

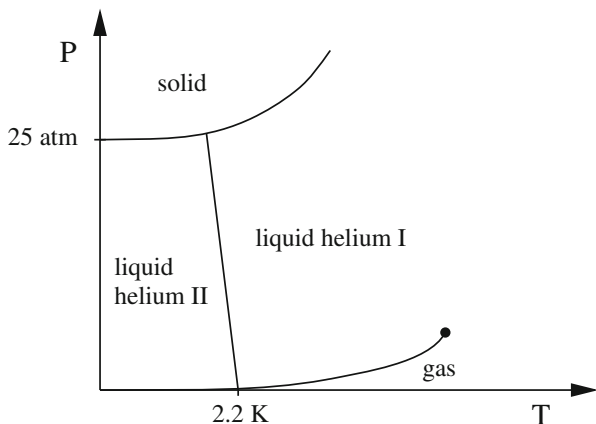
## Chapter 2

# Superfluid Helium

Helium was first liquefied in 1908 by H. Kamerlingh Onnes who cooled it below the liquid/gas transition temperature of 4.2 K.<sup>1</sup> Later, in 1927, M. Wolfke and W.H. Keesom realized that there is another phase transition at lower temperatures, around 2.17 K. This phase transition had manifested itself in a discontinuity of the specific heat, whose curve as a function of temperature resembles the Greek letter  $\lambda$ , and thus the transition point was called  $\lambda$  point. The two phases of liquid helium were termed “helium I” and “helium II”. The remarkable superfluid properties of liquid helium below the  $\lambda$  point, helium II, were experimentally established by P. Kapitza in 1938 [10], and independently by J.F. Allen and A.D. Misener in the same year [3]. Kapitza had set up an experiment with two cylinders that were connected by a thin tube with a thickness of 0.5  $\mu\text{m}$ . Only below the  $\lambda$  point, helium was flowing easily through the tube, suggesting a strikingly low viscosity. Kapitza received the Nobel prize for this discovery in 1978 (interestingly, together with Penzias and Wilson who received it for the completely unrelated discovery of the cosmic microwave background radiation). Kapitza coined the term “superfluidity” in his paper of 1938, having some intuition about a deep connection to superconductivity. This is remarkable because, although superconductivity had been observed much earlier in 1911, a microscopic understanding was only achieved much later in 1957. Only then, with the help of the microscopic theory of Bardeen, Cooper, and Schrieffer it was possible to appreciate the deep connection between electronic superconductivity and superfluidity in  $^4\text{He}$ . For the case of a bosonic superfluid such as  $^4\text{He}$ , the theoretical background of Bose-Einstein condensation was already known since 1924 [5, 7], and F. London proposed shortly after the discovery that helium undergoes a Bose-Einstein condensation [14]. Other early theoretical developments, such as the phenomenological two-fluid model, were put forward by L. Landau [11] and L. Tisza [19]. More details about the interesting history of the discovery of superfluidity can be found in [4, 8].

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<sup>1</sup>In this chapter, helium is always synonymous to  $^4\text{He}$ , which is bosonic. I will write  $^4\text{He}$  only when I want to emphasize the bosonic nature. The fermionic counterpart  $^3\text{He}$  can also become superfluid, see Sect. 5.4.



**Fig. 2.1** Schematic phase diagram of  ${}^4\text{He}$  in the plane of pressure  $P$  and temperature  $T$ . Below a certain pressure, helium remains liquid for arbitrarily small temperatures, allowing for a superfluid phase below a critical temperature, sometimes called  $\lambda$ -temperature. Superfluid and normal fluid phases are denoted by helium II and helium I, respectively. This terminology has historical origin and was given to the two phases after the discovery of the phase transition, but before the discovery of superfluidity of helium II

What is special about helium, i.e., why can it become superfluid at low temperatures? Superfluidity is a quantum effect, so if we were to invent a liquid that becomes superfluid, one thing we would have to make sure is that it remains a liquid for very small temperatures, because only there quantum effects become important. Helium is special in this sense. All other elements solidify at some point when they are cooled down. The reason is that the kinetic energy of their atoms becomes sufficiently small at small temperatures to confine the atoms within their lattice sites. For very small temperatures, the kinetic energy is solely given by the zero-point motion. It turns out that the zero-point motion for helium atoms is sufficiently large to prevent them from forming a solid. Only under strong pressure does helium solidify. Although hydrogen atoms are lighter, their inter-atomic attractive interactions are much stronger, so hydrogen does solidify. The phase diagram of helium is shown in Fig. 2.1.

## 2.1 Landau's Critical Velocity

To explain why helium can be superfluid, we need to explain why it transports charge (here: mass, or helium atoms) without friction. The most important ingredient is the Bose-Einstein condensate. It carries charge and can flow without losing energy. The excitations on top of the condensate potentially lead to dissipation. Landau came up with a very general argument that results in a condition for these excitations in order to allow for superfluidity: let us consider a superfluid moving

through a capillary with velocity  $\mathbf{v}_s$ . In the rest frame of the fluid (where the capillary moves with velocity  $-\mathbf{v}_s$ ), let the energy of such an excitation and its corresponding momentum be  $\epsilon_p > 0$  and  $\mathbf{p}$ . Now, in the rest frame of the capillary, the energy of the fluid is given by the kinetic energy  $E_{\text{kin}}$  plus the energy of the elementary excitations, transformed into the new frame,<sup>2</sup>

$$E = E_{\text{kin}} + \epsilon_p + \mathbf{p} \cdot \mathbf{v}_s. \quad (2.1)$$

The fluid loses energy through dissipation if

$$\epsilon_p + \mathbf{p} \cdot \mathbf{v}_s < 0. \quad (2.2)$$

The left-hand side can only be negative if its minimum is negative,  $\epsilon_p - pv_s < 0$ . Consequently, the system transports charge without dissipation for velocities smaller than the critical velocity

$$v_c = \min_p \frac{\epsilon_p}{p}. \quad (2.3)$$

This simple argument by Landau is of fundamental importance for the understanding of a superfluid. A direct consequence is that systems where  $\min_p \frac{\epsilon_p}{p} = 0$  cannot be superfluid since then  $v_c = 0$  and an arbitrarily small velocity would result in dissipation. We can write the minimum of  $\epsilon_p/p$  as the solution of

$$0 = \frac{\partial}{\partial p} \frac{\epsilon_p}{p} \Rightarrow \frac{\partial \epsilon_p}{\partial p} = \frac{\epsilon_p}{p}. \quad (2.4)$$

For a given point on the curve  $\epsilon_p$  we are thus asking whether the slope of the curve is identical to the slope of a straight line from the origin through the given point. Or, in other words, to check the superfluidity of a system, take a horizontal line through the origin in the  $\epsilon_p$ - $p$  plane and rotate it upwards. If you can do so by a finite amount before touching the dispersion curve, the system supports superfluidity. The slope of the line at the touching point is the critical velocity according to Landau above which superfluidity is destroyed. In particular, any gapless dispersion with slope zero in the origin must lead to dissipation for any nonzero velocity. It is important to remember that the criterion for superfluidity is not only a requirement for the excitations of the system. Otherwise one might incorrectly conclude that a free gas of relativistic particles with dispersion  $\epsilon_p = \sqrt{p^2 + m^2}$  is a superfluid. The criterion rather requires a nonzero critical velocity  $v_c$  and the existence of a

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<sup>2</sup>Here, in the context of superfluid helium, we change frames by a Galilei transformation, and do not use the more general Lorentz transformation. Later we shall discuss relativistic excitations whose transformation reduces in the low-velocity limit to Eq. (2.1), see Eq. (4.33) and discussion below that equation.

condensate. Without a condensate, there is nothing to transport the charge without friction.

We shall see later that Bose-Einstein condensation is always accompanied by a gapless mode  $\epsilon_{p=0} = 0$  due to the Goldstone theorem, and this gapless mode is called Goldstone mode. One might think that the Goldstone mode can very easily be excited. And this is true in some sense. For instance, due to the gaplessness, such a mode becomes populated for arbitrarily small temperatures. Landau's argument, however, shows that even a gapless mode is sufficiently difficult to excite by forcing the fluid to move through a capillary: if for instance the dispersion of the Goldstone mode is linear,  $\epsilon_p \propto p$ , the mode is gapless but Landau's critical velocity is nonzero, and in fact identical to the slope of the Goldstone mode. Typically, the slope of a Goldstone mode is indeed linear for small momenta. This is true for instance in superfluid helium. On the other hand, if we had  $\epsilon_p \propto p^2$  for small momenta, the slope of the dispersion at the origin would be zero and as a consequence  $v_c = 0$ .

## 2.2 Thermodynamics of Superfluid Helium

While the existence of a Goldstone mode and the linearity at small  $p$  are very general features, the details of the complete dispersion of this mode depend on the details of the interactions in a given system. In superfluid helium, it turns out that the mode has a dispersion of the form shown in Fig. 2.2. For low energies, it can effectively be described by two different excitations, one accounting for the linear low-momentum part—this is called the “phonon”—and one accounting for the vicinity of the local minimum at a finite value of  $p$ —this is called the “roton”. We can write these two dispersions as

$$\epsilon_p = cp \quad (\text{“phonon”}), \quad (2.5a)$$

$$\epsilon_p = \Delta + \frac{(p - p_0)^2}{2m} \quad (\text{“roton”}). \quad (2.5b)$$

with parameters  $c$ ,  $\Delta$ ,  $p_0$ ,  $m$ , whose values are specified in Fig. 2.3.

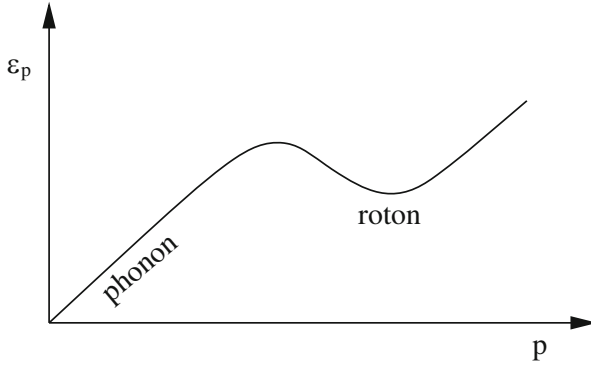
Let us first compute some of the thermodynamic properties given by the Goldstone mode. We start from the general expression for the pressure,

$$P = -T \int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln(1 - e^{-\epsilon_p/T}) = \frac{1}{3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} p \frac{\partial \epsilon_p}{\partial p} f(\epsilon_p), \quad (2.6)$$

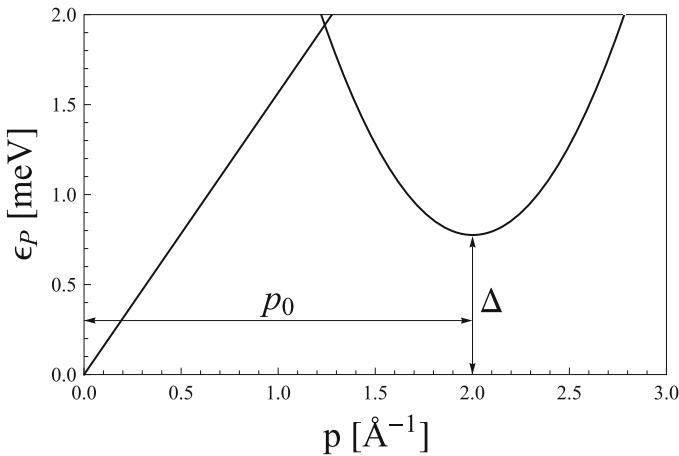
where, in the second step, we have used partial integration, where  $T$  is the temperature, and where

$$f(\epsilon_p) = \frac{1}{e^{\epsilon_p/T} - 1} \quad (2.7)$$

is the Bose distribution function.



**Fig. 2.2** Schematic plot of the Goldstone dispersion for superfluid helium. This mode is often modelled in terms of two different modes, the phonon and the roton, see Fig. 2.3



**Fig. 2.3** Dispersions for phonons and rotons from Eq. (2.5) with parameters  $m = 1.72 \times 10^{-24}$  g,  $p_0 = 2.1 \times 10^{-19}$  g cm s $^{-1}$ ,  $\Delta = 8.9$  K,  $c = 238$  m s $^{-1}$

Consequently, the phonon contribution to the pressure is

$$P_{\text{ph}} = \frac{c}{6\pi^2} \int_0^\infty dp \frac{p^3}{e^{cp/T} - 1} = \frac{T^4}{6\pi^2 c^3} \underbrace{\int_0^\infty dy \frac{y^3}{e^y - 1}}_{\pi^4/15} = \frac{\pi^2 T^4}{90 c^3}. \quad (2.8)$$

If the dispersion were linear for all  $p$ , this result would be valid for any  $T$ . However, the dispersion is linear only for small  $p$ . Since the corrections to the linear behavior become important at larger temperatures, this result cannot be trusted for all  $T$ . (Obviously, the critical temperature for superfluidity is another, absolute, limit above which this result is inapplicable). The result for the pressure is similar to

the Stefan-Boltzmann pressure of blackbody radiation. The reason is that a photon has the same linear dispersion as the superfluid phonon, but moves with the speed of light. Thus, we recover the Stefan-Boltzmann pressure if we set  $c$  equal to the speed of light and multiply the result by 2 because a photon has two degrees of freedom.

We can now compute the entropy and the specific heat per unit volume from the usual thermodynamic definitions,

$$s_{\text{ph}} = \frac{\partial P_{\text{ph}}}{\partial T} = \frac{2\pi^2 T^3}{45c^3}, \quad (2.9)$$

and

$$c_{V,\text{ph}} = T \frac{\partial s_{\text{ph}}}{\partial T} = \frac{2\pi^2 T^3}{15c^3} = 3s_{\text{ph}}, \quad (2.10)$$

where the subscript  $V$  indicates that the specific heat is computed at fixed volume (as opposed to fixed pressure).

The calculation of the roton contribution is a bit more complicated,

$$P_{\text{rot}} = \frac{1}{6\pi^2 m} \int_0^\infty dp p^3 \frac{p - p_0}{e^{\epsilon_p/T} - 1}. \quad (2.11)$$

In general, this integral has to be solved numerically. Here we proceed by making the assumption  $T \ll \Delta$ , such that we can approximate

$$\frac{1}{e^{\frac{\Delta}{T} + \frac{(p-p_0)^2}{2mT}} - 1} \simeq e^{-\Delta/T} e^{-\frac{(p-p_0)^2}{2mT}}, \quad (2.12)$$

and thus

$$P_{\text{rot}} \simeq \frac{e^{-\Delta/T}}{6\pi^2 m} \int_0^\infty dp p^3 (p - p_0) e^{-\frac{(p-p_0)^2}{2mT}}. \quad (2.13)$$

This expression shows that the contribution of the rotons is exponentially suppressed for temperatures much smaller than  $\Delta$ . To obtain the subleading temperature dependence, we introduce the new integration variable  $y = (p - p_0)/\sqrt{2mT}$ ,

$$\begin{aligned} P_{\text{rot}} &= \frac{e^{-\Delta/T} T (2mT)^{3/2}}{3\pi^2} \int_{-\frac{p_0}{\sqrt{2mT}}}^\infty dy y \left( y + \frac{p_0}{\sqrt{2mT}} \right)^3 e^{-y^2} \\ &\simeq \frac{e^{-\Delta/T} T p_0^2 (2mT)^{1/2}}{\pi^2} \underbrace{\int_{-\infty}^\infty dy y^2 e^{-y^2}}_{\sqrt{\pi}/2} \\ &= \sqrt{\frac{m}{2\pi^3}} p_0^2 T^{3/2} e^{-\Delta/T}, \end{aligned} \quad (2.14)$$

where we have assumed  $T \ll p_0^2/(2m)$ . With the parameters given in Fig. 2.3 we have  $p_0^2/(2m) \simeq 93 \text{ K}$ , i.e., since we already have assumed that  $T$  is much smaller than  $\Delta \simeq 8.9 \text{ K}$ ,  $T$  is also much smaller than  $p_0^2/(2m)$ . Again we may compute entropy and specific heat,

$$s_{\text{rot}} \simeq \sqrt{\frac{m}{2\pi^3}} \frac{p_0^2 \Delta}{T^{1/2}} e^{-\Delta/T}, \quad (2.15)$$

and

$$c_{V,\text{rot}} \simeq \sqrt{\frac{m}{2\pi^3}} \frac{p_0^2 \Delta^2}{T^{3/2}} e^{-\Delta/T}, \quad (2.16)$$

where we have neglected terms of higher order in  $T/\Delta$ .

## 2.3 Two-Fluid Model

The hydrodynamics of a superfluid is often described within a so-called two-fluid model, suggested by Tisza [19] and Landau [11] shortly after the discovery of superfluidity. A priori, this was a purely phenomenological description. We shall discuss later how it emerges as a kind of effective theory from a microscopic description. In the two-fluid picture, the system is formally divided into two fluids, the *superfluid* and *normal fluid*, which interpenetrate each other. The superfluid component consists of the condensate, while the normal component contains the elementary excitations, i.e., the phonon and roton excitations in the case of superfluid helium. This picture suggests that at zero temperature there is only a superfluid. Then, upon heating up the system, the normal fluid will start to appear and become more and more dominant until the superfluid completely vanishes at and above the critical temperature. Originally, the model served to explain the ‘‘viscosity paradox’’ which had appeared from two apparently contradicting behaviors of superfluid helium: damping times of the oscillations of a torsion pendulum in liquid helium suggested a viscosity [20], in apparent contrast to the dissipationless flow through a thin capillary [10]. In the two-fluid picture it is only the superfluid component that can flow through the thin tube while the pendulum sees both fluid components, i.e., the excitations of the normal fluid were responsible for the damping of the pendulum. The model predicts the existence of a second sound mode, see Sect. 2.4, which was indeed observed after the two-fluid picture was suggested.

The flow of the system is described by two fluids with independent velocity fields. The momentum density  $\mathbf{g}$  receives contribution from both fluids,

$$\mathbf{g} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n, \quad (2.17)$$

where  $\mathbf{v}_s$  and  $\mathbf{v}_n$  are the velocities of the superfluid and the normal fluid, respectively, and  $\rho_s$  and  $\rho_n$  are the superfluid and normal-fluid mass densities, such that the total mass density is

$$\rho = \rho_n + \rho_s. \quad (2.18)$$

To compute the normal-fluid density, we consider the rest frame of the superfluid, in which the normal fluid is moving with velocity  $\mathbf{w} \equiv \mathbf{v}_n - \mathbf{v}_s$ . In this frame, the momentum density only receives a contribution from the normal fluid and is given by  $\rho_n \mathbf{w}$ . We can also express the momentum density of the normal fluid as

$$\rho_n \mathbf{w} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \mathbf{p} f(\epsilon_p - \mathbf{p} \cdot \mathbf{w}), \quad (2.19)$$

where we have taken into account that the distribution function of the elementary excitations depends on the relative velocity between the two fluids. As in the previous subsections,  $\epsilon_p$  is the dispersion of the elementary excitations measured in the superfluid rest frame. In particular, we recover the Galilei transformed excitations from Eq. (2.1) for  $\mathbf{v}_n = 0$ . Multiplying both sides of Eq. (2.19) with  $\mathbf{w}$ , we obtain an expression for the normal-fluid density,

$$\rho_n = \frac{1}{w} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \hat{\mathbf{w}} \cdot \mathbf{p} f(\epsilon_p - \mathbf{p} \cdot \mathbf{w}), \quad (2.20)$$

where  $\hat{\mathbf{w}} \equiv \mathbf{w}/w$ . It is important to realize that the concept of normal-fluid and superfluid densities only makes sense in the presence of a (at least infinitesimal) relative velocity. In general,  $\rho_s$  and  $\rho_n$  are functions of this relative velocity. For many applications one is interested in the limit of small relative velocities. To compute  $\rho_n$  in the limit  $w \rightarrow 0$ , we insert the Taylor expansion

$$f(\epsilon_p - \mathbf{p} \cdot \mathbf{w}) = f(\epsilon_p) - \mathbf{p} \cdot \mathbf{w} \left. \frac{\partial f}{\partial \epsilon_p} \right|_{\mathbf{w}=0} + \mathcal{O}(w^2) \quad (2.21)$$

into Eq. (2.20). The integral over the first term of this expansion vanishes, and we obtain

$$\begin{aligned} \rho_n(\mathbf{w} \rightarrow 0) &= - \int \frac{d^3 \mathbf{p}}{(2\pi)^3} (\mathbf{p} \cdot \hat{\mathbf{w}})^2 \frac{\partial f}{\partial \epsilon_p} \\ &= - \frac{1}{3} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} p^2 \frac{\partial f}{\partial \epsilon_p} \\ &= \frac{1}{3T} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{p^2 e^{\epsilon_p/T}}{(e^{\epsilon_p/T} - 1)^2}. \end{aligned} \quad (2.22)$$



Since we have modelled the dispersion of the Goldstone mode by two separate excitations, we can compute their contribution to the normal-fluid density separately. Let us start with the phonon contribution.

With the  $z$ -axis of our coordinate system pointing in the direction of  $\mathbf{w}$  and  $x = \cos \theta$  with  $\theta$  being the angle between  $\mathbf{w}$  and the momentum  $\mathbf{p}$ , the phonon contribution from Eq. (2.20) is

$$\begin{aligned}
 \rho_{n,\text{ph}} &= \frac{1}{4\pi^2 w} \int_{-1}^1 dx x \int_0^\infty dp \frac{p^3}{e^{p(c-wx)/T} - 1} \\
 &= \frac{T^4}{4w\pi^2} \underbrace{\int_{-1}^1 dx \frac{x}{(c-wx)^4}}_{\frac{8cw}{3(c^2-w^2)^3} \text{ for } w < c} \int_0^\infty dy \frac{y^3}{e^y - 1} \\
 &= \frac{2\pi^2 T^4}{45c^5} \left(1 - \frac{w^2}{c^2}\right)^{-3}. \tag{2.23}
 \end{aligned}$$

The condition  $w < c$  is necessary to ensure superfluidity: we can repeat the argument for Landau's critical velocity from Sect. 2.1, replacing the rest frame of the capillary with the rest frame of the normal fluid. This shows that the relative velocity  $w$  has an upper limit given by Eq. (2.3) above which dissipation sets in. In the absence of rotons, this limit would be given by  $c$ . As Fig. 2.3 shows, the presence of the rotons only decreases the limit.

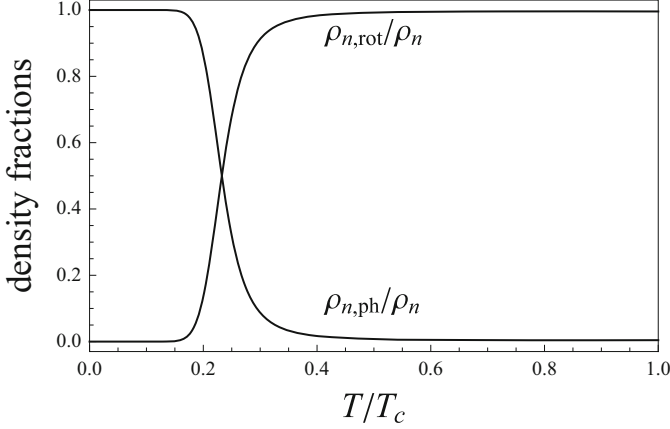
For small relative velocities  $w$  we find

$$\rho_{n,\text{ph}} = \frac{2\pi^2 T^4}{45c^5} \left[ 1 + \mathcal{O}\left(\frac{w^2}{c^2}\right) \right] \simeq \frac{s_{\text{ph}} T}{c^2}, \tag{2.24}$$

where the result for the entropy density (2.9) has been used. One can check that the same  $w \rightarrow 0$  result is obtained by directly using Eq. (2.22).

For the roton contribution we find for small temperatures

$$\begin{aligned}
 \rho_{n,\text{rot}} &\simeq \frac{e^{-\Delta/T}}{4\pi^2 w} \int_0^\infty dp p^3 e^{-\frac{(p-p_0)^2}{2mT}} \int_{-1}^1 dx x e^{\frac{pw}{T}} \\
 &= \frac{T e^{-\Delta/T}}{2\pi^2 w^2} \int_0^\infty dp p^2 e^{-\frac{(p-p_0)^2}{2mT}} \left( \cosh \frac{pw}{T} - \frac{T}{wp} \sinh \frac{pw}{T} \right) \\
 &\simeq \frac{T e^{-\Delta/T} p_0^2}{2\pi^2 w^2} \left( \cosh \frac{p_0 w}{T} - \frac{T}{wp_0} \sinh \frac{p_0 w}{T} \right) \int_{-p_0}^\infty dq e^{-\frac{q^2}{2mT}}
 \end{aligned}$$



**Fig. 2.4** Contributions of phonons and rotons to the normal-fluid density for all temperatures up to the critical temperature in the absence of a relative flow,  $\mathbf{w} = 0$ , with the parameters of Fig. 2.3

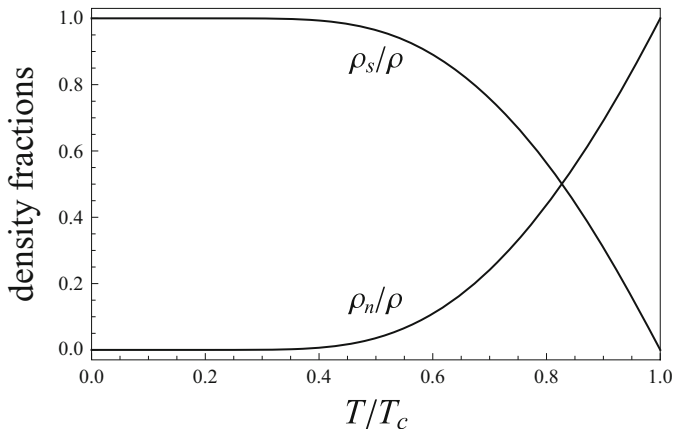
$$\simeq \sqrt{\frac{m}{2\pi^3}} \frac{T^{3/2} e^{-\Delta/T} p_0^2}{w^2} \left( \cosh \frac{p_0 w}{T} - \frac{T}{w p_0} \sinh \frac{p_0 w}{T} \right). \quad (2.25)$$

In the limit  $w \rightarrow 0$  this becomes

$$\rho_{n,\text{rot}} = \sqrt{\frac{m}{2\pi^3}} \frac{e^{-\Delta/T} p_0^4}{3T^{1/2}} \left[ 1 + \mathcal{O}\left(\frac{p_0^2 w^2}{T^2}\right) \right] \simeq \frac{p_0^2}{3T^2} P_{\text{rot}}. \quad (2.26)$$

Again, we can check that this result is obtained from the general expression (2.22). This is left as a small exercise to the reader.

We can now compute the total normal-fluid density from the two separate contributions. Together with a given total density  $\rho$ , this allows us to compute the density fractions for superfluid and normal-fluid components for all temperatures up to the critical temperature. Since at the critical temperature  $T_c$  all mass sits in the normal fluid, we can compute  $T_c$  by solving  $\rho = \rho_{n,\text{ph}}(T_c) + \rho_{n,\text{rot}}(T_c)$  numerically for  $T_c$ . For the limit  $w \rightarrow 0$  and with  $\rho = 0.147 \text{ g cm}^{-3}$  one obtains  $T_c \simeq 2.47 \text{ K}$ . This number is obtained by using the full temperature dependence of  $\rho_n$ : the phonon contribution (2.24) is exact for all temperatures (under the assumption that the phonon dispersion continues linearly for all momenta), while for the roton contribution the full expression (2.22) has been used, including a numerical momentum integral. (Using the low-temperature approximation (2.26) gives a slightly larger critical temperature.) The discrepancy to the actual value of the critical temperature of  $T_c \simeq 2.17 \text{ K}$  is due to the model assumption of separate phonon and roton excitations, which differs from the correct quasiparticle spectrum, see Fig. 2.3. This difference is important for large temperatures. Within the given model, we show the phonon and roton contributions to the normal-fluid density in Fig. 2.4, and the superfluid and normal-fluid density fractions in Fig. 2.5.



**Fig. 2.5** Normal-fluid and superfluid density fractions in the absence of a relative flow,  $\mathbf{w} = 0$ , with the parameters of Fig. 2.3 and the total mass density  $\rho = 0.147 \text{ g cm}^{-3}$ . For all temperatures above  $T_c$ , we have  $\rho_n = \rho$ ,  $\rho_s = 0$

## 2.4 First and Second Sound

One interesting consequence of the two-fluid model is the existence of two sound modes. The second sound mode was first observed in superfluid helium [13, 15], for a nice popular article about second sound and its significance for developments in the theory of superfluid helium see [6]. Much more recently, second sound was also measured in an ultra-cold Fermi gas [17]. Before we can discuss first and second sound, we need to discuss some hydrodynamics. We shall give a very brief introduction to single-fluid hydrodynamics before we add a second fluid in order to describe the superfluid. Here we shall only discuss ideal, i.e., dissipationless, hydrodynamics. This is sufficient for the discussion of first and second sound, which can propagate non-dissipatively. If you are interested in a much more detailed account of hydrodynamics, see for instance [12].

### 2.4.1 Single-Fluid Hydrodynamics

We shall start from the relativistic form of hydrodynamics and then take the non-relativistic limit. The conservation equations for charge and (four-)momentum are

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad (2.27)$$

where

$$j^\mu = nv^\mu \quad (2.28)$$

is the current associated with the conserved charge, and

$$T^{\mu\nu} = (\epsilon + P)v^\mu v^\nu - g^{\mu\nu} P \quad (2.29)$$

is the stress-energy tensor for an ideal fluid. Here,  $n$ ,  $\epsilon$ , and  $P$  are number density, energy density, and pressure, measured in the rest frame of the fluid, and  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$  is the metric tensor. Moreover,

$$v^\mu = \gamma(1, \mathbf{v}) \quad (2.30)$$

is the four-velocity of the fluid, expressed in terms of the three-velocity  $\mathbf{v}$  and the Lorentz factor  $\gamma = (1 - v^2)^{-1/2}$ . Here,  $v^2$  denotes the square of the modulus of the three-velocity. This form of the four-velocity ensures

$$v_\mu v^\mu = 1. \quad (2.31)$$

Even though we have omitted the arguments, in general all quantities of course depend on space-time, i.e.,  $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$  etc.

The various components of the stress-energy tensor are

$$T^{00} = \frac{\epsilon + P v^2}{1 - v^2}, \quad (2.32a)$$

$$T^{0i} = T^{i0} = \frac{\epsilon + P}{1 - v^2} v_i, \quad (2.32b)$$

$$T^{ij} = \frac{\epsilon + P}{1 - v^2} v_i v_j + \delta_{ij} P, \quad (2.32c)$$

where  $v_i$  are the components of the three-velocity  $\mathbf{v}$ , not of the four-velocity. In particular, the stress-energy tensor is symmetric. We define the rest frame of the fluid by  $\mathbf{v} = 0$ , i.e.,  $v^\mu = (1, 0, 0, 0)$ . In this particular frame, the stress-energy tensor assumes the simple form

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \quad (2.33)$$

As a simple exercise, one can check that the general stress-energy tensor (2.29) can be obtained via a Lorentz transformation from the stress-energy tensor (2.33). In a single-fluid system with uniform fluid velocity, it is obviously always possible to choose a global frame in which the three-velocity vanishes. This is the rest frame of the fluid. This is not possible in a two-fluid system, even if the two velocities of

the two fluids are uniform. In that case, one may choose to work in the rest frame of one of the fluids, and the direction of the velocity of the other fluid will necessarily break rotational invariance.

In order to take the non-relativistic limit, we introduce the rest mass density  $\rho$  via the temporal component of the four-current,

$$\rho = mj^0. \quad (2.34)$$

With Eq. (2.28), this means that  $mn = \rho\sqrt{1-v^2}$ , i.e.,  $\rho\sqrt{1-v^2}$  is the rest mass density in the fluid rest frame, while  $\rho$  is the rest mass density in the frame where the fluid moves with velocity  $\mathbf{v}$ . Eventually, after having derived the non-relativistic limit, the mass density  $\rho$  will be assumed to be frame independent.

The spatial components of Eq. (2.28) now give

$$m\mathbf{j} = \rho\mathbf{v}. \quad (2.35)$$

Next, we need the non-relativistic version of the stress-energy tensor. To this end, we introduce the non-relativistic energy density  $\epsilon_0$  in the fluid rest frame by separating the rest energy,

$$\epsilon = \rho\sqrt{1-v^2} + \epsilon_0. \quad (2.36)$$

Neglecting terms of order  $v^4$ , we can write

$$\begin{aligned} T^{00} &\simeq \epsilon + (\epsilon + P)v^2 \\ &\simeq \rho + \epsilon_0 + \left(\frac{\rho}{2} + \epsilon_0 + P\right)v^2. \end{aligned} \quad (2.37)$$

We now remove the rest mass density  $\rho$  and assume that  $\epsilon_0 + P \ll \rho$  in the kinetic term to obtain the non-relativistic version

$$T_{\text{non-rel.}}^{00} = \epsilon_0 + \frac{\rho v^2}{2}, \quad (2.38)$$

which contains the energy density in the fluid rest frame plus a kinetic term which has the usual non-relativistic form. We proceed analogously for the other components. First, we write

$$\begin{aligned} T^{0i} &\simeq (\epsilon + P)(1 + v^2)v_i \\ &\simeq \rho v_i + \left(\epsilon_0 + \frac{\rho v^2}{2} + P\right)v_i, \end{aligned} \quad (2.39)$$

where, in the  $\mathcal{O}(v^3)$  terms, we have again neglected  $\epsilon_0 + P$  compared to  $\rho$ . Then, we define the *momentum density* by

$$g_i \equiv T_{\text{non-rel.}}^{0i} = \rho v_i, \quad (2.40)$$

and the *energy flux* by

$$q_i \equiv T_{\text{non-rel.}}^{i0} = \left( \epsilon_0 + \frac{\rho v^2}{2} + P \right) v_i. \quad (2.41)$$

As a consequence, in the non-relativistic version the stress-energy tensor is not symmetric,  $T_{\text{non-rel.}}^{0i} \neq T_{\text{non-rel.}}^{i0}$ . Finally,

$$\begin{aligned} T^{ij} &\simeq (\epsilon + P)v_i v_j + \delta_{ij}P \\ &\simeq (\rho + \epsilon_0 + P)v_i v_j + \delta_{ij}P, \end{aligned} \quad (2.42)$$

and, again using  $\epsilon_0 + P \ll \rho$ , we define the non-relativistic stress tensor

$$\Pi_{ij} \equiv T_{\text{non-rel.}}^{ij} = \rho v_i v_j + \delta_{ij}P. \quad (2.43)$$

We are now prepared to formulate the conservation equations (2.27) in the non-relativistic limit,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{g} = 0, \quad (2.44a)$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{q} = 0, \quad (2.44b)$$

$$\frac{\partial g_i}{\partial t} + \partial_j \Pi_{ji} = 0, \quad (2.44c)$$

where we have defined the energy density<sup>3</sup>

$$\epsilon \equiv \epsilon_0 + \frac{\rho v^2}{2} \quad (2.45)$$

(remember that  $\epsilon_0$  is the energy density in the rest frame of the fluid). The first equation is the current conservation  $\partial_\mu j^\mu = 0$  multiplied by  $m$ , the second equation is the  $\nu = 0$  component of the four-momentum conservation  $\partial_\mu T^{\mu\nu} = 0$ , and the third equation is the  $\nu = i$  component of the four-momentum conservation. In summary, we repeat the definitions of the non-relativistic quantities that appear in these equations,

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<sup>3</sup>The  $\epsilon$  defined here is *not* the relativistic  $\epsilon$  used above. But since for the rest of the chapter we shall work in the non-relativistic framework, this slight abuse of notation should not cause any confusion.

$$\mathbf{g} = \rho \mathbf{v}, \quad \mathbf{q} = (\epsilon + P) \mathbf{v}, \quad \Pi_{ij} = \rho v_i v_j + \delta_{ij} P. \quad (2.46)$$

Using these definitions, we can also bring the hydrodynamic equations in the following form<sup>4</sup>

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2.47a)$$

$$\frac{\partial s}{\partial t} + \nabla \cdot (s \mathbf{v}) = 0, \quad (2.47b)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla P}{\rho} \quad (\text{Euler equation}). \quad (2.47c)$$

The continuity equation for the mass current (2.47a) is simply copied from above; Eq. (2.47c) is straightforwardly obtained by inserting Eq. (2.46) into Eq. (2.44c) and using the continuity equation (2.47a). To derive Eq. (2.47b)—which is a continuity equation for the entropy current—start from Eq. (2.44b) and write

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left( \epsilon_0 + \frac{\rho v^2}{2} \right) + \nabla \cdot \left( \epsilon_0 + \frac{\rho v^2}{2} + P \right) \mathbf{v} \\ &= \frac{\partial \epsilon_0}{\partial t} + \frac{v^2}{2} \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \\ &\quad + (\mu \rho + Ts) \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \epsilon_0 + \nabla \cdot \left( \frac{\rho v^2}{2} \mathbf{v} \right) + \mathbf{v} \cdot \nabla P, \end{aligned} \quad (2.48)$$

where we have used the relation

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<sup>4</sup>Dissipative effects are included by adding the following terms to the energy flux and the stress tensor (the momentum density remains unchanged),

$$q_i = (\epsilon + P)v_i + v_j \delta \Pi_{ij} + Q_i, \quad \Pi_{ij} = \rho v_i v_j + \delta_{ij} P + \delta \Pi_{ij},$$

where

$$\mathbf{Q} \equiv -\kappa \nabla T, \quad \delta \Pi_{ij} \equiv -\eta \left( \partial_i v_j - \partial_j v_i - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) - \zeta \delta_{ij} \nabla \cdot \mathbf{v},$$

with the thermal conductivity  $\kappa$ , the shear viscosity  $\eta$ , and the bulk viscosity  $\zeta$ . In the presence of dissipation, the entropy current is no longer conserved, i.e., the right-hand side of Eq. (2.47b) is not zero. With dissipative terms, the Euler equation (2.47c) is known as the Navier-Stokes equation. Existence and smoothness of general solutions to the Navier-Stokes equation (and also to the Euler equation) are an unsolved problem in mathematical physics and its solution is worth a million dollars, see <http://www.claymath.org/millennium-problems/navier-stokes-equation>.

$$\epsilon_0 + P = \mu \rho + Ts, \quad (2.49)$$

where  $s$  is the entropy density and  $\mu$  the chemical potential.<sup>5</sup> Now we remember the thermodynamic relations

$$d\epsilon_0 = \mu d\rho + Tds, \quad (2.50)$$

and

$$dP = \rho d\mu + s dT. \quad (2.51)$$

These two thermodynamic relations reflect the fact that  $\epsilon_0$  and  $P$  are related via two Legendre transforms with respect to the pairs  $(T, s)$  and  $(\mu, \rho)$ . We shall need Eq. (2.51) later. Here we make use of Eq. (2.50) which we insert into Eq. (2.48) to obtain

$$\begin{aligned} 0 &= \mu \frac{\partial \rho}{\partial t} + T \frac{\partial s}{\partial t} + (\mu \rho + Ts) \nabla \cdot \mathbf{v} + \mu \mathbf{v} \cdot \nabla \rho + T \mathbf{v} \cdot \nabla s \\ &\quad + \underbrace{\frac{v^2}{2} \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \left( \frac{\rho v^2}{2} \mathbf{v} \right)}_{=0} + \mathbf{v} \cdot \nabla P \\ &= T \left[ \frac{\partial s}{\partial t} + \nabla \cdot (s \mathbf{v}) \right], \end{aligned} \quad (2.52)$$

where we have used the continuity equation (2.47a) twice and the Euler equation (2.47c). The result is the entropy conservation (2.47b). The entropy current is only conserved in the absence of dissipation.

## 2.4.2 Two-Fluid Hydrodynamics

In view of the two-fluid model discussed in Sect. 2.3, we have to modify the single-fluid hydrodynamics because each of the fluid components of the superfluid acquires its own, independent velocity field. Let us distinguish two (local) reference frames in the following way. Imagine a superfluid flowing through a tube. Then, our first reference frame is the frame where the tube is at rest and where superfluid and

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<sup>5</sup>In this non-relativistic context, we work with the chemical potential per unit mass  $\mu$ , which has the same units as a velocity squared (i.e., it is dimensionless if the speed of light is set to one). In the relativistic treatment, starting in Chap. 3,  $\mu$  will denote the chemical potential per unit charge, which has the same units as energy.



normal fluid have velocities  $\mathbf{v}_s$  and  $\mathbf{v}_n$ , respectively. The second reference frame is the one where the superfluid is at rest, i.e., the tube moves with velocity  $-\mathbf{v}_s$  and the superfluid and normal fluid move with velocities zero and  $\mathbf{v}_n - \mathbf{v}_s$ , respectively. We denote quantities in the superfluid rest frame with a subscript 0 and quantities in the rest frame of the tube without additional subscript.

In the rest frame of the tube, the momentum density is given by the sum of both fluids, as already stated in Eq. (2.17). In the superfluid rest frame, the momentum density is only given by the normal fluid which has mass density  $\rho_n$  and which moves with velocity  $\mathbf{v}_n - \mathbf{v}_s$ . Consequently,

$$\mathbf{g} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s, \quad \mathbf{g}_0 = \rho_n (\mathbf{v}_n - \mathbf{v}_s) \quad (2.53a)$$

$$\Rightarrow \quad \mathbf{g} = \mathbf{g}_0 + \rho \mathbf{v}_s, \quad (2.53b)$$

where  $\rho = \rho_n + \rho_s$  is the total mass density, as above. The stress tensor in the two frames reads

$$\Pi_{ij} = \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + \delta_{ij} P,$$

$$\Pi_{0ij} = \rho_n (v_{ni} - v_{si})(v_{nj} - v_{sj}) + \delta_{ij} P \quad (2.54a)$$

$$\Rightarrow \quad \Pi_{ij} = \Pi_{0ij} + \rho v_{si} v_{sj} + v_{si} g_{0j} + v_{sj} g_{0i}. \quad (2.54b)$$

For completeness, although we shall not need this in the following, let us also write down the energy density and the energy density current in the two frames. We have

$$\epsilon = \epsilon_n + \epsilon_s + \frac{\rho_n v_n^2}{2} + \frac{\rho_s v_s^2}{2}, \quad \epsilon_0 = \epsilon_n + \epsilon_s + \frac{\rho_n (\mathbf{v}_n - \mathbf{v}_s)^2}{2} \quad (2.55a)$$

$$\Rightarrow \quad \epsilon = \epsilon_0 + \mathbf{v}_s \cdot \mathbf{g}_0 + \frac{\rho v_s^2}{2}, \quad (2.55b)$$

where

$$\epsilon_n = -P_n + \mu \rho_n + T s, \quad \epsilon_s = -P_s + \mu \rho_s \quad (2.56)$$

are the energy densities of normal fluid and superfluid, measured in their respective rest frames. Analogously,  $P_n$  and  $P_s$  are the pressures of the normal fluid and superfluid, and with  $P = P_n + P_s$  the relations (2.56) imply  $\epsilon_n + \epsilon_s = P + \mu \rho + T s$ . In the absence of a normal fluid, we have  $\epsilon_s = \epsilon_0$ , which makes the connection to the notation of the previous subsection. In Eq. (2.56) we have used that only the normal fluid carries entropy.

Finally, for the energy flux we find

$$\mathbf{q} = \left( \epsilon_n + P_n + \frac{\rho_n v_n^2}{2} \right) \mathbf{v}_n + \left( \epsilon_s + P_s + \frac{\rho_s v_s^2}{2} \right) \mathbf{v}_s,$$

$$\mathbf{q}_0 = \left[ \epsilon_n + P_n + \frac{\rho_n (\mathbf{v}_n - \mathbf{v}_s)^2}{2} \right] (\mathbf{v}_n - \mathbf{v}_s) \quad (2.57a)$$

$$\Rightarrow \quad q_i = q_{0i} + \left( \epsilon_0 + \mathbf{v}_s \cdot \mathbf{g}_0 + \frac{\rho_s v_s^2}{2} \right) v_{si} + \frac{v_s^2}{2} g_{0i} + v_{sj} \Pi_{0ij}. \quad (2.57b)$$

Equations (2.53b), (2.54b), (2.55b), and (2.57b) are the Galilei transforms of momentum density, stress tensor, energy density, and energy flux from the superfluid rest frame into the corresponding quantities in the rest frame of the tube. Notice that they are expressed solely in terms of quantities measured in the superfluid rest frame and the superfluid velocity. Since the relative velocity between the two reference frames is given by  $\mathbf{v}_s$ , the normal-fluid velocity  $\mathbf{v}_n$  does not appear in the Galilei transform, as it should be.

We write the hydrodynamic equations in the rest frame of the tube as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{g} = 0, \quad (2.58a)$$

$$\frac{\partial s}{\partial t} + \nabla \cdot (s \mathbf{v}_n) = 0, \quad (2.58b)$$

$$\frac{\partial \mathbf{g}}{\partial t} + \mathbf{v}_s (\nabla \cdot \mathbf{g}) + (\mathbf{g} \cdot \nabla) \mathbf{v}_s + \mathbf{g}_0 (\nabla \cdot \mathbf{v}_n) + (\mathbf{v}_n \cdot \nabla) \mathbf{g}_0 + \nabla P = 0. \quad (2.58c)$$

The first two equations have the same form as for the single fluid case, see Eq.(2.47), with  $\rho$  and  $\mathbf{g}$  now being the total mass and momentum densities, receiving contributions from both fluids, and the entropy density  $s \mathbf{v}_n$  solely coming from the normal fluid. To derive Eq.(2.58c) from (2.44c) one first easily checks that the stress tensor from Eq.(2.54a) can be written as

$$\Pi_{ij} = v_{sj} g_i + v_{ni} g_{0j} + \delta_{ij} P. \quad (2.59)$$

(Although not manifest in this form, the stress tensor is of course still symmetric.) Inserting Eq.(2.59) into Eq.(2.44c) immediately yields Eq.(2.58c).

Before we turn to the sound modes we derive one more useful relation. Using the hydrodynamic equations, the thermodynamic relations (2.50) and (2.51), and the explicit two-fluid form of  $\epsilon$  and  $\mathbf{q}$ , a rather tedious calculation yields

$$\begin{aligned} \frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{q} = & -\rho_s (\mathbf{v}_n - \mathbf{v}_s) \left[ \nabla \mu + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \frac{\partial \mathbf{v}_s}{\partial t} \right] \\ & + \frac{(\mathbf{v}_n - \mathbf{v}_s)^2}{2} \left[ \frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s) \right]. \end{aligned} \quad (2.60)$$

Now we use that the left-hand side of this equation is zero due to (the two-fluid version of) Eq. (2.44b) and neglect the term quadratic in the relative velocity  $\mathbf{v}_n - \mathbf{v}_s$  on the right-hand side to obtain the following relation for the superfluid velocity,

$$(\mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \mu. \quad (2.61)$$

### 2.4.3 Sound Modes

Imagine both fluid components to be at rest and the system to be in thermodynamic equilibrium. A sound wave is a (small) oscillation in the thermodynamic quantities like entropy, pressure etc and in the velocities of the two fluids. We thus imagine adding small deviations from equilibrium to the thermodynamic quantities like  $s(\mathbf{x}, t) = s_0 + \delta s(\mathbf{x}, t)$ ,  $P(\mathbf{x}, t) = P_0 + \delta P(\mathbf{x}, t)$  etc and small deviations (from zero) to the velocities,  $\mathbf{v}_n(\mathbf{x}, t) = \delta \mathbf{v}_n(\mathbf{x}, t)$  and  $\mathbf{v}_s(\mathbf{x}, t) = \delta \mathbf{v}_s(\mathbf{x}, t)$ . Here, the subscript 0 denotes thermodynamic equilibrium. In general, one might also compute the sound modes in the presence of a relative velocity of the two fluids, i.e., one may choose nonzero values of  $\mathbf{v}_{n,0}$  and  $\mathbf{v}_{s,0}$ . Here we restrict ourselves to the isotropic situation  $\mathbf{v}_{n,0} = \mathbf{v}_{s,0} = 0$ .

Since we are interested in small deviations from equilibrium, we neglect terms quadratic in the deviations, for instance

$$\begin{aligned} \nabla \cdot \mathbf{g} = & (\rho_{n,0} + \delta \rho_n) \nabla \cdot \delta \mathbf{v}_n + (\rho_{s,0} + \delta \rho_s) \nabla \cdot \delta \mathbf{v}_s + \delta \mathbf{v}_n \cdot \nabla \delta \rho_n + \delta \mathbf{v}_s \cdot \nabla \delta \rho_s \\ \simeq & \rho_{n,0} \nabla \cdot \delta \mathbf{v}_n + \rho_{s,0} \nabla \cdot \delta \mathbf{v}_s. \end{aligned} \quad (2.62)$$

The linearized hydrodynamic equations (2.58) thus become

$$\frac{\partial \rho}{\partial t} + \rho_n \nabla \cdot \mathbf{v}_n + \rho_s \nabla \cdot \mathbf{v}_s \simeq 0, \quad (2.63a)$$

$$\frac{\partial s}{\partial t} + s \nabla \cdot \mathbf{v}_n \simeq 0, \quad (2.63b)$$

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \nabla P \simeq 0, \quad (2.63c)$$

and Eq. (2.61) simplifies to

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu \simeq 0. \quad (2.64)$$

A usual sound wave is a density oscillation and is described by a wave equation that relates a spatial second derivative to a temporal second derivative. Due to the presence of two fluids, we will now obtain a second wave equation for the entropy. The two wave equations are

$$\frac{\partial^2 \rho}{\partial t^2} = \Delta P, \quad (2.65a)$$

$$\frac{\partial^2 S}{\partial t^2} = \frac{S^2 \rho_s}{\rho_n} \Delta T, \quad (2.65b)$$

where  $S$  is the entropy per unit mass, such that  $s = \rho S$ . The equations are derived as follows. Equation (2.65a) is immediately obtained by taking the time derivative of Eq. (2.63a) and the divergence of Eq. (2.63c). Equation (2.65b) requires some more work. From the thermodynamic relation (2.51) we obtain  $\nabla P = \rho \nabla \mu + s \nabla T$ . Inserting  $\nabla P$  from Eq. (2.63c) and  $\nabla \mu$  from Eq. (2.64) into this relation, taking the divergence on both sides, and keeping only terms linear in the deviations from equilibrium yields

$$\rho_n \frac{\partial}{\partial t} \nabla \cdot (\mathbf{v}_n - \mathbf{v}_s) \simeq -s \Delta T. \quad (2.66)$$

In order to replace the divergence on the right-hand side of this equation we observe

$$\begin{aligned} \frac{\partial S}{\partial t} &= \frac{1}{\rho} \frac{\partial s}{\partial t} - \frac{S}{\rho} \frac{\partial \rho}{\partial t} \\ &= -S \nabla \cdot \mathbf{v}_n + \frac{S}{\rho} (\rho_n \nabla \cdot \mathbf{v}_n + \rho_s \nabla \cdot \mathbf{v}_s) \\ &= -\frac{S \rho_s}{\rho} \nabla \cdot (\mathbf{v}_n - \mathbf{v}_s), \end{aligned} \quad (2.67)$$

where, in the second step, we have used Eqs. (2.63a) and (2.63b). Inserting this result into Eq. (2.66) and again using the linear approximation yields the second wave equation (2.65b).

Next, we solve the wave equations. We take  $T$  and  $P$  as independent variables, such that  $S$  and  $\rho$  are functions of  $T$  and  $P$ ,

$$\delta S = \frac{\partial S}{\partial T} \delta T + \frac{\partial S}{\partial P} \delta P, \quad \delta \rho = \frac{\partial \rho}{\partial T} \delta T + \frac{\partial \rho}{\partial P} \delta P, \quad (2.68)$$

where all derivatives are evaluated in equilibrium. Inserting this into Eqs. (2.65a) and (2.65b) yields to linear order in  $\delta T$ ,  $\delta P$ ,

$$\frac{\partial \rho}{\partial P} \frac{\partial^2 \delta P}{\partial t^2} + \frac{\partial \rho}{\partial T} \frac{\partial^2 \delta T}{\partial t^2} = \Delta(\delta P), \quad (2.69a)$$

$$\frac{\partial S}{\partial P} \frac{\partial^2 \delta P}{\partial t^2} + \frac{\partial S}{\partial T} \frac{\partial^2 \delta T}{\partial t^2} = \frac{S^2 \rho_s}{\rho_n} \Delta(\delta T). \quad (2.69b)$$

The deviations from equilibrium are assumed to be harmonic oscillations,  $\delta P(\mathbf{x}, t) = \delta P_0 e^{-i(\omega t - kx)}$ ,  $\delta T(\mathbf{x}, t) = \delta T_0 e^{-i(\omega t - kx)}$ , where the amplitudes  $\delta P_0$ ,  $\delta T_0$  are constant in time and space, and where  $\omega$  and  $k$  are frequency and wave number of the oscillation. Without loss of generality, we have chosen the sound waves to propagate in the  $x$ -direction. We define the sound velocity

$$u = \frac{\omega}{k}, \quad (2.70)$$

such that the wave equations become

$$\left( u^2 \frac{\partial \rho}{\partial P} - 1 \right) \delta P_0 + u^2 \frac{\partial \rho}{\partial T} \delta T_0 = 0, \quad (2.71a)$$

$$u^2 \frac{\partial S}{\partial P} \delta P_0 + \left( u^2 \frac{\partial S}{\partial T} - \frac{S^2 \rho_s}{\rho_n} \right) \delta T_0 = 0. \quad (2.71b)$$

For this system of equations to have nontrivial solutions, we must require the determinant to vanish,

$$u^4 |J_f(T, P)| - u^2 \left( \frac{\partial \rho}{\partial P} \frac{S^2 \rho_s}{\rho_n} + \frac{\partial S}{\partial T} \right) + \frac{S^2 \rho_s}{\rho_n} = 0, \quad (2.72)$$

where  $|J_f(T, P)|$  is the determinant of the Jacobian matrix of the function  $f(T, P) \equiv (S(T, P), \rho(T, P))$ . The Jacobian matrix of  $f$  is

$$J_f(T, P) \equiv \frac{\partial(S, \rho)}{\partial(T, P)} = \begin{pmatrix} \frac{\partial S}{\partial T} & \frac{\partial S}{\partial P} \\ \frac{\partial \rho}{\partial T} & \frac{\partial \rho}{\partial P} \end{pmatrix}. \quad (2.73)$$

Now remember that the derivative of the inverse function  $f^{-1}(S, \rho) = (T(S, \rho), P(S, \rho))$  is given by the inverse of the Jacobian matrix of  $f$ ,  $J_{f^{-1}}[f(T, P)] = [J_f(T, P)]^{-1}$ . Therefore,

$$J_{f^{-1}}(S, \rho) \equiv \frac{\partial(T, P)}{\partial(S, \rho)} = \begin{pmatrix} \frac{\partial T}{\partial S} & \frac{\partial T}{\partial \rho} \\ \frac{\partial P}{\partial S} & \frac{\partial P}{\partial \rho} \end{pmatrix} = \frac{1}{|J_f(T, P)|} \begin{pmatrix} \frac{\partial \rho}{\partial P} & -\frac{\partial S}{\partial P} \\ -\frac{\partial \rho}{\partial T} & \frac{\partial S}{\partial T} \end{pmatrix}.$$

Consequently, from the diagonal elements of this matrix equation we read off

$$\frac{\partial T}{\partial S} = \frac{1}{|J_f(T, P)|} \frac{\partial \rho}{\partial P}, \quad \frac{\partial P}{\partial \rho} = \frac{1}{|J_f(T, P)|} \frac{\partial S}{\partial T}. \quad (2.74)$$

After dividing Eq. (2.72) by  $|J_f(T, P)|$  we can use these relations to write

$$u^4 - u^2 \left( \frac{\partial T}{\partial S} \frac{S^2 \rho_s}{\rho_n} + \frac{\partial P}{\partial \rho} \right) + \frac{1}{|J_f(T, P)|} \frac{S^2 \rho_s}{\rho_n} = 0. \quad (2.75)$$

Up to now we have worked with the function  $f(T, P) = [S(T, P), \rho(T, P)]$  and its inverse  $f^{-1}(S, \rho) = [T(S, \rho), P(S, \rho)]$ . In Eq. (2.75), derivatives of  $f^{-1}$  appear, and thus the derivatives of  $T$  with respect to  $S$  and of  $P$  with respect to  $\rho$  are obviously taken at fixed  $\rho$  and  $S$ , respectively. We further simplify the polynomial for  $u$  as follows. Take the first component of  $f^{-1}$ ,  $T(S, \rho)$ , and read it as a function of  $S$  only, with a fixed  $\rho$ . Inversion of this function then yields a function  $S(T, \rho)$ . Now do the same with the second component of  $f$ ,  $\rho(P, T)$ , i.e., invert this function at a fixed  $T$ . This yields

$$\frac{\partial T}{\partial S} = \left( \frac{\partial S}{\partial T} \right)^{-1} = \frac{T\rho}{c_V} \quad \text{at fixed } \rho, \quad (2.76a)$$

$$\left( \frac{\partial \rho}{\partial P} \right)^{-1} = \frac{\partial P}{\partial \rho} \quad \text{at fixed } T, \quad (2.76b)$$

with the definition for the specific heat per unit mass

$$\frac{c_V}{\rho} = T \frac{\partial S}{\partial T}, \quad (2.77)$$

where  $c_V$  is the specific heat per unit volume, and the derivative is taken at fixed  $\rho$ . With the help of Eq. (2.76) we write the first relation of Eq. (2.74) as

$$\frac{1}{|J_f(T, P)|} = \frac{T\rho}{c_V} \frac{\partial P}{\partial \rho}, \quad (2.78)$$

with the derivative taken at fixed  $T$ . Now we insert Eqs. (2.76a) and (2.78) into the polynomial (2.75) to obtain

$$u^4 - u^2 \left( \frac{\rho S^2 T \rho_s}{c_V \rho_n} + \frac{\partial P}{\partial \rho} \right) + \frac{\rho S^2 T \rho_s}{c_V \rho_n} \frac{\partial P}{\partial \rho} = 0. \quad (2.79)$$

The two derivatives of  $P$  with respect to  $\rho$  appearing here were not identical originally because the derivatives are taken at fixed  $S$  and at fixed  $T$ . However, we approximate these derivatives to be equal, which is equivalent to approximating the specific heat at constant pressure  $c_P$  by the specific heat at constant volume  $c_V$ . This approximation turns out to be a good approximation for superfluid helium.

The resulting equation has the simple structure  $u^4 - u^2(a + b) + ab = 0$  with solutions  $u^2 = a, b$ . Consequently, the two positive solutions for  $u$  are

$$u_1 = \sqrt{\frac{\partial P}{\partial \rho}}, \quad (2.80a)$$

$$u_2 = \sqrt{\frac{\rho S^2 T \rho_s}{c_V \rho_n}} = \sqrt{\frac{s^2 T \rho_s}{\rho c_V \rho_n}}. \quad (2.80b)$$

These are the velocities of first and second sound. Since we have not worked with a relative velocity  $\mathbf{v}_n - \mathbf{v}_s$  between normal fluid and superfluid (except for the small oscillations that constitute the sound waves), the sound velocities are pure numbers, i.e., they do not depend on the direction of propagation.

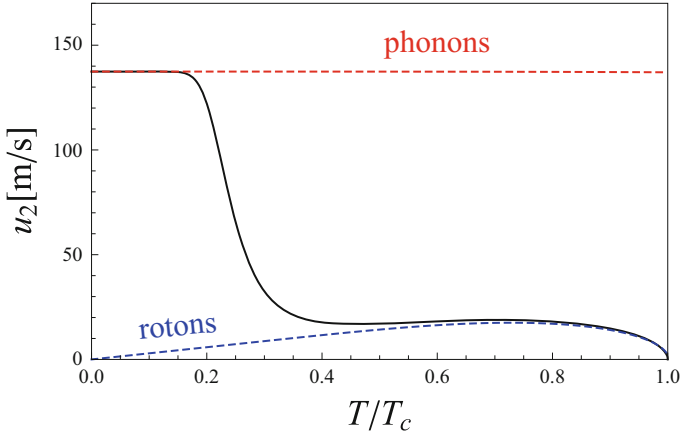
At low temperatures, as we shall see in the next section in a microscopic model,

$$\frac{\partial P}{\partial \rho} \simeq c^2, \quad (2.81)$$

i.e., the speed of first sound is given by the slope of the Goldstone dispersion,  $u_1(T \rightarrow 0) = c$ . For the speed of second sound, we may use the results from our thermodynamic calculations in Sect. 2.2. At low temperatures, the roton contribution is irrelevant, and we use  $c_{V,\text{ph}} = 3s_{\text{ph}}$  and  $\rho_{n,\text{ph}} = s_{\text{ph}}T/c^2$ , see Eqs. (2.10) and (2.24), respectively. We may also approximate  $\rho_s \simeq \rho$ . Inserting all this into our expression for  $u_2$  we find

$$u_2(T \rightarrow 0) = \frac{c}{\sqrt{3}} = \frac{u_1(T \rightarrow 0)}{\sqrt{3}}. \quad (2.82)$$

The full temperature dependence of  $u_2$ , within the present phonon/roton model, is shown in Fig. 2.6. Remember that second sound is only possible due to the presence of the second fluid. Therefore, it is easy to understand that  $u_2$  goes to zero at the critical temperature, because at that point  $\rho_s \rightarrow 0$  and the system becomes a single-fluid system. Interestingly, the behavior at small temperatures is different. Had we set  $T = 0$  exactly, there would have been no normal fluid and thus no second sound. However, starting with two fluids and then taking the limit  $T \rightarrow 0$  leads to a nonzero



**Fig. 2.6** Speed of second sound  $u_2$  as a function of temperature in superfluid helium from Eq. (2.80) with the parameters given in Fig. 2.3. The dashed lines are obtained by only taking into account phonon and roton contributions

speed of second sound. The figure also shows the characteristic behavior of  $u_2$  due to the presence of the phonons and rotons. As expected from the discussion above, see in particular Fig. 2.4, the phonons dominate at low temperatures,  $T \lesssim 0.25 T_c$ , while the rotons dominate for all larger temperatures below  $T_c$ . This characteristic behavior is special for helium, and superfluids that have no rotons show a different behavior. In contrast, the ratio of first and second sound at low temperatures given in Eq. (2.82) is more universal because it only depends on the linear behavior of the Goldstone mode at small momenta. If you are interested in recent theoretical studies about sound waves in superfluids, for instance in the context of superfluid atomic gases or relativistic superfluids, see [1, 2, 9, 16, 18].

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