2. Stress and Strain Analysis and Measurement

The engineering design of structures using polymers requires a thorough knowledge of the basic principles of stress and strain analysis and measurement. Readers of this book should have a fundamental knowledge of stress and strain from a course in elementary solid mechanics and from an introductory course in materials. Therefore, we do not rigorously derive from first principles all the necessary concepts. However, in this chapter we provide a review of the fundamentals and lay out consistent notation used in the remainder of the text. It should be emphasized that the interpretations of stress and strain distributions in polymers and the properties derived from the standpoint of the traditional analysis given in this chapter are approximate and not applicable to viscoelastic polymers under all circumstances. By comparing the procedures discussed in later chapters with those of this chapter, it is therefore possible to contrast and evaluate the differences.

2.1. Some Important and Useful Definitions

In elementary mechanics of materials (Strength of Materials or the first under-graduate course in solid mechanics) as well as in an introductory graduate elasticity course five fundamental assumptions are normally made about the characteristics of the materials for which the analysis is valid. These assumptions require the material to be,

- **Linear**
- **Homogeneous**
- **Isotropic**
- **Elastic**
- **Continuum**

Provided that a material has these characteristics, be it a metal or polymer, the elementary stress analysis of bars, beams, frames, pressure vessels, columns, etc. using these assumptions is quite accurate and useful. However, when these assumptions are violated serious errors can occur if the same analysis approaches are used. It is therefore incumbent upon engineers to thoroughly

understand these fundamental definitions as well as how to determine if they are appropriate for a given situation. As a result, the reader is encouraged to gain a thorough understanding of the following terms:

Linearity: Two types of linearity are normally assumed: Material linearity (Hookean stress-strain behavior) or linear relation between stress and strain; Geometric linearity or small strains and deformation.

Elastic: Deformations due to external loads are completely and instantaneously reversible upon load removal.

Continuum: Matter is continuously distributed for all size scales, i.e. there are no holes or voids.

Homogeneous: Material properties are the same at every point or material properties are invariant upon translation.

Inhomogeneous or Heterogeneous: Material properties are not the same at every point or material properties vary upon translation.

Amorphous: Chaotic or having structure without order. An example would be glass or most metals on a macroscopic scale.

Crystalline: Having order or a regular structural arrangement. An example would be naturally occurring crystals such as salt or many metals on the microscopic scale within grain boundaries.

Isotropic: Materials which have the same mechanical properties in all directions at an arbitrary point or materials whose properties are invariant upon rotation of axes at a point. Amorphous materials are isotropic.

Anisotropic: Materials which have mechanical properties which are not the same in different directions at a point or materials whose properties vary with rotation at a point. Crystalline materials are anisotropic.

Plastic: The word comes from the Latin word plasticus, and from the Greek words plastikos which in turn is derived from plastos (meaning molded) and from plassein (meaning to mold). Unfortunately, this term is often used as a generic name for a polymer (see definition below) probably because many of the early polymers (cellulose, polyesters, etc.) appear to yield and/or flow in a similar manner to metals and could be easily molded. However, not all polymers are moldable, exhibit plastic flow or a definitive yield point.

Viscoelasticity or Rheology: The study of materials whose mechanical properties have characteristics of both solid and fluid materials. Viscoelasticity is a term often used by those whose primary interest is solid

mechanics while rheology is a term often used by those whose primary interest is fluid mechanics. The term also implies that mechanical properties are a function of time due to the intrinsic nature of a material and that the material possesses a memory (fading) of past events. The latter separates such materials from those with time dependent properties due primarily to changing environments or corrosion. All polymers (fluid or solid) have time or temperature domains in which they are viscoelastic.

Polymer: The word Polymer originates from the Greek word "polymeros" which means many-membered (Clegg and Collyer 1993). Often the word polymer is thought of as being composed of the two words; "poly" meaning many and "mer" meaning unit. Thus, the word polymer means many units and is very descriptive of a polymer molecule.

Several of these terms will be reexamined in this chapter but the intent of the remainder of this book is to principally consider aspects of the last three.

2.2. Elementary Definitions of Stress, Strain and Material Properties

This section will describe the most elementary definitions of stress and strain typically found in undergraduate strength of materials texts. These definitions will serve to describe some basic test methods used to determine elastic material properties. A later section will revisit stress and strain, defining them in a more rigorous manner.

Often, stress and strain are defined on the basis of a simple uniaxial tension test. Typically, a "dogbone" specimen such as that shown in **Fig. 2.1a** is used and material properties such as Young's modulus, Poisson's ratio, failure (yield) stress and strain are found therefrom. The specimen may be cut from a thin flat plate of constant thickness or may be machined from a cylindrical bar. The "dogbone" shape is to avoid stress concentrations from loading machine connections and to insure a homogeneous state of stress and strain within the measurement region. The term homogeneous here indicates a uniform state of stress or strain over the measurement region, i.e. the throat or reduced central portion of the specimen. **Figure 2.1b** shows the uniform or constant stress that is present and that is calculated as given below.
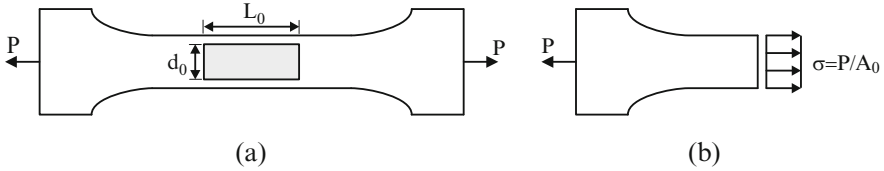
Fig. 2.1 "Dogbone" tensile specimen

The engineering (average) stress can be calculated by dividing the applied tensile force, **P**, (normal to the cross section) by the area of the original cross sectional area A_0 as follows,

$$\sigma_{av} = \frac{P}{A_0} \qquad (2.1)$$

The engineering (average) strain in the direction of the tensile load can be found by dividing the change in length, ΔL, of the inscribed rectangle by the original length L_0,

$$\varepsilon_{av} = \int_{L_0}^{L} \frac{dL}{L_0} = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} \qquad (2.2)$$

or

$$\varepsilon_{av} = \frac{L}{L_0} - 1 = \lambda - 1 \qquad (2.3)$$

The term λ in the above equation is called the extension ratio and is sometimes used for large deformations such as those which may occur with low modulus rubbery polymers.

True stress and strain are calculated using the instantaneous (deformed at a particular load) values of the cross-sectional area, **A**, and the length of the rectangle, **L**,

$$\sigma_t = \frac{F}{A} \qquad (2.4)$$

and

$$\varepsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln\frac{L}{L_0} = \ln(1 + \varepsilon) \qquad (2.5)$$

True strain is related to the engineering strain as indicated in **Eq. 2.5** and can also be shown to differ from the engineering strain by higher order terms (ε^2, ε^3, etc.) which are negligible for the small (linear) strain regime. Thus in the limit of small strain, the true strain and engineering strain are identical. These and other nonlinear measures are used for polymers and other materials undergoing large deformations.

Hooke's law is valid provided the stress varies linearly with strain and Young's modulus, **E**, may be determined from the slope of the stress-strain curve or by dividing stress by strain,

$$E = \frac{\sigma_{av}}{\varepsilon_{av}} \tag{2.6}$$

or

$$E = \frac{P/A_0}{\Delta L/L_0} \tag{2.7}$$

and the axial deformation over length $\mathbf{L_0}$ is,

$$\delta = \Delta L = \frac{PL_0}{A_0 E} \tag{2.8}$$

Poisson's ratio, $\boldsymbol{\nu}$, is defined as the absolute value of the ratio of strain transverse, $\boldsymbol{\varepsilon_y}$, to the load direction to the strain in the load direction, $\boldsymbol{\varepsilon_x}$,

$$\nu = \frac{\varepsilon_y}{\varepsilon_x} \tag{2.9}$$

The transverse strain $\boldsymbol{\varepsilon_y}$, of course can be found from,

$$\varepsilon_y = \frac{d - d_0}{d_0} \tag{2.10}$$

and is negative for an applied tensile load.

Shear properties can be found from a right circular cylinder loaded in torsion as shown in **Fig. 2.2**, where the shear stress, $\boldsymbol{\tau}$, angle of twist, $\boldsymbol{\theta}$, and shear strain, $\boldsymbol{\gamma}$, are given by,

$$\tau = \frac{Tr}{J}, \ \theta = \frac{TL}{JG}, \ \gamma = \frac{\delta}{L} = \frac{r\theta}{L} \tag{2.11}$$
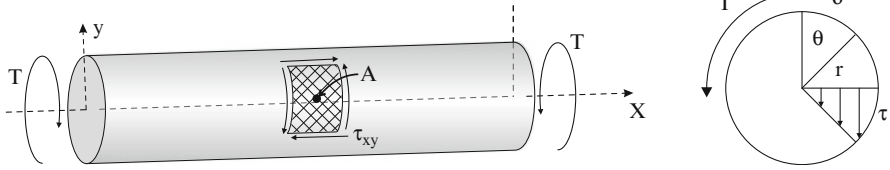
Fig. 2.2 Typical torsion test specimen to obtain shear properties

Herein, **L** is the length of the cylinder, **T** is the applied torque, **r** is the radial distance, **J** is the polar second moment of area and **G** is the shear modulus. These equations are developed assuming a linear relation between shear stress and strain as well as homogeneity and isotropy. With these assumptions, the shear stress and strain vary linearly with the radius and a pure shear stress state exists on any circumferential plane as shown on the surface at point **A** in **Fig. 2.2**. The shear modulus, **G**, is the slope of the shear stress-strain curve and may be found from,

$$G = \frac{\tau}{\gamma} \tag{2.12}$$

where the shear strain is easily found by measuring only the angular rotation, $\boldsymbol{\theta}$, in a given length, **L**. The shear modulus is related to Young's modulus and can also be calculated from,

$$G = \frac{E}{2(1 + \nu)} \tag{2.13}$$

As Poisson's ratio, $\boldsymbol{\nu}$, varies between **0.3** and **0.5** for most materials, the shear modulus is often approximated by, **G ~ E/3**.

While tensile and torsion bars are the usual methods to determine engineering properties, other methods can be used to determine material properties such as prismatic beams under bending or flexure loads similar to those shown in **Fig. 2.3**.

The elementary strength of materials equations for bending (flexural) stress, $\boldsymbol{\sigma_x}$, shear stress, $\boldsymbol{\tau_{xy}}$, due to bending and vertical deflection, **v**, for a beam loaded in bending are,

$$\sigma_x = \frac{M_{zz}y}{I_{zz}}, \quad \tau_{xy} = \frac{VQ}{I_{zz}b}, \quad \frac{d^2v}{dx^2} = \frac{M_{zz}}{EI_{zz}} \tag{2.14}$$

where **y** is the distance from the neutral plane to the point at which stress is calculated, $\mathbf{M_{zz}}$ is the applied moment, $\mathbf{I_{zz}}$ is the second moment of the cross-sectional area about the neutral plane, **b** is the width of the beam at the point of

calculation of the shear stress, **Q** is the first moment of the area about the neutral plane (see a strength of materials text for a more explicit definition of each of these terms), and other terms are as defined previously.

For a beam with a rectangular cross-section, the bending stress, σ_x, varies linearly and shear stress, τ_{xy}, varies parabolically over the cross-section as shown in **Fig. 2.4**.
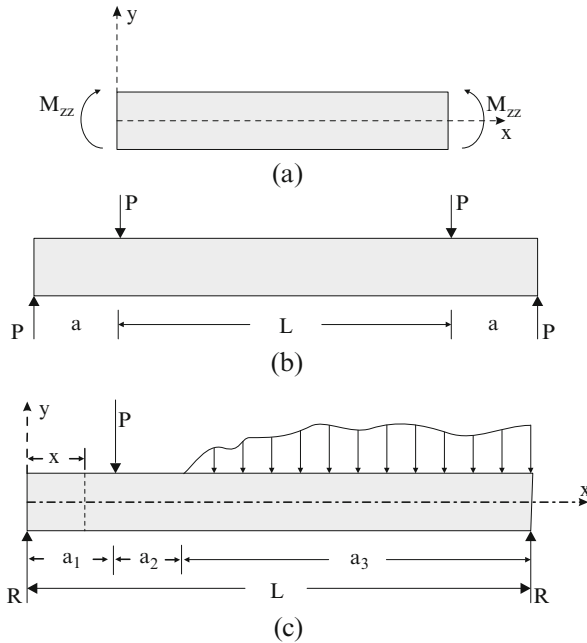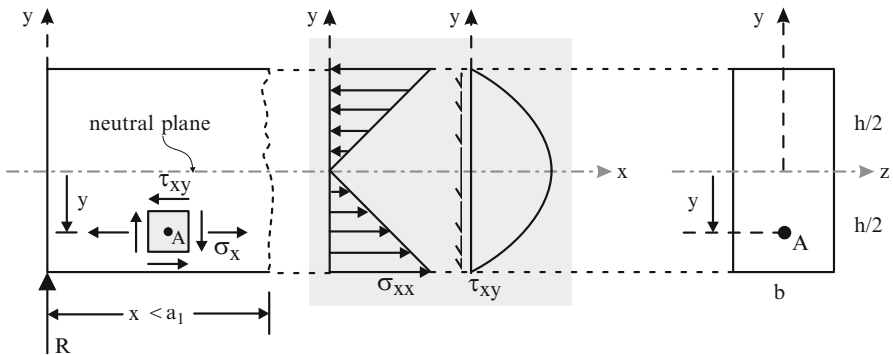
Fig. 2.3 Beams in bending

Fig. 2.4 Normal and shear stress variation in a rectangular beam in flexure

Using **Eq. 2.14**, given the applied moment, **M**, geometry of the beam, and deflection at a point, it is possible to calculate the modulus, **E**. Strictly speaking, the equations for bending stress and beam deflections are only valid for pure bending as depicted in **Fig. 2.3a, b** but give good approximations for other types of loading such as that shown in **Fig. 2.3c** as long as the beam is not very short. Very short beams require a shear correction factor for beam deflection.

As an example, a beam in three-point bending as shown in **Fig. 2.5** is often used to determine a "flex (or flexural) modulus" which is reported in industry specification sheets describing a particular polymer.
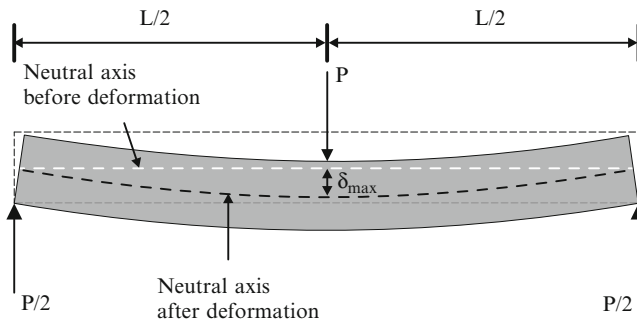

Fig. 2.5 Three-point bend specimen

The maximum deflection can be shown to be,

$$\delta_{max} = \frac{PL^3}{48EI} \tag{2.15}$$

from which the flexural (flex) modulus is found to be,

$$E = \frac{PL^3}{48I} \frac{1}{\delta_{max}} \tag{2.16}$$

Fundamentally, any structure under load can be used to determine properties provided the stress can be calculated and the strain can be measured at the same location. However, it is important to note that no method is available to measure stress directly. Stresses can only be calculated through the determination of forces using Newton's laws. On the other hand, strain can be determined directly from measured deformations. That is, displacement or motion is the physically measured quantity and force (and hence stress) is a defined, derived or calculated quantity. Some might argue that photoelastic techniques may qualify for the direct measurement of stress but it can also be argued that this effect is due to

interaction of light on changes in the atomic and molecular structure associated with a birefringent material, usually a polymer, caused by load induced displacements or strain.

It is clear that all the specimens used to determine properties such as the tensile bar, torsion bar and a beam in pure bending are special solid mechanics boundary value problems (BVP) for which it is possible to determine a "closed form" solution of the stress distribution using only the loading, the geometry, equilibrium equations and an assumption of a linear relation between stress and strain. It is to be noted that the same solutions of these BVP's from a first course in solid mechanics can be obtained using a more rigorous approach based on the Theory of Elasticity.

While the basic definitions of stress and strain are unchanged regardless of material, it should be noted that the elementary relations used above are often not applicable to polymers. As will be discussed in detail in the next chapters, polymers are inherently viscoelastic. For example, the rate of loading in a simple tension test will change the value measured for E in a viscoelastic material since modulus is inherently a function of time.

2.3. Typical Stress-Strain Properties

Properties of materials can be determined using the above elementary approaches. Often, for example, static tensile or compression tests are performed with a modern computer driven servo-hydraulic testing system such as the one shown in **Fig. 2.6**. The applied load is measured by a load cell (shown in (a) just above the grips) and deformation is found by either an extensometer (shown in (b) attached to the specimen) or an electrical resistance strain gage shown in (c). The latter is glued to the specimen and the change in resistance is measured as the specimen and the gage elongate. (Many additional methods are available to measure strain, including laser extensometers, moiré techniques, etc.). The cross-sectional area of the specimen and the gage length are input into the computer and the stress strain diagram is printed as the test is being run or can be stored for later use. The reason for a homogeneous state of stress and strain is now obvious. If a homogeneous state of stress and strain do not exist, it is only possible to determine the average strain value over the gage length region with this procedure and not the true properties of the material at a point.

Typical stress-strain diagrams for brittle and ductile materials are shown in **Fig. 2.7**. For brittle materials such as cast iron, glass, some epoxy resins, etc., the stress strain diagram is linear from initial loading (point 0) nearly to rupture

(point B) when average strains are measured. As will be discussed subsequently, stress and strain are "point" quantities if the correct mathematical definition of each is used. As a result, if the strain were actually measured at a single point, i.e., the point of final failure, the stress and strain at failure even for a brittle material might be slightly higher than the average values shown in **Fig. 2.7**.
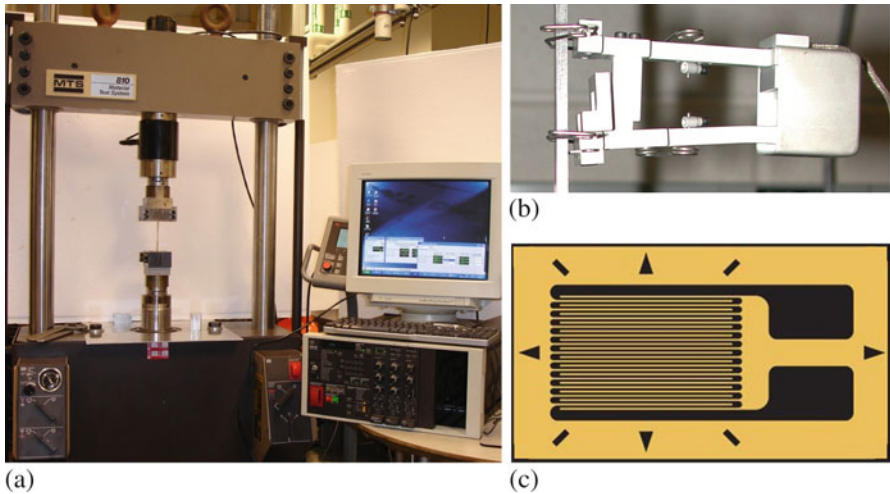

(a)   (b)

                                       (c)

Fig. 2.6 (**a**) Servo-hydraulic testing system: (**b**) extensometer, (**c**) electrical resistance strain gage

For ductile materials such as many aluminum alloys, copper, etc., the stress-strain diagram may be nonlinear from initial loading until final rupture. However, for small stresses and strains, a portion may be well approximated by a straight line and an approximate proportional limit (point A) can be determined. For many metals and other materials, if the stress exceeds the proportional limit a residual or permanent deformation may remain when the specimen is unloaded and the material is said to have "yielded". The exact yield point may not be the same as the proportional limit and if this is the case the location is difficult to determine. As a result, an arbitrary "0.2 % offset" procedure is often used to determine the yield point in metals. That is, a line parallel to the initial tangent to the stress-strain diagram is drawn to pass through a strain of 0.002 in./in. The yield point is then defined as the point of intersection of this line and the stress-strain diagram (point C in **Fig. 2.7**). This procedure can be used for polymers but the offset must be much larger than 0.2 % definition used for metals. Procedures to find the yield point in polymers will be discussed in Chaps. 3 and 11.
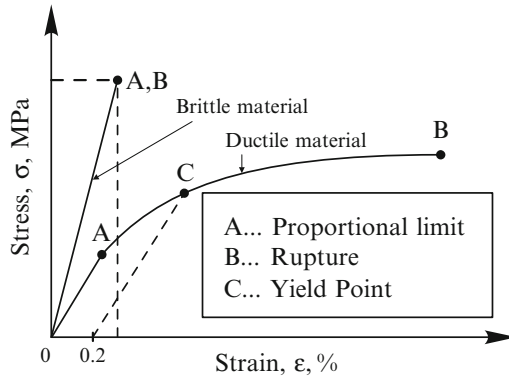
Fig. 2.7 Stress-strain diagrams for brittle and ductile materials

An approximate sketch of the stress-strain diagram for mild steel is shown in **Fig. 2.8a**. The numbers given for proportional limit, upper and lower yield points and maximum stress are taken from the literature, but are only approximations. Notice that the stress is nearly linear with strain until it reaches the upper yield point stress which is also known as the elastic-plastic tensile instability point. At this point the load (or stress) decreases as the deformation continues to increase. That is, less load is necessary to sustain continued deformation. The region between the lower yield point and the maximum stress is a region of strain hardening, a concept that is discussed in the next section. Note that if true stress and strain are used, the maximum or ultimate stress is at the rupture point.

The elastic-plastic tensile instability point in mild steel has received much attention and many explanations. Some polymers, such as polycarbonate, exhibit a similar phenomenon. Both steel and polycarbonate not only show an upper and lower yield point but visible striations of yielding; plastic flow or slip lines (Luder's bands), at an approximate angle of 54.7° to the load axis, occur in each for stresses equivalent to the upper yield point stress. (For a description and an example of Luder's band formation in polycarbonate, see **Fig. 3.7c**). It has been argued that this instability point (and the appearance of an upper and lower yield point) in metals is a result of the testing procedure and is related to the evolution of internal damage. That this is the case for polycarbonate will be shown in Chap. 3. For a discussion of these factors for metals, see Drucker (1962) and Kachanov (1986).

If the strain scale of **Fig. 2.8a** is expanded as illustrated in **Fig. 2.8b**, the stress-strain diagram of mild steel is approximated by two straight lines; one for the linear elastic portion and one which is horizontal at a stress level of the lower yield point. This characteristic of mild steel to "flow", "neck" or "draw" without

rupture when the yield point has been exceeded has led to the concepts of plastic, limit or ultimate design. That is, just because the yield point has been exceeded does not mean that the material cannot support load. In fact, it can be shown that economy of design and weight savings can be obtained using limit design concepts. Concepts of plasticity and yielding date back to St. Venant in about 1870 but the concepts of plastic or limit design including computational plasticity evolved primarily in the latter half of the twentieth century (see Westergaard (1964) for a discussion of the history of solid mechanics including comments on the evolution of plasticity). Also, an excellent discussion of plasticity and metal forming is given by Osakada (2010). Computational plasticity has its origins associated with calculations of deformations beyond the yield point for stress-strain diagrams similar to that of mild steel and will be briefly discussed in Chap. 11 in the context of polymers.
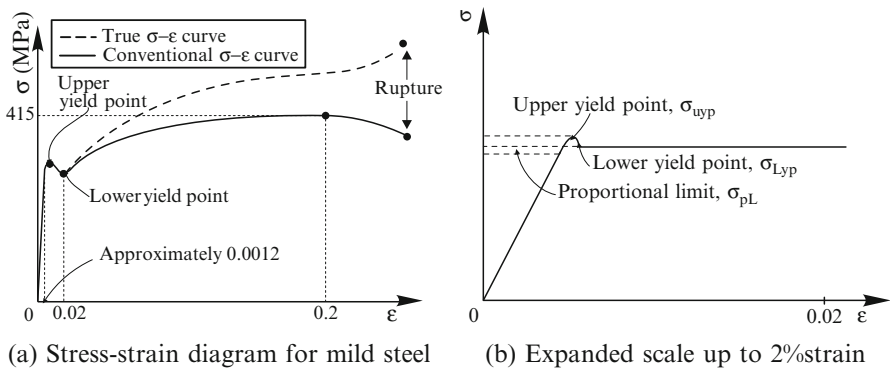

(a) Stress-strain diagram for mild steel   (b) Expanded scale up to 2%strain

Fig. 2.8 Typical tensile stress-strain diagrams (not to scale)

As will be discussed in Chap. 3, the same procedures discussed in the present chapter are used to determine the stress-strain characteristics of polymers. If only a single rate of loading is used, similar results will be obtained. On the other hand, if polymers are loaded at various strain rates, the behavior varies significantly from that of metals. Generally, metals do not show rate effects at ambient temperatures. They do, however, show considerable rate effects at elevated temperatures but the molecular mechanisms responsible for such effects are very different in polymers and metals.

It is appropriate to note that industry specification sheets often give the elastic modulus, yield strength, strain to yield, ultimate stress and strain to failure as determined by these elementary techniques. One objective of this text is to emphasize the need for approaches to obtain more appropriate specifications for the engineering design of polymers.

2.4. Idealized Stress-Strain Diagrams

The stress-strain diagrams discussed in the last section are often approximated by idealized diagrams. For example, a linear elastic perfectly brittle material is assumed to have a stress-strain diagram similar to that given in **Fig. 2.9a**. On the other hand, the stress-strain curve for mild steel can be approximated as a perfectly elastic-plastic material with the stress-strain diagram given in **Fig. 2.9b**. Metals (and polymers) often have nonlinear stress-strain behavior as shown in **Fig. 2.10a**. These are sometimes modeled with a bilinear diagram as shown in **Fig. 2.10b** and are referred to as a perfectly linear elastic strain hardening material. Here the 0.2 % offset method for determining the yield point for metals is used as an illustration. For polymers a different method must be used (See Chap. 3).
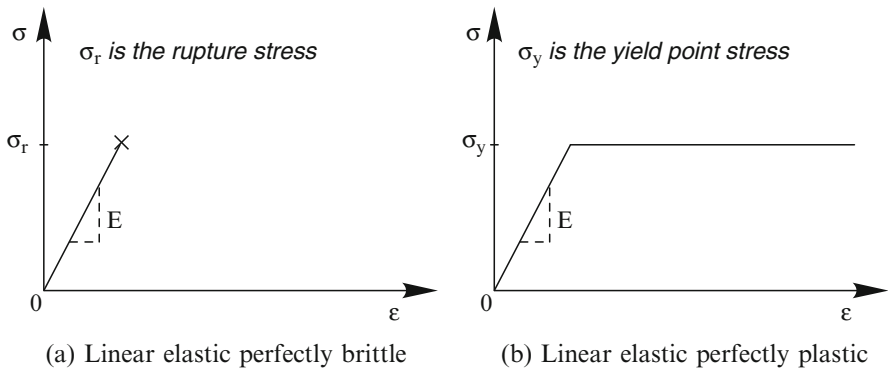

(a) Linear elastic perfectly brittle   (b) Linear elastic perfectly plastic

Fig. 2.9 Idealized uniaxial stress-strain diagrams

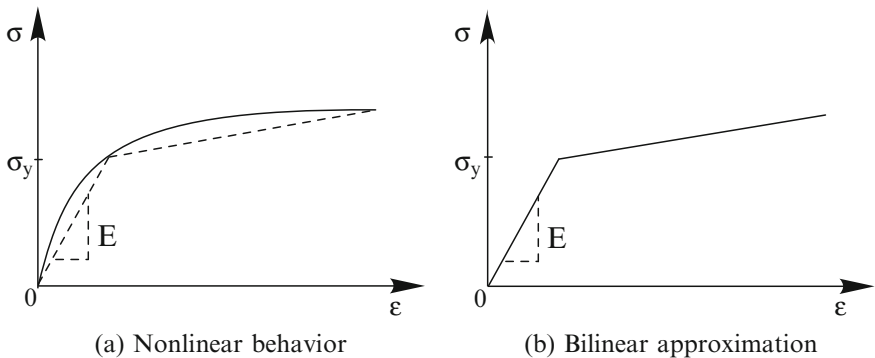(a) Nonlinear behavior   (b) Bilinear approximation

Fig. 2.10 Nonlinear stress-strain diagram with linear elastic strain hardening approximation (σ_y is the yield point stress)

2.5. Mathematical Definitions of Stress, Strain and Material Characteristics

The previous sections give a brief review of some elementary concepts of solid mechanics which are often used to determine basic properties of most engineering materials. However, these approaches are sometimes not adequate and more advanced concepts from the theory of elasticity or the theory of plasticity are needed. Herein, a brief discussion is given of some of the more exact modeling approaches for linear elastic materials. Even these methods need to be modified for viscoelastic materials but this section will only give some of the basic elasticity concepts.

Definition of a Continuum A basic assumption of elementary solid mechanics is that a material can be approximated as a continuum. That is, the material (of mass ΔM) is continuously distributed over an arbitrarily small volume, ΔV, such that,

$$\lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV} = \text{const.} = \rho = (\text{density at a point}) \qquad \textbf{(2.17)}$$

Quite obviously such an assumption is at odds with our knowledge of the atomic and molecular nature of materials but is an acceptable approximation for most engineering applications. The principles of linear elasticity, though based upon the premise of a continuum, have been shown to be useful in estimating the stress and strain fields associated with dislocations and other non-continuum microstructural details.

Physical and Mathematical Definition of Normal Stress and Shear Stress Consider a body in equilibrium under the action of external forces $\mathbf{F_i}$ as shown in **Fig. 2.11a**. If a cutting plane is passed through the body as shown in **Fig. 2.11b**, equilibrium is maintained on the remaining portion by internal forces distributed over the newly exposed internal surface.
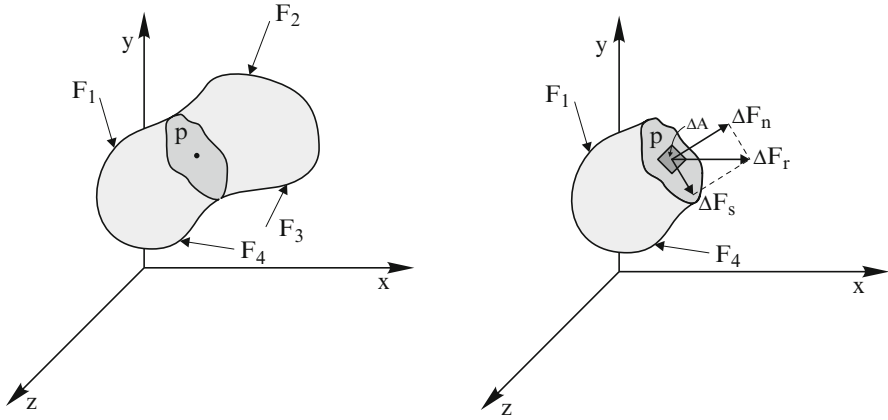
Fig. 2.11 Physical definition of normal force and shear force

At any arbitrary point **p**, the incremental resultant force, $\mathbf{\Delta F_r}$, on the cut surface can be broken up into a normal force in the direction of the normal, **n**, to surface **S** and a tangential force parallel to surface **S**. The normal stress and the shear stress at point **p** is mathematically defined as,

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} \quad \tau_s = \lim_{\Delta A \to 0} \frac{\Delta F_s}{\Delta A} \qquad (2.18)$$

where $\mathbf{\Delta F_n}$ and $\mathbf{\Delta F_s}$ are the normal and shearing forces on the area $\mathbf{\Delta A}$ surrounding point **p**.

Alternatively, the resultant force, $\mathbf{\Delta F_r}$, at point **p** can be divided by the area, $\mathbf{\Delta A}$, and the limit taken to obtain the stress resultant $\mathbf{\sigma_r}$ as shown in **Fig. 2.12**. Normal and tangential components of this stress resultant will then be the normal stress $\mathbf{\sigma_n}$ and shear stress $\mathbf{\tau_s}$ at point **p** on the infinitesimal area $\mathbf{\Delta A}$.
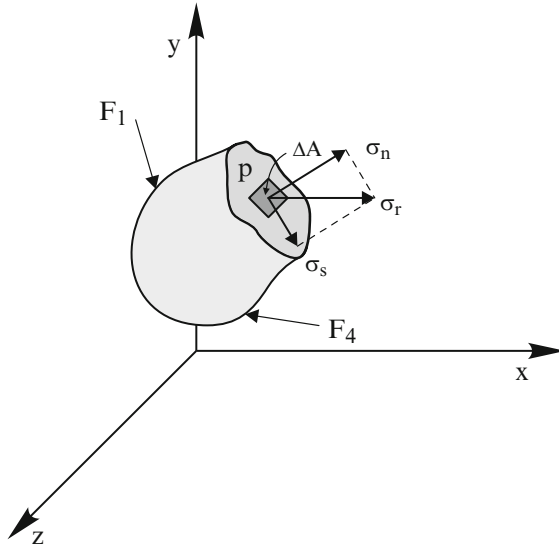
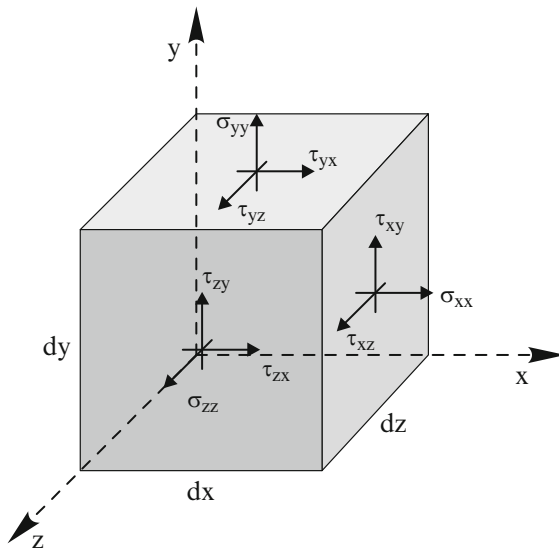**Fig. 2.12** Stress resultant definition

Fig. 2.13 Cartesian components of internal stresses

If a pair of cutting planes a differential distance apart are passed through the body parallel to each of the three coordinate planes, a cube will be identified. Each plane will have normal and tangential components of the stress resultants. The tangential or shear stress resultant on each plane can further be represented by two components in the coordinate directions. The internal stress state is then represented by three stress components on each coordinate plane as shown in **Fig. 2.13**. (Note that equal and opposite components will exist on the unexposed faces). Therefore at any point in a body there will be nine stress components. These are often identified in matrix form such that,

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \tag{2.19}$$

Using equilibrium, it is easy to show that the stress matrix is symmetric or,

$$\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy} \tag{2.20}$$

leaving only six independent stresses existing at a material point.

Physical and Mathematical Definition of Normal Strain and Shear Strain If a differential element is acted upon by stresses as shown in **Fig. 2.14a** both normal and shearing deformations will result. The resulting deformation in the x-y plane is shown in **Fig. 2.14b**, where **u** is the displacement component in the x direction and **v** is the displacement component in the y direction.
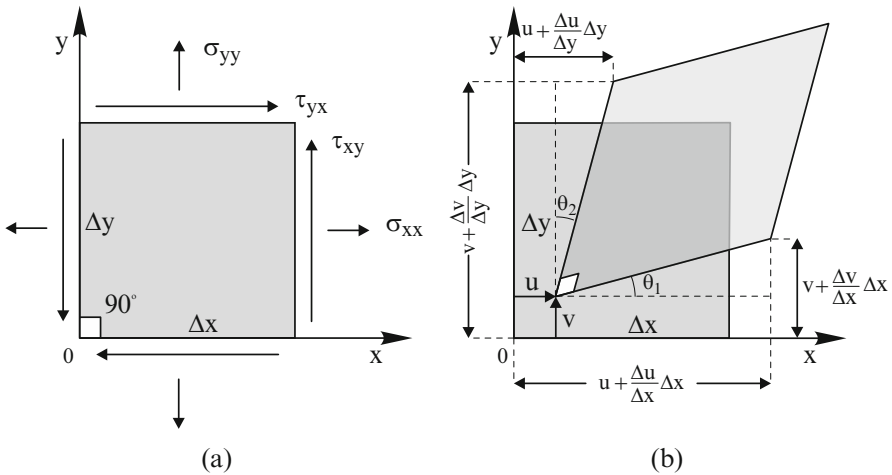

Fig. 2.14 Definitions of displacements **u** and **v** and corresponding shear and normal strains

The unit change in the $\mathbf{\Delta x}$ dimension will be the strain ε_{xx} and is given by,

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \left\{ \frac{\left(u + \frac{\Delta u}{\Delta x} \Delta x \right) - u}{\Delta x} \right\} \tag{2.21}$$

with similar definitions for the unit change in the y and z directions. (The assumption of small strain and linear behavior is implicit here with the assumption that $\mathbf{\theta}$ is small and thus its impact on $\mathbf{\Delta u}$ is ignored). Therefore the normal strains in the three coordinate directions are defined as,

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \quad, \quad \varepsilon_{yy} = \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y} = \frac{\partial v}{\partial y} \quad,$$
$$\varepsilon_{zz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial w}{\partial z} \tag{2.22}$$

where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the displacement components in the three coordinate directions at a point. Shear strains are defined as the distortion of the original $90°$ angle at the origin or the sum of the angles $\mathbf{\theta_1 + \theta_2}$. That is, again using the small deformation assumption,

$$\tan(\theta_1 + \theta_2) \approx (\theta_1 + \theta_2) = \lim_{\Delta x, \Delta y \to 0} \left[\frac{(v + \frac{\Delta v}{\Delta x} \Delta x) - v}{\Delta x} + \frac{\left(u + \frac{\Delta u}{\Delta y} \right) - u}{\Delta y} \right] \tag{2.23}$$

which leads to the three shear strains,

$$\gamma_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \quad \gamma_{xz} = \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), \quad \gamma_{yz} = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \tag{2.24}$$

Stresses and strains are often described using tensorial mathematics but in order for strains to transform as tensors, the definition of shear strain must be modified to include a factor of one half as follows,

$$\varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \quad \varepsilon_{xz} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), \quad \varepsilon_{yz} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \tag{2.25}$$

The difference between the latter two sets of equations can lead to very erroneous values of stress when attempting to use an electrical strain gage rosette to determine the state of stress experimentally. In **Eq. 2.25** the traditional symbol ε

with mixed indices has been used to identity tensorial shear strain. The symbol γ with mixed indices will be used to describe non-tensorial shear strain, also called engineering strain.

In general, as with stresses, nine components of strain exist at a point and these can be represented in matrix form as,

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \tag{2.26}$$

Again, it is possible to show that the strain matrix is symmetric or that,

$$\varepsilon_{xy} = \varepsilon_{yx}, \quad \varepsilon_{xz} = \varepsilon_{zx}, \quad \varepsilon_{yz} = \varepsilon_{zy} \tag{2.27}$$

Hence there are only six independent strains.

Generalized Hooke's Law As noted previously, Hooke's law for one dimension or for the condition of uniaxial stress and strain for elastic materials is given by $\sigma = E\,\varepsilon$. Using the principle of superposition, the generalized Hooke's law for a three dimensional state of stress and strain in a homogeneous and isotropic material can be shown to be,

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right], & \gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right], & \gamma_{yz} &= \frac{\tau_{yz}}{G} \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right], & \gamma_{xz} &= \frac{\tau_{xz}}{G}
\end{aligned} \tag{2.28}$$

where E, G and ν are Young's modulus, the shear modulus and Poisson's ratio respectively. Only two are independent and as indicated earlier,

$$G = \frac{E}{2(1+\nu)} \tag{2.29}$$

The proof for **Eq. 2.29** may be found in many elementary books on solid mechanics.

Other forms of the generalized Hooke's law can be found in many texts. The relation between various material constants for linear elastic materials are shown below in **Table 2.1** where E, G and ν are previously defined and where K is the bulk modulus and λ is known as Lame's constant.

Table 2.1 Relation between various elastic constants. λ and G are often termed Lame' constants and K is the bulk modulus

$$\dagger A \equiv \sqrt{(E + \lambda)^2 + 8\lambda^2}$$

	Lamé's modulus, λ	Shear modulus, G	Young's modulus, E	Poisson's ratio, ν	Bulk modulus, K
λ, G			$\dfrac{G(3\lambda + 2G)}{\lambda + G}$	$\dfrac{\lambda}{2(\lambda + G)}$	$\dfrac{3\lambda + 2G}{3}$
λ, E		$\dfrac{A^\dagger + (E - 3\lambda)}{4}$		$\dfrac{A^\dagger - (E + \lambda)}{4\lambda}$	$\dfrac{A^\dagger + (3\lambda + E)}{6}$
λ, ν		$\dfrac{\lambda(1 - 2\nu)}{2\nu}$	$\dfrac{\lambda(1 + \nu)(1 - 2\nu)}{\nu}$		$\dfrac{\lambda(1 + \nu)}{3\nu}$
λ, K		$\dfrac{3(K - \lambda)}{2}$	$\dfrac{9K(K - \lambda)}{3K - \lambda}$	$\dfrac{\lambda}{3K - \lambda}$	
G, E	$\dfrac{(2G - E)G}{E - 3G}$			$\dfrac{E - 2G}{2G}$	$\dfrac{GE}{3(3G - E)}$
G, ν	$\dfrac{2G\nu}{1 - 2\nu}$		$2G(1 + \nu)$		$\dfrac{2G(1 + \nu)}{3(1 - 2\nu)}$
G, K	$\dfrac{3K - 2G}{3}$		$\dfrac{9KG}{3K + G}$	$\dfrac{3K - 2G}{2(3K + G)}$	
E, ν	$\dfrac{\nu E}{(1 + \nu)(1 - 2\nu)}$	$\dfrac{E}{2(1 + \nu)}$			$\dfrac{E}{3(1 - 2\nu)}$
E, K	$\dfrac{3K(3K - E)}{(9K - E)}$	$\dfrac{3EK}{9K - E}$		$\dfrac{3K - E}{6K}$	
ν, K	$\dfrac{3K\nu}{1 + \nu}$	$\dfrac{3K(1 - 2\nu)}{2(1 + \nu)}$	$3K(1 - 2\nu)$		

Hooke's law is a mathematical statement of the linear relation between stress and strain and usually implies both small strains ($\varepsilon^2 << \varepsilon$) and small deformations. It is also to be noted that in general elasticity solutions in two and three dimensions, the displacement, stress and strain variables are functions of spatial position, $\mathbf{x_i}$. This will be handled more explicitly in Chap. 9.

Again, it is important to note that stress and strain are point quantities, yet methods for strain measurement are not capable of measuring strain at an infinitesimal point. Thus, average values are measured and moduli are obtained using stresses calculated at a point. For this reason, strains are best measured where no gradients exist or are so small that an average is a good approximation. One approach when large gradients exist is to try to measure the gradient and extrapolate to a point. The development of methods to measure strains within very small regions has become a topic of great importance due to the

development of micro-devices and machines. Further, such concerns as interface or interphase properties in multi-phase materials also creates the need for new micro strain measurement techniques.

Indicial Notation and Compact Form of Generalized Hooke's Law Because of the cumbersome form of the generalized Hooke's Law for material constitutive response in three dimensions (**Eq. 2.28**), a shorthand notation referred to as indicial or index notation is extensively used. Here we provide a brief summary of indicial notation and further details may be found in many books on continuum mechanics (e.g., Flügge 1972). In indicial notation, the subscripts on tensors are used with very precise rules and conventions and provide a compact way to relate and manipulate tensorial expressions.

The conventions are as follows:
- Subscripts indicating coordinate direction (**x, y, z**) can be generally represented by a roman letter variable that is understood to take on the values of **1, 2**, or **3**. For example, the stress tensor can be written as σ_{ij} which then gives reference to the entire 3×3 matrix. That is the stress and strain matrices given by **Eqs. 2.19** and **2.26** become,

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \varepsilon_{ij} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \tag{2.30}$$

- Summation convention: if the same index appears twice in any term, summation is implied over that index (unless suspended by the phrase "no sum"). For example,

$$\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{2.31}$$

- Free index: non-repeated subscripts are called free subscripts since they are free to take on any value in 3D space. The count of the free indices on a variable indicates the order of the tensor. e.g. $\mathbf{F_i}$ is a vector (first order tensor), σ_{ij} is a second order tensor.
- Dummy index: repeated subscripts are called dummy subscripts, since they can be changed freely to another letter with no effect on the equation.
- Rule 1: The same subscript cannot appear more than twice in any term.
- Rule 2: Free indices in each term (both sides of the equation) must agree (all terms in an equation must be of the same order).
  Example of valid expression: $v_i = a_{ij}u_j - \lambda e_{kl}d_{ikl}$
- Rule 3: Both free and repeated indices may be replaced with others subject to the rules.

Example of valid expression: $a_{ij}u_j + d_i = a_{ik}u_k + d_i$

- Unlike in vector algebra, the order of the variables in a term is unimportant, as the bookkeeping is done by the subscripts. For example consider the inner product of a second order tensor and a vector:

$$A_{ij}u_j = u_j A_{ij} \tag{2.32}$$

- Differentiation with respect to spatial coordinates is represented by a comma, for example

$$\frac{dv_i}{dx_j} = v_{i,j} \tag{2.33}$$

- The identity matrix is also referred to as the Kronecker Delta function and is represented by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{2.34}$$

The properties of $\boldsymbol{\delta_{ij}}$ are thus

$$\begin{aligned} \delta_{ii} &= 3 \\ \delta_{ij}v_j &= v_i \\ \delta_{ij}\delta_{jk} &= \delta_{ik} \\ \delta_{ij}\sigma_{jk} &= \sigma_{ik} \end{aligned} \tag{2.35}$$

Although the conventions listed above may seem tedious at first, with a little practice index notation provides many advantages including easier manipulations of matrix expressions. Additionally, it is a very compact notation and the rules listed above can often be used during manipulation to reduce errors in derivations.

The generalized Hooke's Law from **Eq. 2.28** may be rewritten to relate tensorial stress and strain in index notation as follows:

$$\varepsilon_{ij} = \frac{1 + \nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \tag{2.36}$$

or

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} \tag{2.37}$$

Additionally, the strain-displacement relations, **Eqs. 2.22** and **2.25**, can be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2.38}$$

where $\mathbf{u_i}$ are the three displacement components, represented as **u**, **v**, and **w** earlier (e.g., $\mathbf{u_2 = v}$).

These expressions will be used extensively later in Chap. 9 when dealing with viscoelasticity problems in two and three dimensions.

Consequences of Homogeneity and Isotropy Assumptions It is interesting to examine the consequences if a material is linearly elastic but not homogeneous or isotropic. For such a material, the generalized Hooke's law is often expressed using index notation as,

$$\sigma_{ij} = E_{ijkq}\varepsilon_{kq} \tag{2.39}$$

For a material that is nonhomogeneous, the material properties are a function of spatial position and $\mathbf{E_{ijkq}}$ becomes $\mathbf{E_{ijkq}(x,y,z)}$. The nonhomogeneity for a particular material determines exactly how the moduli vary across the material. The geometry of the material on an atomic or even microscale determines symmetry relationships that govern the degree of anisotropy of the material. Without regard to symmetry constraints, **Eq. 2.39** could have 81 independent proportionality properties relating stress components to strain components.

The complete set of nine equations (one for each stress) each with nine coefficients (one for each strain term) can be found from **Eq. 2.39**. This is accomplished using the summation convention over repeated indices. That is, **Eq. 2.39** is understood to be a double summation as follows,

$$\sigma_{ij} = \sum_{k=1}^{3}\sum_{q=1}^{3} E_{ijkq}\varepsilon_{kq} \tag{2.40}$$

(The expansion is left as an exercise for the reader. See problem 2.4).

If a material is nonlinear elastic as well as heterogeneous and anisotropic, **Eq. 2.39** becomes,

$$\sigma_{ij} = E_{ijkl}(x, y, z)\varepsilon_{kl} + E_{ijkl}'(x, y, z)\varepsilon_{kl}^2 + \cdots \tag{2.41}$$

Again each term on the right hand side of **Eq. 2.40** represents a double summation and each coefficient of strain is an independent set of material parameters. Thus, many more than 81 parameters may be required to represent a nonlinear heterogeneous and anisotropic material. Further, for viscoelastic materials, these material parameters are time dependent. The introduction of the assumption of linearity reduces the number of parameters to 81 while homogeneity removes their spatial variation (i.e., the E_{ijkq} parameters are now constants). Symmetry of the stress and strain tensors (matrices) reduces the number of constants to 36. The existence of a strain energy potential reduces the number of constants to 21. Material symmetry reduces the number of constants further. For example, an orthotropic material, one with three planes of material symmetry, has only 9 constants and an isotropic material, one with a center of symmetry, has only two independent constants (and **Eq. 2.39** reduces to **Eq. 2.28**). Now it is easy to see why the assumptions of linearity, homogeneity and isotropy are used for most engineering analyses.

A plane of material symmetry exists within a material when the material properties (elastic moduli) at mirror-imaged points across the plane are identical. For example, in the sketch given in **Fig. 2.15a**, the yz plane is a plane of symmetry and the elastic moduli would be the same at the material points A and B.
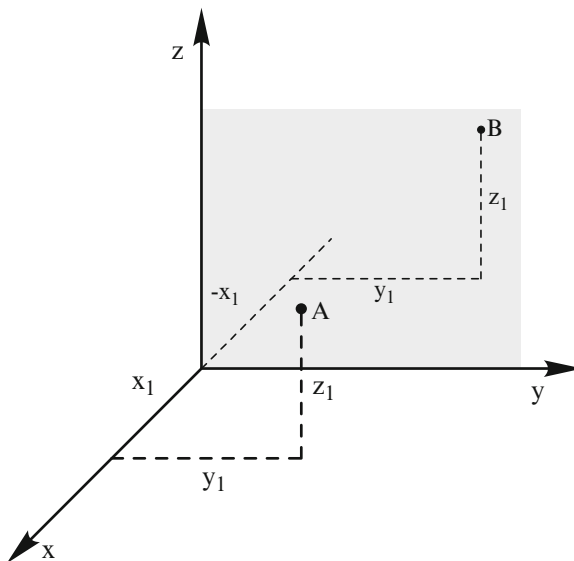

Fig. 2.15 (**a**) Definition of a plane of material symmetry

Experimentation is needed to determine if a material is homogeneous or isotropic. One approach is to cut small tensile coupons from a three-dimensional body and perform a uniaxial tensile or compressive test as well as a torsion test for shear. Obviously, to obtain a statistical sample of specimens at a single point would require exact replicas of the same material or a large number of near replicas. Assuming that such could be accomplished for a body with points A and B as in **Fig. 2.15a**, the following relationships would hold for homogeneity,

$$E_{xxxx}\big|_A = E_{xxxx}\big|_B \quad E_{yyyy}\big|_A = E_{yyyy}\big|_B \quad E_{zzzz}\big|_A = E_{zzzz}\big|_B \tag{2.42}$$

That is, the modulus components are invariant (constant) for all directions at a point (See Problem 2.5).

The above measurement approach illustrates the influence of heterogeneity and anisotropy on moduli but is not very practical. A sonic method of measuring properties, though not as precise as tensile or torsion tests, is often used and is based upon the fact that the speed of sound, v_s, in a medium is related to its modulus of elasticity, **E**, and density, **ρ**, such that (Kolsky 1963),

$$v_s = \sqrt{\frac{E}{\rho}} \tag{2.43}$$

The above is adequate for a thin long bar of material but for three-dimensional bodies the velocity is related to both dilatational (volume change – see subsequent section for definition) and shear effects as well as geometry effects, etc.

It is to be noted that the condition of heterogeneity and anisotropy are confronted when considering many materials used in engineering design. For example, while many metals are isotropic on a macroscopic scale, they are crystalline on a microscopic scale. Crystalline materials are at least anisotropic and may be heterogeneous as well. Wood is both heterogeneous and anisotropic as are many ceramic materials. Modern polymer, ceramic or metal matrix composites such as fiberglass, etc. are both heterogeneous and anisotropic. The mathematical analysis of such materials often neglects the effect of heterogeneity but does include anisotropic effects. (See Lekhnitskii (1963), Daniel and Ishai (2005)).

2.6. Principal Stresses

In the study of viscoelasticity as in the study of elasticity, it is mandatory to have a thorough understanding of methods to determine principal stresses and strains. Principal stresses are defined as the normal stresses on the planes oriented such that the shear stresses are zero – the maximum and minimum normal stresses at a point are principal stresses. The determination of stresses and strains in two dimensions is well covered in elementary solid mechanics both analytically and semi-graphically using Mohr's circle. However, practical stress analysis problems frequently involve three dimensions. The basic equations for transformation of stresses in three-dimensions, including the determination of principal stresses, will be given and the interested reader can find the complete development in many solid mechanics texts.

Often in stress analysis it is necessary to determine the stresses (strains) in a new coordinate system after calculating or measuring the stresses (strains) in another coordinate system. In this connection, the use of index notation is very helpful as it can be shown that the stress σ_{ij}' in a new coordinate system, $\mathbf{x_i'}$, can be easily obtained from the σ_{ij} in the old coordinate system, $\mathbf{x_i}$, by the equation,

$$\sigma_{ij}' = a_{ik} a_{jq} \sigma_{kq} \tag{2.44}$$

where the quantities $\mathbf{a_{ij}}$ are the direction cosines between the axes $\mathbf{x_i'}$ and $\mathbf{x_i}$ and may be given in matrix form as,

$$a_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{2.45}$$

Figure 2.15b illustrates coordinate transformation for stress at a material point in two dimensions, showing the primed and unprimed axis systems where the angle between them is defined as θ. In **Eq. 2.44**, the repeated indices on the right again indicate summation over the three coordinates, **x,y,z** or the indices **1,2,3**. It is left as an exercise for the reader to show that this process leads to the familiar two-dimensional expressions found in the first course in solid mechanics (see Problem 2.6),

$$\sigma_x' = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \tag{2.46a}$$

or

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \tag{2.46b}$$
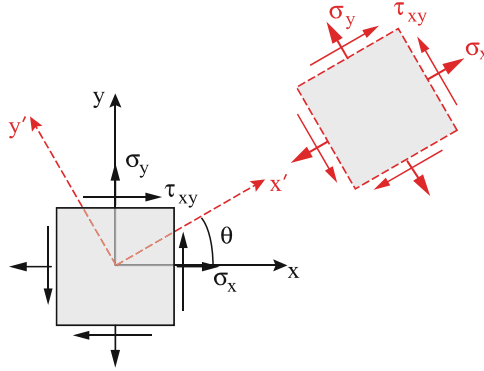
Fig. 2.15 (b) Illustration of coordinate transformation in two dimensions

$$\tau'_{xy} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) \tag{2.47a}$$

or

$$\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{2.47b}$$

Using **Eq. 2.44** it is possible to show that the three principal stresses (strains) can be calculated from the following cubic equation,

$$\sigma_i^3 - I_1\sigma_i^2 + I_2\sigma_i - I_3 = 0 \tag{2.48}$$

where the principal stresses, $\boldsymbol{\sigma_i}$, are given by one of the three roots $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$ or $\boldsymbol{\sigma_3}$ and,

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{xz}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \tag{2.49}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2 + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} = \sigma_1\sigma_2\sigma_3$$

The quantities I_1, I_2, and I_3 are the same for any arbitrary coordinate system located at the same point and are therefore called invariants.

In two-dimensions when $\boldsymbol{\sigma_{zz} = 0}$ and a state of plane stress exists, **Eq. 2.48** reduces to the familiar form,

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \left(\tau_{xy}\right)^2} \tag{2.50}$$

where the comma does not indicate differentiation in this case, but is here used to emphasize the similarity in form of the two principle stresses by writing them in one equation. The proof of **Eq. 2.50** is left as an exercise for the reader.

The directions of principal stresses (strains) are also very important. However, the development of the necessary equations will not be presented here but it might be noted that the procedure is an eigenvalue problem associated with the diagonalization of the stress (strain) matrix.

2.7. Deviatoric and Dilatational Components of Stress and Strain

A general state of stress at a point or the stress tensor at a point can be separated into two components, one of which results in a change of shape (deviatoric) and one which results in a change of volume (dilatational). Shape changes due to a pure shear stress such as that of a bar in torsion given in **Fig. 2.2** are easy to visualize and are shown by the dashed lines in **Fig. 2.16a** (assuming only a horizontal motion takes place).
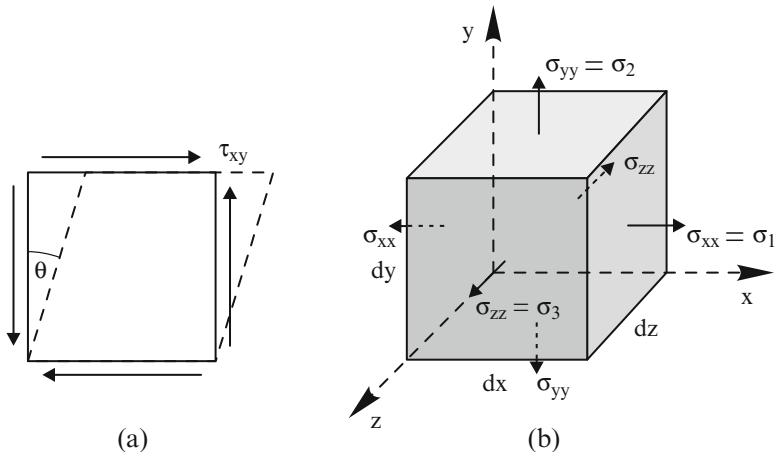

Fig. 2.16 (**a**) Shape changes due to pure shear. (**b**) Normal stresses leading to a pure volume change

Shear Modulus Because only shear stresses and strains exist for the case of pure shear, the shear modulus can easily be determined from a torsion test by measuring the angle of twist over a prescribed length under a known torque, i.e.,

$$T = \theta\frac{JG}{L} \qquad (2.51)$$

where all terms are as previously defined in **Eq. 2.11**.

Bulk Modulus. Volume changes are produced only by normal stresses. For example, consider an element loaded with only normal stresses (principal stresses) as shown in **Fig. 2.16b**. The change in volume can be shown to be (for small strains),

$$\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \qquad (2.52)$$

Substituting the values of strains from the generalized Hooke's law, **Eq. 2.28**, gives,

$$\frac{\Delta V}{V} = \frac{1 - 2\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) \qquad (2.53)$$

If Poisson's ratio is $\nu = 0.5$, the change in volume is zero or the material is incompressible. Here it is important to note that Poisson's ratio for metals and many other materials in the linear elastic range is approximately 0.33 (i.e., $\nu \sim 1/3$). However, near and beyond the yield point, Poisson's ratio is approximately 0.5 (i.e., $\nu \sim 1/2$). That is, when materials yield, neck or flow, they do so at constant volume.

In the case when all the stresses on the element in **Fig. 2.16b** are equal ($\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$), a spherical state of stress (hydrostatic stress) is said to exist and,

$$\frac{\Delta V}{V} = \frac{1 - 2\nu}{E}(3\sigma) \qquad (2.54)$$

By equating **Eqs. 2.52** and **2.54** the Bulk Modulus can be defined as the ratio of the hydrostatic stress, $\boldsymbol{\sigma}$, to volumetric strain or unit change in volume ($\boldsymbol{\Delta V/V}$),

$$K = \frac{E}{3(1 - 2\nu)} \qquad (2.55)$$

Notice that the bulk modulus becomes infinite, $\mathbf{K} \sim \infty$, if the material is incompressible and Poisson's ratio is, $\nu \sim 1/2$.

Obviously, one method for obtaining the bulk modulus of a material would be to create a hydrostatic compression (or tension) state of stress and measure the resulting volume change.

Dilatational and Deviatoric Stresses for a General State of Stress For a general stress state, the dilatational or volumetric component is defined by the mean stress or the average of the three normal stress components shown in **Fig. 2.13**,

$$\sigma = \sigma_m = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{1}{3}\sigma_{kk} \qquad (2.56)$$

In **Eq. 2.56** care has been taken to provide three different symbolic ways of indicating the volumetric stress, σ, σ_m, or $\sigma_{kk}/3$ to emphasize the many notations found in the literature. Since the sum of the normal stresses is the first Invariant, I_1, the mean stress, σ_m, will be the same for any axis orientation at a point including the principal axes as shown in **Eq. 2.56**. Thus, independent of axis orientation the general stress state can be separated into a volumetric component plus a shear component as shown in **Fig. 2.17**. That is, if the stresses responsible for volumetric changes are subtracted from a general stress state, only stresses responsible for shape changes remain. This statement can be expressed in matrix form as,

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{pmatrix} + \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix} \qquad (2.57)$$

or in index notation as

$$\sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + s_{ij} \qquad (2.58)$$

where s_{ij} are the deviator (shape change) components of stress and δ_{ij} is the Kronecker Delta function as defined earlier (**Eq. 2.34**).
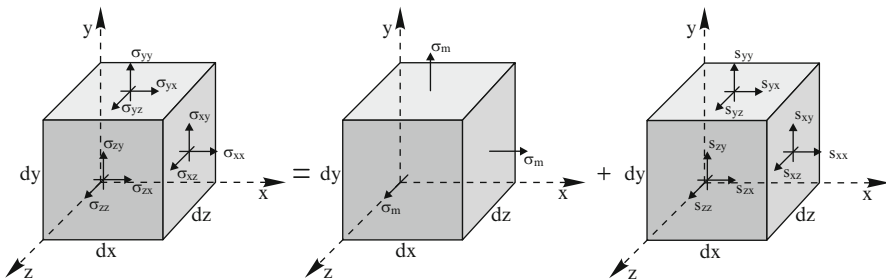

Fig. 2.17 Separation of a general stress state into dilatational and deviator stresses

Since the trace of the first two matrices in **Eq. 2.52** are the same, i.e.,

$$\sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 3\sigma_m \tag{2.59}$$

the trace of the third matrix is zero, i.e.,

$$s_{kk} = s_{xx} + s_{yy} + s_{zz} = 0 \tag{2.60}$$

Using **Eq. 2.60**, the deviator matrix can be separated into five simple shear stress systems,

$$
\begin{pmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{pmatrix} = \begin{pmatrix} 0 & s_{xy} & 0 \\ s_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_{yz} \\ 0 & s_{zy} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & s_{xz} \\ 0 & 0 & 0 \\ s_{zx} & 0 & 0 \end{pmatrix}
$$
$$
+ \begin{pmatrix} s_{xx} & 0 & 0 \\ 0 & -s_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -s_{zz} & 0 \\ 0 & 0 & s_{zz} \end{pmatrix}
\tag{2.61}
$$

That the stress states given by the first three matrices on the right side of **Eq. 2.61** are pure shear states is obvious. The last two are also pure shear states but at 45° to the indicated axis as shown in **Fig. 2.18**.
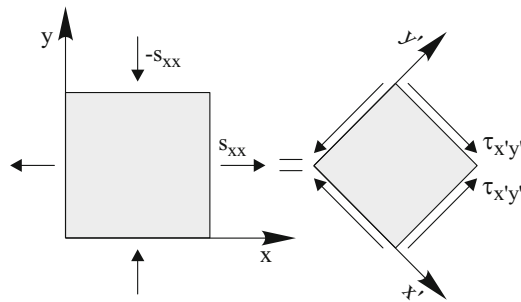

Fig. 2.18 Pure shear state

Therefore each term in **Eq. 2.61** represents a pure shear state and results in only shape changes with no volume change.

Strains can also be separated into dilatational and deviatoric components and the equation for strain analogous to **Eq. 2.58** is,

$$\varepsilon_{ij} = e_{ij} + \varepsilon_m \delta_{ij} \quad \textbf{or} \quad \varepsilon_{ij} = e_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{2.62}$$

where e_{ij} are the deviatoric strains and $e_m = \frac{1}{3}\varepsilon_{kk}$ is the dilatational component. Clearly the trace of the strain tensor equivalent to **Eq. 2.59** can be recovered from **Eq. 2.62**.

The generalized Hooke's law given by **Eq. 2.28** or **Eq. 2.36** can now be written in terms of deviatoric and dilatational stresses and strains using the equations above as well as **Eqs. 2.52–2.55**

$$\begin{aligned} s_{ij} &= 2Ge_{ij} \\ \sigma_{kk} &= 3K\varepsilon_{kk} \end{aligned} \tag{2.63}$$

The importance of the concept of a separating the stress (and strain) tensors into dilatational and deviatoric components is due to the observation that viscoelastic and/or plastic (meaning yielding, not polymers) deformations in materials are predominately due to changes in shape. For this reason, volumetric effects can often be neglected and, in fact, the assumption of incompressibility is often invoked. If this assumption is used, the solution of complex boundary value problems (BVP) are often greatly simplified. Such an assumption is often made in analyses using the theory of plasticity and theory of viscoelasticity and each will be discussed in later chapters.

Further, the observation that deformations in viscoelastic materials such as polymers is more related to changes of shape than changes of volume suggest that shear and volumetric tests may be more valuable than the traditional uniaxial test.

It can be shown that additional invariants exist for both dilatational and deviatoric stresses. For a derivation and description of these see Fung (1965) and Shames et al. (1992). The invariants for the deviator state will be used briefly in Chap. 11 and are therefore given below.

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3 = 0$$

$$J_2 = 3\sigma_m^2 - I_2 \tag{2.64}$$

$$J_3 = I_3 - J_2\sigma_m - \sigma_m^3$$

All invariants have many different forms other than those given herein.

2.8. Failure (Rupture or Yield) Theories

Simply stated, failure theories are attempts to have a method by which the failure of a material can be predicted and thereby prevented. Most often the physical property to be limited is determined by experimental observations and then a mathematical theory is developed to accommodate observations. To date, no set of universal failure criteria has been determined which is suitable for all materials. Because of the large interest in light weight but strong materials such as polymer, metal and ceramic matrix composites (PMC, MMC and CMC respectively) that will operate at high temperatures or under other adverse conditions there has been much activity in developing special failure criteria appropriate for individual materials. As a result, the number of failure theories now is in the hundreds. Here we will only give the essential features of the classical theories, which were primarily developed for metals. For this reason, it is suggested that the reader keep an open mind and be extremely careful when investigating the behavior of polymers using these traditional methods. It is virtually certain that actual behavior will not always be well represented using any of the following theories due to the time dependent nature of polymer based materials. The same statement is likely true for most of the current popular theories used for composites.

Ductile materials often have a stress-strain diagram similar to that of mild steel shown in **Fig. 2.8** and can be approximated by a linear elastic-perfectly plastic material with a stress-strain diagram such as that given in **Fig. 2.9b**. Failure for ductile materials is assumed to occur when stresses or strains exceed those at the yield point. Materials such as cast iron, glass, concrete and epoxy are very brittle and can often be approximated as perfectly linear elastic-perfectly brittle materials similar to that given in **Fig. 2.9a**. Failure for brittle materials is assumed to occur when stresses or strains reach a value for which rupture (separation) will occur.

The following are the simple statements and expressions for three well known and often used failure theories. They are described in terms of principal stresses, where $\sigma_1 > \sigma_2 > \sigma_3$, and a failure stress in a uniaxial tensile test, $\sigma_f|_{tensile}$, which is either the rupture stress or the yield stress as appropriate for the material. Typically, tensile and compression properties as found in a uniaxial test are assumed to be the same.

Maximum Normal Stress Theory (Lame-Navier) Failure occurs when the largest principal stress (either tension or compression) is equal to the maximum tensile stress at failure (rupture or yield) in a uniaxial tensile test.

$$\sigma_1 = \sigma_f|_{\textbf{tensile}} \tag{2.65}$$

Maximum Shear Stress Theory (Tresca) Failure occurs when the maximum shear stress at an arbitrary point in a stressed body is equal to the maximum shear stress at failure (rupture or yield) in a uniaxial tensile test.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \tau_{max}|_{\textbf{tensile}} = \frac{\sigma_f|_{\textbf{tensile}}}{2}$$

$$\sigma_1 - \sigma_3 = \sigma_f|_{\textbf{tensile}} \tag{2.66}$$

Maximum Distortion Energy (or Maximum Octahedral Shear Stress) Theory (von Mises) Failure occurs when the maximum distortion energy (or maximum octahedral shear stress) at an arbitrary point in a stressed medium reaches the value equivalent to the maximum distortion energy (or maximum octahedral shear stress) at failure (yield) in simple tension

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = 2\sigma_f^2|_{\textbf{tensile}} \tag{2.67}$$

Development of the octahedral shear stress can be found in many texts and will not be given here. However, it is appropriate to note the geometry of the octahedral plane. That is, if a diagonal plane is identified for stressed element as shown in **Fig. 2.19a** such that the normal to the diagonal plane makes an angle of 54.7°, the stress state will be as shown in **Fig. 2.19b**. The resultant shear stress on this octahedral plane, so named because there are eight such planes at a point, is the octahedral shear stress. The octahedral shear stress and the octahedral plane is very important especially for polymers as the majority of viscoelastic behavior is associated with shear or deviatoric response as opposed bulk or dilational response. This is discussed in more detail in the next section.
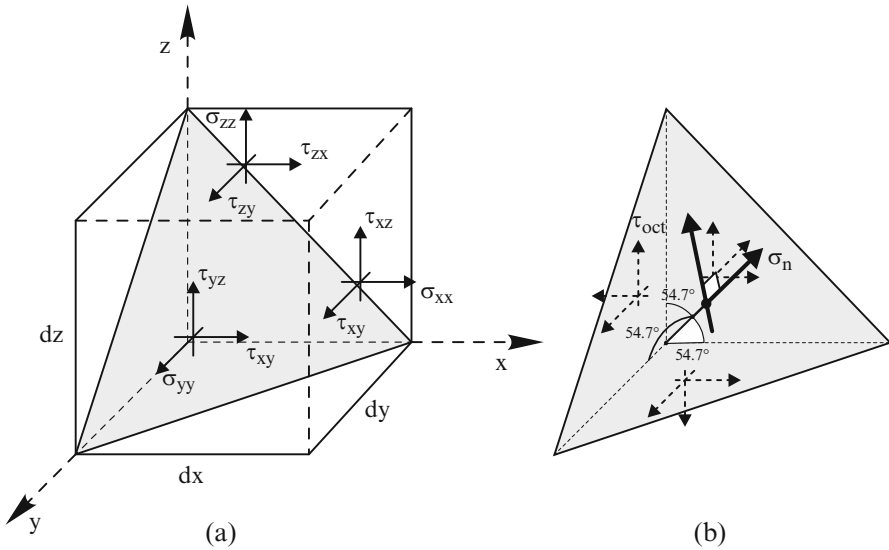
Fig. 2.19 Definition of the octahedral shear stress

Comparison Between Theory and Experiment Comparisons between theory and experiment have been made for many materials. Shown in **Fig. 2.20** are the graphs in stress space for the equations for the three theories given above. Also shown is experimental data on five different metals as well as four different polymers. It will be noted that cast iron, a very brittle material agrees well with the maximum normal stress theory while the ductile materials of steel and aluminum tend to agree best with the von Mises criteria. Polymers tend to be better represented by von Mises than the other theories.
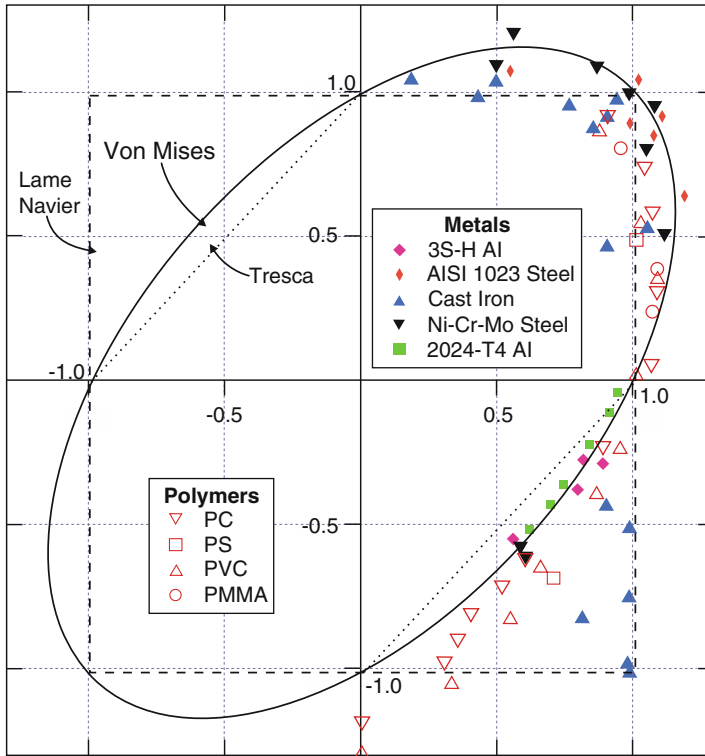
Fig. 2.20 Comparison between failure theories and experiment (Data from Dowling (1993): metal p. 252, polymer p. 254)

2.9. Atomic Bonding Model for Theoretical Mechanical Properties

Materials scientists and engineers have long sought methods to determine the mechanical properties of materials from knowledge of the bonding properties of individual atoms, which, of course, hold materials together. Observation of elastic behavior suggests the existence of both attractive and repulsive forces between individual atoms. Stretching an elastic bar in tension, stretches the atomic bonds and release of the load allows the bonds to return to their original equilibrium positions. Likewise, compression causes atoms to move closer together and release of the load allows the atoms to return to their equilibrium position. A hypothetical tensile (or compressive) bar composed of perfectly packed atoms is shown in **Fig. 2.21**. The distances between the centers of four

neighboring atoms, mnpq, form a rhombus. When stretched, the strains in the vertical and horizontal directions, ε_x and ε_y, can be calculated from geometrical changes in the position of the spheres and the ratio can be shown to give a Poisson's ratio of $\nu = 1/3$, which is close to the measured value for metals and many materials. The proof is left as an exercise for the reader (see Problem 2.8). This simple calculation tends to give some confidence in the use of an atomic model to represent mechanical behavior.

Now consider just two atoms in equilibrium with each other as shown in **Fig. 2.22**. Application of a tensile force, $\mathbf{F_T}$, will induce an attractive force, $\mathbf{F_A}$, between the two atoms in order to maintain equilibrium. Application of a compressive force will induce a repulsive force, $\mathbf{F_R}$, between the two atoms to maintain equilibrium. These attractive and repulsive forces will vary depending upon the separation distance. It is to be noted that the attractive forces in interatomic bonds are largely electrostatic in nature. For example, Coulomb's law for electrostatic charges indicates that the force is inversely proportional to the square of the spacing. The repulsive forces are caused by the interactions of the electron shells of the atoms and is somewhat difficult to estimate directly.

The variation of attractive and repulsive forces and energies with separation distance are given in **Fig. 2.22d–e**, where $\mathbf{r_0}$ is the equilibrium spacing. The forms of the equations agree with physical observations but the values of the constants $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{m}$ and $\mathbf{n}$ vary for different materials. Obviously, the effect of dislocations, vacancies, grain boundaries, etc. complicates the picture in metals and the long molecular chains, entanglements and other defects complicate the picture in polymers. The energy equations and diagrams given in **Fig. 2.22** can be simply calculated from the force diagram using the basic definitions of work and energy given in elementary mechanics. This proof is left as an exercise for the reader.

Obviously, if the tensile forces are large enough, the distance between atoms can become so great that the attractive force will tend to zero and no force would be required for the atom to be in equilibrium. On the other hand, the application of a compressive forces can not force the two atoms to merge and the repulsive force will increase without bound. For this reason, it should be possible to calculate the theoretical strength of a material if sufficient information is known about the bonding forces in atoms of a particular material. This interpretation has been used by many (see for example, (Courtney (1990), McClintock and Argon (1966), Richards (1961), Shames and Cozzarelli (1992)) to formulate nonlinear stress-strain relations, laws for creep, plasticity effects, etc. However, as far as is known by the authors, no direct experimental verification has ever been made and, at best, such deduction must be termed empirical.
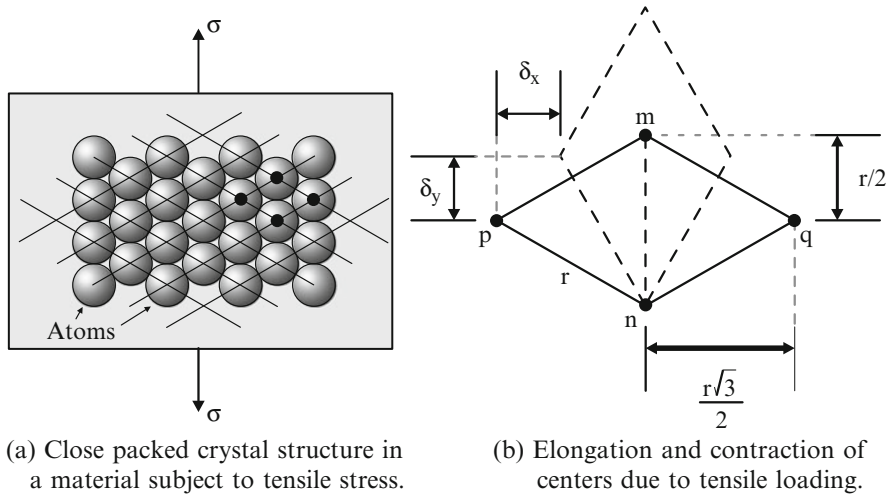
(a) Close packed crystal structure in
    a material subject to tensile stress.

(b) Elongation and contraction of
    centers due to tensile loading.

Fig. 2.21 Atomic deformations in a material composed of perfectly packed atoms

Not withstanding the empirical nature of the force and energy variations in **Fig. 2.22**, this approach does give insight to the strength limitations of materials. For example, by examination of **Fig. 2.22d** it can be shown that for a perfect crystalline arrangement of atoms as in **Fig. 2.21** that the strength of a material should be the same order of magnitude as its elastic modulus (see Richards 1961). The fact that no material has such high strength properties is an indication of weaknesses caused by imperfections in their molecular structure (e.g. imperfections such as dislocations, vacancies, etc.). Even near perfect crystalline materials do not have such high strength properties. On the other hand, it has been recognized that it is possible to increase strength properties drastically by developing processing approaches to create more nearly perfect crystalline structure and to minimize imperfections in molecular structure. Most of these processing improvements (directional solidification, powder metallurgy, etc.) are used for metals and ceramic type materials. Indeed, it is recognized that the large number of secondary bonds as opposed to primary bonds in polymers gives rise to their relatively modest properties when compared with most metals. Never-the-less, as will be noted in the following chapters, the properties of polymers can also be improved greatly by increasing crystallinity, using additives and developing improved processing techniques.
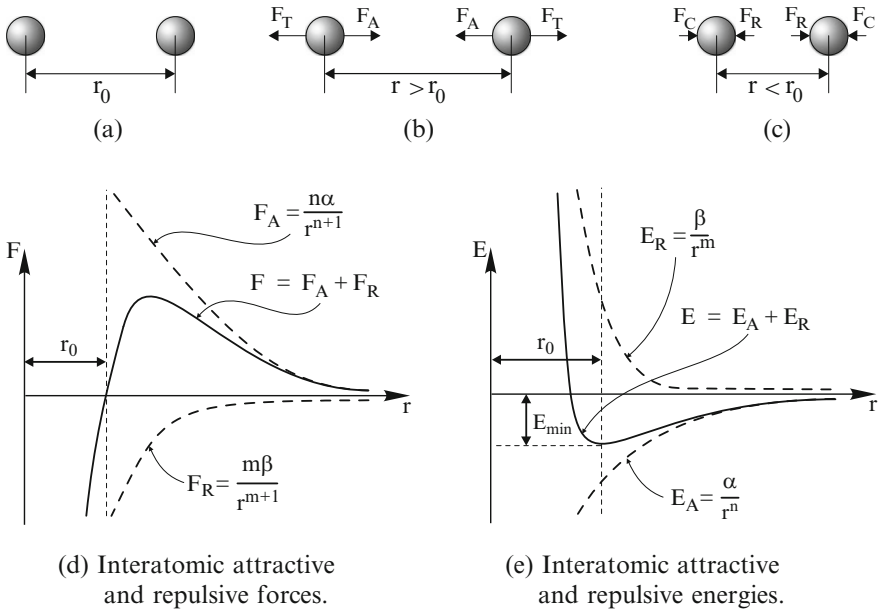
(a) (b) (c)

$$F_A = \frac{n\alpha}{r^{n+1}}$$

$$F = F_A + F_R$$

$$F_R = \frac{m\beta}{r^{m+1}}$$

(d) Interatomic attractive and repulsive forces.

$$E_R = \frac{\beta}{r^m}$$

$$E = E_A + E_R$$

$$E_A = \frac{\alpha}{r^n}$$

(e) Interatomic attractive and repulsive energies.

Fig. 2.22 Attractive and repulsive forces and energies between atoms

2.10. Review Questions

2.1. Name five assumption that are normally made to solve problems in elementary solid mechanics.

2.2. Name two types of nonlinearities encountered in solid mechanics.

2.3. Describe a heterogeneous or an inhomogeneous material. Name several materials that are inhomogeneous

2.4. Describe an anisotropic material. Name several materials that are anisotropic.

2.5. Give a mathematical definition for a continuum.

2.6. Define crystallinity, amorphous, anisotropic and material symmetry.

2.7. Define true stress and true strain and write an appropriate equation for each.

2.8. Discuss the characteristics one would seek in developing a test specimen to determine material properties.

2.9. What is a Luder's band? At what angle do they occur? Name two materials in which they are known to occur.

2.10. Explain the difference between engineering shear strain and the tensorial alternative.

2.11. How many material constants are needed to characterize a linear elastic homogeneous isotropic material? How many material constants are needed to characterize a linear elastic homogeneous anisotropic material?

2.12. Describe a plane of material symmetry. What type of symmetry does an isotropic material possess?

2.13. Define a stress invariant and give the proper expression for the first invariant of stress.

2.14. Define deviatoric and dilatational stresses.

2.15. Give a definition for the classical failure theories of Tresca and von Mises.

2.16. A brittle material is likely to follow which failure theory? On what plane would a brittle material tested in uniaxial tension fail?

2.17. A ductile material is likely to follow which failure theory?

2.18. What is the octahedral shear stress.

2.19. At what angle does a slip band form for a Tresca material tested in uniaxial tension.

2.20. At what angle does a slip band form for a von Mises material tested in uniaxial tension.

2.21. The strength of a material for a perfect arrangement of atoms might be expected to be on the order of what other material parameter?

2.22. Poisson's ratio can be shown to be equal to what value for a perfect arrangement of atoms?

2.11. Problems

2.1. If the engineering strain in a tensile bar is 0.0025 and Poisson's ratio is 0.33, find the original length and the original diameter if the length and diameter under load are 2.333 ft. and 1.005 in. respectively.

2.2. Find the true strain for the circumstances described in problem 2.1.

2.3. A circular tensile bar of a ductile material with an original cross-sectional area of 0.5 in.^{2} is stressed beyond the yield point until a neck is formed. The area of the neck is 0.25 in.^{2} Find the average engineering strain in the necked region. Calculate also the true strain. (Hint: Assume yielding occurs with no volume change).

2.4. The generalized Hooke's law in tensor (matrix) notation is given as $\sigma_{ij} = E_{ijkq}\, \varepsilon_{kq}$. Expand and find the algebraic expansion for σ_{12}.

2.5. From a thin plate of material small tensile coupons are cut at points A, B and C as shown and the following moduli properties are determined

$$E_x|_A, E_x|_B, E_x|_C, E_y|_A, E_y|_B, E_y|_C, E_\theta|_A, E_\theta|_B, E_\theta|_C$$

Give a correct relationship among the moduli for a homogeneous material.
Give a correct relationship among the moduli for an anisotropic material.
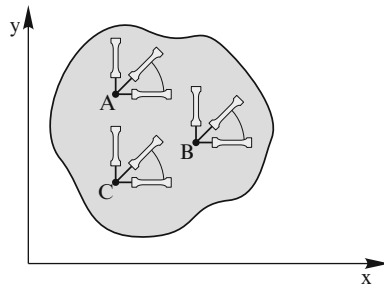
2.6. For a 2-D state of stress show that the tensorial transformation relation given by $\sigma'_{ij} = a_{ik}a_{jq}\sigma_{kq}$ reduces to the form

$$\sigma'_x = \sigma_x \, \cos^2\theta + \sigma_y \, \sin^2\theta + 2\tau_{xy} \, \sin\theta \, \cos\theta$$

2.7. Expand **Eq. 2.58** and show that the matrix given below is recovered.

$$s_{ij} = \begin{pmatrix} \sigma_{xx} - \sigma_m & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma_m & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma_m \end{pmatrix}$$

2.8. Using the geometry given in **Fig. 2.21** show that the ratio of lateral to longitudinal strain is 1/3. (Hint: spheres at **m** and **n** that are initially in contact stretch vertically when a stress is applied resulting in a separation of the spheres at **m** and **n**. Also, spheres at **p** and **q** will move inward to maintain contact with spheres at **m** and **n**).