumed in deriving Eq. (126) may be satisfied in the real world. For discussions of theoretical and observational evidences for such possibility in astrophysical environments, for example in intergalactic gas, see e.g., [21, 22] and the references therein. See also the recent publication [23].

1.4.2 Discussion of Self-Adjointness of the Kompaneets Equation

For unifying the notation, the dependent variable n in Eq. (126) will be denoted by u and Eq. (126) will be written further in the form

$$u_t = \frac{1}{x^2} D_x [x^4(u_x + u + u^2)]. \quad (128)$$

Writing it in the expanded form

$$u_t = x^2 u_{xx} + (x^2 + 4x + 2x^2 u) u_x + 4x(u + u^2) \quad (129)$$

we have the following formal Lagrangian for Eq. (128)

$$\mathcal{L} = v[-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u) u_x + 4x(u + u^2)].$$

Working out the variational derivative of this formal Lagrangian

$$\frac{\delta \mathcal{L}}{\delta u} = D_t(v) + D_x^2(x^2 v) - D_x[(x^2 + 4x + 2x^2 u)v] + 2x^2 v u_x + 4x(1 + 2u)v$$

we obtain the adjoint equation to equation (128)

$$\frac{\delta \mathcal{L}}{\delta u} \equiv v_t + x^2 v_{xx} - x^2(1 + 2u)v_x + 2(x + 2xu - 1)v = 0. \quad (130)$$

If $v = \varphi(u)$, then

$$v_t = \varphi'(u)u_t, \quad v_x = \varphi'(u)u_x, \quad v_{xx} = \varphi'(u)u_{xx} + \varphi''(u)u_x^2.$$

It follows that the quasi self-adjointness condition (36)

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=\varphi(u)} = \lambda[-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2)]$$

is not satisfied.

Let us check if this condition is satisfied in the more general form (72)

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=\varphi(t,x,u)} = \lambda[-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2)]. \quad (131)$$

In this case

$$\begin{aligned} v_t &= D_t[\varphi(t, x, u)] = \varphi_u u_t + \varphi_t \\ v_x &= D_x[\varphi(t, x, u)] = \varphi_u u_x + \varphi_x \\ v_{xx} &= D_x(v_x) = \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}. \end{aligned} \quad (132)$$

Inserting (132) in the expression for the variational derivative given by (130) and singling out in Eq. (131) the terms containing u_t and u_{xx} , we obtain the following equation

$$\varphi_u [u_t + x^2 u_{xx}] = \lambda[-u_t + x^2 u_{xx}].$$

Since this equation should be satisfied identically in u_t and u_{xx} , it yields $\lambda = \varphi_u = 0$. Hence $\varphi = \varphi(t, x)$ and Eq. (131) becomes

$$\varphi_t + x^2 \varphi_{xx} - x^2(1 + 2u)\varphi_x + 2(x + 2xu - 1)\varphi = 0. \quad (133)$$

This equation should be satisfied identically in t, x and u . Therefore we nullify the coefficient for u and obtain

$$x\varphi_x - 2\varphi = 0$$

whence

$$\varphi(t, x) = c(t)x^2.$$

Substitution in Eq. (133) yields $c'(t) = 0$. Hence, $v = \varphi(t, x) = Cx^2$ with arbitrary constant C . Since $\lambda = 0$ in (131) and the adjoint Eq. (130) is linear and homogeneous in v , one can let $C = 1$. Thus, we have demonstrated the following statement.

Proposition 6 *The adjoint Eq. (130) has the solution*

$$v = x^2 \quad (134)$$

for any solution u of Eq. (128). In another words, the Kompaneets Eq. (128) is non-linearly self-adjoint with the substitution (70) given by (134).

Remark 8 The substitution (134) does not depend on u . The question arises on existence of a substitution $v = \varphi(t, x, u)$ involving u if we rewrite Eq. (128) in an equivalent form

$$\alpha(t, x, u)[-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2)] = 0 \quad (129')$$

with an appropriate multiplier $\alpha \neq 0$. This question is investigated in next section for a more general model.

1.4.3 The Generalized Model

In the original derivation of Eq. (126) the following more general equation appears accidentally (see [18], Eqs. (9), (10) and their discussion)

$$\frac{\partial n}{\partial t} = \frac{1}{g(x)} \frac{\partial}{\partial x} \left[g^2(x) \left(\frac{\partial n}{\partial x} + f(n) \right) \right] \quad (135)$$

with undetermined functions $f(u)$ and $g(x)$. Then, using a physical reasoning, Kompaneets takes $f(u) = n(1 + n)$ and $g(x) = x^2$. This choice restricts the symmetry properties of the model significantly. Namely, Eq. (126) has only the time-translational symmetry with the generator

$$X = \frac{\partial}{\partial t}. \quad (136)$$

The symmetry (136) provides only one invariant solution, namely the stationary solution $n = n(x)$ defined by the Riccati equation

$$\frac{dn}{dx} + n^2 + n = \frac{C}{x^4}.$$

The generalized model (135) can be used for extensions of symmetry properties via the methods of *preliminary group classification* [24, 25]. In this way, exact solutions known for particular approximations to the Kompaneets equation can be obtained. This may also lead to new approximations of the solutions by taking into account various inevitable perturbations of the idealized situation assumed in the Kompaneets model (126).

So, we will take with minor changes in notation the generalized model (135)

$$u_t = \frac{1}{h(x)} D_x \{ h^2(x) [u_x + f(u)] \}, \quad h'(x) \neq 0. \quad (137)$$

It is written in the expanded form as follows

$$u_t = h(x)(u_{xx} + f'(u)u_x) + 2h'(x)(u_x + f(u)). \quad (138)$$

We will write Eq. (138) in the equivalent form similar to (129')

$$\alpha(t, x, u) \left[-u_t + h(x)(u_{xx} + f'(u)u_x) + 2h'(x)(u_x + f(u)) \right] = 0 \quad (139)$$

where $\alpha \neq 0$. This provides the following formal Lagrangian

$$\mathcal{L} = v \alpha(t, x, u) \left[-u_t + h(x)(u_{xx} + f'(u)u_x) + 2h'(x)(u_x + f(u)) \right] \quad (140)$$

where v is a new dependent variable. For this Lagrangian, we have

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} &= D_t(v\alpha) + D_x^2[h(x)v\alpha] - D_x[h(x)f'(u)v\alpha + 2h'(x)v\alpha] \\ &\quad + h(x)f''(u)v\alpha u_x + 2h'(x)f'(u)v\alpha \\ &\quad + v\alpha_u \left[-u_t + h(x)(u_{xx} + f'(u)u_x) + 2h'(x)(u_x + f(u)) \right]. \end{aligned}$$

The reckoning shows that

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} &= D_t(v\alpha) + hD_x^2(v\alpha) - hf'D_x(v\alpha) + (h'f' - h'')v\alpha \\ &\quad + v\alpha_u \left[-u_t + (u_{xx} + f'u_x)h + 2(u_x + f)h' \right]. \end{aligned} \quad (141)$$

Now we write the condition for the self-adjointness of Eq. (138) in the form

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=\varphi(t,x,u)} = \lambda \left[-u_t + (u_{xx} + f'u_x)h + 2(u_x + f)h' \right] \quad (142)$$

with an undetermined coefficient λ . Substituting (141) in (142) we have

$$\begin{aligned} &D_t(\varphi\alpha) + hD_x^2(\varphi\alpha) - hf'D_x(\varphi\alpha) + (h'f' - h'')\varphi\alpha \\ &\quad + \varphi\alpha_u \left[-u_t + (u_{xx} + f'u_x)h + 2(u_x + f)h' \right] \\ &= \lambda \left[-u_t + (u_{xx} + f'u_x)h + 2(u_x + f)h' \right]. \end{aligned} \quad (143)$$

Here $\varphi = \varphi(t, x, u)$, $\alpha = \alpha(t, x, u)$ and consequently (see (132))

$$\begin{aligned} D_t(\varphi\alpha) &= (\varphi\alpha)_u u_t + (\varphi\alpha)_t \\ D_x(\varphi\alpha) &= (\varphi\alpha)_u u_x + (\varphi\alpha)_x \\ D_x^2(\varphi\alpha) &= (\varphi\alpha)_u u_{xx} + (\varphi\alpha)_{uu} u_x^2 + 2(\varphi\alpha)_{xu} u_x + (\varphi\alpha)_{xx}. \end{aligned} \quad (144)$$

We substitute (141) in Eq. (143), equate the coefficients for u_t in both sides of the resulting equation and obtain $(\varphi\alpha)_u - \varphi\alpha_u = -\lambda$. Hence

$$\lambda = -\alpha\varphi_u.$$

Using this expression for λ and equating the coefficients for hu_{xx} in both sides of Eq. (143) we get $(\varphi\alpha)_u + \varphi\alpha_u = -\alpha\varphi_u$. It follows that $(\varphi\alpha)_u = 0$ and hence

$$\alpha\varphi = k(t, x).$$

Now Eq. (143) becomes

$$k_t + h(x)k_{xx} - h''(x)k + f'(u)[h'(x)k - h(x)k_x] = 0.$$

If $f''(u) \neq 0$, the above equation splits into two equations

$$h'(x)k - h(x)k_x = 0, \quad k_t + h(x)k_{xx} - h''(x)k.$$

The first of these equations yields $k(t, x) = c(t)h(x)$, and then the second equation shows that $c'(t) = 0$. Hence, $k = Ch(x)$ with $C = \text{const}$. Letting $C = 1$, we have

$$\alpha\varphi = h(x). \quad (145)$$

Equation (145) can be satisfied by taking, e.g.,

$$\alpha = \frac{h(x)}{u}, \quad \varphi = u. \quad (146)$$

Thus, we have proved the following statement.

Proposition 7 Equation (137) written in the equivalent form

$$\frac{h(x)}{u} u_t = \frac{1}{u} D_x \{h^2(x)[u_x + f(u)]\} \quad (147)$$

is strictly self-adjoint. In another words, the adjoint equation to equation (147) coincides with (147) upon the substitution

$$v = u. \quad (148)$$

In particular, let us verify by direct calculations that the original Eq. (128) becomes strictly self-adjoint if we rewrite it in the equivalent form

$$\frac{x^2}{u} u_t = \frac{1}{u} D_x [x^4(u_x + u + u^2)]. \quad (149)$$

Equation (149) reads

$$-\frac{x^2}{u} u_t + \frac{x^4}{u} u_{xx} + \left[(x^4 + 4x^3) \frac{1}{u} + 2x^4 \right] u_x + 4x^3(1 + u) = 0 \quad (150)$$

and has the formal Lagrangian

$$\mathcal{L} = -x^2 \frac{v}{u} u_t + x^4 \frac{v}{u} u_{xx} + \left[(x^4 + 4x^3) \frac{v}{u} + 2x^4 v \right] u_x + 4x^3(v + uv).$$

Accordingly, the adjoint equation to equation (150) is written

$$D_t \left(x^2 \frac{v}{u} \right) + D_x^2 \left(x^4 \frac{v}{u} \right) - D_x \left[(x^4 + 4x^3) \frac{v}{u} + 2x^4 v \right] \\ + x^2 \frac{v}{u^2} u_t - x^4 \frac{v}{u^2} u_{xx} - (x^4 + 4x^3) \frac{v}{u^2} u_x + 4x^3 v = 0.$$

Letting here $v = u$ one has $v/u = 1$ and after simple calculations arrives at Eq. (150).

1.5 Quasi Self-Adjoint Reaction-Diffusion Models

Let us consider the one-dimensional reaction-diffusion model described by the following system (see e.g., [26])

$$\begin{aligned} \frac{\partial u}{\partial t} &= f(u, v) + A \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left(\phi(u, v) \frac{\partial v}{\partial x} \right) \\ \frac{\partial v}{\partial t} &= g(u, v) + B \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x} \left(\psi(u, v) \frac{\partial u}{\partial x} \right). \end{aligned} \quad (151)$$

It is convenient to write Eq. (151) in the form

$$\begin{aligned} D_t(u) &= AD_x^2(u) + D_x[\phi(u, v)D_x(v)] + f(u, v) \\ D_t(v) &= BD_x^2(v) + D_x[\psi(u, v)D_x(u)] + g(u, v). \end{aligned} \quad (152)$$

The total differentiations have the form

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} + \dots \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_t} + v_{xx} \frac{\partial}{\partial v_x} + \dots \end{aligned} \quad (153)$$

and Eq. (152) are written

$$\begin{aligned} u_t &= Au_{xx} + \phi v_{xx} + [\phi_u u_x + \phi_v v_x] v_x + f \\ v_t &= Bv_{xx} + \psi u_{xx} + [\psi_u u_x + \psi_v v_x] u_x + g. \end{aligned} \quad (154)$$

The formal Lagrangian for the system (154) is

$$\begin{aligned} \mathcal{L} &= z(Au_{xx} - u_t + \phi v_{xx} + \phi_u u_x v_x + \phi_v v_x^2 + f) \\ &+ w(Bv_{xx} - v_t + \psi u_{xx} + \psi_u u_x^2 + \psi_v u_x v_x + g) \end{aligned} \quad (155)$$

where z and w are new dependent variables. Equation (8) are written

$$F_1^* = \frac{\delta \mathcal{L}}{\delta u} = D_x^2 \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) - D_t \left(\frac{\partial \mathcal{L}}{\partial u_t} \right) - D_x \left(\frac{\partial \mathcal{L}}{\partial u_x} \right) + \frac{\partial \mathcal{L}}{\partial u}$$

$$F_2^* = \frac{\delta \mathcal{L}}{\delta v} = D_x^2 \left(\frac{\partial \mathcal{L}}{\partial v_{xx}} \right) - D_t \left(\frac{\partial \mathcal{L}}{\partial v_t} \right) - D_x \left(\frac{\partial \mathcal{L}}{\partial v_x} \right) + \frac{\partial \mathcal{L}}{\partial v}.$$

Substituting here the expression (155) for \mathcal{L} we obtain after simple calculations the following adjoint Eq. (69) to the system (154)

$$Az_{xx} + z_t + \psi_v v_x w_x - \phi_u v_x z_x + \psi w_{xx} + z f_u + w g_u = 0 \quad (156)$$

$$Bw_{xx} + w_t + \phi_u u_x z_x - \psi_v u_x w_x + \phi z_{xx} + z f_v + w g_v = 0. \quad (157)$$

Let us investigate the system (154) for quasi self-adjointness (Definition 3). We write the left-hand sides of Eqs. (156) and (157) as linear combinations of the left-hand sides of Eq. (154)

$$\begin{aligned} & Az_{xx} + z_t + \psi_v v_x w_x - \phi_u v_x z_x + \psi w_{xx} + z f_u + w g_u \\ &= (Au_{xx} - u_t + \phi v_{xx} + \phi_u u_x v_x + \phi_v v_x^2 + f)P \\ &+ (Bv_{xx} - v_t + \psi u_{xx} + \psi_u u_x^2 + \psi_v u_x v_x + g)Q \end{aligned} \quad (158)$$

$$\begin{aligned} & Bw_{xx} + w_t + \phi_u u_x z_x - \psi_v u_x w_x + \phi z_{xx} + z f_v + w g_v \\ &= (Au_{xx} - u_t + \phi v_{xx} + \phi_u u_x v_x + \phi_v v_x^2 + f)M \\ &+ (Bv_{xx} - v_t + \psi u_{xx} + \psi_u u_x^2 + \psi_v u_x v_x + g)N \end{aligned} \quad (159)$$

where P , Q , M and N are undetermined coefficients. We write the substitution (42) in the form

$$z = Z(u, v), \quad w = W(u, v) \quad (160)$$

and insert in the left-hand sides of Eqs. (158) and (159) these expressions for z , w together with their derivatives

$$\begin{aligned} z_t &= Z_u u_t + Z_v v_t, & z_x &= Z_u u_x + Z_v v_x \\ z_{xx} &= Z_u u_{xx} + Z_v v_{xx} + Z_{uu} u_x^2 + 2Z_{uv} u_x v_x + Z_{vv} v_x^2 \\ w_t &= W_u u_t + W_v v_t, & w_x &= W_u u_x + W_v v_x \\ w_{xx} &= W_u u_{xx} + W_v v_{xx} + W_{uu} u_x^2 + 2W_{uv} u_x v_x + W_{vv} v_x^2. \end{aligned}$$

Equating the coefficients in front of u_t and v_t in both sides of Eqs.(158) and (159) we obtain

$$P = -Z_u, \quad Q = -Z_v, \quad N = -W_v, \quad M = -W_u. \quad (161)$$

Now we calculate the coefficients for u_{xx} and v_{xx} , take into account Eq.(161) and arrive at the following system of equations

$$\begin{aligned} 2AZ_u + \psi Z_v + \psi W_u &= 0, & (A+B)Z_v + \phi Z_u + \psi W_v &= 0 \\ 2BW_v + \phi Z_v + \psi W_u &= 0, & (A+B)W_u + \phi Z_u + \psi W_v &= 0. \end{aligned} \quad (162)$$

Equation (162) provide a linear homogeneous algebraic equations for the quantities

$$Z_u, \quad Z_v, \quad W_u, \quad W_v$$

with the matrix

$$\begin{pmatrix} 2A & \psi & \psi & 0 \\ \phi & A+B & 0 & \psi \\ 0 & \phi & \phi & 2B \\ \phi & 0 & A+B & \psi \end{pmatrix}.$$

This matrix has an inverse because its determinant is equal to

$$4(A+B)^2(\phi\psi - AB)$$

which does not vanish in the case of arbitrary A , B , ϕ and ψ . Hence, Eq.(162) yield

$$Z_u = Z_v = W_u = W_v = 0. \quad (163)$$

It follows that $Z(u, v) = C_1$, $W(u, v) = C_2$. Thus, the substitution (42) has the form

$$z = C_1, \quad w = C_2 \quad (164)$$

with arbitrary constants C_1 , C_2 . Then Eqs.(158) and (159) become

$$(C_1f + C_2g)_u = 0, \quad (C_1f + C_2g)_v = 0$$

and yield

$$\tilde{f} + \tilde{g} = C,$$

where $\tilde{f} = C_1f$, $\tilde{g} = C_2g$, and $C = \text{const}$. Since \tilde{f} and \tilde{g} , along with f and g , are arbitrary functions, we can omit the ‘‘tilde’’ and write

$$f + g = C. \quad (165)$$

Equation (165) provides the necessary and sufficient condition for the quasi self-adjointness of the system (151). Thus, we have proved the following statement.

Theorem 2 *The system (151) is quasi self-adjoint if and only if it has the form*

$$\begin{aligned}\frac{\partial u}{\partial t} &= f(u, v) + A \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left(\phi(u, v) \frac{\partial v}{\partial x} \right) \\ \frac{\partial v}{\partial t} &= C - f(u, v) + B \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x} \left(\psi(u, v) \frac{\partial u}{\partial x} \right)\end{aligned}\quad (166)$$

where $\phi(u, v)$, $\psi(u, v)$, $f(u, v)$ are arbitrary functions and A, B, C are arbitrary constants. The substitution (42) is given by (164).

Remark 9 If we replace (160) by the general substitution (70), i.e., take

$$z = Z(t, x, u, v), \quad w = W(t, x, u, v) \quad (167)$$

then Eq. (164) will be replaced by

$$z = Z(t, x), \quad w = W(t, x) \quad (168)$$

with functions $Z(t, x)$, $W(t, x)$ satisfying the following equations

$$(\psi_v W - \phi_u Z)_x = 0 \quad (169)$$

$$\begin{aligned}AZ_{xx} + Z_t + \psi W_{xx} + (fZ + gW)_u &= 0 \\ BW_{xx} + W_t + \phi Z_{xx} + (fZ + gW)_v &= 0.\end{aligned}\quad (170)$$

1.6 A Model of an Irrigation System

Let us consider the second-order nonlinear partial differential equation

$$C(\psi)\psi_t = [K(\psi)\psi_x]_x + [K(\psi)(\psi_z - 1)]_z - S(\psi). \quad (171)$$

It serves as a mathematical model for investigating certain irrigation systems (see [15–17], vol. 2, Sect. 9.8 and references therein). The dependent variable ψ denotes the soil moisture pressure head, $C(\psi)$ is the specific water capacity, $K(\psi)$ is the unsaturated hydraulic conductivity, $S(\psi)$ is a source term. The independent variables are the time t , the horizontal axis x and the vertical axis z which is taken to be positive downward.

The adjoint equation (69) to equation (171) has the form

$$C(\psi)v_t + K(\psi)[v_{xx} + v_{zz}] + K'(\psi)v_z - S'(\psi)v = 0. \quad (172)$$

It follows from (172) that Eq. (171) is not nonlinearly self-adjoint if $C(\psi)$, $K(\psi)$ and $S(\psi)$ are arbitrary functions. Indeed, using Definition 5 of the nonlinear self-adjointness and nullifying in (172) the term with $S'(\psi)$ we obtain $v = 0$. Hence, the condition (71) of the nonlinear self-adjointness is not satisfied.

However, Eq. (171) can be nonlinearly self-adjoint if there are certain relations between the functions $C(\psi)$, $K(\psi)$ and $S(\psi)$. For example, let us suppose that the specific water capacity $C(\psi)$ and the hydraulic conductivity $K(\psi)$ are arbitrary, but the source term $S(\psi)$ is related with $C(\psi)$ by the equation

$$S'(\psi) = aC(\psi), \quad a = \text{const}. \quad (173)$$

Then Eq. (172) becomes $C(\psi)[v_t - av] + K(\psi)[v_{xx} + v_{zz}] + K'(\psi)v_z = 0$ and yields

$$v_z = 0, \quad v_{xx} = 0, \quad v_t - av = 0. \quad (174)$$

We solve the first two Eq. (174) and obtain

$$v = p(t)x + q(t).$$

We substitute this in the third Eq. (174)

$$[p'(t) - ap(t)]x + q'(t) - aq(t) = 0$$

split it with respect to x and obtain $p'(t) - ap(t) = 0$, $q'(t) - aq(t) = 0$ whence

$$p(t) = be^{at}, \quad q(t) = le^{at}, \quad b, l = \text{const}.$$

Thus, Eq. (171) satisfying the condition Eq. (173) is nonlinearly self-adjoint, and the substitution (70) has the form

$$v = (bx + l)e^{at}. \quad (175)$$

One can obtain various nonlinearly self-adjoint equations (171) by considering other relations between $C(\psi)$, $K(\psi)$ and $S(\psi)$ different from (173).

1.7 Krichever–Novikov Equation

Let us consider the Krichever–Novikov equation [27] written in the form

$$F \equiv u_t - u_{xxx} + \frac{3}{2} \frac{u_{xx}^2}{u_x} - \frac{P(u)}{u_x} = 0 \quad (176)$$

where $P(u)$ a polynomial of degree four with distinct roots. The nonlinear self-adjointness of this equation has been investigated recently in [28]. Namely, it is shown that Eq. (176) satisfies the nonlinear self-adjointness condition in the form

$$F^*|_{v=\varphi} = \mu_0 F + \mu_1 D_x(F) + \mu_2 D_x^2(F) + \mu_3 D_x^3(F) + \mu_4 D_x^4(F) \quad (177)$$

with

$$\begin{aligned} \mu_0 &= 2 \frac{u_{xx}}{u_x^4} P'(u) - \frac{1}{u_x^2} P''(u) \\ \mu_1 &= 2 \frac{u_{xxxx}}{u_x^3} - 12 \frac{u_{xxx} u_{xx}}{u_x^4} + 12 \frac{u_{xx}^3}{u_x^5} + \frac{2}{u_x^3} P'(u) - 8 \frac{u_{xx}}{u_x^5} P(u) \\ \mu_2 &= 4 \frac{u_{xxx}}{u_x^3} - 9 \frac{u_{xx}^2}{u_x^4} + \frac{2}{u_x^4} P(u), \quad \mu_3 = 4 \frac{u_{xx}}{u_x^3}, \quad \mu_4 = -\frac{1}{u_x^2} \end{aligned}$$

where $v = \varphi$ is the differential substitution (76) with the following φ

$$\varphi = \frac{u_{xxxx}}{u_x^2} + 3 \frac{u_{xx}^3}{u_x^4} - 4 \frac{u_{xx} u_{xxx}}{u_x^3} - 2 \frac{P(u) u_{xx}}{u_x^4} + \frac{P'(u)}{u_x^2}. \quad (178)$$

2 Construction of Conservation Laws Using Symmetries

The well-known Noether's theorem [29] states that if a variational integral is invariant with respect to a one-parameter group of transformations then a certain formula (see further Theorem 3) provides a conservation law for the corresponding Euler–Lagrange equation. Thus, according to Noether's theorem, the invariance of the variational integral is a *sufficient condition* for existence of the conservation law. It has been proved in [30] that the *necessary and sufficient condition* for existence of this conservation law is the invariance of the extremal values (the values on the *solutions of the Euler–Lagrange equations*) of the variational integral. The result extends to multi-parameter symmetry groups.

The new approach based on the concept of nonlinear self-adjointness allows to extend a connection between symmetries and conservation laws significantly.

2.1 Discussion of the Operator Identity

2.1.1 Operator Identity and Alternative Proof of Noether's Theorem

Let us discuss some consequences of the operator identity⁴

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i \mathbf{N}^i. \quad (179)$$

Here

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots \quad (180)$$

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, \dots, m \quad (181)$$

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m \quad (182)$$

and

$$\mathbf{N}^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s=1}^{\infty} D_{i_1} \dots D_{i_s} (W^\alpha) \frac{\delta}{\delta u_{i i_1 \dots i_s}^\alpha}, \quad i = 1, \dots, n \quad (183)$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (182) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{j_1} \dots D_{j_s} \frac{\partial}{\partial u_{i j_1 \dots j_s}^\alpha}. \quad (184)$$

The coefficients ξ^i , η^α in (180) are arbitrary *differential functions* (see Sect. 1.1.1) and the other coefficients are determined by the prolongation formulae

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \quad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \dots \quad (185)$$

The derivation of Eq. (179) is essentially based on Eq. (185).

Remark 10 If we write the operator (180) in the equivalent canonical form⁵

⁴ The operator identity (179) was called in [31] the *Noether identity* and used for simplifying the proof of Noether's theorem. A simple proof of the identity (179) can be found in [32] where Eq. (179) is written as Eq. (19).

⁵ One can verify that if one uses the canonical form of the operator X , then the operator identity (179) becomes identical with Eqs. (3), (6) in Noether's paper [29] except for the notation. Noether

$$X = W^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots \tag{186}$$

then the prolongation formulae (185) become simpler

$$\zeta_i^\alpha = D_i(W^\alpha), \quad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(W^\alpha), \dots \tag{187}$$

The original proof of Noether’s theorem [29] is based on the calculus of variations. An alternative proof of this theorem has been given in [31] (see also [4, 10]). Let us outline the latter proof based on the identity (179).

We consider the Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1, \dots, m. \tag{188}$$

If we assume that the operator (180) is admitted by Eq. (188) and that the variational integral

$$\int \mathcal{L}(x, u, u_{(1)}, \dots) dx$$

is invariant under the transformations of the group with the generator X then the following equation holds

$$X(\mathcal{L}) + D_i(\xi^i)\mathcal{L} = 0. \tag{189}$$

Therefore, if we act on \mathcal{L} by both sides of the identity (179)

$$X(\mathcal{L}) + D_i(\xi^i)\mathcal{L} = W^\alpha \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i[\mathbf{N}^i(\mathcal{L})]$$

and take into account Eqs. (188) and (189), we see that the vector with the components

$$C^i = \mathbf{N}^i(\mathcal{L}), \quad i = 1, \dots, n \tag{190}$$

satisfies the conservation equation

$$D_i(C^i) \Big|_{(188)} = 0. \tag{191}$$

For practical applications, when we deal with low order Lagrangians \mathcal{L} , it is convenient to restrict the operator (183) on the derivatives involved in \mathcal{L} and write the expressions (190) in the expanded form

(Footnote 5 continued)
 comments that in the case of the first-order Lagrangians her Eq.(3) is identical with the *central equation of Lagrange* (Eqs. (4) and (5) in [29]).

$$\begin{aligned}
 C^i = \xi^i \mathcal{L} + W^\alpha & \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \dots \right] \\
 + D_j (W^\alpha) & \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] + D_j D_k (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots \right].
 \end{aligned}
 \tag{192}$$

Thus, Noether’s theorem can be formulated as follows.

Theorem 3 *If the operator (180) is admitted by Eq. (188) and satisfies the condition (189) of the invariance of the variational integral, then the vector (192) constructed by Eq. (192) satisfies the conservation law (191).*

Remark 11 The identity (179) is valid also in the case when the coefficients ξ^i, η^α of the operator X involve not only the local variables $x, u, u_{(1)}, u_{(2)}, \dots$ but also nonlocal variables (see Sect. 2.5.6). Accordingly, the formula (192) associates conserved vectors with nonlocal symmetries as well.

Remark 12 If the invariance condition (189) is replaced by the divergence condition

$$X(\mathcal{L}) + D_i(\xi^i)\mathcal{L} = D_i(B^i)$$

then the identity (179) leads to the conservation law (191) where the conserved vector (190) is replaced with

$$C^i = N^i(\mathcal{L}) - B^i, \quad i = 1, \dots, n.
 \tag{193}$$

2.1.2 Test for Total derivative and for Divergence

I recall here the well-known necessary and sufficient condition for a differential function to be divergence, or total derivative in the case of one independent variable.

One can easily derive from the Definition (1) of the total differentiation D_i the following lemmas (see also [4], Sect. 8.4.1).

Lemma 1 *The following infinite series of equations hold*

$$\begin{aligned}
 \frac{\partial}{\partial u^\alpha} D_i &= D_i \frac{\partial}{\partial u^\alpha} \\
 D_j \frac{\partial}{\partial u_j^\alpha} D_i &= D_i \frac{\partial}{\partial u^\alpha} + D_i D_j \frac{\partial}{\partial u_j^\alpha} \\
 D_j D_k \frac{\partial}{\partial u_{jk}^\alpha} D_i &= D_i D_k \frac{\partial}{\partial u_k^\alpha} + D_i D_j D_k \frac{\partial}{\partial u_{jk}^\alpha} \\
 &\dots\dots\dots
 \end{aligned}$$

Lemma 2 *The following operator identity holds for every i and α*

$$\frac{\delta}{\delta u^\alpha} D_i = 0.$$

Proof Using Lemma 1 and manipulating with summation indices we obtain

$$\begin{aligned} \frac{\delta}{\delta u^\alpha} D_i &= \left(\frac{\partial}{\partial u^\alpha} - D_j \frac{\partial}{\partial u_j^\alpha} + D_j D_k \frac{\partial}{\partial u_{jk}^\alpha} - D_j D_k D_l \frac{\partial}{\partial u_{jkl}^\alpha} + \dots \right) D_i \\ &= \frac{\partial}{\partial u^\alpha} D_i - D_i \frac{\partial}{\partial u^\alpha} - D_i D_j \frac{\partial}{\partial u_j^\alpha} + D_i D_k \frac{\partial}{\partial u_k^\alpha} + D_i D_j D_k \frac{\partial}{\partial u_{jk}^\alpha} \\ &\quad - D_i D_k D_l \frac{\partial}{\partial u_{kl}^\alpha} - \dots = 0. \end{aligned} \quad \square$$

Proposition 8 *A differential function $f(x, u, u_{(1)}, \dots, u_{(s)}) \in \mathcal{A}$ is divergence*

$$f = D_i(h^i), \quad h^i(x, u, \dots, u_{(s-1)}) \in \mathcal{A} \quad (194)$$

if and only if the following equations hold identically in $x, u, u_{(1)}, \dots$

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1, \dots, m. \quad (195)$$

The statement that the relation (194) implies (195) follows immediately from Lemma 2. For the proof of the inverse statement that (195) implies (194), see [33], Chap. 4, Sect. 3.5, and [32]. See also [4], Sect. 8.4.1.

We will use Proposition 8 also in the particular case of one independent variable x and one dependent variable $u = y$. Then it is formulated as follows.

Proposition 9 *A differential function $f(x, y, y', \dots, y^{(s)}) \in \mathcal{A}$ is the total derivative*

$$f = D_x(g), \quad g(x, y, y', \dots, y^{(s-1)}) \in \mathcal{A} \quad (196)$$

if and only if the following equation holds identically in x, y, y', \dots

$$\frac{\delta f}{\delta y} = 0. \quad (197)$$

Here $\delta f / \delta y$ is the Euler-Lagrange operator (184)

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'} + D_x^2 \frac{\partial}{\partial y''} - D_x^3 \frac{\partial}{\partial y'''} + \dots \quad (198)$$

2.1.3 Adjoint Equation to Linear ODE

Let us consider an arbitrary s th-order linear ordinary differential operator

$$L[y] = a_0 y^{(s)} + a_1 y^{(s-1)} + \cdots + a_{s-2} y'' + a_{s-1} y' + a_s y \quad (199)$$

where $a_i = a_i(x)$. We know from Sect. 1.1.5 that the adjoint operator to (199) can be calculated by using Eq. (8). I give here the independent proof based on the operator identity (179).

Proposition 10 *The adjoint operator to (199) can be calculated by the formula*

$$L^*[z] = \frac{\delta(zL[y])}{\delta y}. \quad (200)$$

Proof Let

$$X = w \frac{\partial}{\partial y} + w' \frac{\partial}{\partial y'} + w'' \frac{\partial}{\partial y''} + \cdots \quad (201)$$

be the operator (186) with one independent variable x and one dependent variable $u = y$, where the prolongation formulae (187) are written using the notation

$$w' = D_x(w), \quad w'' = D_x^2(w), \dots \quad (202)$$

In this notation the operator (183) is written

$$\mathbf{N} = w \frac{\delta}{\delta y'} + w' \frac{\delta}{\delta y''} + w'' \frac{\delta}{\delta y'''} + \cdots.$$

Having in mind its application to the differential function $L[y]$ given by (199) we consider the following restricted form of \mathbf{N}

$$\mathbf{N} = w \frac{\delta}{\delta y'} + w' \frac{\delta}{\delta y''} + \cdots + w^{(s-1)} \frac{\delta}{\delta y^{(s)}}. \quad (203)$$

The identity (179) has the form

$$X = w \frac{\delta}{\delta y} + D_x \mathbf{N}. \quad (204)$$

We act by both sides of this identity on $zL[y]$, where z is a new dependent variable:

$$X(zL[y]) = w \frac{\delta(zL[y])}{\delta y} + D_x \mathbf{N}(zL[y]). \quad (205)$$

Since the operator (201) does not act on the variables x and z , we have

$$X(zL[y]) = zX(L[y]). \tag{206}$$

Furthermore we note that

$$X(L[y]) = L[w]. \tag{207}$$

Inserting (206) and (207) in Eq. (205) we obtain

$$zL[w] - w \frac{\delta(zL[y])}{\delta y} = D_x(\Psi) \tag{208}$$

where Ψ is a quadratic form $\Psi = \Psi[w, z]$ defined by

$$\Psi = N(zL[y]). \tag{209}$$

After replacing w with y Eq. (208) coincides with Eq. (3) for the adjoint operator

$$zL[y] - yL^*[z] = D_x(\psi) \tag{210}$$

where $L^*[z]$ is given by the formula (200) and $\psi = \psi[y, z]$ is defined by

$$\psi[y, z] = \Psi[w, z]|_{w=y} \equiv N(zL[y])|_{w=y}. \tag{211}$$

□

Remark 13 Let us find the explicit formula for ψ in Eq. (210) We write the operator N given by Eq. (203) in the expanded form

$$\begin{aligned} N = & w \left[\frac{\partial}{\partial y'} - D_x \frac{\partial}{\partial y''} + \dots + (-D_x)^{s-1} \frac{\partial}{\partial y^{(s)}} \right] \\ & + w' \left[\frac{\partial}{\partial y''} - D_x \frac{\partial}{\partial y'''} + \dots + (-D_x)^{s-2} \frac{\partial}{\partial y^{(s)}} \right] \\ & + w^{(s-2)} \left[\frac{\partial}{\partial y^{(s-1)}} - D_x \frac{\partial}{\partial y^{(s)}} \right] + w^{(s-1)} \frac{\delta}{\delta y^{(s)}} \end{aligned}$$

act on $zL[y]$ written in the form

$$zL[y] = a_s y z + a_{s-1} y' z + a_{s-2} y'' z + \dots + a_1 y^{(s-1)} z + a_0 y^{(s)} z$$

and obtain Ψ . We replace w with y in $\Psi = \Psi[w, z]$ and $\psi = \psi[y, z]$

$$\begin{aligned} \psi[y, z] = & y \left[a_{s-1} z - (a_{s-2} z)' + \dots + (-1)^{s-1} (a_0 z)^{(s-1)} \right] \\ & + y' \left[a_{s-2} z - (a_{s-3} z)' + \dots + (-1)^{s-2} (a_0 z)^{(s-2)} \right] \\ & + y^{(s-2)} \left[a_1 z - (a_0 z)' \right] + y^{(s-1)} a_0 z. \end{aligned} \tag{212}$$

The expression (212) is obtained in the classical literature using integration by parts (see e.g., [34, Chap. 5, Sect. 4, Eq. (31')]).

2.1.4 Conservation Laws and Integrating Factors for Linear ODEs

Consider an s th-order homogeneous linear ordinary differential equation

$$L[y] = 0 \quad (213)$$

where $L[y]$ is the operator defined by Eq. (199). If $L[y]$ is a total derivative

$$L[y] = D_x \left(\psi(x, y, y', \dots, y^{(s-1)}) \right) \quad (214)$$

the Eq. (213) can be written as a conservation law

$$D_x \left(\psi(x, y, y', \dots, y^{(s-1)}) \right) = 0$$

whence upon integration one obtains a linear equation of order $s - 1$

$$\psi(x, y, y', \dots, y^{(s-1)}) = C_1. \quad (215)$$

We can also reduce the order of the non-homogeneous equation

$$L[y] = f(x) \quad (216)$$

by rewriting it in the conservation form

$$D_x \left[\psi(x, y, y', \dots, y^{(s-1)}) - \int f(x) dx \right] = 0. \quad (217)$$

Integrating it once we obtain the non-homogeneous linear equation of order $s - 1$

$$\psi(x, y, y', \dots, y^{(s-1)}) = C_1 + \int f(x) dx.$$

Example 15 Consider the second-order equation

$$y'' + y' \sin x + y \cos x = 0.$$

We have

$$y'' + y' \sin x + y \cos x = D_x(y' + y \sin x).$$

Therefore the second-order equation in question reduces to the first-order equation

$$y' + y \sin x = C_1.$$

Integrating the latter equation we obtain the general solution

$$y = \left[C_2 + C_1 \int e^{-\cos x} dx \right] e^{\cos x}$$

to our second-order equation. Dealing likewise with the non-homogeneous equation

$$y'' + y' \sin x + y \cos x = 2x$$

we obtain its general solution

$$y = \left[C_2 + \int (C_1 + x^2) e^{-\cos x} dx \right] e^{\cos x}.$$

If $L[y]$ in Eq. (213) is not a total derivative, one can find an appropriate factor $\phi(x) \neq 0$, called an *integrating factor*, such that $\phi(x)L[y]$ becomes a total derivative

$$\phi(x)L[y] = D_x \left(\psi(x, y, y', \dots, y^{(s-1)}) \right). \quad (218)$$

A connection between integrating factors and the adjoint equations for linear equations is well known in the classical literature (see e.g., [34], Chap. 5, Sect. 4). Proposition 9 gives a simple way to establish this connection and prove the following statement.

Proposition 11 *A function $\phi(x)$ is an integrating factor for Eq. (213) if and only if*

$$z = \phi(x), \quad \phi(x) \neq 0 \quad (219)$$

is a solution of the adjoint equation ⁶ to Eq. (213)

$$L^*[z] = 0. \quad (220)$$

Knowledge of a solution (219) to the adjoint equation (220) allows to reduce the order of Eq. (213) by integrating Eq. (218)

$$\psi(x, y, y', \dots, y^{(s-1)}) = C_1. \quad (221)$$

Here C_1 is an arbitrary constants and ψ defined according to Eqs. (209) and (210), i.e.,

$$\psi = \mathbf{N}(zL[y])|_{w=y}. \quad (222)$$

⁶ This statement is applicable to nonlinear ODEs as well, see [35].

Proof If (219) is a solution of the adjoint Eq. (220), we substitute it in Eq. (210) and arrive at Eq. (218). Hence $\phi(x)$ is an integrating factor for Eq. (213). Conversely, if $\phi(x)$ is an integrating factor for Eq. (213), then Eq. (218) is satisfied. Now Proposition 9 yields

$$\frac{\delta(\phi(x)L[y])}{\delta y} = 0.$$

Hence (219) is a solution of the adjoint Eq. (220). Finally, Eq. (222) follows from (210). \square

Example 16 Let us apply the above approach to the first-order equation

$$y' + P(x)y = Q(x). \quad (223)$$

Here $L[y] = y' + P(x)y$. The adjoint Eq. (220) is written

$$z' - P(x)z = 0.$$

Solving it we obtain the integrating factor

$$z = e^{\int P(x)dx}.$$

Therefore we rewrite Eq. (223) in the equivalent form

$$[y' + P(x)y]e^{\int P(x)dx} = Q(x)e^{\int P(x)dx} \quad (224)$$

and compute the function Ψ given by Eq. (209)

$$\Psi = \mathbf{N}(zL[y]) = w \frac{\partial}{\partial y'} [z(y' + P(x)y)] = wz = we^{\int P(x)dx}.$$

Equation (222) yields

$$\psi = ye^{\int P(x)dx}. \quad (225)$$

Now we can take (224) instead of Eq. (216) and write it in the form (217) with ψ given by (225). Then we obtain

$$D_x \left[ye^{\int P(x)dx} - \int Q(x)e^{\int P(x)dx} dx \right] = 0$$

whence

$$ye^{\int P(x)dx} = C_1 + \int Q(x)e^{\int P(x)dx} dx.$$

Solving the latter equation for y we obtain the general solution of Eq. (223)

$$y = \left[C_1 + \int Q(x)e^{\int P(x)dx} dx \right] e^{-\int P(x)dx}. \quad (226)$$

Example 17 Let us consider the second-order homogeneous equation

$$y'' + \frac{\sin x}{x^2} y' + \left(\frac{\cos x}{x^2} - \frac{\sin x}{x^3} \right) y = 0. \quad (227)$$

Its left-hand side does not satisfy the total derivative condition (197) because

$$\frac{\delta}{\delta y} \left[y'' + \frac{\sin x}{x^2} y' + \left(\frac{\cos x}{x^2} - \frac{\sin x}{x^3} \right) y \right] = \frac{\sin x}{x^2}.$$

Therefore we will apply Proposition 11. The adjoint equation to equation (227) is written

$$z'' - \frac{\sin x}{x^2} z' + \frac{\sin x}{x^3} z = 0.$$

We take its obvious solution $z = x$, substitute it in Eq. (209) and using (211) find

$$\Psi = \mathbf{N} \left[xy'' + \frac{\sin x}{x} y' + \left(\frac{\cos x}{x} - \frac{\sin x}{x^2} \right) y \right] = \frac{\sin x}{x} w - w + xw'.$$

Therefore Eq. (221) is written

$$xy' + \left(\frac{\sin x}{x} - 1 \right) y = C_1.$$

Integrating this first-order linear equation we obtain the general solution of Eq. (227)

$$y = \left(C_2 + C_1 \int \frac{1}{x^2} e^{\int \frac{\sin x}{x^2} dx} dx \right) x e^{-\int \frac{\sin x}{x^2} dx}. \quad (228)$$

2.1.5 Application of the Operator Identity to Linear PDEs

Using the operator identity (179) one can easily extend the Eqs. (210) and (211) for linear ODEs to linear partial differential equations and systems. Let us consider the second-order linear operator

$$L[u] = a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u \quad (229)$$

considered in Sect. 1.1.5, Remark 1. The adjoint operator is

$$L^*[v] \equiv \frac{\delta(vF[u])}{\delta u} = D_i D_j (a^{ij} v) - D_i (b^i v) + cv. \quad (230)$$

Let us take the operator identity (179)

$$X = W \frac{\delta}{\delta u} + D_i N^i \quad (231)$$

where X is the operator (186) with one dependent variable u

$$X = W \frac{\partial}{\partial u} + W_i \frac{\partial}{\partial u_i} + W_{ij} \frac{\partial}{\partial u_{ij}}$$

and N^i are the operators (183)

$$N^i = W \frac{\delta}{\delta u_i} + W_j \frac{\delta}{\delta u_{ij}} = W \left[\frac{\partial}{\partial u_i} - D_j \frac{\partial}{\partial u_{ij}} \right] + W_j \frac{\partial}{\partial u_{ij}}.$$

We use above the notation $W_i = D_i(W)$, $W_{ij} = D_i D_j(W)$. Now we proceed as in Sect. 2.1.3. Namely, we act on $vL[u]$ by both sides of the identity (231)

$$X(vL[u]) = W \frac{\delta(vL[u])}{\delta u} + D_i N^i(vL[u])$$

take into account that X does not act on the variables x^i , v , and that $X(L[u]) = L[W]$, use Eq. (230) and obtain

$$vL[W] - WL^*[v] = D_i N^i(vL[u]).$$

Letting here $W = u$ we arrive at the following generalization of Eq. (210)

$$vL[u] - uL^*[v] = D_i(\psi^i) \quad (232)$$

where ψ^i are defined as in (211) and (212)

$$\psi^i = N^i(vL[u])|_{W=u} \equiv a^{ij}(x)[vu_i - uv_i] + [b^i(x) - D_i(a^{ij}(x))]uv. \quad (233)$$

Equation (232) with ψ^i given by (233) is called *Lagrange's identity* (see [36], p. 80).

2.1.6 Application of the Operator Identity to Nonlinear Equations

Let us apply the constructions of Sect. 2.1.5 to nonlinear equation (6)

$$F_\alpha(x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad \alpha = 1, \dots, m. \tag{234}$$

We write the operator (186) in the form

$$X = W^\alpha \frac{\partial}{\partial u^\alpha} + W_i^\alpha \frac{\partial}{\partial u_i^\alpha} + W_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \dots$$

where $W_i^\alpha = D_i(W^\alpha)$, $W_{ij}^\alpha = D_i D_j(W^\alpha)$, \dots . Then the operator (183) is written

$$N^i = W_j^\alpha \frac{\delta}{\delta u_i^\alpha} + W^\alpha \frac{\delta}{\delta u_{ij}^\alpha} + \dots$$

We act on $v^\beta F_\beta$ by both sides of the operator identity (179)

$$X = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i$$

denote by $F_\alpha^*[v]$ the adjoint operator defined by Eq. (8) and obtain

$$v^\beta \hat{F}_\beta[W] - W^\alpha F_\alpha^*[v] = D_i(\Psi^i) \tag{235}$$

where

$$\Psi^i = N^i(v^\beta F_\beta)$$

and $\hat{F}_\beta[W]$ is the *linear approximation* to F_β defined by (see also Sect. 1.1.3)

$$\hat{F}_\beta[W] = X(F_\beta) \equiv W^\alpha \frac{\partial F_\beta}{\partial u^\alpha} + W_i^\alpha \frac{\partial F_\beta}{\partial u_i^\alpha} + W_{ij}^\alpha \frac{\partial F_\beta}{\partial u_{ij}^\alpha} + \dots$$

Remark 14 Equation (235) shows that $F_\alpha^*[v] = \hat{F}_\beta^*[W]$, i.e., the adjoint operator F_α^* to nonlinear equation (234) is the usual adjoint operator \hat{F}_β^* to the linear operator $\hat{F}_\beta[W]$ (see also [37]). But the linear self-adjointness of $\hat{F}_\beta[W]$ is not identical with the nonlinear self-adjointness of Eq. (234). For example, the KdV equation $F \equiv u_t - u_{xxx} - uu_x = 0$ is nonlinearly self-adjoint (see Example 2 in Sect. 1.1.6). But its linear approximation $\hat{F}[W] = W_t - W_{xxx} - uW_x - Wu_x$ is not a self-adjoint linear operator. Moreover, *all linear equations are nonlinearly self-adjoint*.

2.2 Conservation Laws: Generalities and Explicit Formula

2.2.1 Preliminaries

Let us consider a system of \bar{m} differential equations

$$F_{\bar{\alpha}}(x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad \bar{\alpha} = 1, \dots, \bar{m} \quad (236)$$

with m dependent variables u^1, \dots, u^m and n independent variables x^1, \dots, x^n .

A conservation law for Eq. (236) is written

$$\left[D_i(C^i) \right]_{(236)} = 0. \quad (237)$$

The subscript $|_{(236)}$ means that the left-hand side of (237) is restricted on the solutions of Eq. (236). In practical calculations this restriction can be achieved by solving Eq. (236) with respect to certain derivatives of u and eliminating these derivatives from the left-hand side of (237). For example, if (236) is an evolution equation

$$u_t = \Phi(t, x, u, u_x, u_{xx})$$

the restriction $|_{(236)}$ can be understood as the elimination of u_t . The n -dimensional vector

$$C = (C^1, \dots, C^n) \quad (238)$$

satisfying Eq. (237) is called a *conserved vector* for the system (236). If its components are functions $C^i = C^i(x, u, u_{(1)}, \dots)$ of x, u and derivatives $u_{(1)}, \dots$ of a finite order, the conserved vector (238) is called a *local conserved vector*.

Since the conservation equation (237) is linear with respect to C^i , any linear combination with constant coefficients of a finite number of conserved vectors is again a conserved vector. It is obvious that if the divergence of a vector (238) vanishes identically, it is a conserved vector for any system of differential equations. This is a *trivial* conserved vectors for all differential equations. Another type of *trivial conserved vectors* for Eq. (236) are provided by those vectors whose components C^i vanish on the solutions of the system (236). One ignores both types of trivial conserved vectors. In other words, conserved vectors (238) are simplified by considering them up to addition of these trivial conserved vectors.

The following less trivial operation with conserved vectors is particularly useful in practice. Let

$$C^1|_{(236)} = \tilde{C}^1 + D_2(H^2) + \dots + D_n(H^n) \quad (239)$$

the conserved vector (238) can be replaced with the equivalent conserved vector

$$\tilde{C} = (\tilde{C}^1, \tilde{C}^2, \dots, \tilde{C}^n) = 0 \quad (240)$$

with the components

$$\tilde{C}^1, \quad \tilde{C}^2 = C^2 + D_1(H^2), \quad \dots, \quad \tilde{C}^n = C^n + D_1(H^n). \quad (241)$$

The passage from (238) to the vector (240) is based on the commutativity of the total differentiations. Namely, we have

$$D_1 D_2(H^2) = D_2 D_1(H^2), \quad D_1 D_n(H^n) = D_n D_1(H^n)$$

and therefore the conservation Eq. (237) for the vector (238) is equivalent to the conservation equation

$$\left[D_i(\tilde{C}^i) \right]_{(236)} = 0$$

for the vector (240). If $n \geq 3$, the simplification (241) of the conserved vector can be iterated: if \tilde{C}^2 contains the terms

$$D_3(\tilde{H}^3) + \dots + D_n(\tilde{H}^n)$$

one can subtract them from \tilde{C}^2 and add to $\tilde{C}^3, \dots, \tilde{C}^n$ the corresponding terms

$$D_2(\tilde{H}^3), \dots, D_2(\tilde{H}^n).$$

Note that the conservation law (237) for equations (236) can be written in the form

$$D_i(C^i) = \mu^{\bar{\alpha}} F_{\bar{\alpha}}(x, u, u_{(1)}, \dots, u_{(s)}) \quad (242)$$

with undetermined coefficients $\mu^{\bar{\alpha}} = \mu^{\bar{\alpha}}(x, u, u_{(1)}, \dots)$ depending on a finite number of variables $x, u, u_{(1)}, \dots$. If C^i depend on higher-order derivatives, Eq. (242) is replaced with

$$D_i(C^i) = \mu^{\bar{\alpha}} F_{\bar{\alpha}} + \mu^{i\bar{\alpha}} D_i(F_{\bar{\alpha}}) + \mu^{ij\bar{\alpha}} D_i D_j(F_{\bar{\alpha}}) + \dots \quad (243)$$

It is manifest from Eq. (242) or Eq. (243) that the total differentiations of a conserved vector (238) provide again conserved vectors. Therefore, e.g., the vector

$$D_1(C) = (D_1(C^1), \dots, D_1(C^n)) \quad (244)$$

obtained from the known vector (238) is not considered as a new conserved vector.

If one of the independent variables is time, e.g., $x^1 = t$, then the conservation Eq. (237) is often written, using the divergence theorem, in the integral form

$$\frac{d}{dt} \int_{\mathbb{R}^{n-1}} C^1 dx^2 \cdots dx^n = 0. \quad (245)$$

But the differential form (237) of conservation laws carries, in general, more information than the integral form (245). Using the integral form (245) one may even lose some nontrivial conservation laws. As an example, consider the two-dimensional Boussinesq equations

$$\begin{aligned} \Delta\psi_t - g\rho_x - f v_z &= \psi_x \Delta\psi_z - \psi_z \Delta\psi_x \\ v_t + f\psi_z &= \psi_x v_z - \psi_z v_x \\ \rho_t + \frac{N^2}{g} \psi_x &= \psi_x \rho_z - \psi_z \rho_x \end{aligned} \quad (246)$$

used in geophysical fluid dynamics for investigating uniformly stratified incompressible fluid flows in the ocean. Here Δ is the two-dimensional Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

and ψ is the stream function so that the x, z -components u, w of the velocity (u, v, w) of the fluid are given by

$$u = \psi_z, \quad w = -\psi_x. \quad (247)$$

Equation (246) involve the physical constants: g is the gravitational acceleration, f is the Coriolis parameter, and N is responsible for the density stratification of the fluid. Each equation of the system (246) has the conservation form (237), namely

$$\begin{aligned} D_t(\Delta\psi) + D_x(-g\rho + \psi_z \Delta\psi) + D_z(-f v - \psi_x \Delta\psi) &= 0 \\ D_t(v) + D_x(v\psi_z) + D_z(f\psi - v\psi_x) &= 0 \\ D_t(\rho) + D_x\left(\frac{N^2}{g} \psi + \rho\psi_z\right) + D_z(-\rho\psi_x) &= 0. \end{aligned} \quad (248)$$

In the integral form (245) these conservation laws are written

$$\frac{d}{dt} \iint \Delta\psi dx dz = 0, \quad \frac{d}{dt} \iint v dx dz = 0, \quad \frac{d}{dt} \iint \rho dx dz = 0. \quad (249)$$

We can rewrite the differential conservation equations (248) in an equivalent form by using the operations (239)–(241) of the conserved vectors. Namely, let us apply these operations to the first Eq. (248), i.e., to the conserved vector

$$C^1 = \Delta\psi, \quad C^2 = -g\rho + \psi_z \Delta\psi, \quad C^3 = -f v - \psi_x \Delta\psi. \quad (250)$$

Noting that

$$C^1 = D_x(\psi_x) + D_z(\psi_z)$$

and using the operations (239)–(241) we transform the vector (250) to the form

$$\tilde{C}^1 = 0, \quad \tilde{C}^2 = -g\rho + \psi_{tx} + \psi_z \Delta\psi, \quad \tilde{C}^3 = -fv + \psi_{tz} - \psi_x \Delta\psi. \quad (251)$$

The integral conservation equation (245) for the vector in (251) is trivial, $0 = 0$. Thus, after the transformation of the conserved vector (250) to the equivalent form (251) we have lost the first integral conservation law in (249). But it does not mean that the conserved vector (251) has no physical significance. Indeed, if we write the differential conservation equation with the vector (251), we again obtain the first equation of the system (246)

$$D_x(\tilde{C}^2) + D_z(\tilde{C}^3) = \Delta\psi_t - g\rho_x - fv_z - \psi_x \Delta\psi_z + \psi_z \Delta\psi_x.$$

Let us assume that Eq.(236) have a nontrivial local conserved vector satisfying Eq.(242). Then not all $\mu^{\tilde{\beta}}$ vanish simultaneously due to non-triviality of the conserved vector. Furthermore, since $\mu^{\tilde{\beta}} F_{\tilde{\beta}}$ depends on x, u and a finite number of derivatives $u_{(1)}, u_{(2)}, \dots$ (i.e., it is a *differential function*) and has a divergence form, the following equations hold (for a detailed discussion see [4], Sect. 8.4.1)

$$\frac{\delta}{\delta\alpha} \left[\mu^{\tilde{\beta}} F_{\tilde{\beta}}(x, u, u_{(1)}, \dots, u_{(s)}) \right] = 0, \quad \alpha = 1, \dots, m. \quad (252)$$

Note that Eq.(252) are identical with Eq.(69) where the differential substitution (76) is made with $\varphi^{\tilde{\alpha}} = \mu^{\tilde{\alpha}}$. Hence, the system (236) is nonlinearly self-adjoint. I formulate this simple observation as a theorem since it is useful in applications (see Sect. 2.5).

Theorem 4 *Any system of differential equation (236) having a nontrivial local conserved vector satisfying Eq.(242) is nonlinearly self-adjoint.*

2.2.2 Explicit Formula for Conserved Vectors

Using Definition 4 of nonlinear self-adjointness and the theorem on conservation laws proved in [1] by using the operator identity (179), we obtain the explicit formula for constructing conservation laws associated with symmetries of any nonlinearly self-adjoint system of equations. The method is applicable independently on the number of equations in the system and the number of dependent variables. The result is as follows.

Theorem 5 *Let the system of differential equation (236) be nonlinearly self-adjoint. Specifically, let the adjoint system (69)–(236) be satisfied for all solutions of Eq. (236) upon a substitution (70)*

$$v^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x, u), \quad \bar{\alpha} = 1, \dots, \bar{m}. \quad (253)$$

Then any Lie point, contact or Lie-Bäcklund symmetry

$$X = \xi^i(x, u, u_{(1)}, \dots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \dots) \frac{\partial}{\partial u^\alpha} \quad (254)$$

as well as a nonlocal symmetry of Eq. (236) leads to a conservation law (237) constructed by the following formula

$$\begin{aligned} C^i = & \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \dots \right] \\ & + D_j (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] + D_j D_k (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots \right] \end{aligned} \quad (255)$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha \quad (256)$$

and \mathcal{L} is the formal Lagrangian for the system (236)

$$\mathcal{L} = v^{\bar{\beta}} F_{\bar{\beta}}. \quad (257)$$

In (255) the formal Lagrangian \mathcal{L} should be written in the symmetric form with respect to all mixed derivatives u_{ij}^α , u_{ijk}^α , ... and the “non-physical variables” $v^{\bar{\alpha}}$ should be eliminated via Eq. (253).

One can omit in (255) the term $\xi^i \mathcal{L}$ when it is convenient. This term provides a trivial conserved vector mentioned in Sect. 2.2.1 because \mathcal{L} vanishes on the solutions of Eq. (236). Thus, the conserved vector (255) can be taken in the following form

$$\begin{aligned} C^i = & W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \dots \right] \\ & + D_j (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] + D_j D_k (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots \right]. \end{aligned} \quad (258)$$

Remark 15 One can use Eq. (258) for constructing conserved vectors even if the system (236) is not self-adjoint, in particular, if one cannot find explicit formulae (253) or (76) for point or differential substitutions, respectively. The resulting conserved vectors will be nonlocal in the sense that they involve the variables v connected with the physical variables u via differential equations, namely, adjoint equations to (236).

Remark 16 Theorem 5, unlike Nother's theorem 3, does not require additional restrictions such as the invariance condition (189) or the divergence condition mentioned in Remark 12.

2.3 A Nonlinearly Self-Adjoint Irrigation System

Let us apply Theorem 5 to Eq. (171) satisfying the condition (173)

$$C(\psi)\psi_t = [K(\psi)\psi_x]_x + [K(\psi)(\psi_z - 1)]_z - S(\psi) \quad (259)$$

$$S'(\psi) = aC(\psi), \quad a = \text{const.} \quad (260)$$

The formal Lagrangian (257) for Eq. (259) has the form

$$\mathcal{L} = \left[-C(\psi)\psi_t + K(\psi)(\psi_{xx} + \psi_{zz}) + K'(\psi)(\psi_x^2 + \psi_z^2 - \psi_z) - S(\psi) \right] v. \quad (261)$$

We will use the substitution (175) of the particular form

$$v = e^{at}. \quad (262)$$

Denoting $t = x^1, x = x^2, z = x^3$ we write the conservation equation (237) in the form

$$D_t(C^1) + D_x(C^2) + D_z(C^3) = 0. \quad (263)$$

This equation should be satisfied on the solutions of Eq. (259).

The formal Lagrangian (261) does not contain derivatives of order higher than two. Therefore in our case Eq. (258) take the simple form

$$C^i = W \left[\frac{\partial \mathcal{L}}{\partial \psi_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial \psi_{ij}} \right) \right] + D_j(W) \frac{\partial \mathcal{L}}{\partial \psi_{ij}} \quad (264)$$

and yield

$$\begin{aligned} C^1 &= W \frac{\partial \mathcal{L}}{\partial \psi_t} \\ C^2 &= W \left[\frac{\partial \mathcal{L}}{\partial \psi_x} - D_x \left(\frac{\partial \mathcal{L}}{\partial \psi_{xx}} \right) \right] + D_x(W) \frac{\partial \mathcal{L}}{\partial \psi_{xx}} \\ C^3 &= W \left[\frac{\partial \mathcal{L}}{\partial \psi_z} - D_z \left(\frac{\partial \mathcal{L}}{\partial \psi_{zz}} \right) \right] + D_z(W) \frac{\partial \mathcal{L}}{\partial \psi_{zz}}. \end{aligned}$$

Substituting here the expression (261) for \mathcal{L} we obtain

$$\begin{aligned}
C^1 &= -WC(\psi)v \\
C^2 &= W[2K'(\psi)v\psi_x - D_x(K(\psi)v)] + D_x(W)K(\psi)v \\
C^3 &= W[K'(\psi)v(2\psi_z - 1) - D_z(K(\psi)v)] + D_z(W)K(\psi)v
\end{aligned}$$

where v should be eliminated by means of the substitution (262). So, we have

$$\begin{aligned}
C^1 &= -WC(\psi)e^{at} \\
C^2 &= [WK'(\psi)\psi_x + D_x(W)K(\psi)]e^{at} \\
C^3 &= [WK'(\psi)(\psi_z - 1) + D_z(W)K(\psi)]e^{at}.
\end{aligned} \tag{265}$$

Since Eq. (259) does not explicitly involve the independent variables t, x, z , it is invariant under the translations of these variables. Let us construct the conserved vector (265) corresponding to the time translation group with the generator

$$X = \frac{\partial}{\partial t}. \tag{266}$$

For this operator Eq. (256) yields

$$W = -\psi_t. \tag{267}$$

Substituting (267) in Eq. (265) we obtain

$$\begin{aligned}
C^1 &= C(\psi)\psi_t e^{at} \\
C^2 &= -[K'(\psi)\psi_t\psi_x + K(\psi)\psi_{tx}]e^{at} \\
C^3 &= -[K'(\psi)\psi_t(\psi_z - 1) + K(\psi)\psi_{tz}]e^{at}.
\end{aligned} \tag{268}$$

Now we replace in C^1 the term $C(\psi)\psi_t$ by the right-hand side of Eq. (259) to obtain

$$C^1 = -S(\psi)e^{at} + D_x(K(\psi)\psi_x e^{at}) + D_z(K(\psi)(\psi_z - 1)e^{at}).$$

When we substitute this expression in the conservation Eq. (263), we can write

$$D_t(D_x(K(\psi)\psi_x e^{at})) = D_x(D_t(K(\psi)\psi_x e^{at})).$$

Therefore we can transfer the terms $D_x(\dots)$ and $D_z(\dots)$ from C^1 to C^2 and C^3 , respectively (see 241). Thus, we rewrite the vector (268), changing its sign, as follows

$$\begin{aligned}
C^1 &= S(\psi)e^{at} \\
C^2 &= [K'(\psi)\psi_t\psi_x + K(\psi)\psi_{tx}]e^{at} - D_t(K(\psi)\psi_x e^{at}) \\
C^3 &= [K'(\psi)\psi_t(\psi_z - 1) + K(\psi)\psi_{tz}]e^{at} - D_t(K(\psi)(\psi_z - 1)e^{at}).
\end{aligned}$$

Working out the differentiation D_t in the last terms of C^2 and C^3 we finally arrive at the following vector

$$C^1 = S(\psi)e^{at}, \quad C^2 = aK(\psi)\psi_x e^{at}, \quad C^3 = aK(\psi)(\psi_z - 1)e^{at}. \quad (269)$$

The reckoning shows that the vector (269) satisfies the conservation equation (263) due to the condition (260). Note that C^1 is the *density* of the conserved vector (269).

The use of the general substitution (175) instead of its particular case (262) leads to the conserved vector with the density

$$C^1 = S(\psi)(bx + l)e^{at}.$$

This approach opens a new possibility to find a variety of conservation laws for the irrigation model (171) by considering other self-adjoint cases of the model and using the extensions of symmetry Lie algebras (see [15–17], vol. 2, Sect. 9.8).

2.4 Utilization of Differential Substitutions

2.4.1 Equation $u_{xy} = \sin u$

We return to Sect. 1.3.2 and calculate the conservation laws for Eq. (78)

$$u_{xy} = \sin u \quad (270)$$

using the differential substitution (79)

$$v = A_1[xu_x - yu_y] + A_2u_x + A_3u_y \quad (271)$$

and the admitted three-dimensional Lie algebra with the basis

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \quad (272)$$

The conservation equation for Eq. (270) will be written in the form

$$D_x(C^1) + D_y(C^2) = 0.$$

We write the formal Lagrangian for Eq. (270) in the symmetric form

$$\mathcal{L} = \left(\frac{1}{2} u_{xy} + \frac{1}{2} u_{yx} - \sin u \right) v. \quad (273)$$

Equation (258) yield

$$C^1 = \frac{1}{2} D_y(W)v - \frac{1}{2} Wv_y, \quad C^2 = \frac{1}{2} D_x(W)v - \frac{1}{2} Wv_x \quad (274)$$

where we have to eliminate the variable v via the differential substitution (271).

Substituting in (274) $W = -u_x$ corresponding to the operator X_1 from (272), replacing v with (271) and u_{xy} with $\sin u$, then transferring the terms of the form $D_y(\dots)$ from C^1 to C^2 (see the simplification (241)) we obtain

$$C^1 = A_1 \cos u, \quad C^2 = \frac{1}{2} A_1 u_x^2.$$

We let $A_1 = 1$ and conclude that the application of Theorem 5 to the symmetry X_1 yields the conserved vector

$$C^1 = \cos u, \quad C^2 = \frac{1}{2} u_x^2. \quad (275)$$

The similar calculations with the operator X_2 from (272) lead to the conserved vector

$$C^1 = \frac{1}{2} u_y^2, \quad C^2 = \cos u. \quad (276)$$

The third symmetry, X_3 from (272), does not lead to a new conserved vector. Indeed, in this case $W = yu_y - xu_x$. Substituting it in the first formula (274) we obtain after simple calculations

$$C^1 = \frac{1}{2} A_3 u_y^2 - A_2 \cos u + D_y \left[(A_2 y + A_3 x) \left(\frac{1}{2} u_x u_y + \cos u \right) \right].$$

Hence, upon transferring the term $D_y(\dots)$ from C^1 to C^2 the resulting C^1 will be a linear combination with constant coefficients of the components C^1 of the conserved vectors (275) and (276). The same will be true for C^2 . Therefore the conserved vector provided by the symmetry X_3 will be a linear combination with constant coefficients of the conserved vectors (275) and (276).

One can also use the Noether theorem because Eq.(270) has the classical Lagrangian, namely

$$L = -\frac{1}{2} u_x u_y + \cos u. \quad (277)$$

Then the symmetries X_1 and X_2 provide again the conserved vectors (275) and (276), respectively. But now we obtain one more conserved vector using X_3 , namely

$$C^1 = x \cos u - \frac{y}{2} u_y^2, \quad C^2 = \frac{x}{2} u_x^2 - y \cos u. \quad (278)$$

2.4.2 Short Pulse Equation

The differential equation (up to notation and appropriate scaling the physical variables)

$$D_t D_x(u) = u + \frac{1}{6} D_x^2(u^3) \quad (279)$$

was suggested in [38] (see there Eq.(11), also [39]) as a mathematical model for the propagation of ultra-short light pulses in media with nonlinearities, e.g., in silica fibers. The mathematical model is derived in [38] by considering the propagation of linearly polarized light in a one-dimensional medium and assuming that the light propagates in the infrared range. The final step in construction of the model is based on the method of multiple scales.

Equation (279) is connected with Eq. (270) by a non-point transformation which is constructed in [40] as a chain of differential substitutions (given also in [39] by Eq. (2)). Using this connection, an exact solitary wave solution (a *pulse solution*) to Eq. (279) is constructed in [39]. One can also find in [40] a Lax pair and a recursion operator for Eq. (279).

Note that Eq. (279) does not have a conservation form. I will find a conservation law of equation (279) thus showing that it can be rewritten in a conservation form. A significance of this possibility is commonly known and is not discussed here.

We write the *short pulse Eq. (279)* in the expanded form

$$u_{xt} = u + \frac{1}{2} u^2 u_{xx} + uu_x^2 \quad (280)$$

so that the formal Lagrangian is written

$$\mathcal{L} = v \left[u_{xt} - u - \frac{1}{2} u^2 u_{xx} - uu_x^2 \right]. \quad (281)$$

Substituting (281) in (69) we obtain the following *adjoint equation* to equation (280)

$$v_{xt} = v + \frac{1}{2} u^2 v_{xx}. \quad (282)$$

We first demonstrate the following statement.

Proposition 12 *Equation (279) is not nonlinearly self-adjoint with a substitution*

$$v = \varphi(t, x, u) \quad (283)$$

but it is nonlinearly self-adjoint with the differential substitution

$$v = u_t - \frac{1}{2} u^2 u_x. \quad (284)$$

Proof We write the nonlinear self-adjointness condition (72)

$$\left[v_{xt} - v - \frac{1}{2} u^2 v_{xx} \right]_{(283)} = \lambda [u_{xt} - u - \frac{1}{2} u^2 u_{xx} - uu_x^2]$$

substitute here the expression (283) for v and its derivatives

$$\begin{aligned} v_{xx} &= \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx} \\ v_{xt} &= \varphi_u u_{xt} + \varphi_{uu} u_x u_t + \varphi_{xu} u_t + \varphi_{tu} u_x + \varphi_{xt} \end{aligned} \quad (285)$$

and first obtain $\lambda = \varphi_u$ by comparing the terms with the second-order derivatives of u . This reduces the nonlinear self-adjointness condition to the following equation

$$\begin{aligned} \varphi_{uu} u_x u_t + \varphi_{xu} u_t + \varphi_{tu} u_x + \varphi_{xt} - \varphi + \varphi_{xx} \\ - \frac{1}{2} u^2 (\varphi_{uu} u_x^2 + 2\varphi_{xu} u_x = -\varphi_u [u + uu_x^2]. \end{aligned} \quad (286)$$

The terms with u_t in Eq. (286) yield $\varphi_{uu} = \varphi_{xu} = 0$. Then we take the term with u_x^2 and obtain $\varphi_u = 0$. Hence

$$\varphi = a(t, x).$$

Now Eq. (286) gives $a_{xx} = 0$, $a_{xt} - a = 0$, whence $a = 0$. Thus

$$\varphi = 0$$

i.e., the substitution (283) is trivial. This proves the first part of Proposition 12. Its second part is proved by similar calculations with the substitution

$$v = \varphi(t, x, u, u_x, u_t). \quad \square$$

I will not reproduce these rather lengthy calculations, but instead we will verify that the substitution (284) maps any solution of Eq. (270) into a solution of the adjoint Eq. (282). First we calculate

$$v_x = u_{xt} - \frac{1}{2} u^2 u_{xx} - uu_x^2$$

and see that on the solutions of Eq. (270) we have $v_x = u$. Now we calculate other derivatives and verify that on the solutions of Eq. (270) the following equations hold

$$v_x = u, \quad v_t = u_{tt} - \frac{1}{2} u^2 u_{xt} - uu_x u_t, \quad v_{xt} = u_t, \quad v_{xx} = u_x. \quad (287)$$

It is easily seen that Eq. (282) is satisfied. Namely, using (284) and (287) we have

$$v_{xt} - v - \frac{1}{2} u^2 v_{xx} = u_t - \left(u_t - \frac{1}{2} u^2 u_x \right) - \frac{1}{2} u^2 u_x = 0.$$

The maximal Lie algebra of point symmetries of Eq. (279) is the three-dimensional algebra spanned by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = u \frac{\partial}{\partial u} + x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t}. \quad (288)$$

Let us construct the conservation laws

$$D_t(C^1) + D_x(C^2) = 0 \quad (289)$$

for the basis operators (288).

Since the formal Lagrangian (281) does not contain derivatives of order higher than two, Eq. (258) are written (see 264)

$$C^i = W \left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}} \right) \right] + D_j(W) \frac{\partial \mathcal{L}}{\partial u_{ij}}.$$

In our case we have

$$\begin{aligned} C^1 &= -W D_x \left(\frac{\partial \mathcal{L}}{\partial u_{tx}} \right) + D_x(W) \frac{\partial \mathcal{L}}{\partial u_{tx}} \\ C^2 &= W \left[\frac{\partial \mathcal{L}}{\partial u_x} - D_t \left(\frac{\partial \mathcal{L}}{\partial u_{xt}} \right) - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \right] \\ &\quad + D_t(W) \frac{\partial \mathcal{L}}{\partial u_{xt}} + D_x(W) \frac{\partial \mathcal{L}}{\partial u_{xx}}. \end{aligned} \quad (290)$$

Substituting in (290) the expression (281) for \mathcal{L} written in the symmetric form

$$\mathcal{L} = v \left[\frac{1}{2} u_{tx} + \frac{1}{2} u_{xt} - u - \frac{1}{2} u^2 u_{xx} - u u_x^2 \right] \quad (291)$$

we obtain

$$\begin{aligned} C^1 &= -\frac{1}{2} W v_x + \frac{1}{2} v D_x(W) \\ C^2 &= -W \left[u v u_x + \frac{1}{2} v_t - \frac{1}{2} u^2 v_x \right] + \frac{1}{2} v D_t(W) - \frac{1}{2} u^2 v D_x(W). \end{aligned} \quad (292)$$

Since v should be eliminated via the differential substitution (284), we further simplify this vector by replacing v_x with u according to the first Eq. (287) and obtain

$$\begin{aligned}
C^1 &= -\frac{1}{2}Wu + \frac{1}{2}vD_x(W) \\
C^2 &= -W\left[uvu_x + \frac{1}{2}v_t - \frac{1}{2}u^3\right] + \frac{1}{2}vD_t(W) - \frac{1}{2}u^2vD_x(W)
\end{aligned} \tag{293}$$

where v and v_t should be replaced with their values given in Eqs. (284) and (287).

Let us construct the conserved vectors using the symmetries (288). Their commutators are

$$[X_1, X_3] = -X_1, \quad [X_2, X_3] = X_2.$$

Hence, according to [10], Sect.22.4, the operator X_3 plays a distinguished role. Namely, the conserved vectors associated with X_1 and X_2 can be obtained from the conserved vector provided by X_3 using the adjoint actions of the operators X_1 and X_2 , respectively. Therefore we start with X_3 . Substituting in (293) the expression

$$W = u + tu_t - xu_x$$

corresponding to the symmetry X_3 , eliminating the terms of the form $D_x(A)$ from C^1 and adding them to C^2 in the form $D_t(A)$ according to the simplification (241), we obtain after routine calculations the following conserved vector

$$C^1 = u^2, \quad C^2 = u^2u_xu_t - u_t^2 - \frac{1}{4}u^4 - \frac{1}{4}u^4u_x^2. \tag{294}$$

The conservation Eq. (289) for the vector (294) holds in the form

$$D_t(C^1) + D_x(C^2) = 2\left(u_t - \frac{1}{2}u^2u_x\right)\left(u + \frac{1}{2}u^2u_{xx} + uu_x^2 - u_{xt}\right). \tag{295}$$

Let us turn now to the operators X_1 and X_2 from (288). To simplify the calculations it is useful to modify Eq. (293) as follows. Noting that

$$vD_x(W) = D_x(vW) - Wv_x$$

we rewrite the vector (292) in the form

$$C^1 = -Wv_x, \quad C^2 = -W\left[uvu_x - \frac{1}{2}u^2v_x\right] + vD_t(W) - \frac{1}{2}u^2vD_x(W).$$

Then (293) is replaced with

$$\begin{aligned}
C^1 &= -uW \\
C^2 &= -W\left[uvu_x - \frac{1}{2}u^3\right] + vD_t(W) - \frac{1}{2}u^2vD_x(W).
\end{aligned} \tag{296}$$

Substituting in the first formula (296) to expression $W = -u_t$ corresponding the operator X_1 we obtain $C^1 = uu_t$. This is the time derivative of C^1 from (294). Hence the symmetry X_1 leads to a trivial conserved vector obtained from the vector (294) by the differentiation D_t , in accordance with [10]. Likewise, it is manifest from (296) that the operator X_2 leads to a trivial conserved vector obtained from the conserved vector (294) by the differentiation D_x . Thus we have demonstrated the following statement.

Proposition 13 *The Lie point symmetries (288) of Eq. (280) yield one non-trivial conserved vector (294). Accordingly, the short pulse Eq. (280) can be written in the following conservation form*

$$D_t(u^2) + D_x(u^2 u_x u_t - u_t^2 - \frac{1}{4}u^4 - \frac{1}{4}u^4 u_x^2) = 0. \quad (297)$$

2.5 Gas Dynamics

2.5.1 Classical Symmetries and Conservation Laws

Let us consider the polytropic gasdynamic equations

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho} \nabla p &= 0 \\ \rho_t + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} &= 0 \\ p_t + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} &= 0 \end{aligned} \quad (298)$$

where γ is a constant known as the polytropic (or adiabatic) exponent. The independent variables are the time and the space coordinates

$$t, \quad \mathbf{x} = (x^1, \dots, x^n), \quad n \leq 3. \quad (299)$$

The dependent variables are the velocity, the density and the pressure

$$\mathbf{v} = (v^1, \dots, v^n), \quad \rho, \quad p. \quad (300)$$

Equation (298) with arbitrary γ have the Lie algebra of point symmetries spanned by

$$\begin{aligned} X_0 &= \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad Y_0 = t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, \quad Y_i = t \frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i} \\ X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + v^j \frac{\partial}{\partial v^i} - v^i \frac{\partial}{\partial v^j}, \quad i < j \\ Z_0 &= \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}, \quad Z_1 = t \frac{\partial}{\partial t} - v^i \frac{\partial}{\partial v^i} + 2\rho \frac{\partial}{\partial \rho}, \quad i, j = 1, \dots, n \end{aligned} \quad (301)$$

and the following classical conservation laws

$$\begin{aligned} \frac{d}{dt} \int_{\Omega(t)} \rho d\omega &= 0 && \text{-- Conservation of mass} \\ \frac{d}{dt} \int_{\Omega(t)} \left(\frac{1}{2} \rho |\mathbf{v}|^2 + \frac{p}{\gamma - 1} \right) d\omega &= - \int_{S(t)} p \mathbf{v} \cdot \boldsymbol{\nu} dS && \text{-- Energy} \\ \frac{d}{dt} \int_{\Omega(t)} \rho \mathbf{v} d\omega &= - \int_{S(t)} p \boldsymbol{\nu} dS && \text{-- Momentum} \\ \frac{d}{dt} \int_{\Omega(t)} \rho (\mathbf{x} \times \mathbf{v}) d\omega &= - \int_{S(t)} p (\mathbf{x} \times \boldsymbol{\nu}) dS && \text{-- Angular momentum} \\ \frac{d}{dt} \int_{\Omega(t)} \rho (t\mathbf{v} - \mathbf{x}) d\omega &= - \int_{S(t)} t p \boldsymbol{\nu} dS && \text{-- Center-of-mass.} \end{aligned}$$

The conservation laws are written in the integral form by using the standard symbols

$$\begin{aligned} \Omega(t) &\text{-- arbitrary } n\text{-dimensional volume, moving with fluid} \\ S(t) &\text{-- boundary of the volume } \Omega(t) \\ \boldsymbol{\nu} &\text{-- unit (outer) normal vector to the surface } S(t). \end{aligned}$$

If we write the above conservation laws in the general form

$$\frac{d}{dt} \int_{\Omega(t)} T d\omega = - \int_{S(t)} (\boldsymbol{\chi} \cdot \boldsymbol{\nu}) dS \quad (302)$$

then the differential form of these conservation laws will be

$$D_t(T) + \nabla \cdot (\boldsymbol{\chi} + T\mathbf{v}) = 0. \quad (303)$$

2.5.2 Adjoint Equations and Self-Adjointness When $n = 1$

Theorem 4 from Sect. 2.2.1 shows that the system of gasdynamic Eq. (298) is non-linearly self-adjoint. Let us illustrate this statement in the one-dimensional case

$$v_t + vv_x + \frac{1}{\rho} p_x = 0, \quad \rho_t + v\rho_x + \rho v_x = 0, \quad p_t + vp_x + \gamma p v_x = 0. \quad (304)$$

We write the formal Lagrangian in the form

$$\mathcal{L} = U \left(v_t + vv_x + \frac{1}{\rho} p_x \right) + R(\rho_t + v\rho_x + \rho v_x) + P(p_t + vp_x + \gamma p v_x) \quad (305)$$

and obtain the following adjoint system for the new dependent variables U, R, P

$$\begin{aligned}\frac{\delta \mathcal{L}}{\delta v} &\equiv -U_t - vU_x - \rho R_x + (1 - \gamma)Pp_x - \gamma pP_x = 0 \\ \frac{\delta \mathcal{L}}{\delta \rho} &\equiv -R_t - vR_x - \frac{1}{\rho^2}Up_x = 0 \\ \frac{\delta \mathcal{L}}{\delta p} &\equiv -P_t - \frac{1}{\rho}U_x + \frac{1}{\rho^2}U\rho_x + (\gamma - 1)Pv_x - vP_x = 0.\end{aligned}\tag{306}$$

Let us take, e.g., the conservation of energy from Sect. 2.2.1. Then we have

$$T = \frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1}, \quad \chi = pv$$

and using the differential form (303) of the energy conservation we obtain following the Eq. (242)

$$\begin{aligned}D_t \left(\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} \right) + D_x \left(\frac{1}{2}\rho v^3 + \frac{\gamma}{\gamma - 1}pv \right) \\ = \rho v \left(v_t + vv_x + \frac{1}{\rho}p_x \right) + \frac{v^2}{2}(\rho_t + v\rho_x + \rho v_x) + \frac{1}{\gamma - 1}(p_t + vp_x + \gamma pv_x).\end{aligned}\tag{307}$$

Hence, the adjoint Eq. (306) are satisfied for all solutions of the gasdynamic Eq. (298) upon the substitution

$$U = \rho v, \quad R = \frac{v^2}{2}, \quad P = \frac{1}{\gamma - 1}.\tag{308}$$

This conclusion can be easily verified by the direct substitution of (308) in the adjoint system (306). Namely, we have

$$\begin{aligned}\left. \frac{\delta \mathcal{L}}{\delta v} \right|_{(308)} &= -\rho \left(v_t + vv_x + \frac{1}{\rho}p_x \right) - v(\rho_t + v\rho_x + \rho v_x) \\ \left. \frac{\delta \mathcal{L}}{\delta \rho} \right|_{(308)} &= -v \left(v_t + vv_x + \frac{1}{\rho}p_x \right) \\ \left. \frac{\delta \mathcal{L}}{\delta p} \right|_{(308)} &= 0.\end{aligned}\tag{309}$$

2.5.3 Adjoint System to Equations (298) with $n \geq 2$

For gasdynamic Eq. (298) with two and three space variables x^i the formal Lagrangian (305) is replaced by

$$\begin{aligned} \mathcal{L} = & \mathbf{U} \cdot \left(\mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \nabla p \right) + R(\rho_t + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}) \\ & + P(p_t + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v}) \end{aligned} \quad (310)$$

where the vector $\mathbf{U} = (U^1, \dots, U^n)$ and the scalars R, P are new dependent variables. Using this formal Lagrangian, we obtain the following adjoint system instead of (306)

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \mathbf{v}} & \equiv -\mathbf{U}_t - (\mathbf{v} \cdot \nabla) \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{v} - (\nabla \cdot \mathbf{v}) \mathbf{U} \\ & \quad - \rho \nabla R + (1 - \gamma) P \nabla p - \gamma p \nabla P = 0 \\ \frac{\delta \mathcal{L}}{\delta \rho} & \equiv -R_t - \mathbf{v} \cdot \nabla R - \frac{1}{\rho^2} \mathbf{U} \cdot \nabla p = 0 \\ \frac{\delta \mathcal{L}}{\delta p} & \equiv -P_t - \frac{1}{\rho} (\nabla \cdot \mathbf{U}) + \frac{1}{\rho^2} \mathbf{U} \cdot \nabla \rho + (\gamma - 1) P (\nabla \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla P = 0. \end{aligned} \quad (311)$$

The nonlinear self-adjointness of the system (298) can be demonstrated as in the one-dimensional case discussed in Sect. 2.5.2.

2.5.4 Application to Nonlocal Symmetries of the Chaplygin Gas

The Chaplygin gas is described by the one-dimensional gasdynamic Eq. (304) with $\gamma = -1$, i.e.,

$$v_t + vv_x + \frac{1}{\rho} p_x = 0, \quad \rho_t + v\rho_x + \rho v_x = 0, \quad p_t + vp_x - pv_x = 0. \quad (312)$$

Equation (312) have the same maximal Lie algebra of Lie point symmetries as Eq. (304) with arbitrary γ . This algebra is spanned by the symmetries (301) in the one-dimensional case, namely

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \quad X_4 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \\ X_5 &= \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}, \quad X_6 = t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v} + 2\rho \frac{\partial}{\partial \rho}. \end{aligned} \quad (313)$$

But the Chaplygin gas has more symmetries than an arbitrary one-dimensional polytropic gas upon rewriting it in Lagrange's variables obtained by replacing x and ρ with τ and q , respectively, obtained by the following *nonlocal transformation*

$$\tau = \int \rho dx, \quad q = \frac{1}{\rho}. \quad (314)$$

Then the system (312) becomes

$$q_t - v_\tau = 0, \quad v_t + p_\tau = 0, \quad p_t - \frac{p}{q} v_\tau = 0 \quad (315)$$

and admits the 8-dimensional Lie algebra with the basis

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t}, & Y_2 &= \frac{\partial}{\partial \tau}, & Y_3 &= \frac{\partial}{\partial v}, & Y_4 &= t \frac{\partial}{\partial t} + \tau \frac{\partial}{\partial \tau} \\ Y_5 &= \tau \frac{\partial}{\partial \tau} + p \frac{\partial}{\partial p} - q \frac{\partial}{\partial q}, & Y_6 &= v \frac{\partial}{\partial v} + p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q} \\ Y_7 &= \frac{\partial}{\partial p} + \frac{q}{p} \frac{\partial}{\partial q}, & Y_8 &= t \frac{\partial}{\partial v} - y \frac{\partial}{\partial p} - \frac{yq}{p} \frac{\partial}{\partial q}. \end{aligned} \quad (316)$$

It is shown in [24] that the operators Y_7, Y_8 from (316) lead to the following *nonlocal symmetries* for Eq. (312)

$$\begin{aligned} X_7 &= \sigma \frac{\partial}{\partial x} - \frac{\partial}{\partial p} + \frac{\rho}{p} \frac{\partial}{\partial \rho} \\ X_8 &= \left(\frac{t^2}{2} + s \right) \frac{\partial}{\partial x} + t \frac{\partial}{\partial v} - \tau \frac{\partial}{\partial p} + \frac{\rho \tau}{p} \frac{\partial}{\partial \rho} \end{aligned} \quad (317)$$

where τ, s, σ are the following *nonlocal variables*

$$\tau = \int \rho dx, \quad s = - \int \frac{\tau}{p} dx, \quad \sigma = - \int \frac{dx}{p}. \quad (318)$$

They can be equivalently defined by the compatible over-determined systems

$$\begin{aligned} \tau_x &= \rho, & \tau_t + v\tau_x &= 0 \\ s_x &= -\frac{\tau}{p}, & s_t + vs_x &= 0 \\ \sigma_x &= -\frac{1}{p}, & \sigma_t + v\sigma_x &= 0 \end{aligned} \quad (319)$$

or

$$\begin{aligned}
\tau_x &= \rho, & \tau_t &= -v\rho \\
s_x &= -\frac{\tau}{p}, & s_t &= \frac{v\tau}{p} \\
\sigma_x &= -\frac{1}{p}, & \sigma_t &= \frac{v}{p}.
\end{aligned} \tag{320}$$

Let us verify that the operator X_7 is admitted by Eq. (312). Its first prolongation is obtained by applying the usual prolongation procedure and eliminating the partial derivatives σ_x and σ_t via Eq. (320). It has the form

$$\begin{aligned}
X_7 &= \sigma \frac{\partial}{\partial x} - \frac{\partial}{\partial p} + \frac{\rho}{p} \frac{\partial}{\partial \rho} - \frac{vv_x}{p} \frac{\partial}{\partial v_t} + \frac{v_x}{p} \frac{\partial}{\partial v_x} - \frac{vp_x}{p} \frac{\partial}{\partial p_t} + \frac{p_x}{p} \frac{\partial}{\partial p_x} \\
&+ \left(\frac{\rho_t}{p} - \frac{\rho p_t}{p^2} - \frac{v\rho_x}{p} \right) \frac{\partial}{\partial \rho_t} + \left(2\frac{\rho_x}{p} - \frac{\rho p_x}{p^2} \right) \frac{\partial}{\partial \rho_x}.
\end{aligned} \tag{321}$$

The calculation shows that the invariance condition is satisfied in the following form:

$$\begin{aligned}
X_7 \left(v_t + vv_x + \frac{1}{\rho} p_x \right) &= 0 \\
X_7(\rho_t + v\rho_x + \rho v_x) &= \frac{1}{p} (\rho_t + v\rho_x + \rho v_x) - \frac{\rho}{p^2} (p_t + vp_x - pv_x) \\
X_7(p_t + vp_x - pv_x) &= 0.
\end{aligned}$$

One can verify likewise that the invariance test for the operator X_8 is satisfied in the following form

$$\begin{aligned}
X_8(v_t + vv_x + \frac{1}{\rho} p_x) &= 0 \\
X_8(\rho_t + v\rho_x + \rho v_x) &= \frac{\tau}{p} (\rho_t + v\rho_x + \rho v_x) - \frac{\rho\tau}{p^2} (p_t + vp_x - pv_x) \\
X_8(p_t + vp_x - pv_x) &= 0.
\end{aligned}$$

The operators Y_1, \dots, Y_6 from (316) do not add to the operators (313) new symmetries of the system (312).

Thus, the Chaplygin gas described by Eq. (312) admits the eight-dimensional vector space spanned by the operators (313) and (317). However this vector space is not a Lie algebra. Namely, the commutators of the dilation generators X_4, X_5, X_6 from (313) with the operators (317) are not linear combinations of the operators (313) and (317) with constants coefficients. The reason is that the operators X_4, X_5, X_6 are not admitted by the differential equation (319) for the nonlocal variables τ, s, σ . Therefore I will extend the action of the dilation generators to τ, s, σ so that the extended operators will be admitted by Eq. (319).

Let us take the operator X_4 . We write it in the extended form

$$X'_4 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial \tau} + \beta \frac{\partial}{\partial s} + \mu \frac{\partial}{\partial \sigma}$$

where α, β, μ are unknown functions of $t, x, v, \rho, p, \tau, s, \sigma$. Then we make the prolongation of X'_4 to the first-order partial derivatives of the nonlocal variables with respect to t and x by treating τ, s, σ as new dependent variables and obtain

$$\begin{aligned} X'_4 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial \tau} + \beta \frac{\partial}{\partial s} + \mu \frac{\partial}{\partial \sigma} \\ &+ [D_t(\alpha) - \tau_t] \frac{\partial}{\partial \tau_t} + [D_x(\alpha) - \tau_x] \frac{\partial}{\partial \tau_x} \\ &+ [D_t(\beta) - s_t] \frac{\partial}{\partial s_t} + [D_x(\beta) - s_x] \frac{\partial}{\partial s_x} \\ &+ [D_t(\mu) - \sigma_t] \frac{\partial}{\partial \sigma_t} + [D_x(\mu) - \sigma_x] \frac{\partial}{\partial \sigma_x}. \end{aligned}$$

Now we require the invariance of the Eq. (319)

$$\begin{aligned} X'_4(\tau_x - \rho) &= 0, & X'_4(\tau_t + v\tau_x) &= 0 \\ X'_4\left(s_x + \frac{\tau}{p}\right) &= 0, & X'_4(s_t + vs_x) &= 0 \\ X'_4\left(\sigma_x + \frac{1}{p}\right) &= 0, & X'_4(\sigma_t + v\sigma_x) &= 0. \end{aligned} \quad (322)$$

As usual, Eq. (322) should be satisfied on the solutions of Eq. (319). Let us solve the equations $X'_4(\tau_x - \rho) = 0$, $X'_4(\tau_t + v\tau_x) = 0$. They are written

$$[D_x(\alpha) - \tau_x]_{(319)} = 0, \quad [D_t(\alpha) - \tau_t + v(D_x(\alpha) - \tau_x)]_{(319)} = 0. \quad (323)$$

Since $\tau_x = D_x(\alpha)$, the first equation in (323) is satisfied if we take

$$\alpha = \tau.$$

With this α the second equation in (323) is also satisfied because $\tau_t + v\tau_x = 0$. Now the first equation in the second line of Eq. (322) becomes

$$\left[D_x(\beta) - s_x + \frac{\tau}{p} \right]_{(319)} = D_x(\beta) - 2s_x = 0$$

and yields

$$\beta = 2s.$$

Table 1 The structure of the Lie algebra L_8

	X_1	X_2	X_3	X'_4	X'_5	X'_6	X_7	X_8
X_1	0	0	X_2	X_1	0	X_1	0	X_3
X_2	0	0	0	X_2	0	0	0	0
X_3	$-X_2$	0	0	0	0	$-X_3$	0	0
X'_4	$-X_1$	$-X_2$	0	0	0	0	0	X_8
X'_5	0	0	0	0	0	0	$-X_7$	0
X'_6	$-X_1$	0	X_3	0	0	0	0	$2X_8$
X_7	0	0	0	0	X_7	0	0	0
X_8	$-X_3$	0	0	$-X_8$	0	$-2X_8$	0	0

The second equation in the second line of Eq.(322) is also satisfied with this β . Applying the same approach to the third line of Eq.(322) we obtain

$$\mu = \sigma.$$

After similar calculations with X_5 and X_6 we obtain the following extensions of the dilation generators

$$\begin{aligned} X'_4 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial \tau} + 2s \frac{\partial}{\partial s} + \sigma \frac{\partial}{\partial \sigma} \\ X'_5 &= \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p} + \tau \frac{\partial}{\partial \tau} - \sigma \frac{\partial}{\partial \sigma} \\ X'_6 &= t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v} + 2\rho \frac{\partial}{\partial \rho} + 2\tau \frac{\partial}{\partial \tau} + 2s \frac{\partial}{\partial s}. \end{aligned} \tag{324}$$

The operators (317) and (324) together with the operators X_1, X_2, X_3 from (313) span the eight-dimensional Lie algebra L_8 admitted by Eqs.(312) and (319). The algebra L_8 has the following commutator Table 1.

Let us apply Theorem 5 to the nonlocal symmetries (317) of the Chaplygin gas. The formal Lagrangian (305) for Eq.(312) has the form

$$\mathcal{L} = U \left(v_t + v v_x + \frac{1}{\rho} p_x \right) + R(\rho_t + v \rho_x + \rho v_x) + P(p_t + v p_x - p v_x). \tag{325}$$

Accordingly, the adjoint system (306) for the Chaplygin gas is written

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta v} &\equiv -U_t - v U_x - \rho R_x + 2P p_x + p P_x = 0 \\ \frac{\delta \mathcal{L}}{\delta \rho} &\equiv -R_t - v R_x - \frac{1}{\rho^2} U p_x = 0 \\ \frac{\delta \mathcal{L}}{\delta p} &\equiv -P_t - \frac{1}{\rho} U_x + \frac{1}{\rho^2} U \rho_x - 2P v_x - v P_x = 0. \end{aligned} \tag{326}$$

Let us proceed as in Sect. 2.5.2. Namely, let us first construct solutions to the adjoint system (326) by using the known conservation laws given in Sect. 2.2.1. Since the one-dimensional equation does not have the conservation of angular momentum, we use the conservation of mass, energy, momentum and center-of-mass and obtain the respective differential conservation equations (see the derivation of Eq. (307))

$$D_t(\rho) + D_x(\rho v) = \rho_t + v\rho_x + \rho v_x \quad (327)$$

$$D_t(\rho v^2 - p) + D_x(pv + \rho v^3) = 2\rho v \left(v_t + vv_x + \frac{1}{\rho} p_x \right) + v^2(\rho_t + v\rho_x + \rho v_x) - (p_t + vp_x - pv_x) \quad (328)$$

$$D_t(\rho v) + D_x(p + \rho v^2) = \rho \left(v_t + vv_x + \frac{1}{\rho} p_x \right) + v(\rho_t + v\rho_x + \rho v_x) \quad (329)$$

$$D_t(t\rho v - x\rho) + D_x(tp + t\rho v^2 - x\rho v) = t\rho \left(v_t + vv_x + \frac{1}{\rho} p_x \right) + (tv - x)(\rho_t + v\rho_x + \rho v_x). \quad (330)$$

Equations (327)–(330) give the following solutions to the system of the adjoint equation (326)

$$U = 0, \quad R = 1, \quad P = 0 \quad (331)$$

$$U = 2\rho v, \quad R = v^2, \quad P = -1 \quad (332)$$

$$U = \rho, \quad R = v, \quad P = 0 \quad (333)$$

$$U = t\rho, \quad R = tv - x, \quad P = 0. \quad (334)$$

The formal Lagrangian (325) contains the derivatives only of the first order. Therefore Eq. (258) for calculating the conserved vectors take the simple form

$$C^i = W^\alpha \frac{\partial \mathcal{L}}{\partial u_i^\alpha}, \quad i = 1, 2. \quad (335)$$

We denote

$$t = x^1, \quad x = x^2, \quad v = u^1, \quad \rho = u^2, \quad p = u^3.$$

In this notation conservation Eq. (237) will be written in the form

$$\left[D_t(C^1) + D_x(C^2) \right]_{(312)} = 0. \quad (336)$$

Writing (335) in the form

$$\begin{aligned} C^1 &= W^1 \frac{\partial \mathcal{L}}{\partial v_t} + W^2 \frac{\partial \mathcal{L}}{\partial \rho_t} + W^3 \frac{\partial \mathcal{L}}{\partial p_t} \\ C^2 &= W^1 \frac{\partial \mathcal{L}}{\partial v_x} + W^2 \frac{\partial \mathcal{L}}{\partial \rho_x} + W^3 \frac{\partial \mathcal{L}}{\partial p_x} \end{aligned}$$

and substituting the expression (325) for \mathcal{L} we obtain the following final expressions for computing the components of conserved vectors

$$C^1 = UW^1 + RW^2 + PW^3 \quad (337)$$

$$C^2 = (vU + \rho R - pP)W^1 + vRW^2 + \left(\frac{1}{\rho}U + vP\right)W^3 \quad (338)$$

where

$$W^\alpha = \eta^\alpha - \xi^i u_i^\alpha, \quad \alpha = 1, 2, 3. \quad (339)$$

We will apply Eqs. (337) and (338) to the nonlocal symmetries (317). First we write the expressions (339) for the operator X_7 from (317)

$$W^1 = -\sigma v_x, \quad W^2 = \frac{\rho}{p} - \sigma \rho_x, \quad W^3 = -(1 + \sigma p_x). \quad (340)$$

Then we substitute (340) in (337) and (338) and obtain four conserved vectors by replacing U, R, P with each of four different solutions (331)–(334) of the adjoint system (326). Some of these conserved vectors may be trivial. We select only the nontrivial ones.

Let us calculate the conserved vector obtained by eliminating U, R, P by using the solution (331), $U = 0, R = 1, P = 0$. In this case (337) and (338) and (340) yield

$$C^1 = W^2 = \frac{\rho}{p} - \sigma \rho_x \quad (341)$$

$$C^2 = \rho W^1 + v W^2 = -\sigma \rho v_x + \frac{\rho}{p} v - \sigma v \rho_x.$$

We write

$$-\sigma \rho_x = -D_x(\sigma \rho) + \rho \sigma_x$$

replace σ_x with $-1/p$ according to Eq. (319) and obtain

$$C^1 = -D_x(\sigma \rho).$$

Therefore application of the operations (239)–(241) yields $\tilde{C}^1 = 0$ and

$$\begin{aligned}\tilde{C}^2 &= -\sigma\rho v_x + \frac{\rho}{p}v - \sigma v\rho_x - D_t(\sigma\rho) \\ &= -\sigma\rho v_x + \frac{\rho}{p}v - \sigma v\rho_x - \sigma\rho_t - \sigma_t\rho \\ &= -\sigma(\rho_t + v\rho_x + \rho v_x).\end{aligned}$$

We have replaced σ_t with v/p according to Eq. (320). The above expression for \tilde{C}^2 vanishes on Eq. (312). Hence, the conserved vector (341) is trivial.

Utilization of the solutions (332) and (333) also leads to trivial conserved vectors only. Finally, using the solution (334)

$$U = t\rho, \quad R = tv - x, \quad P = 0$$

we obtain, upon simplifying by using the operations (239)–(241), the following nontrivial conserved vector

$$C^1 = \sigma\rho, \quad C^2 = \sigma\rho v + t. \quad (342)$$

The conservation Eq. (336) is satisfied in the following form

$$D_t(C^1) + D_x(C^2) = \sigma(\rho_t + v\rho_x + \rho v_x). \quad (343)$$

Note that we can write C^2 in (342) without t since it adds only the trivial conserved vector with the components $C^1 = 0$, $C^2 = t$. Thus, removing t in (342) and using the definition of σ given in (318) we formulate the result.

Proposition 14 *The nonlocal symmetry X_7 of the Chaplygin gas gives the following nonlocal conserved vector*

$$C^1 = -\rho \int \frac{dx}{p}, \quad C^2 = -\rho v \int \frac{dx}{p}. \quad (344)$$

Now we use the operator X_8 from (317). In this case

$$\begin{aligned}W^1 &= t - \left(\frac{t^2}{2} + s\right)v_x \\ W^2 &= \frac{\rho\tau}{p} - \left(\frac{t^2}{2} + s\right)\rho_x \\ W^3 &= -\tau - \left(\frac{t^2}{2} + s\right)p_x.\end{aligned} \quad (345)$$

Substituting in (337) and (338) the expressions (345) and the solution (331) of the adjoint system, i.e., letting $U = 0$, $R = 1$, $P = 0$, we obtain

$$\begin{aligned} C^1 &= W^2 = \frac{\rho\tau}{p} - \left(\frac{t^2}{2} + s\right)\rho_x \\ C^2 &= \rho W^1 + v W^2 = t\rho + \frac{\rho v\tau}{p} - \left(\frac{t^2}{2} + s\right)(\rho v_x + v\rho_x). \end{aligned}$$

Noting that

$$-\left(\frac{t^2}{2} + s\right)\rho_x = -\frac{\rho\tau}{p} - D_x\left(\frac{t^2}{2}\rho + \rho s\right)$$

we reduce the above vector to the trivial conserved vector $\tilde{C}^1 = 0$, $\tilde{C}^2 = 0$.

Taking the solution (332) of the adjoint system, i.e., letting

$$U = 2\rho v, \quad R = v^2, \quad P = -1$$

we obtain

$$\begin{aligned} C^1 &= 2\rho v W^1 + v^2 W^2 - W^3 \\ &= 2t\rho v + \frac{\rho\tau v^2}{p} + \tau - \left(\frac{t^2}{2} + s\right)D_x(\rho v^2 - p) \\ C^2 &= (3\rho v^2 + p)W^1 + v^3 W^2 + v W^3 \\ &= t(3\rho v^2 + p) + \frac{\rho\tau v^3}{p} - v\tau \\ &\quad - \left(\frac{t^2}{2} + s\right)(3\rho v^2 v_x + v^3 \rho_x + p v_x + v p_x). \end{aligned}$$

Then, upon rewriting C^1 in the form

$$C^1 = 2t\rho v + 2\tau - D_x\left[\left(\frac{t^2}{2} + s\right)(\rho v^2 - p)\right]$$

and applying the operations (239)–(241) we arrive at the following conserved vector

$$C^1 = t\rho v + \tau, \quad C^2 = t(\rho v^2 + p). \quad (346)$$

The conservation Eq. (336) is satisfied for (346) in the following form

$$D_t(C^1) + D_x(C^2) = t\rho\left(v_t + v v_x + \frac{1}{\rho} p_x\right) + t v(\rho_t + v\rho_x + \rho v_x). \quad (347)$$

Taking the solution (333) of the adjoint system, i.e., letting

$$U = \rho, \quad R = v, \quad P = 0$$

we obtain

$$C^1 = \rho W^1 + v W^2, \quad C^2 = 2\rho v W^1 + v^2 W^2 + W^3.$$

Substituting the expressions (345) for W^1 , W^2 , W^3 and simplifying as in the previous case we obtain the conserved vector

$$C^1 = t\rho, \quad C^2 = t\rho v - \tau. \quad (348)$$

The conservation Eq. (336) is satisfied for (346) in the following form

$$D_t(C^1) + D_x(C^2) = t(\rho_t + v\rho_x + \rho v_x). \quad (349)$$

Finally, we take the solution (334), $U = t\rho$, $R = tv - x$, $P = 0$, and obtain

$$C^1 = t\rho W^1 + (tv - x)W^2, \quad C^2 = (2t\rho v - x\rho)W^1 + (tv^2 - xv)W^2 + tW^3.$$

Simplifying as above, we arrive at the conserved vector

$$C^1 = \left(\frac{t^2}{2} - s\right)\rho, \quad C^2 = \left(\frac{t^2}{2} - s\right)\rho v - t\tau. \quad (350)$$

The conservation Eq. (336) is satisfied for (346) in the following form

$$D_t(C^1) + D_x(C^2) = \left(\frac{t^2}{2} - s\right)(\rho_t + v\rho_x + \rho v_x). \quad (351)$$

Substituting in the conserved vectors (346), (348) and (350) the definition (318) of the nonlocal variables we formulate the result.

Proposition 15 *The nonlocal symmetry X_8 of the Chaplygin gas gives the following nonlocal conserved vectors*

$$C^1 = t\rho v + \int \rho dx, \quad C^2 = t(\rho v^2 + p) \quad (352)$$

$$C^1 = t\rho, \quad C^2 = t\rho v - \int \rho dx \quad (353)$$

$$\begin{aligned}
C^1 &= \left[\frac{t^2}{2} + \int \frac{1}{p} \left(\int \rho dx \right) dx \right] \rho \\
C^2 &= \left[\frac{t^2}{2} + \int \frac{1}{p} \left(\int \rho dx \right) dx \right] \rho v - t \int \rho dx.
\end{aligned} \tag{354}$$

Theorem 6 Application of Theorem 5 to two nonlocal symmetries (317) gives four nonlocal conservation laws (344) and (352)–(354) for the Chaplygin gas (312).

2.5.5 Steady Two-Dimensional Gas Flow

Consider the steady two-dimensional equations of gasdynamics

$$\begin{aligned}
(\rho u)_x + (\rho v)_y &= 0, & us_x + vs_y &= 0 \\
p_x + \rho(uu_x + vv_y) &= 0, & p_y + \rho(uv_x + vv_y) &= 0
\end{aligned} \tag{355}$$

where s denotes the specific entropy. It is shown in [41] that the Bateman-type reciprocal transformations provide the following nonlocal symmetry of equations (355)

$$X = \alpha \frac{\partial}{\partial x} + \sigma \frac{\partial}{\partial y} - p^2 \frac{\partial}{\partial p} - pu \frac{\partial}{\partial u} - pv \frac{\partial}{\partial v} - (u^2 + v^2) \rho^2 \frac{\partial}{\partial \rho}$$

where α and σ are nonlocal variables determined by the equations

$$\alpha_x = p + \rho v^2, \quad \alpha_y = -\rho uv$$

and

$$\sigma_x = -\rho uv, \quad \sigma_y = p + \rho u^2$$

respectively. Using this nonlocal symmetry and the nonlinear self-adjointness of the system (355), the following nonlocal conserved vector is constructed in [41]

$$\begin{aligned}
C^1 &= \rho pu + u(u^2 + v^2)\rho^2 + \alpha(\rho u)_x + \sigma(\rho u)_y \\
C^2 &= \rho pv + v(u^2 + v^2)\rho^2 + \alpha(\rho v)_x + \sigma(\rho v)_y.
\end{aligned} \tag{356}$$

The vector (356) satisfies the conservation law in the following form

$$\begin{aligned}
D_x(C^1) + D_y(C^2) &= 2[p + \rho(u^2 + v^2)][(\rho u)_x + (\rho v)_y] \\
&\quad + \rho u[p_x + \rho(uu_x + vv_y)] + \rho v[p_y + \rho(uv_x + vv_y)] \\
&\quad + \alpha[(\rho u)_x + (\rho v)_y]_x + \sigma[(\rho u)_x + (\rho v)_y]_y
\end{aligned}$$

and the nonlocal conservation law that this symmetry

2.5.6 The Operator Identity for Nonlocal Symmetries

Example 18 Let us verify that the operator identity (179) is satisfied for the nonlocal symmetry X_7 of the Chaplygin gas. To this end, we have to verify that the coefficients of

$$\frac{\partial}{\partial v}, \frac{\partial}{\partial \rho}, \frac{\partial}{\partial p}, \frac{\partial}{\partial v_t}, \frac{\partial}{\partial v_x}, \frac{\partial}{\partial \rho_t}, \frac{\partial}{\partial \rho_x}, \frac{\partial}{\partial p_t}, \frac{\partial}{\partial p_x} \quad (357)$$

in both sides of (179) are equal. Using the first prolongation (321) of X_7 and the definition of the nonlocal variable σ given in Eq. (320) we see that the left-hand side of the identity (179) is written

$$\begin{aligned} X_7 + D_t(\xi^i) = & \sigma \frac{\partial}{\partial x} - \frac{\partial}{\partial p} + \frac{\rho}{p} \frac{\partial}{\partial \rho} - \frac{v v_x}{p} \frac{\partial}{\partial v_t} + \frac{v_x}{p} \frac{\partial}{\partial v_x} - \frac{v p_x}{p} \frac{\partial}{\partial p_t} \\ & + \frac{p_x}{p} \frac{\partial}{\partial p_x} + \left(\frac{\rho_t}{p} - \frac{\rho p_t}{p^2} - \frac{v \rho_x}{p} \right) \frac{\partial}{\partial \rho_t} + \left(2 \frac{\rho_x}{p} - \frac{\rho p_x}{p^2} \right) \frac{\partial}{\partial \rho_x} - \frac{1}{p}. \end{aligned} \quad (358)$$

Then we use the expressions (340) of W^α for the operator X_7 , substitute them in the definition (183) of N^i and obtain in our approximation

$$\begin{aligned} N^1 = & -\sigma v_x \frac{\partial}{\partial v_t} + \left(\frac{\rho}{p} - \sigma \rho_x \right) \frac{\partial}{\partial \rho_t} - (1 + \sigma p_x) \frac{\partial}{\partial p_t} \\ N^2 = & \sigma - \sigma v_x \frac{\partial}{\partial v_x} + \left(\frac{\rho}{p} - \sigma \rho_x \right) \frac{\partial}{\partial \rho_x} - (1 + \sigma p_x) \frac{\partial}{\partial p_t}. \end{aligned}$$

Now the right-hand side of (179) is written

$$\begin{aligned} & W^1 \frac{\delta}{\delta v} + W^2 \frac{\delta}{\delta \rho} + W^3 \frac{\delta}{\delta p} + D_t N^1 + D_x N^2 \\ & = -\sigma v_x \left[\frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_t \frac{\partial}{\partial v_x} \right] \\ & \quad + \left(\frac{\rho}{p} - \sigma \rho_x \right) \left[\frac{\partial}{\partial \rho} - D_t \frac{\partial}{\partial \rho_t} - D_x \frac{\partial}{\partial \rho_x} \right] \\ & \quad - (1 + \sigma p_x) \left[\frac{\partial}{\partial p} - D_t \frac{\partial}{\partial p_t} - D_x \frac{\partial}{\partial p_x} \right] \\ & \quad + D_t \left[-\sigma v_x \frac{\partial}{\partial v_t} + \left(\frac{\rho}{p} - \sigma \rho_x \right) \frac{\partial}{\partial \rho_t} - (1 + \sigma p_x) \frac{\partial}{\partial p_t} \right] \\ & \quad + D_x \left[\sigma - \sigma v_x \frac{\partial}{\partial v_x} + \left(\frac{\rho}{p} - \sigma \rho_x \right) \frac{\partial}{\partial \rho_x} - (1 + \sigma p_x) \frac{\partial}{\partial p_t} \right]. \end{aligned} \quad (359)$$

Making the changes in two last lines of Eq. (359) such as

$$\begin{aligned}
 D_t \left[-\sigma v_x \frac{\partial}{\partial v_t} \right] &= -\sigma v_x D_t \frac{\partial}{\partial v_t} - D_t(\sigma v_x) \frac{\partial}{\partial v_t} \\
 &= -\sigma v_x D_t \frac{\partial}{\partial v_t} - \left(\frac{v}{p} v_x + \sigma v_{tx} \right) \frac{\partial}{\partial v_t}
 \end{aligned}$$

one can see that the coefficients of the differentiations (357) in (358) and (359) coincide. Inspection of the coefficients of the differentiations in higher derivatives v_{tt} , v_{tx} , v_{xx} , ... requires the higher-order prolongations of the operator X_7 .

Exercise 7 Verify that the operator identity (179) is satisfied in the same approximation as in Example 18 for the nonlocal symmetry operator X_8 from (317).

2.6 Comparison With the “Direct Method”

2.6.1 General Discussion

Theorem 5 allows to construct conservation laws for equations with known symmetries simply by substituting in Eq. (258) the expressions W^α and \mathcal{L} given by Eqs. (256) and (256), respectively.

The “direct method” means the determination of the conserved vectors (238) by solving Eq. (237) for C^i . Upon restricting the highest order of derivatives of u involved in C^i , Eq. (237) splits into several equations. If one can solve the resulting system, one obtains the desired conserved vectors. Existence of symmetries is not required.

To the best of my knowledge, the direct method was used for the first time in 1798 by Laplace [42]. He applied the method to Kepler’s problem in celestial mechanics and found a new vector-valued conserved quantity (see [42], Book II, Chap. III, equations (P)) known as Laplace’s vector.

The application of the direct method to the gasdynamic Eq. (298) allowed to demonstrate in [43] that all conservation laws involving only the independent and dependent variables (299) and (300) were provided by the classical conservation laws (mass, energy, momentum, angular momentum and center-of-mass) given in Sect. 2.5.1 and the following two special conservation laws

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega(t)} \left\{ t(\rho|v|^2 + np) - \rho \mathbf{x} \cdot \mathbf{v} \right\} d\omega &= - \int_{S(t)} p (2t\mathbf{v} - \mathbf{x}) \cdot \boldsymbol{\nu} dS \\
 \frac{d}{dt} \int_{\Omega(t)} \left\{ t^2(\rho|v|^2 + np) - \rho \mathbf{x} \cdot (2t\mathbf{v} - \mathbf{x}) \right\} d\omega &= - \int_{S(t)} 2tp (t\mathbf{v} - \mathbf{x}) \cdot \boldsymbol{\nu} dS
 \end{aligned}$$

that were found in [44] in the case $\gamma = (n + 2)/n$ by using the symmetry ideas.

All local conservation laws for the heat equation $u_t - u_{xx} = 0$ have been found by the direct method in [45] (see [15–17, vol. 1, Sect. 10.1] and [46]). Namely it has been shown by considering the conservation equations of the form

$$D_t[\tau(t, x, u, u_x, u_{xx}, \dots)] + D_x[\psi(t, x, u, u_x, u_{xx}, \dots)] = 0$$

that all such conservation laws are given by

$$D_t[\varphi(t, x)u] + D_x[u\varphi_x(t, x) - \varphi(t, x)u_x] = 0$$

where $v = \varphi(t, x)$ is an arbitrary solution of the adjoint equation $v_t + v_{xx} = 0$ to the heat equation. Similar result can be obtained by applying Theorem 5 for any linear equation, e.g., for the heat equation $u_t - \Delta u = 0$ with any number of spatial variables $x = (x^1, \dots, x^n)$. Namely, applying formula (258) to the scaling symmetry $X = u\partial/\partial u$ we obtain the conservation law

$$D_t[\varphi(t, x)u] + \nabla \cdot [u\nabla\varphi(t, x) - \varphi(t, x)\nabla u] = 0$$

where $v = \varphi(t, x)$ is an arbitrary solution of the adjoint equation $v_t + \Delta v = 0$ to the heat equation. This conservation law embraces the conservation laws associated with all other symmetries of the heat equation.

Various mathematical models for describing the geological process of segregation and migration of large volumes of molten rock were proposed in the geophysical literature (see the papers [47–51] and the references therein). One of them is known as the *generalized magma equation* and has the form

$$u_t + D_z [u^n - u^n D_z (u^{-m} u_t)] = 0, \quad n, m = \text{const.} \quad (360)$$

It is accepted as a reasonable mathematical model for describing melt migration through the Earth's mantle. Several conservation laws for this model have been calculated by the direct method in [47, 48] and interpreted from symmetry point of view in [49]. It is shown in [52] that Eq. (360) is quasi self-adjoint with the substitution (34) given by $v = u^{1-n-m}$ if $m + n \neq 1$ and $v = \ln |u|$ if $m + n = 1$. These substitutions show that Eq. (360) is strictly self-adjoint (Definition 2) if $m + n = 0$. Using the quasi self-adjointness, the conservation laws are easily computed in [52].

Some simplification of the direct method was suggested in [37]. Namely, one writes the conservation equation in the form (242)

$$D_i(C^i) = \mu^{\bar{\alpha}} F_{\bar{\alpha}}(x, u, u_{(1)}, \dots, u_{(s)}) \quad (361)$$

and first finds the undetermined coefficients $\mu^{\bar{\alpha}}$ by satisfying the integrability condition of Eq. (361), i.e., by solving the equations (see Proposition 8 in Sect. 2.1.2)

$$\frac{\delta}{\delta u^{\alpha}} \left[\mu^{\bar{\beta}}(x, u, u_{(1)}, \dots) F_{\bar{\beta}}(x, u, u_{(1)}, \dots, u_{(s)}) \right] = 0, \quad \alpha = 1, \dots, m. \quad (362)$$

Then, for each solution $\mu^{\bar{\alpha}}$ of Eq. (362), the components C^i of the corresponding conserved vector are computed from Eq. (361). In simple situations C^i can be detected merely by looking at the right-hand side of Eq. (361), see further Example 19.

Remark 17 Note that Eq. (362) should be satisfied on the solutions of Eq. (236). Then the left-hand side of (362) can be written as

$$F_{\alpha}^*(x, u, v, \dots, u_{(s)}, v_{(s)}) \Big|_{v=\mu(x,u,u_{(1)},\dots)}$$

with F_{α}^* defined by Eq. (69).

The reader can find a detailed discussion of the direct method in the recent book [14]. I will compare two methods by considering few examples and exercises.

2.6.2 Examples and Exercises

Example 19 (See [14, Sect. 1.3]). Let us consider the KdV Eq. (73)

$$u_t = u_{xxx} + uu_x \quad (73)$$

and write the condition (362) for $\mu = \mu(t, x, u)$. We have

$$\begin{aligned} & \frac{\delta}{\delta u} [\mu(t, x, u)(u_t - u_{xxx} - uu_x)] \\ &= -D_t(\mu) + D_x^3(\mu) + D_x(u\mu) - \mu u_x + (u_t - u_{xxx} - uu_x) \frac{\partial \mu}{\partial u} \\ &= -D_t(\mu) + D_x^3(\mu) + u D_x(\mu) + (u_t - u_{xxx} - uu_x) \frac{\partial \mu}{\partial u}. \end{aligned}$$

In accordance with Remark 17, we consider this expression on the solutions of the KdV equation and see that Eq. (362) coincides with the adjoint Eqs. (73) and (74)

$$D_t(\mu) = D_x^3(\mu) + u D_x(\mu). \quad (363)$$

Its solution is given in Example 11 and has the form (75)

$$\mu = A_1 + A_2 u + A_3(x + tu), \quad A_1, A_2, A_3 = \text{const.}$$

Thus, we have the following three linearly independent solutions of Eq. (363)

$$\mu_1 = 1, \quad \mu_2 = u, \quad \mu_3 = (x + tu).$$

and the corresponding three Eq. (361)

$$D_t(C^1) + D_x(C^2) = u_t - u_{xxx} - uu_x \quad (364)$$

$$D_t(C^1) + D_x(C^2) = u(u_t - u_{xxx} - uu_x) \quad (365)$$

$$D_t(C^1) + D_x(C^2) = (x + tu)(u_t - u_{xxx} - uu_x). \quad (366)$$

In this simple example the components C^1, C^2 of the conserved vector can be easily seen from the right-hand sides of Eqs. (364)–(366). In the case of (364) and (365) it is obvious. Therefore let us consider the right-hand side of Eq. (366). We see that

$$\begin{aligned} (x + tu)u_t &= D_t \left(xu + \frac{1}{2} tu^2 \right) - \frac{1}{2} u^2 \\ -(x + tu)uu_x &= -D_x \left(\frac{1}{2} xu^2 + \frac{1}{3} tu^3 \right) + \frac{1}{2} u^2 \\ -(x + tu)u_{xxx} &= -D_x (xu_{xx} + tuu_{xx}) + u_{xx} + tu_x u_{xx} \\ &= D_x \left(u_x + \frac{1}{2} tu_x^2 - xu_{xx} - tuu_{xx} \right). \end{aligned}$$

Hence, the right-hand side of Eq. (366) can be written in the divergence form

$$\begin{aligned} &(x + tu)(u_t - u_{xxx} - uu_x) \\ &= D_t \left(t \frac{u^2}{2} + xu \right) + D_x \left[u_x + t \left(\frac{u_x^2}{2} - uu_{xx} - \frac{u^3}{3} \right) - x \left(\frac{u^2}{2} + u_{xx} \right) \right]. \end{aligned}$$

The expressions under $D_t(\dots)$ and $D_x(\dots)$ give C^1 and C^2 , respectively, in (366). Note that the corresponding conservation law

$$D_t \left(t \frac{u^2}{2} + xu \right) + D_x \left[u_x + t \left(\frac{u_x^2}{2} - uu_{xx} - \frac{u^3}{3} \right) - x \left(\frac{u^2}{2} + u_{xx} \right) \right] = 0 \quad (367)$$

was derived from the Galilean invariance of the KdV equation (see [10, Sect. 22.5]) and by the direct method (see [14, Sect. 1.3.5]).

The similar treatment of the right-hand sides of the Eqs. (364) and (365) leads to Eq. (73) and to the conservation law

$$D_t(u^2) + D_x \left(u_x^2 - 2uu_{xx} - \frac{2}{3} u^3 \right) = 0 \quad (368)$$

respectively. Theorem 5 associates the conservation law (368) with the scaling symmetry of the KdV equation.

Exercise 8 Apply the direct method to the short pulse Eq. (280) using the differential substitution (284). In this case Eq. (361) is written

$$D_t(C^1) + D_x(C^2) = u_t u_{xt} - \frac{1}{2} u^2 u_x u_{xt} - \left(u + \frac{1}{2} u^2 u_{xx} + u u_x^2 \right) u_t + \frac{1}{2} u^3 u_x + \frac{1}{4} u^4 u_x u_{xx} + \frac{1}{2} u^3 u_x^3. \quad (369)$$

Exercise 9 Consider the Boussinesq Eq. (246). Taking its formal Lagrangian

$$\mathcal{L} = \omega [\Delta \psi_t - g \rho_x - f v_z - \psi_x \Delta \psi_z + \psi_z \Delta \psi_x] + \mu [v_t + f \psi_z - \psi_x v_z + \psi_z v_x] + r [\rho_t + (N^2/g) \psi_x - \psi_x \rho_z + \psi_z \rho_x]$$

where ω, μ, r are new dependent variables, we obtain the adjoint system to Eq. (246)

$$\frac{\delta \mathcal{L}}{\delta \psi} = 0, \quad \frac{\delta \mathcal{L}}{\delta v} = 0, \quad \frac{\delta \mathcal{L}}{\delta \rho} = 0. \quad (370)$$

It is shown in [53] that the system (246) is self-adjoint. Namely, the substitution

$$\omega = \psi, \quad \mu = -v, \quad r = -(g^2/N^2) \rho \quad (371)$$

maps the adjoint system (370) into the system (246). Using the so established self-adjointness, nontrivial conservation laws were constructed via Theorem 5. Apply the direct method to the system (246). Note that knowledge of the substitution (371) gives the following Eq. (361)

$$D_t(C^1) + D_x(C^2) + D_z(C^3) = \psi [\psi_{txx} + \psi_{tzz} - g \rho_x - f v_z - \psi_x (\psi_{zxx} + \psi_{zzz}) + \psi_z (\psi_{xx} + \psi_{xzz})] - v [v_t + f \psi_z - \psi_x v_z + \psi_z v_x] - \frac{g^2}{N^2} \rho [\rho_t + \frac{N^2}{g} \psi_x - \psi_x \rho_z + \psi_z \rho_x]. \quad (372)$$

Example 20 Let us consider the conservation Eq. (343)

$$D_t(C^1) + D_x(C^2) = \sigma(\rho_t + v \rho_x + \rho v_x)$$

where σ is connected with the velocity v and the pressure p of the Chaplygin gas by Eq. (319)

$$\sigma_x = -\frac{1}{p}, \quad \sigma_t + v \sigma_x = 0.$$

In this example Eq. (362) are not satisfied. Indeed, we have

$$\begin{aligned}\frac{\delta}{\delta v} [\sigma(\rho_t + v\rho_x + \rho v_x)] &= \sigma\rho_x - D_x(\sigma\rho) = -\rho\sigma = \rho \int \frac{dx}{p} \neq 0 \\ \frac{\delta}{\delta \rho} [\sigma(\rho_t + v\rho_x + \rho v_x)] &= \sigma_t - D_x(\sigma v) + \sigma v_x = -(\sigma_t + v\sigma_x) = 0 \\ \frac{\delta}{\delta p} [\sigma(\rho_t + v\rho_x + \rho v_x)] &= 0.\end{aligned}$$

Example 21 Let us consider the conservation Eq. (347)

$$D_t(C^1) + D_x(C^2) = t\rho \left(v_t + vv_x + \frac{1}{\rho} p_x \right) + tv(\rho_t + v\rho_x + \rho v_x).$$

Here Eq. (362) are not satisfied. Namely, writing

$$\begin{aligned}t\rho \left(v_t + vv_x + \frac{1}{\rho} p_x \right) + tv(\rho_t + v\rho_x + \rho v_x) \\ = t\rho v_t + 2t\rho vv_x + tp_x + tv\rho_t + tv^2\rho_x\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\delta}{\delta v} [t\rho v_t + 2t\rho vv_x + tp_x + tv\rho_t + tv^2\rho_x] &= -\rho \\ \frac{\delta}{\delta \rho} [t\rho v_t + 2t\rho vv_x + tp_x + tv\rho_t + tv^2\rho_x] &= -v \\ \frac{\delta}{\delta p} [t\rho v_t + 2t\rho vv_x + tp_x + tv\rho_t + tv^2\rho_x] &= 0.\end{aligned}$$

Exercise 10 Check if Eq. (362) are satisfied for the conservation Eqs. (349) and (351).

3 Utilization of Conservation Laws for Constructing Solutions of PDEs

In this part we will discuss a method for constructing exact solutions for systems of nonlinear partial differential equations. The method is based on knowledge of conservation laws of the equations under consideration and therefore it is called the *method of conservation laws*.

Application of the method to the Chaplygin gas allowed to construct new solutions containing several arbitrary parameters. One can verify that these solutions cannot be obtained as group invariant solutions.

3.1 General Discussion of the Method

As mentioned in Sect. 2.1.4, one can integrate or reduce the order of linear ordinary differential equations by rewriting them in a conservation form (217). Likewise one can integrate or reduce the order of a nonlinear ordinary differential equation as well as a system of ordinary differential equations using their conservation laws. Namely, a conservation law

$$D_x \left(\psi(x, y, y', \dots, y^{(s-1)}) \right) = 0 \quad (373)$$

for a nonlinear ordinary differential equation

$$F(x, y, y', \dots, y^{(s)}) = 0 \quad (374)$$

yields the first integral

$$\psi(x, y, y', \dots, y^{(s-1)}) = C_1. \quad (375)$$

We will discuss now an extension of this idea to partial differential equations. Namely, we will apply conservation laws for constructing particular exact solutions of systems of partial differential equations. Detailed calculations are given in examples considered in the next sections.

Let us assume that the system (236)

$$F_{\bar{\alpha}}(x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad \bar{\alpha} = 1, \dots, \bar{m} \quad (376)$$

has a conservation law (237)

$$\left[D_i(C^i) \right]_{(376)} = 0 \quad (377)$$

with a known conserved vector

$$C = (C^1, \dots, C^n) \quad (378)$$

where

$$C^i = C^i(x, u, u_{(1)}, \dots), \quad i = 1, \dots, n.$$

We write the conservation Eq. (377) in the form (361)

$$D_i(C^i) = \mu^{\bar{\alpha}} F_{\bar{\alpha}}(x, u, u_{(1)}, \dots, u_{(s)}). \quad (379)$$

For a given conserved vector (378) the coefficients $\mu^{\bar{\alpha}}$ in Eq. (379) are known functions $\mu^{\bar{\alpha}} = \mu^{\bar{\alpha}}(x, u, u_{(1)}, \dots)$.

We will construct a particular solution of the system (383) using the simplest conservation law (327)

$$D_t(\rho) + D_x(\rho v) = \rho_t + v\rho_x + \rho v_x. \quad (384)$$

The conservation Eq. (384) is written in the form (379) with the following conserved vector (378)

$$C^1 = \rho, \quad C^2 = \rho v. \quad (385)$$

The differential constraints (381) are written as follows

$$\rho = g(x), \quad \rho v = h(t). \quad (386)$$

Thus we look for solutions of the form

$$\rho = g(x), \quad v = \frac{h(t)}{g(x)}. \quad (387)$$

The functions (387) solve the second equation in (383) because the conservation law (384) coincides with the second Eq. (383) (see Remark 18). Therefore it remains to substitute (387) in the first and third equations of the system (383). The result of this substitution can be solved for the derivatives of p

$$p_x = -h' + \frac{h^2 g'}{g^2}, \quad p_t = -\frac{hg'}{g^2} p + \frac{hh'}{g} - \frac{h^3 g'}{g^3}. \quad (388)$$

The compatibility condition $p_{xt} = p_{tx}$ of the system (388) gives the equation

$$\left(g'' - 2\frac{g'^2}{g}\right)p = g^2 \frac{h''}{h} - 2g'h' - h^2 \frac{g''}{g} + 2h^2 \frac{g'^2}{g^2}. \quad (389)$$

For illustration purposes I will simplify further calculations by considering the particular case when the coefficient in front of p in Eq. (389) vanishes

$$g'' - 2\frac{g'^2}{g} = 0. \quad (390)$$

The solution of Eq. (390) is

$$g(x) = \frac{1}{ax + b}, \quad a, b = \text{const.} \quad (391)$$

Substituting (391) in Eq. (389) we obtain

$$h'' + 2ahh' = 0 \quad (392)$$

whence

$$h(t) = k \tan(c - akt) \quad (393)$$

if $a \neq 0$, and

$$h(t) = At + B \quad (394)$$

if $a = 0$.

If the constant a in (391) does not vanish, we substitute (391) and (393) in Eq. (388), integrate them and obtain

$$p = k^2(ax + b) + Q \cos(c - akt), \quad Q = \text{const.} \quad (395)$$

In the case $a = 0$ the similar calculations yield

$$p = -Ax + \frac{b}{2} A^2 t^2 + ABbt + Q, \quad Q = \text{const.} \quad (396)$$

Thus, using the conservation law (384) we have arrived at the solutions

$$\begin{aligned} \rho &= \frac{1}{ax + b} \\ v &= k(ax + b) \tan(c - akt) \\ p &= k^2(ax + b) + Q \cos(c - akt) \end{aligned} \quad (397)$$

and

$$\begin{aligned} \rho &= \frac{1}{b} \\ v &= b(At + B) \\ p &= -Ax + \frac{b}{2} A^2 t^2 + ABbt + Q. \end{aligned} \quad (398)$$

3.2.2 Differential Constraints Provided by Other Conserved Vectors

The conservation laws (328) and (330) give the following differential constraints (381)

$$\rho v^2 - p = g(x), \quad pv + \rho v^3 = h(t) \quad (399)$$

$$\rho v = g(x), \quad p + \rho v^2 = h(t) \quad (400)$$

$$t\rho v - x\rho = g(x), \quad tp + t\rho v^2 - x\rho v = h(t). \quad (401)$$

The nonlocal conserved vectors (346), (348) and (350) lead to the following differential constraints (381)

$$t\rho v + \tau = g(x), \quad p + \rho v^2 = h(t) \quad (402)$$

$$t\rho = g(x), \quad t\rho v - \tau = h(t) \quad (403)$$

$$\left(\frac{t^2}{2} - s\right)\rho = g(x), \quad \left(\frac{t^2}{2} - s\right)\rho v - t\tau = h(t). \quad (404)$$

The constraints (402) are not essentially different from the constraints (400). It is manifest if we write them in the form (382).

3.3 Application to Nonlinear Equation Describing an Irrigation System

The method of Sect. 3.1 can be used for constructing particular solutions not only of a system, but of a single partial differential equations as well.

Let us consider the nonlinear Eq. (171)

$$C(\psi)\psi_t = [K(\psi)\psi_x]_x + [K(\psi)(\psi_z - 1)]_z - S(\psi) \quad (405)$$

satisfying the nonlinear self-adjointness condition (173)

$$S'(\psi) = aC(\psi), \quad a = \text{const.} \quad (406)$$

and apply the method of Sect. 3.1 to the conserved vector (269)

$$C^1 = S(\psi)e^{at}, \quad C^2 = aK(\psi)\psi_x e^{at}, \quad C^3 = aK(\psi)(\psi_z - 1)e^{at}. \quad (407)$$

The conditions (381) are written

$$S(\psi)e^{at} = f(x, z), \quad aK(\psi)\psi_x e^{at} = g(t, z), \quad aK(\psi)(\psi_z - 1)e^{at} = h(t, x).$$

These conditions mean that the left-hand sides of the first, second and third equation do not depend on t , x and z , respectively. Therefore they can be equivalently written as the following differential constraints (see Eq. 382)

$$aS(\psi) + S'(\psi)\psi_t = 0, \quad [K(\psi)\psi_x]_x = 0, \quad [K(\psi)(\psi_z - 1)]_z = 0. \quad (408)$$

The constraints (408) reduce Eqs. (405) and (406). Hence, the particular solutions of Eq. (405) provided by the conserved vector (407) are described by the system

$$\begin{aligned} aC(\psi) - S'(\psi) &= 0, & aS(\psi) + S'(\psi)\psi_t &= 0 \\ [K(\psi)\psi_x]_x &= 0, & [K(\psi)(\psi_z - 1)]_z &= 0. \end{aligned} \quad (409)$$

4 Approximate Self-Adjointness and Approximate Conservation Laws

The methods developed in this paper can be extended to differential equations with a small parameter in order to construct approximate conservation laws using approximate symmetries. I will illustrate this possibility by examples. The reader interested in approximate symmetries can find enough material in [15–17], vol. 3, Chaps. 2 and 9.

4.1 The Van Der Pol Equation

The van der Pol equation has the form

$$F \equiv y'' + y + \varepsilon(y^3 - y') = 0, \quad \varepsilon = \text{const.} \neq 0. \quad (410)$$

4.1.1 Approximately Adjoint Equation

We have

$$\frac{\delta}{\delta y} \left\{ z \left[y'' + y + \varepsilon (y^3 - y') \right] \right\} = z'' + z + \varepsilon D_x (z - 3zy'^2).$$

Thus, the adjoint equation to the van der Pol equation is

$$F^* \equiv z'' + z + \varepsilon (z' - 3z'y'^2 - 6zy'y'') = 0.$$

We eliminate here y'' by using Eq. (410), consider ε as a small parameter and write F^* in the first order of precision with respect to ε . In other words, we write

$$y'' \approx -y. \quad (411)$$

Then we obtain the following *approximately adjoint equation* to equation (410)

$$F^* \equiv z'' + z + \varepsilon (z' - 3z'y'^2 + 6zy'y') = 0. \quad (412)$$

4.1.2 Approximate Self-Adjointness

Let us investigate Eq. (410) for approximate self-adjointness. Specifically, I will call Eq. (410) *approximately self-adjoint* if there exists a non-trivial (not vanishing identically) approximate substitution

$$z \approx f(x, y, y') + \varepsilon g(x, y, y') \quad (413)$$

such that F given by Eq. (410) and F^* defined by Eq. (412) approximately satisfy the condition (72) of nonlinear self-adjointness. In other words, the following equation is satisfied in the first-order of precision in ε

$$F^* \Big|_{z=f+\varepsilon g} = \lambda F. \quad (414)$$

Note, that the unperturbed equation $y'' + y = 0$ is nonlinearly self-adjoint. Namely it coincides with the adjoint equation $z'' + z = 0$ upon the substitution

$$z = \alpha y + \beta \cos x + \gamma \sin x, \quad \alpha, \beta, \gamma = \text{const.} \quad (415)$$

Therefore we will consider the substitution (413) of the following restricted form

$$z \approx f(x, y) + \varepsilon g(x, y, y'). \quad (416)$$

In differentiating $g(x, y, y')$ we will use Eq. (411) because we make out calculations in the first order of precision with respect to ε . Then we obtain

$$\begin{aligned} z' &= D_x(f) + \varepsilon D_x(g) \Big|_{y''=-y} \equiv f_x + y' f_y + \varepsilon(g_x + y' g_y - y g_{y'}) \\ z'' &= D_x^2(f) + \varepsilon D_x^2(g) \Big|_{y''=-y} \equiv f_{xx} + 2y' f_{xy} + y'^2 f_{yy} + y'' f_y \\ &\quad + \varepsilon(g_{xx} + 2y' g_{xy} - 2y g_{xy'} + y'^2 g_{yy} \\ &\quad - 2y y' g_{yy'} + y^2 g_{y'y'} - y g_y - y' g_{y'}). \end{aligned} \quad (417)$$

Substituting (417) in (412) and solving Eq. (414) with $\varepsilon = 0$ we see that f is given by Eq. (415). Then $\lambda = C$ and the terms with ε in Eq. (414) give the following second-order linear partial differential equation for $g(x, y, y')$

$$\begin{aligned} g + D_x^2(g) \Big|_{y''=-y} &= \alpha \left(4y'^3 - 6y^2 y' - 2y' \right) \\ &\quad + \beta \left(\sin x - 3y'^2 \sin x - 6yy' \cos x \right) \\ &\quad + \gamma \left(3y'^2 \cos x - \cos x - 6yy' \sin x \right). \end{aligned} \quad (418)$$

The standard existence theorem guarantees that Eq. (418) has a solution. It is manifest that the solution does not vanish because $g = 0$ does not satisfy Eq. (418). We conclude that the van der Pol equation (410) with a small parameter ε is approximately self-adjoint. The substitution (416) satisfying the approximate self-adjointness condition (414) has the form

$$z \approx \alpha y + \beta \cos x + \gamma \sin x + \varepsilon g(x, y, y') \quad (419)$$

where α, β, γ are arbitrary constants and $g(x, y, y')$ solves Eq. (418).

4.1.3 Exact and Approximate Symmetries

If ε is treated as an arbitrary constant, Eq. (410) has only one point symmetry, namely the one-parameter group of translations of the independent variable x . We will write the generator $X_1 = \partial/\partial x$ of this group in the form (186)

$$X_1 = y' \frac{\partial}{\partial y}. \quad (420)$$

If ε is a small parameter, then Eq. (410) has, along with the exact symmetry (420), the following 7 approximate symmetries ([15–17], vol. 3, Sect. 9.1.3.3)

$$\begin{aligned} X_2 &= \left\{ 4y - \varepsilon \left[y^2 y' + 3xy \left(y^2 + y'^2 \right) \right] \right\} \frac{\partial}{\partial y} \\ X_3 &= \left\{ 8 \cos x + \varepsilon \left[\left(4 - 3y'^2 - 9y^2 \right) x \cos x + 3(xy^2)' \sin x \right] \right\} \frac{\partial}{\partial y} \\ X_4 &= \left\{ 8 \sin x + \varepsilon \left[\left(4 - 3y'^2 - 9y^2 \right) x \sin x - 3(xy^2)' \cos x \right] \right\} \frac{\partial}{\partial y} \\ X_5 &= \left\{ 24y^2 \cos x - 24yy' \sin x + \varepsilon \left[(12yy' + 9yy'^3 + 9y^3 y') x \sin x \right. \right. \\ &\quad \left. \left. + (12y^2 - 9y^2 y'^2 - 6y^4) \sin x - (12y^2 - 9y^2 y'^2 - 9y^4) x \cos x \right. \right. \\ &\quad \left. \left. - 3y^3 y' \cos x \right] \right\} \frac{\partial}{\partial y} \\ X_6 &= \left\{ 24y^2 \sin x + 24yy' \cos x - \varepsilon \left[(12yy' + 9yy'^3 + 9y^3 y') x \cos x \right. \right. \\ &\quad \left. \left. + (12y^2 - 9y^2 y'^2 - 6y^4) \cos x + (12y^2 + 9y^2 y'^2 + 9y^4) x \sin x \right. \right. \\ &\quad \left. \left. + 3y^3 y' \sin x \right] \right\} \frac{\partial}{\partial y} \\ X_7 &= \left\{ 4y \cos 2x - 4y' \sin 2x + \varepsilon \left[3(yy'^2 - y^3) x \cos 2x \right. \right. \\ &\quad \left. \left. - 3y^2 y' \cos 2x + 6y^2 y' x \sin 2x + 2(y - y^3) \sin 2x \right] \right\} \frac{\partial}{\partial y} \\ X_8 &= \left\{ 4y \sin 2x + 4y' \cos 2x - \varepsilon \left[3(y^3 - yy'^2) x \sin 2x \right. \right. \end{aligned} \quad (421)$$

$$+ 3y^2y' \sin 2x + 6y^2y'x \cos 2x + 2(y - y^3) \cos 2x \Big] \frac{\partial}{\partial y}.$$

4.1.4 Approximate Conservation Laws

We can construct now approximate conserved quantities for the van der Pol equation using the formula (258) and the approximate substitution (419). Inserting in (258) the formal Lagrangian

$$\mathcal{L} = z \left[y'' + y + \varepsilon (y'^3 - y') \right]$$

we obtain

$$C = W \left[-z' + \varepsilon (3y'^2z - z) \right] + W'z. \quad (422)$$

Let us calculate the conserved quantity (422) for the operator X_1 given by Eq. (420). In this case $W = y'$, $W' = y''$, and therefore (422) has the form

$$C = -y'z' + \varepsilon (3y'^3 - y')z + y''z.$$

We eliminate here y'' via Eq. (410), use the approximate substitution (419) and obtain (in the first order of precision with respect to ε) the following approximate conserved quantity

$$C = -\alpha (y^2 + y'^2) + \beta (y' \sin x - y \cos x) - \gamma (y' \cos x + y \sin x) + \varepsilon (2\alpha y y'^3 + 2\beta y'^3 \cos x + 2\gamma y'^3 \sin x - yg - y'D_x(g)|_{y''=-y}). \quad (423)$$

Differentiating it and using the Eqs. (410) and (411) we obtain

$$D_x(C) = \varepsilon y' \left[\alpha (4y'^3 - 6y^2y' - 2y') + \beta (\sin x - 3y'^2 \sin x - 6yy' \cos x) + \gamma (3y'^2 \cos x - \cos x - 6yy' \sin x) - g - D_x^2(g)|_{y''=-y} \right] + o(\varepsilon) \quad (424)$$

where $o(\varepsilon)$ denotes the higher-order terms in ε . The Eqs. (418) and (424) show that the quantity (423) satisfies the approximate conservation law

$$D_x(C)|_{(410)} \approx 0. \quad (425)$$

Let us consider the operator X_2 from (421). In this case we have

$$\begin{aligned} W &= 4y - \varepsilon \left[y^2 y' + 3xy \left(y^2 + y'^2 \right) \right] \\ W' &\approx 4y' - \varepsilon \left[2y^3 + 5yy'^2 + 3x \left(y^2 y' + y'^3 \right) \right]. \end{aligned} \quad (426)$$

Proceeding as above we obtain the following approximate conserved quantity

$$\begin{aligned} C &= 4y'(\beta \cos x + \gamma \sin x) - 4y(\gamma \cos x - \beta \sin x) \\ &+ \varepsilon \left\{ 2\alpha y^2 \left(4y'^2 - y^2 - 2 \right) + 4y'g - 4yD_x(g) \Big|_{y''=-y} \right. \\ &+ \left. \left[7yy'^2 - 3xy'(y^2 + y'^2) - 2y^3 - 4y \right] (\beta \cos x + \gamma \sin x) \right. \\ &+ \left. \left[y^2 y' + 3xy(y^2 + y'^2) \right] (\gamma \cos x - \beta \sin x) \right\}. \end{aligned} \quad (427)$$

The calculation shows that the quantity (427) satisfies the approximate conservation law (425) in the following form

$$\begin{aligned} D_x(C) &= 4(\beta \cos x + \gamma \sin x) \left[y'' + y + \varepsilon \left(y'^3 - y' \right) \right] \\ &+ 4\varepsilon y \left[\alpha \left(4y'^3 - 6y^2 y' - 2y' \right) + \beta \left(\sin x - 3y'^2 \sin x - 6yy' \cos x \right) \right. \\ &+ \left. \gamma \left(3y'^2 \cos x - \cos x - 6yy' \sin x \right) - g - D_x^2(g) \Big|_{y''=-y} \right] + o(\varepsilon). \end{aligned} \quad (428)$$

Continuing this procedure, one can construct approximate conservation laws for the remaining approximate symmetries (421).

4.2 Perturbed KdV Equation

Let us consider again the KdV equation (73)

$$u_t = u_{xxx} + uu_x$$

and the following perturbed equation

$$F \equiv u_t - u_{xxx} - uu_x - \varepsilon u = 0. \quad (429)$$

We will follow the procedure described in Sect. 4.1.

4.2.1 Approximately Adjoint Equation

Let us write the formal Lagrangian for Eq. (429) in the form

$$\mathcal{L} = v [-u_t + u_{xxx} + uu_x + \varepsilon u]. \quad (430)$$

Then

$$\frac{\delta \mathcal{L}}{\delta u} = v_t - v_{xxx} - D_x(uv) + vu_x + \varepsilon v = v_t - v_{xxx} - uv_x + \varepsilon v.$$

Hence, the approximately adjoint equation to equation (429) has the form

$$F^* \equiv v_t - v_{xxx} - uv_x + \varepsilon v = 0. \quad (431)$$

4.2.2 Approximate Self-Adjointness

As mentioned in Sect. 1.3.1, Example 11, the KdV Eq. (73) is nonlinearly self-adjoint with the substitution (75)

$$v = A_1 + A_2u + A_3(x + tu). \quad (75)$$

Therefore in the case of the perturbed Eq. (429) we look for the substitution

$$v = \phi(t, x, u) + \varepsilon\psi(t, x, u)$$

satisfying the nonlinear self-adjointness condition

$$F^* \Big|_{v=\phi+\varepsilon\psi} = \lambda F \quad (432)$$

in the first-order of precision in ε , in the following form

$$v = A_1 + A_2u + A_3(x + tu) + \varepsilon\psi(t, x, u). \quad (433)$$

When we substitute the expression (433) in the definition (431) of F^* , the terms without ε in Eq. (432) disappear by construction of the substitution (75) and give $\lambda = A_2 + A_3t$. Then we write Eq. (432), rearranging the terms, in the form

$$\begin{aligned} & \varepsilon\psi_u[u_t - u_{xxx} - uu_x] - 3\varepsilon u_{xx}[u_x\psi_{uu} + \psi_{xu}] \\ & - \varepsilon u_x[u_x^2\psi_{uuu} + 3u_x\psi_{xuu} + 3\psi_{xXu}] + \varepsilon[\psi_t - \psi_{xxx} - u\psi_x \\ & + A_1 + A_2u + A_3(x + tu)] = -\varepsilon(A_2 + A_3t)u. \end{aligned} \quad (434)$$

In view Eq. (429), the first term in the first line of Eq. (434) is written $\varepsilon^2 u \psi_u$. Hence, this term vanishes in our approximation. The terms with u_{xx} in the first line of Eq. (434) yield

$$\psi_{uu} = 0, \quad \psi_{xu} = 0$$

whence

$$\psi = f(t)u + g(t, x).$$

The third bracket in the first line of Eq. (434) vanishes, and Eq. (434) becomes

$$[f'(t) - g_x(t, x)]u + g_t(t, x) - g_{xx}(t, x) + 2[A_2 + A_3t]u + A_1 + A_3x = 0.$$

After rather simple calculations we solve this equation and obtain

$$g(t, x) = A_4 - A_1t + (A_5 + 2A_2 - A_3t)x, \quad f(t) = A_6 + A_5t - \frac{3}{2}A_3t^2.$$

We conclude that the perturbed KdV equation (429) is approximately self-adjoint. The approximate substitution (433) has the following form

$$v \approx A_1 + A_2u + A_3(x + tu) + \varepsilon \left[\left(A_6 + A_5t - \frac{3}{2}A_3t^2 \right) u + A_4 - A_1t + (A_5 + 2A_2 - A_3t)x \right]. \quad (435)$$

4.2.3 Approximate Symmetries

Recall that the Lie algebra of point symmetries of the KdV equation (73) is spanned by the following operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= t \frac{\partial}{\partial x} - \frac{\partial}{\partial u} \\ X_4 &= 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u}. \end{aligned} \quad (436)$$

Following the method for calculating approximate symmetries and using the terminology presented in [15–17], vol. 3, Chap. 2, we can prove that all symmetries (436) are stable. Namely the perturbed equation (429) inherits the symmetries (436) of the KdV equation in the form of the following approximate symmetries

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= t \frac{\partial}{\partial x} - \frac{\partial}{\partial u} + \varepsilon \left(\frac{1}{2} t^2 \frac{\partial}{\partial x} - t \frac{\partial}{\partial u} \right) \\
X_4 &= 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} - \varepsilon \left[\frac{9}{2} t^2 \frac{\partial}{\partial t} + 3tx \frac{\partial}{\partial x} - (6tu + 3x) \frac{\partial}{\partial u} \right].
\end{aligned} \quad (437)$$

4.2.4 Approximate Conservation Laws

We can construct now the approximate conservation laws

$$\left[D_t(C^1) + D_x(C^2) \right]_{(429)} \approx 0 \quad (438)$$

for the perturbed KdV equation (429) using its approximate symmetries (437), the general formula (258) and the approximate substitution (435). Inserting in (258) the formal Lagrangian (430) we obtain

$$C^1 = -Wv, \quad C^2 = W[uv + v_{xx}] - v_x D_x(W) + v D_x^2(W). \quad (439)$$

I will calculate here the conserved vector (439) for the operator X_4 from (437). In this case we have

$$W = -2u - 3tu_t - xu_x + \varepsilon \left(6tu + 3x + \frac{9}{2} t^2 u_t + 3txu_x \right). \quad (440)$$

We further simplify the calculations by taking the particular substitution (435) with $A_2 = 1$, $A_1 = A_3 = \dots = A_6 = 0$. Then

$$v = u + 2\varepsilon x. \quad (441)$$

Substituting (440) and (441) in the first component of the vector (439) and then eliminating u_t via Eq. (429) we obtain

$$\begin{aligned}
C^1 &\approx (2u + 3tu_t + xu_x)(u + 2\varepsilon x) - \varepsilon \left(6tu + 3x + \frac{9}{2} t^2 u_t + 3txu_x \right) u \\
&= 2u^2 + 3tuu_{xxx} + 3tu^2 u_x + xuu_x + \varepsilon \left(xu + 6txu_{xxx} + 3txuu_x \right. \\
&\quad \left. + 2x^2 u_x - 3tu^2 - \frac{9}{2} t^2 uu_{xxx} - \frac{9}{2} t^2 u^2 u_x \right).
\end{aligned}$$

Upon singling out the total derivatives in x , it is written

$$\begin{aligned}
C^1 &\approx \frac{3}{2} u^2 - 3\varepsilon \left(xu + \frac{3}{2} tu^2 \right) + D_x \left[\frac{1}{2} xu^2 + tu^3 - \frac{3}{2} tu_x^2 + 3tuu_{xx} \right. \\
&\quad \left. + \varepsilon \left(2x^2 u + \frac{3}{2} txu^2 - \frac{3}{2} t^2 u^3 - 6tu_x + 6txu_{xx} + \frac{9}{4} t^2 u_x^2 - \frac{9}{2} t^2 uu_{xx} \right) \right].
\end{aligned} \quad (442)$$

Then we substitute (440) and (441) in the second component of the vector (423), transfer the term $D_x(\dots)$ from C^1 to C^2 , multiply the resulting vector (C^1, C^2) by $2/3$ and arrive at the following vector

$$\begin{aligned} C^1 &= u^2 - 2\varepsilon \left[xu + \frac{3}{2} tu^2 \right] \\ C^2 &= u_x^2 - \frac{2}{3} u^3 - 2uu_{xx} + \varepsilon \left[xu^2 - 2u_x + 2xu_{xx} + 2tu^3 - 3tu_x^2 + 6tuu_{xx} \right]. \end{aligned} \quad (443)$$

The approximate conservation law (438) for the vector (443) is satisfied in the following form

$$\begin{aligned} D_t(C^1) + D_x(C^2) &= 2u(u_t - u_{xxx} - uu_x - \varepsilon u) \\ &\quad - 2\varepsilon(x + 3tu)(u_t - u_{xxx} - uu_x) + o(\varepsilon). \end{aligned}$$

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