Chapter 2
Balanced Growth in Decentralized Economies

Abstract In this chapter, we consider decision making algorithms that ensure balanced growth in decentralized economic systems. We assume that the economic system under consideration consists of a finite number monoproduct sectors whose outputs in any time period are defined by Leontief production functions. We show that there exists a dynamic system of prices where all the sectors ensure balanced growth for the whole economic system by aiming to maximize their profit. Prices have the following characteristic property: They depend on the volume of production so that the price of a product goes down as its output grows. The dependencies under consideration ensure that profit is a unimodal function of the volume of production in the next technological cycle. This ensures that each of the sectors selects its production plans in a unique way. We also consider the case where prices are set individually for each buyer depending on the order size. This is usual for wholesale trade, where the price of a product goes down as the order size grows.

For the purpose of studying management mechanisms that ensure balanced growth in economic systems that do not have a control center, we consider a mathematical model of an economic system that uses Leontief technologies. The main difference between this model and the simple dynamic Leontief model is that the latter, classical, model implicitly assumes the existence of a control center endowed with the rights of a dictator. In our case, the system's sectors are assumed to be completely independent economic agents that make management decisions based on their own interests. We assume that the model under consideration is closed in terms of production and finances. Since the sectors are autonomous economic agents, we potentially have a balance-of-payments problem. For this reason, we must identify conditions that ensure that payments between all the sectors are balanced.

We devote a separate section to a financial mechanism that helps counteract decreases in profit. (Such decreases can be seen in the case of the basic dynamics of prices.) We also consider a modification of this dynamics such that the profit of all

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the industries and the gross product of the system expressed in current prices grow as output grows.

Note that the mathematical model considered in this chapter is intended solely to demonstrate that multisector economic systems that use Leontief technologies and that do not use centralized management can theoretically function in balanced growth mode. This model is in no way intended to be applied in practice to real-world economic systems, because such economic systems would have to satisfy a great number of severe constraints.

2.1 Model Description

Let us consider a closed, dynamic model of production and exchange that has n sectors. Each of the sectors produces only one product and each of the products is produced by one sector only. The states of the economic system are observed at discrete points in time, denoted by t, $t = 0, 1, 2, \ldots$. We will use the same notation for the model's time periods (i.e., the periods of time between two consecutive points in time); the index of each time period corresponds to the right endpoint. Each time period has the same duration, equal to one *production cycle* (in each of the sectors). The time needed for exchanging products among the sectors is assumed to be negligibly short.

The commodities produced in a given time period must be used by the end of the next one: unused remains are considered not fit for consumption in later time periods. The closed model considered here does not take into account final consumption and the dynamics of production assets. For this reason, all the commodities produced in a given time period serve as resources for the next production cycle.

Let us introduce the following notation:

i = the sector index ($i = 1, \ldots, n$),
N_i^+ = the subset of the sectors whose products are needed for sector i to produce its commodity,
N_i^- = the subset of the sectors that consume (use as a resource) the commodity produced by sector i,
$x_i(t)$ = the output of sector i in time period t,
$y_{ji}(t)$ = the amount of resource j, $j \in N_i^+$, that sector i has at the beginning of time period t.

We assume that $N_i^+ \neq \emptyset \neq N_i^-$, $i = 1, \ldots, n$.

Suppose that the *production function* of sector i is defined as follows:

$$x_i(t) = \min_{j \in N_i^+} \left\{ \frac{y_{ji}(t)}{a_{ji}} \right\}, \tag{2.1}$$

where $a_{ji} > 0$ is the minimum amount of resource j needed for producing one unit of commodity i. This type of function is referred to as the *Leontief production function*, or the *zero-elasticity-of-substitution (ZES) production function*, or the *fixed-proportion production function* (Nadiri 1982).

Remark 2.1. In economics, some types of resources can, to a certain degree, substitute for other types (for example, labor and capital in the constant-elasticity-of-substitution (CES) production function (Nadiri 1982; Solow 1956). However, the higher the level of detail used to describe production processes, the less flexible you are. The Leontief production function represents a limiting case where no product can be substituted for any other product.

Remark 2.2. Some authors distinguish between production resources and factors of production. In that case, the latter include labor and production assets. In this book, we do not make such a distinction: We refer to the inputs of a technological process as resources or factors of production and we refer to its output as its product.

Function (2.1) has a special property: Among its arguments, there is at least one *limiting resource (limiting factor of production)* that determines the sector's volume of output. It is clear that in a given time period a sector uses its production resources in the most efficient way if all of the sector's resources are limiting. If the above is true, we say that the corresponding production process is *balanced* in that time period. This means that if in time period t the operation of sector i is balanced, then all the arguments in the right part of (2.1) are equal.

If sector i distributes all of its commodity produced in time period t among its consumers, then the following holds:

$$x_i(t) = \sum_{j \in N_i^-} y_{ij}(t+1).$$

We say that an economic system is *balanced* in time period t if it satisfies the following conditions:

(a) all its production processes are balanced in time period t,
(b) all the outputs in the previous time period, $x_i(t-1), i = 1, \ldots, n$, are completely distributed.

Remark 2.3. In the sequel, we never consider the trivial case of a balanced economic system where all outputs and inputs equal zero.

If the number of coefficients a_{ji} that appear in the n Leontief production functions (2.1) is less than n^2, then if $j \notin N_i^+$ we can define these coefficients as follows: $a_{ji} = 0$. Here, the zero value of the coefficient a_{ji} means that sector i does not use the product of sector j.

In any case, the set of n^2 nonnegative coefficients a_{ij} allows us (in an obvious way) to form a nonnegative square matrix of order n, denoted by A. In the literature, the matrix A has many names—consumption matrix, technology matrix, input–output matrix, input matrix and so on. In this book, we shall call A the *technology* matrix. It completely describes all n technologies of this model in terms of input (resources).

Note that the matrix A is constant in most of the models considered in this book. Since, in real economies, new technologies are continually introduced and existing ones get modernized, this assumption that all technological coefficients remain constant limits the applicability of the models to relatively short time-frames. We discuss issues related to the dynamics of the matrix A in Chap. 6.

Remark 2.4. It is easy to see that A corresponds to the direct input matrix in the Leontief model. In the latter, the production of a unit of any product requires the minimum possible amount of inputs and these inputs form the columns of A. This means that in the Leontief model the vector of outputs uniquely determines the vector of inputs. On the other hand, if we use function (2.1) as a basis, the vector of inputs uniquely determines the volume of output; here, resources are allowed to be used inefficiently.

Note that we can use the matrix A to see the technological relationships between the sectors. If $a_{ij} > 0$, then sector j uses the product of sector i directly. Evidently, if $a_{ij} = 0$, then there is no such direct relationship. However, if A is *irreducible* (Gantmacher 1959; Horn and Johnson 1985) *(indecomposable)* (Ashmanov 1984; Lancaster 1968), then each sector directly or indirectly consumes the products of all the sectors. An irreducible matrix has no zero rows or columns. In terms of technology, no zero columns means that the production of any product requires at least one type of resource and no zero rows means that the product of any sector is used by at least one sector.

For a system that is balanced in time period t, the following is obviously true:

$$x(t-1) = Ax(t), \tag{2.2}$$

where $x(t-1)$ and $x(t)$ are n-dimensional column vectors of outputs.

Remark 2.5. In the general case, instead of (2.2), the following inequality is true for the vectors of outputs in any two consecutive time periods:

$$x(t-1) \geqslant Ax(t). \tag{2.3}$$

This corresponds to the dynamic variant of the Leontief input–output model. This model is a special case of the von Neumann growth model such that product i is produced by sector i and no other sector (see Ashmanov 1984). Inequality (2.3) has an obvious economic interpretation: The inputs in the current time period cannot exceed the volume of output of the previous production cycle.

If the economic system is balanced in every time period, then by applying induction to (2.2), we get

$$x(0) = A^t x(t), \qquad t \geqslant 1,$$

where $x(0)$ is the initial level of stocks.

As the following proposition shows, the requirement of being balanced places tight constraints on the system's parameters.

Proposition 2.1. *Let the matrix A be primitive; then the economic system can be balanced for all $t \geqslant 1$ if and only if $x(0)$ is the (right) Frobenius vector of A.*

Proof. The properties of dynamic equations in reverse time in the form of (2.2) are studied in details in Nikaido (1968). See the same work for a proof of the necessity of the stated condition. Next, let us distribute the resources that the economic system has at the end of time period $t-1$ as follows:

$$y_{ij}(t) = \frac{1}{\lambda_A} a_{ij} x_j(t-1), \qquad j \in N_i^-, \quad t \geqslant 1, \tag{2.4}$$

where λ_A is the Frobenius eigenvalue of A. Elementary reasoning shows that the outputs are completely distributed and there are no surplus resources in any production in any time period. Moreover, the vectors of outputs in any two consecutive time periods are related as follows: $x(t) = (1/\lambda_A)x(t-1)$, $t \geqslant 1$. This means that the condition is sufficient. $\square$

Let us recall the main definitions related to the dynamics of outputs. The ratio $x_i(t)/x_i(t-1)$ is usually called the *growth factor* (Nikaido 1968) or *expansion rate* (Gale 1960) of output for product i in time period t. The sequence of outputs $\{x(t)\}$, $t = 0, 1, 2, \ldots$ is called an *admissible trajectory of outputs* or a *feasible path* (Nikaido 1968) if the members of the sequence satisfy system of inequalities (2.3).

Further, let us consider an admissible trajectory such that it is not identically equal to zero over an infinite time period. This trajectory is called a *steady states trajectory (path)* (Ashmanov 1984) or a *balanced growth trajectory (path)*, if there exists a scalar ν such that $\nu > 0$ and $x(t) = \nu x(t-1)$ for any $t \geqslant 1$. It is clear that in this case we have $x(t) = \nu^t x(0)$.

Example 2.1. Consider a two-sector economic system that has the following technology matrix:

$$A = \begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}.$$

It can easily be checked that $\lambda_A = 0.8$. Obviously, here the following is a steady states trajectory: the sequence of outputs $x(t) = \nu^t [1,\ 1]^T$ where the value of ν belongs to the right-closed interval $(0,\ 1.25]$. If $\nu < 1.25$, then, for any of the products, its loss in any time period equals $(1 - 0.8\nu)$. In the general case where the initial vector is $[x_1^0,\ x_2^0]^T$ and $x_1^0 \leqslant x_2^0$, a steady states trajectory exists only if we draw the values of ν from the right-closed interval $(0,\ 2.5x_1^0/(x_1^0 + x_2^0)]$. $\square$

Remark 2.6. In the above example and all the examples that follow, the expansion rates does not necessarily correspond to those in the real economic world. This is done to make it easier for the reader to perceive the numeric data.

Out of all steady states trajectories, those where ν is maximum are of the greatest interest. This maximum value is called the *von Neumann expansion rate* (Ashmanov 1984). A sequence of outputs that corresponds to it is called the *maximum balanced growth trajectory (path)* or the *von Neumann path* (Lancaster 1968). In Example 2.1, the von Neumann expansion rate equals $1.25 = 1/\lambda_A$.

Proposition 2.2. *If the technology matrix A is irreducible, then the von Neumann expansion rate ν of the economic system is related to the Frobenius eigenvalue λ_A of A as follows:*

$$\nu = \lambda_A^{-1} \tag{2.5}$$

Moreover, any von Neumann path of the system belongs to the ray that corresponds to the Frobenius vector of A.

Proof. If A is irreducible, then the system has a steady states trajectory only if the vector $x(0)$ is positive. For this reason, we can rewrite the inequality $\nu Ax(0) \leqslant x(0)$ as follows:

$$\nu \leqslant \frac{x_i(0)}{(Ax(0))_i}, \qquad i = 1, \ldots, n.$$

This yields that,

$$\nu \leqslant \min_i \frac{x_i(0)}{(Ax(0))_i} = \left(\max_i \frac{(Ax(0))_i}{x_i(0)} \right)^{-1} \leqslant \frac{1}{\lambda_A},$$

where we used the right inequality from double-sided bound (A.2).

On the other hand, if $x(0)$ is the Frobenius vector of A and the resources are distributed according to (2.4), the system has a steady states trajectory with an expansion rate of $1/\lambda_A$. It is obvious that this trajectory, which we denote by $\{x(t)\}$, belongs to the ray that corresponds to $x(0)$.

Suppose there exists a von Neumann path $\{\tilde{x}(t)\}$ whose points belong to a different ray. Then, we can find a scalar μ such that the sequence $\{x(t) + \mu\tilde{x}(t)\}$ consists of nonnegative vectors, all of them having the same zero component. It is easy to check that this sequence of vectors satisfies the definition of a von Neumann path. On the other hand, this trajectory cannot be constructed, because A is irreducible. Indeed, since all the sectors of the economic system need (directly or indirectly) a product that is never produced, each of the sectors must stop production within a finite number of time periods. This contradiction proves the uniqueness of the ray to which all the von Neumann paths belong. □

The ray to which all the von Neumann paths belong is called the *von Neumann ray* (Ashmanov 1984; Lancaster 1968).

We shall show that if the technology matrix of an economic system is reducible, then this economic system can have more than one von Neumann ray.

Example 2.2. Consider a two-sector economic system that has the following technology matrix:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}.$$

It is easy to see that here the von Neumann expansion rate equals 1.25 (as in Example 2.1) and that the maximum balanced growth trajectory is the sequence of outputs $x(t) = (1.25)^t\,[x_1^0, x_2^0]^T$, where x_1^0 and x_2^0 are any nonnegative numbers that satisfy the condition $x_1^0 + x_2^0 > 0$. Hence, here every point of the nonnegative quadrant (zero excluded) belongs to a von Neumann ray. □

On the other hand, its is obvious that even if the technology matrix of an economic system is reducible, this system has a unique von Neumann ray if this system has a unique isolated subset that corresponds to an irreducible submatrix. The simple example illustrates this statement.

Example 2.3. Consider a three-sector economic system that has the following technology matrix:

$$A = \begin{bmatrix} 0.4 & 0.4 & 0.1 \\ 0.4 & 0.4 & 0.1 \\ 0 & 0 & 0.4 \end{bmatrix}.$$

Here, the isolated subset is formed by the first two sectors. The Frobenius eigenvalue of the corresponding submatrix equals 0.8. It is equal to the Frobenius eigenvalue of A. The von Neumann expansion rate of the model is equal to 1.25. We can construct a von Neumann path as follows: $x(t) = (1.25)^t[1, 1, 0]^T$. This trajectory belong to the unique von Neumann ray that corresponds to the vector $[1, 1, 0]^T$. □

Let us show that the expansion rate for a product can sometimes exceed the von Neumann expansion rate.

Example 2.4. Consider the economic system from Example 2.1. Suppose sector 1 receives 0.8 units of each of the resources from the output $x(1) = [1, 1]^T$. Hence, $x_1(2) = 2$, which means that the expansion rate for product 1 in time period 2 equals two. This exceeds the model's von Neumann expansion rate $\nu = 1.25$. □

However, as the following statement shows, this can be reached only at the expense of other sectors, where the expansion rate goes down.

Proposition 2.3. *Let the matrix A be irreducible. If there exists a time period t such that here the expansion rate in one of the system's sectors exceeds the von Neumann expansion rate ν, it follows that one can find another sector whose expansion rate in this period is lower than ν.*

Proof. Suppose the contrary, i.e., that the following holds for time period t:

$$x_i(t) \geqslant \nu x_i(t-1), \qquad i = 1, \dots, n.$$

Moreover, suppose that for some i its corresponding inequality is strict. In this case, using (2.3), we get $x(t) \geqslant \nu A x(t)$ and $x(t) \neq \nu A x(t)$. Multiplying the first of the inequalities by the left Frobenius vector p_A of A, we obtain

$$\langle p, x(t) \rangle > \nu \lambda_A \langle p, x(t) \rangle.$$

Since $\langle p, x(t) \rangle > 0$, we have $\nu \lambda_A < 1$. This means that $\nu < \lambda_A^{-1}$, which contradicts (2.5). □

Thus, in any Leontief-type model whose technology matrix A is irreducible (the subject of this book), the von Neumann paths belong to a unique half-line that corresponds to the right Frobenius vector x_A of A. This half-line (the von Neumann ray), is also called the *turnpike* (Nikaido 1968). Accordingly, the maximum balanced growth mode can also be referred to as the *turnpike mode*. Even though the latter term was used for the first time in economic dynamics, in its field of optimization modeling, we may use it here, because in optimization problems a maximum balanced growth mode that has been observed for a certain period of time is usually referred to as a turnpike mode.

Finally, since this book does not consider steady states trajectories that have expansion rates not equal to the von Neumann expansion rate, we will use the term *balanced growth* to mean maximum balanced growth.

Taking into consideration the terminology introduced above, we can infer the following statement from Proposition 2.1:

Corollary 2.1. *Let the technology matrix A of an economic system be primitive. If this economic system is balanced in every time period, then all its sectors have the same constant growth factor equal to $\nu = 1/\lambda_A$. The corresponding trajectory of outputs is a von Neumann path. If $\nu > 1$, then the volume of output increases in every time period (expanded reproduction); if $\nu < 1$, then the volume of output decreases; and if $\nu = 1$, then the volume of output remains constant (simple reproduction).*

Note that an economic system such that $\nu \leqslant 1$ is of no interest for obvious reasons. In the sequel, we assume that $\nu > 1$.

In Proposition 2.1 let us substitute the condition that the matrix A must be primitive with the condition that A must be irreducible, i.e., we shall consider the more general case. Then, as the following example shows, the economic system can be completely balanced even if the initial vector is not equal to the Frobenius vector of A (i.e., this requirement imposed on the vector $x(0)$ is no longer necessary).

Table 2.1 The dynamics of outputs and expansion rates

t	0	1	2	3	4	...
$x_1(t)$	4	20	25	125	156.25	...
$x_2(t)$	4	5	25	31.25	156.25	...
$\nu_1(t)$	–	5	1.25	5	1.25	...
$\nu_2(t)$	–	1.25	5	1.25	5	...

Example 2.5. Consider a two-sector economic system that has the following imprimitive technology matrix:

$$A = \begin{bmatrix} 0 & 0.8 \\ 0.2 & 0 \end{bmatrix},$$

where $\lambda_A = 0.4$ and the Frobenius vector $x_A = [2, 1]^T$. Suppose that the initial vector $x(0)$ does not equal the Frobenius vector: $x(0) = [4, 4]^T$. Then, the economic system is completely balanced if the outputs $x_i(t)$ are equal to the values shown in Table 2.1.

It is obvious that this trajectory is not a steady states trajectory. Note that here we can introduce the notion of a *cycle-long* expansion rate: In this example, the cycle is two time periods long, because each sector's volume of output grows by a factor of $5 \times 1.25 = 6.25$ every two time periods. This notion can be applied to any Leontief-type economy whose technology matrix is imprimitive. On the other hand, if $x(0)$ is the Frobenius vector, then the condition of being completely balanced generates a steady states trajectory that has the following expansion rate: $1/\lambda_A = 2.5$. Note that the expansion rate per two time periods equals 6.25 in this case as well. □

2.2 Planning Based on Profit Maximization

When some of the sectors of an economic system are technologically interrelated, the interaction among these sectors requires organization. For example, at the end of every production cycle, its outputs must be distributed among the consumers.

Traditionally, Gale-type models, which are a generalization of models that use Leontief technologies, implicitly assume the existence of a control center endowed with the rights of a dictator. This center controls production and product distribution, and all its orders are executed with perfect accuracy. However, as the cases of the former USSR, Cuba, North Korea and other countries show, it is impossible in practice to plan the operation of thousands of enterprises—producing millions of different product—from a single control center. Totally centralized planning and management lead to imbalances in the economy, which in turn lead to the inefficient use of physical, financial, and labor resources.

Moreover, the entire history of the world demonstrates the ineffectiveness of economic systems where economic agents are not provided with incentives to produce cheap and quality products (because, for example, there systems lack market mechanisms that would allow consumers to “grade” the commodities). For these reasons, it is important to study dynamic economic models that do not assume centralized management.

Naturally, we will start with the most simple models, namely from Leontief-type dynamic models. We will consider sectors as autonomous economic agents that plan and organize their operation themselves. The purpose of the model presented in this section is to demonstrate that economic systems that do not have a control center can function in balanced growth mode.

Thus, we assume that sectors sell products and procure resources themselves (i.e., they are completely autonomous economic agents). We will refer to such an economic system as a *decentralized economic system*.

The economic autonomy of a sector means that the sector has certain targets on which it bases its management decisions. Consider a case where each of the sectors aims to maximize its profit at the next production cycle (Abramov 2006).

In order to formally define the target functions of the sectors, we now introduce appropriate financial indicators. Let $p_i(t)$ denote the price that sector i sets for its commodity produced in time period t. We assume that this price is the same for all the consumers of sector i and does not depend on the order size. If we further assume that sector i sells all of its commodity produced in time period t, then the sector's profit in time period t, $\Pi_i(t)$, equals the volume of sales at t, expressed in value terms, minus the production costs. In the model under consideration, the latter equal the resource procurement costs, which includes product i if this product appears in the production function. Hence the profit of sector i in time period t is given by

$$\Pi_i(t) = p_i(t)x_i(t) - \sum_{j \in N_i^+} p_j(t-1)y_{ji}(t). \tag{2.6}$$

We assume that the volume of sales, which is here considered equal to the volume of production $x_i(t)$, and the price $p_i(t)$ are linearly related as follows:

$$p_i(t) = b_i(t) + d_i(t)x_i(t), \qquad t \geqslant 1, \tag{2.7}$$

where the values of $b_i(t) > 0$ and $d_i(t) < 0$ are fixed for given i and t. The above inequality, combined with (2.7), means that the price decreases as the volume of sales grows. Hence, the model explicitly takes into consideration the *price elasticity of demand* (For an introduction to the price elasticity of demand and the price elasticity of supply, see Heyne 1997).

It is clear that the coefficient $b_i(t)$ equals the limit value of the price, where the volume of sales (production) for product i tends to zero. The coefficient $d_i(t)$ shows by how much the price decreases as the volume of sales (production) grows by one unit. We will refer to $b_i(t)$ and $d_i(t)$ individually as the *base price* and the unit *discount coefficient*, and collectively as the *price coefficients*.

Remark 2.7. In practice, the dependencies of the form (2.7) are established statistically by analyzing time series. A range is selected where the obtained approximation of the demand function is considered satisfactory. It is clear that this range cannot contain volumes of sales that lead to negative prices.

We will use (2.7) for determining the prices $p_i(0)$ as well. However, here the parameter $x_i(0)$ must be interpreted as the volume of sales of part of the initial resource stock x_i^0; here $x_i(0) \leqslant x_i^0$, $i = 1, \ldots, n$. Since negative prices do not have any meaningful interpretation and zero prices are not allowed in this model, the price coefficients must satisfy the following constraints:

$$b_i(t) + d_i(t)x_i(t) > 0, \qquad t \geqslant 0.$$

If follows from (2.6) and (2.7) that the profit of sector i in time period $t \geqslant 1$ can be calculated as

$$\Pi_i(t) = (b_i(t) + d_i(t)x_i(t))x_i(t) - \sum_{j \in N_i^+} (b_j(t-1) + d_j(t-1)x_j(t-1))y_{ji}(t). \tag{2.8}$$

If a sector decides to buy its resources in amounts that exceed the minimum amount needed for producing a certain amount of product, the sector's profit decreases. This yields that,

$$y_{ji}(t) = a_{ji}x_i(t), \qquad i = 1, \ldots, n; \quad j \in N_i^+, \quad t \geqslant 1. \tag{2.9}$$

If we take these relationships into account, we can rewrite (2.8) as follows:

$$\Pi_i(t) = (b_i(t) + d_i(t)x_i(t))x_i(t) - \sum_{j \in N_i^+} (b_j(t-1) + d_j(t-1)x_j(t-1))a_{ji}x_i(t). \tag{2.10}$$

Before we find the maximum of this function by differentiating with respect to $x_i(t)$, we must justify that this operation is well defined in the present case. The point is that the variables $x_j(t-1)$, $j \in N_i^+$, and $x_i(t)$ that appear in (2.10) are, generally speaking, related with a functional relationship. For example, in balanced growth mode we have

$$x_j(t-1) = \sum_{i \in N_j^-} a_{ji}x_i(t).$$

Formally, this means that if $x_i(t)$ grows, then the suppliers of sector i see their volumes of sales grow in the previous time period. This in turn leads to a decrease in the prices of the resources being bought by sector i. We can eliminate this "impact" of $x_i(t)$ on the prices $p_j(t-1)$, $j \in N_i^+$, by using either of the two methods presented below.

First, we can limit ourselves to considering such economic systems where for each consumer the total amount of its orders for any product is negligibly small compared with the product's volume of production. As a result, the price of any product will not depend on one single consumer.

Second, the economic system under consideration works as follows: By the time when the output plan for $x_i(t)$ is being determined, the variables of the previous time period have already assumed their values. For this reason, the choice of a value for $x_i(t)$ does not affect those variables. Consequently, the prices for the previous time period are uniquely determined by the actual volumes of production. Hence, the choice of a value for $x_i(t)$ does not affect the values of $p_j(t-1)$, $j \in N_i^+$.

We will adopt the second approach: We will find the unconditional maximum of function (2.10) without taking any resource constraints into account. As we intend to show, these constraints will not be violated in the case under consideration.

Thus, now that we assume that profit depends solely on the volume of production, we can easily obtain the optimal production plan, which maximizes function (2.10):

$$x_i(t) = -\frac{1}{2d_i(t)}\left(b_i(t) - \sum_{j\in N_i^+}(b_j(t-1) + d_j(t-1)x_j(t-1))a_{ji}\right). \quad (2.11)$$

Note that the maximum does indeed exist, because the coefficient $d_i(t)$ at the square term of parabola (2.10) has a negative value.

Remark 2.8. Using (2.7), we can rewrite (2.11) such that

$$x_i(t) = -\frac{1}{2d_i(t)}\left(b_i(t) - \sum_{j\in N_i^+} p_j(t-1)a_{ji}\right).$$

This means that in order to calculate the optimal plan of sector i for time period t one must know

(a) the values of the financial coefficients $b_i(t)$ and $d_i(t)$,
(b) the prices $p_j(t-1)$ of the resources $j \in N_i^+$ being bought. This formula also shows that the base price $b_i(t)$ of commodity i must exceed its unit cost of production.

If a system operates in balanced growth mode and its expansion rate is ν, then we have in (2.11):

$$x_i(t) = \nu^t x_i(0), \quad x_j(t-1) = \nu^{t-1}x_j(0), \qquad j \in N_i^+. \quad (2.12)$$

Using (2.11) and (2.12), we obtain that under the condition of balanced growth mode the coefficients $b_i(t)$ and $d_i(t)$ must, for all i and t, satisfy the following system of equations:

$$b_i(t) + 2\nu^t d_i(t)x_i(0) - \sum_{j\in N_i^+} (b_j(t-1) + \nu^{t-1} d_j(t-1)x_j(0))a_{ji} = 0. \quad (2.13)$$

For this, it suffices if the variables $b_i(t)$ satisfy the following system of equations:

$$b_i(t) = \sum_{j\in N_i^+} b_j(t-1)a_{ji}, \qquad t \geqslant 1 \quad (2.14)$$

and in addition, if variables $d_i(t)$ satisfy the following system of equations:

$$d_i(t) = \frac{1}{2\nu x_i(0)} \left(\sum_{j\in N_i^+} d_j(t-1)x_j(0)a_{ji} \right), \qquad t \geqslant 1. \quad (2.15)$$

Note that we can rewrite the latter system as

$$d_i(t) = \frac{1}{2\nu x_i(t-1)} \left(\sum_{j\in N_i^+} d_j(t-1)x_j(t-1)a_{ji} \right), \qquad t \geqslant 1.$$

Now let us consider the economic interpretation of systems (2.14) and (2.15). As noted above, the coefficient $b_i(t)$ equals the limit of the price as the volume of sales for product i tends to zero. Conditions (2.14) can then be interpreted as follows: The revenues that sector i receives in time period t by selling one unit of its product for $b_i(t)$ must equal the cost of procuring, at the base prices of the previous time period, the minimum amount of resources needed to produce this unit of commodity.

In order to be able to interpret system (2.15), we must multiply the equation that corresponds to sector i by $\nu^{2t-1}x_i(0)$. The result is

$$(d_i(t)x_i(t))x_i(t) = \frac{1}{2} \sum_{j\in N_i^+} \big(d_j(t-1)x_j(t-1)\big)\, y_{ji}(t), \quad (2.16)$$

where the variables $x_i(t)$, $x_j(t-1)$, and $y_{ji}(t)$ correspond to the balanced growth mode with the initial vector $x(0)$ such that $x(0) = x^0$.

The left side of this equation represents the financial losses that are suffered by sector i in time period t. These losses and caused by a decrease in price as compared with the original value $b_i(t)$. In the equation's right side, the absolute value of the product $d_j(t-1)x_j(t-1)$, $j \in N_i^+$ equals the amount of discount on $b_j(t-1)$, i.e., on the base price of product j produced in time period $t-1$. Then, the absolute value of the sum that appears in the equation's right side equals the amount of money that sector i "save" when buying the resources need for the output $x_i(t)$. Hence, Eq. (2.16) means that the absolute value of the "losses" sustained by sector i in time period t equals half of what i saves when buying its resources. It is the difference of these values that make up the profit of sector i in time period t. In order to make sure that this is the case, let us calculate $\Pi_i(t)$ using (2.10), (2.14), and (2.16):

$$\Pi_i(t) = -d_i(t)x_i^2(t). \tag{2.17}$$

Since the value of $d_i(t)$ is negative, the profit is positive.

Note that the fact that each of the sectors make a positive profit in each time period does not contradict the condition that the model is closed: the resources needed to produce a commodity sold in a given time period are bought at the previous one. At the same time, all sectors of the economy as a whole spend as much on buying the resources needed for a given production cycle as they receive as profit from selling the commodities produced during the previous cycle:

$$\sum_{i=1}^{n} (b_i(t) + d_i(t)x_i(t))\, x_i(t) = \sum_{i=1}^{n} \sum_{j \in N_i^+} \big(b_j(t) + d_j(t)x_j(t)\big)\, y_{ji}(t+1).$$

Thus, the considered algorithm for determining the output plan for sector i requires that at the end of a given time period $t-1$ sector i determine the price coefficients of the next time period, $b_i(t)$ and $d_i(t)$. For calculating these coefficients (and for determining the output plan), sector i must receive from its suppliers the current values of the base prices and discount coefficients. Besides that, when determining its discount coefficient according to (2.15), sector i must know the von Neumann expansion rate ν and the vector of initial stocks $x(0)$.

Since the system operates in balanced growth mode, all the output plans for time period t are provided with all the resources necessary. Then the consumers receive the resources from their suppliers; the time needed for this is assumed to be negligibly short. After that, the production cycles of time period t start in all the sectors. After the cycles finish, the sectors determine their output plans for time period $t+1$, etc.

Note that this planning algorithm works in balanced growth mode only. For this reason, it also makes perfect sense for sector i to set the value for its production plan for time period t as $\nu x_i(t-1)$. Both planning methods produce the same production plans, with the required amount of resources guaranteed.

Hence, the model under consideration serves a single purpose: to demonstrate that decentralized economic systems where each sector plans its work based on its own interests can function in balanced growth mode.

2.3 Payment Balances of Sectors

Let us now assume that the economic system under consideration is closed in terms of finances (as well as in terms of production). In addition, we will assume that the economic system uses a *clearing* system for payments and that its sectors do not provide *commodity credits* to each other, even for a single production cycle. Under such conditions, a sector's revenues received from selling its product in a given time period must equal the sector's expenses for buying the resources needed for the next production cycle.

We will refer to the difference between the revenues and the expenses of sector i after the set-offs have been performed at the end of time period t as the *payment balance* in time period t, denoted by $B_i(t)$. By definition,

$$B_i(t) = (b_i(t) + d_i(t)x_i(t))\, x_i(t) - \sum_{j \in N_i^+} \left(b_j(t) + d_j(t)x_j(t)\right) y_{ji}(t+1). \tag{2.18}$$

If we assume that the base prices and discount parameters of sector i must alone ensure that $B_i(t)$ equals zero, then the coefficients $b_i(t)$ and $d_i(t)$, $i = 1, \ldots, n$, must satisfy the following equations for all t:

$$\begin{aligned} b_i(t)x_i(t) &= \sum_{j \in N_i^+} b_j(t)y_{ji}(t+1), \\ d_i(t)x_i^2(t) &= \sum_{j \in N_i^+} d_j(t)x_j(t)y_{ji}(t+1). \end{aligned} \tag{2.19}$$

For the balanced growth mode, assuming that the dynamics of the price coefficients is as given by (2.14) and (2.15), we can rewrite (2.18) such that

$$B_i(t) = \nu^t\, (b_i(t) - \nu b_i(t+1))\, x_i(0) + \nu^{2t} \left(d_i(t) - 2\nu^2 d_i(t+1)\right) x_i^2(0). \tag{2.20}$$

Similarly, (2.19) can be rewritten as follows:

$$b_i(t) = \nu \sum_{j \in N_i^+} b_j(t)a_{ji}, \tag{2.21}$$

$$d_i(t)x_i(0) = \nu \sum_{j \in N_i^+} d_j(t)x_j(0)a_{ji}. \tag{2.22}$$

If we compare this system of equations with system (2.14) and (2.15), we can see that in balanced growth mode the price coefficients must satisfy the following dynamic equations:

$$\begin{aligned} b_i(t) &= \frac{1}{\nu} b_i(t-1), \\ d_i(t) &= \frac{1}{2\nu^2} d_i(t-1). \end{aligned} \tag{2.23}$$

In this case, as can be seen from (2.20), $B_i(t) = 0$ for all i and t.

Proposition 2.4. *Suppose the initial values of the base prices and discount parameters for sector i, $i = 1, \dots, n$, satisfy the following system of equations:*

$$\begin{aligned} b_i(0) &= \nu \sum_{j \in N_i^+} b_j(0) a_{ji}, \\ d_i(0) x_i(0) &= \nu \sum_{j \in N_i^+} x_j(0) d_j(0) a_{ji}, \end{aligned} \tag{2.24}$$

where λ_A and $x(0)$ are the Frobenius eigenvalue and the Frobenius vector of the matrix A, respectively, and $\nu = 1/\lambda_A$,

Then, the price coefficients $b_i(t)$ and $d_i(t)$, as given by (2.23), satisfy dynamic equations (2.14), (2.21) and (2.15), (2.22), respectively.

The proof is by induction.

It is obvious that in our case we have the balanced growth mode and, at the same time, the zero payment balances for all sectors if and only if there exists a strictly positive solution to system of equations (2.24).

Proposition 2.5. *If the technology matrix A is irreducible, then system of equations (2.24) has a strictly positive solution.*

Proof. Let us introduce the following n-dimensional row vectors:

$$b(0) = [b_1(0), \dots, b_n(0)], \qquad d(0) = [d_1(0), \dots, d_n(0)].$$

Now we can rewrite (2.24) as two matrix equations:

$$\frac{1}{\nu} b(0) = b(0)A, \qquad \frac{1}{\nu} d(0) = d(0)\tilde{A},$$

where the elements $\tilde{a}_{ij}$ of the matrix $\tilde{A}$ are related to the elements a_{ij} of the matrix A such that

$$\tilde{a}_{ij} = \frac{x_i(0)}{x_j(0)} a_{ij}, \qquad i, j = 1, \dots, n. \tag{2.25}$$

If we consider the basic properties of determinants, we can see that the matrices A and $\tilde{A}$ have the same eigenvalues. Therefore, $1/\nu$ is the Frobenius eigenvalue of the irreducible matrix $\tilde{A}$. At this point, the statement that we need to prove directly follows from the Frobenius theorem.

Finally, the obtained left Frobenius vector $d(0)$ is strictly positive. Since the price parameters $d_i(0)$, $i = 1, \dots, n$, must be negative, we must change the sign of this vector. □

By assumption, we consider an economic system whose expansion rate in balanced growth mode is greater than 1 (i.e., $\nu > 1$). For this reason, it follows from (2.23) that the parameters $b_i(t)$ and $d_i(t)$ tend to zero as $t \to \infty$. It is easy to see that the price of any product decreases monotonically and tends to zero as t grows. Note that, at the same time, the prices in two consecutive time periods are related such that

$$p_i(t) = \frac{1}{\nu}\left(p_i(t-1) - \frac{1}{2}d_i(t-1)x_i(t-1)\right), \qquad t \geqslant 1. \tag{2.26}$$

It is readily seen that the prices are positive in all time periods if $p_i(0) > 0$.

Remark 2.9. Consider the dynamic variant of the Leontief model. In this case, the analog of (2.26) for dual variables is

$$p_i(t) = \frac{1}{\nu}\,p_i(t-1), \qquad t \geqslant 1.$$

Given that in the case under consideration we have $d_i(t) = \left(1/(2\nu^2)\right)^t d_i(0)$, the profit trend (2.17) is determined as follows:

$$\Pi_i(t) = -\frac{1}{2^t}d_i(0)x_i^2(0). \tag{2.27}$$

This means that in balanced growth mode this indicator decreases in a geometric progression.

2.4 Counteracting Decreases in Profit

The continuous decrease in profit mentioned in the previous section may cause considerable "psychological discomfort" to the sectors' top managers. Moreover, if we calculate the *gross product* of the system in current prices, i.e., the indicator $G(t) = \sum_{i=1}^{n} p_i(t)x_i(t)$, then it follows from (2.23) that in balanced growth mode we have

$$G(t) = \sum_{i=1}^{n}\left(b_i(0) + \frac{1}{2^t}d_i(0)x_i(0)\right)x_i(0).$$

Since $d_i(0) < 0$, we see that the indicator increases monotonically and, as t grows, tends to $\sum_{i=1}^{n} b_i(0)x_i(0)$ (i.e., the value of all the initial resource stocks expressed in the base prices for $t = 0$). Hence, in balanced growth mode, the unlimited growth in production is not accompanied by the same growth in the gross product (in nominal prices).

We will now consider one way to counteract this decrease in profit. For this, we modify systems of equations (2.14) and (2.15) such that

$$b_i(t) = \sum_{j \in N_i^+} a_{ji} b_j(t-1) + \psi_i(t), \tag{2.28}$$

$$d_i(t) = \frac{1}{2\nu x_i(0)} \left(\sum_{j \in N_i^+} a_{ji} x_j(0) d_j(t-1) \right) - \frac{\psi_i(t)}{2\nu^t x_i(0)}, \tag{2.29}$$

where $\psi_i(t), i = 1, \ldots, n$, are some positive parameters. It is easy to check that the original system of equations (2.13) is invariant under this modification. Note it is up to sector i to select a value for $\psi_i(t)$.

Now let us relate the price of product i in time period t with the prices for the previous period in balanced growth mode:

$$p_i(t) = \sum_{j \in N_i^+} \left(p_j(t-1) - \frac{1}{2} d_j(t-1) x_j(t-1) \right) a_{ji} + \frac{1}{2} \psi_i(t).$$

Since $p_i(0) > 0, i = 1, \ldots, n$, by assumption, we can see that the prices are positive for all time periods.

Elementary reasoning shows that this modification to the dynamics of the price parameters in balanced growth mode does not affect the profit $\Pi_i(t)$ of sector i and this indicator can also be calculated by (2.17). However, now the sector can actively influence the value of this indicator. For example, let sector i define the dynamics of $\psi_i(t)$ such that $\psi_i(t) = \nu^t \psi_i(0)$, $t = 1, 2, \ldots$, where $\psi_i(0)$ is some positive constant. If we combine this with (2.29), we get $|d_i(t)| > \psi_i(0)/(2x_i(0))$. Hence the profit of sector i can be estimated as follows: $\Pi_i(t) > (\psi_i(0)/2x_i(0))\, x_i^2(t)$.

Consider another example: Sector i fixes the value of $\psi_i(t)$ at ψ_i^0 (i.e., now $\psi_i(t) = \psi_i^0$). It is readily seen that $|d_i(t)| > \psi_i^0/(2\nu^t x_i(0))$. In this case we have $\Pi_i(t) > (\psi_i^0/2) x_i(t)$.

Let us now consider the dynamics price parameters such that, in balanced growth mode, the gross product of the system (in current prices) grows at a rate equal to the one of the volume of production. In order to obtain this, it suffices if each sector's volume of sales has an expansion rate of ν in each time period t:

$$p_i(t) x_i(t) = \nu p_i(t-1) x_i(t-1).$$

This yields that in balanced growth mode the prices of all the products must be constant:

$$p_i(t) = p_i^0 = const.$$

In our case, the price coefficients must determine plans that correspond to the balanced growth mode such that all sectors have zero payment balances for all time periods. Therefor the system's parameters must satisfy the following conditions:

(a) the vectors $x(0)$ and $b(0)$ are the right and the left Frobenius vectors, respectively, of the matrix A,
(b) the vector $-d(0)$ is the left Frobenius vector of the matrix $\tilde{A}$ (2.25),
(c) the dynamics of the price parameters for all $t \geqslant 1$ is

$$b_i(t) = b_i(0), \qquad d_i(t) = \frac{1}{\nu} d_i(t-1), \tag{2.30}$$

(d) for $t = 0$, the coefficients $b_i(0)$ and $d_i(0)$ and the component $x_i(0)$ satisfy the following equation:

$$(\nu - 1)\, b_i(0) = (1 - 2\nu)\, d_i(0) x_i(0), \tag{2.31}$$

(e) the parameters $\psi_i(t)$ are calculated as follows:

$$\psi_i(t) = \left(1 - \frac{1}{\nu}\right) b_i(t-1) = -\left(2 - \frac{1}{\nu}\right) d_i(t-1) x_i(t-1).$$

Using (2.10), we get in this case that $\Pi_i(t) = -\nu^t d_i(0) x_i^2(0)$, i.e., the profit of each sector grows with an expansion rate of ν as well.

We can now use (2.28) and (2.29) to express the payment balance of sector i for time period t in balanced growth mode:

$$B_i(t) = \nu^t \left(b_i(t) - \nu \sum_{j \in N_i^+} b_j(t) a_{ji} \right) x_i(0) + \nu^{2t} \left(d_i(t) - \frac{\nu}{x_i(0)} \sum_{j \in N_i^+} d_j(t) x_j(0) a_{ji} \right) x_i^2(0).$$

It is clear that if the row vectors $b(t)$ and $-d(t)$, where

$$b(t) = [b_1(t), \ldots, b_n(t)], \qquad d(t) = [d_1(t), \ldots, d_n(t)]$$

are the left Frobenius vectors of the matrices A and $\tilde{A}$ (2.25), respectively, then we have $B_i(t) = 0$. Hence, (2.28) and (2.29) ensure zero payment balances for all the sectors in all time periods if the row vectors $b(0)$, $\psi(t) = [\psi_1(t), \ldots, \psi_n(t)]$, $t \geqslant 1$, are the left Frobenius vectors of the matrix A, and $-d(0)$ is the left Frobenius vector of the matrix $\tilde{A}$. Indeed, let $\psi(t)$ be the left Frobenius vector of A. It can easily be checked that the row vector whose components are $\psi_i(t)/x_i(0)$ is the Frobenius vector of $\tilde{A}$. This means that the sums in the right-hand sides of (2.28) and (2.29) are the components of the left Frobenius vectors of their corresponding matrices. Note that in this case the values of $\psi_i(t)$ must be agreed on by all the sectors for each time period.

Relations (2.30) and (2.31) hold for the case where the system's gross product (expressed in current prices) grows at a constant expansion rate of ν in balanced growth mode. This case is undoubtedly important, but we should also consider the general case such that

$$p_i(t)x_i(t) = \varphi p_i(t-1)x_i(t-1),$$

where $\varphi > 0$ is some constant. Note that the case where $\varphi > \nu$ can be interpreted as an *inflation* with a constant inflation index of φ/ν; similarly, the case where $\varphi < \nu$ can be interpreted as *deflation* with a constant deflation index of ν/φ.

For the dynamics of the price coefficients (2.30), the corresponding generalization is

$$b_i(t) = \frac{\varphi}{\nu} b_i(t-1), \qquad d_i(t) = \frac{\varphi}{\nu^2} d_i(t-1).$$

Here, the components of the initial Frobenius vectors must satisfy the following equations:

$$(\varphi - 1)\, b_i(0) = (1 - 2\varphi)\, d_i(0)x_i(0).$$

It is easy to obtain the formula for calculating the profit of sector i per time period t:

$$\Pi_i(t) = -\varphi^t d_i(0)x_i^2(0).$$

This means that the profit grows at an expansion rate of φ.

2.5 Personalized Prices for Consumers

In the previous sections, we assumed that the price of a product in a given time period was the same for all the consumers of the product. However, in real-world wholesale trade, product prices usually depend on the order size: the larger the order, the cheaper each unit of the product. For this reason, different order sizes for the same commodity produced during a given time period correspond to different prices, which means that the price is *personalized* for each consumer.

In this section, we will see whether and how a decentralized economic system can function in balanced growth mode if for any time period the price of each product is set individually for each consumer of the product. In the case under consideration, we will again assume that prices depend linearly on a certain parameter. However, this time the price will depend linearly not on the volume of production [as in (2.7)], but on the order size. This means that instead of the variables $p_i(t)$, we now consider the personalized prices $p_{ij}(t)$ defined such that

$$p_{ij}(t) = b_i(t) + d_i(t)y_{ij}(t+1), \qquad j \in N_i^-, \tag{2.32}$$

where the values of $b_i(t) > 0$ and $d_i(t) < 0$ are fixed for given i and t. It is clear that the amount of discount is directly proportional to the order size.

We can now use (2.32) to express the revenue of sector i that i receives by selling its commodity produced in time period t:

$$\sum_{j \in N_i^-} \big(b_i(t) + d_i(t)y_{ij}(t+1)\big)\, y_{ij}(t+1).$$

Similarly, we can express costs of producing in time period t:

$$\sum_{j \in N_i^+} (b_j(t-1) + d_j(t-1)y_{ji}(t))y_{ji}(t).$$

As before, $\Pi_i(t)$ denotes the profit of sector i in time period t. Now we have [compare with (2.8)]

$$\begin{aligned}\Pi_i(t) = \sum_{j \in N_i^-} &(b_i(t) + d_i(t)y_{ij}(t+1))y_{ij}(t+1) \\ &- \sum_{j \in N_i^+} (b_j(t-1) + d_j(t-1)y_{ji}(t))y_{ji}(t).\end{aligned} \tag{2.33}$$

Formally, this function's arguments are $|N_i^-|$ variables of the form $y_{ij}(t+1)$ and $|N_i^+|$ variables of the form $y_{ji}(t)$. Since surplus resources reduce profit, we can assume that the order sizes are uniquely determined by the planned output such that we have (2.9).

We consider the variables $y_{ij}(t+1)$, $j \in N_i^-$, to be functions of $x_i(t)$. For this, we must assume that sector i knows both the Frobenius vector x_A of the technology matrix A and the elements a_{ij} of A, where $j \in N_i^-$. Besides that, all the sectors must know the expansion rate in balanced growth mode.

Remark 2.10. The above assumptions are essential for the planning algorithm under consideration. This means that in order to use this algorithm the sectors must have more information in this case than in the case where the prices are the same for all the consumers of a sector.

It is readily seen that, when the above assumptions hold, the sizes of the orders $y_{ij}(t+1)$, $j \in N_i^-$, for commodity i are uniquely determined by the volume of production $x_i(t)$:

$$y_{ij}(t+1) = a_{ij}x_j(t+1) = a_{ij}\frac{(x_A)_j}{(x_A)_i}\,\nu\, x_i(t) = \beta_{ij}x_i(t),$$

where $\beta_{ij} = \nu a_{ij}\left((x_A)_j/(x_A)_i\right)$, $j \in N_i^-$, are constants. Hence, we can rewrite (2.33) (the planned profit of sector i in time period t) such that

$$\begin{aligned}\Pi_i(t) = \sum_{j\in N_i^-}(b_i(t) + d_i(t)\beta_{ij}x_i(t))\beta_{ij}x_i(t) \\ - \sum_{j\in N_i^+}(b_j(t-1) + d_j(t-1)a_{ji}x_i(t))a_{ji}x_i(t).\end{aligned} \tag{2.34}$$

Let us assume again that for any time period each sector chooses such volume of output that maximizes its planned profit. If the coefficient at the square term in (2.34) is negative, i.e., if

$$d_i(t)\sum_{j\in N_i^-}\beta_{ij}^2 - \sum_{j\in N_i^+}d_j(t-1)a_{ji}^2 < 0, \tag{2.35}$$

then it is easy to see that the profit is maximum when $x_i(t)$ is as follows:

$$x_i(t) = \frac{1}{2}\,\frac{b_i(t)\sum\limits_{j\in N_i^-}\beta_{ij} - \sum\limits_{j\in N_i^+}b_j(t-1)a_{ji}}{\sum\limits_{j\in N_i^+}d_j(t-1)a_{ji}^2 - d_i(t)\sum\limits_{j\in N_i^-}\beta_{ij}^2}. \tag{2.36}$$

It is obvious that a plan defined like this has a meaningful interpretation only when the numerator of the above fraction is positive, i.e., when we have

$$b_i(t)\sum_{j\in N_i^-}\beta_{ij} - \sum_{j\in N_i^+}b_j(t-1)a_{ji} > 0.$$

Since the parameters β_{ij} satisfy the evident condition $\sum\limits_{j\in N_i^-}\beta_{ij} = 1$, we can simplify the above inequality:

$$b_i(t) - \sum_{j\in N_i^+}b_j(t-1)a_{ji} > 0. \tag{2.37}$$

Let us now show that the system of inequalities (2.35), (2.37) is, generally speaking, incompatible with the dynamics of the price coefficients as given by (2.23). Recall that the latter was determined for the case where the price of a given product is the same for all its consumers.

Example 2.6. Suppose all the elements of a technology matrix A have the same value: $a_{ij} = a$. In this case, we can choose the Frobenius vector of A equals the all-one vector $[1, \dots, 1]^T$. It is clear that $\lambda_A = na$, $\nu = 1/(na)$, $\beta_{ij} = 1/n$. Accordingly, inequalities (2.35), (2.37) are here as follows:

$$\frac{1}{n} d_i(t) - \sum_{j=1}^{n} d_j(t-1)a^2 < 0,$$

$$b_i(t) - \left(\sum_{j=1}^{n} b_j(t-1) \right) a > 0.$$

The special form of A unifies the base prices and discount coefficients for any given time period. Thus we can rewrite the above inequalities for each sector separately:

$$d_i(t) < d_i(t-1)(na)^2 = \frac{1}{\nu^2} d_i(t-1),$$

$$b_i(t) > b_i(t-1)na = \frac{1}{\nu} b_i(t-1).$$

Hence the obtained constraints make it no longer possible to determine the dynamics of the price coefficients using (2.23). □

Since in balanced growth mode equalities (2.12) hold, we have from (2.36) that the price coefficients $b_i(t)$ and $d_i(t)$ must satisfy the following system of equations for any $t \geqslant 1$:

$$b_i(t) - \sum_{j \in N_i^+} b_j(t-1)a_{ji} + 2\nu^t \left(d_i(t) \sum_{j \in N_i^-} \beta_{ij}^2 - \sum_{j \in N_i^+} d_j(t-1)a_{ji}^2 \right) x_i(0) = 0. \tag{2.38}$$

Note that here the equations analogous to (2.14) and (2.15) that define the independent dynamics of the base prices and discount parameters are

$$b_i(t) = \sum_{j \in N_i^+} b_j(t-1)a_{ji},$$

$$d_i(t) = \frac{1}{\sum_{j \in N_i^-} \beta_{ij}^2} \sum_{j \in N_i^+} d_j(t-1)a_{ji}^2.$$

However these equations do not allows us to calculate the volume of output $x_i(t)$, because if we use them we obtain the indeterminate form $0/0$ in the right-hand side of (2.36). One constructive way out of this situation is to express each equation in (2.38) as a sum of the following two equations:

$$b_i(t) - \sum_{j \in N_i^+} b_j(t-1)a_{ji} - \psi_i(t) = 0,$$

$$2\nu^t \left(d_i(t) \sum_{j \in N_i^-} \beta_{ij}^2 - \sum_{j \in N_i^+} d_j(t-1)a_{ji}^2 \right) x_i(0) + \psi_i(t) = 0,$$

where $\psi_i(t)$ is some positive parameter. In this case the dynamics of the variables $b_i(t)$ and $d_i(t)$ for $t \geqslant 1$ is

$$\begin{aligned} b_i(t) &= \sum_{j \in N_i^+} b_j(t-1)a_{ji} + \psi_i(t), \\ d_i(t) &= \frac{1}{\sum_{j \in N_i^-} \beta_{ij}^2} \left(\sum_{j \in N_i^+} d_j(t-1)a_{ji}^2 - \frac{\psi_i(t)}{2\nu^t x_i(0)} \right). \end{aligned} \tag{2.39}$$

Note that here condition (2.35) holds for all i and t, because the left-hand side of that inequality equals $-\psi_i(t)/(2x_i(t))$. It is also obvious that condition (2.37) holds.

In order to provide an economic interpretation for the equations in system (2.39), we multiply them by $x_i(t)$ and $x_i^2(t)$, respectively, and rewrite them such that

$$b_i(t) \sum_{j \in N_i^-} y_{ij}(t+1) = \sum_{j \in N_i^+} b_j(t-1)y_{ji}(t) + \psi_i(t)x_i(t),$$

$$d_i(t) \sum_{j \in N_i^-} y_{ij}^2(t+1) = \sum_{j \in N_i^+} d_j(t-1)y_{ji}^2(t) - \frac{1}{2}\psi_i(t)x_i(t).$$

The first equation shows that if no discounts are given the revenues that sector i receives from selling its commodity produced in time period t exceed the expenses for buying the resources needed for that by $\psi_i(t)x_i(t)$. The second equation means that the absolute value of the loss from giving the discounts to the consumers exceeds the total savings from the discounts given to the sector i by its suppliers by $(1/2)\psi_i(t)x_i(t)$. These two imbalances ensure the positive profit. We can check that using (2.33) and (2.39) to calculate the profit $\Pi_i(t)$ in balanced growth mode:

$$\Pi_i(t) = \frac{1}{2}\psi_i(t)x_i(t).$$

This means that if $\psi_i(t) = const$, then the profit of sector i has the expansion rate equals the production expansion rate ν. On the other hand, the profit is the same in all time periods if $\psi_i(t) = \nu^{-t}\psi_i(0)$, where $\psi_i(0)$ is some constant.

Let us now turn to the payment balances of the sectors in the context of the personalized pricing algorithm under consideration. Keeping the same notation for this indicator, we can write the following formula for calculating its value for sector i in time period t:

$$B_i(t) = \sum_{j \in N_i^-} (b_i(t) + d_i(t) y_{ij}(t+1)) y_{ij}(t+1) \\ - \sum_{j \in N_i^+} (b_j(t) + d_j(t) y_{ji}(t+1)) y_{ji}(t+1).$$

Since we consider the balanced growth mode, we get

$$B_i(t) = \left(b_i(t) - \nu \sum_{j \in N_i^+} b_j(t) a_{ji} \right) x_i(t) \\ + \left(d_i(t) \sum_{j \in N_i^-} \beta_{ij}^2 - \nu^2 \sum_{j \in N_i^+} d_j(t) a_{ji}^2 \right) x_i^2(t).$$

It is clear that all the sectors have zero payment balances in time period t if the base prices form the left Frobenius vector of the technology matrix A and the discount coefficients form a nontrivial solution to the following system of linear equations:

$$d_i(t) \sum_{j \in N_i^-} \beta_{ij}^2 - \nu^2 \sum_{j \in N_i^+} d_j(t) a_{ji}^2 = 0, \qquad i = 1, \ldots, n.$$

However, a simple analysis (omitted here) shows that in the general case these solutions are incompatible with the dynamics of the parameters (2.39) and the constraint that $b_i(t)$ must form the left Frobenius vector of A for all t.

Let us now compare the financial indicators of a sector in this model with those when the prices are the same for all its consumers (i.e., in the model considered in Sect. 2.2). We perform this comparison for the same trajectory of outputs that corresponds to the balanced growth mode.

Let $V_i^c(t)$ and $V_i^v(t)$ be the revenues of sector i in time period t when the price is the same for all its consumers and when the price depends on the order size, respectively:

$$V_i^c(t) = b_i(t) x_i(t) + d_i(t) x_i^2(t),$$

$$V_i^v(t) = b_i(t) x_i(t) + d_i(t) \sum_{j \in N_i^-} y_{ij}^2(t+1),$$

where the values of $b_i(t)$ (and $d_i(t)$) are assumed to be the same in both equations. By definition, put $\Delta V_i(t) = V_i^c(t) - V_i^v(t)$. It now follows that

$$\begin{aligned}\Delta V_i(t) &= d_i(t)\Big(x_i^2(t) - \sum_{j\in N_i^-} y_{ij}^2(t+1)\Big)\\ &= d_i(t)\Big(x_i^2(t) - \Big(\sum_{j\in N_i^-} y_{ij}(t+1)\Big)^2 + \sum_{\substack{j,k\in N_i^-\\ j\neq k}} 2y_{ij}(t+1)y_{ik}(t+1)\Big)\\ &= 2d_i(t)\sum_{\substack{j,k\in N_i^-\\ j\neq k}} y_{ij}(t+1)y_{ik}(t+1).\end{aligned}$$

As we can see, when all other conditions are equal, switching to the personalized prices leads to an increase in profit for a given sector if the sector sells its product to at least two consumers. The reason for this is that the absolute value of the amounts of the discounts decreases.

Let $C_i^c(t)$ and $C_i^v(t)$ be the expenses of sector i for buying its resources at the end of time period t in the first case and the second case, respectively:

$$C_i^c(t) = \sum_{j\in N_i^+} (b_j(t) + d_j(t)x_j(t))y_{ji}(t+1),$$

$$C_i^v(t) = \sum_{j\in N_i^+} (b_j(t) + d_j(t)y_{ji}(t+1))y_{ji}(t+1).$$

Let us now calculate the difference $\Delta C_i(t) = C_i^c(t) - C_i^v(t)$:

$$\Delta C_i(t) = \sum_{j\in N_i^+} d_j(t)(x_j(t) - y_{ji}(t+1))y_{ji}(t+1).$$

Here, when all other conditions are equal, switching to the personalized prices leads to an increase in expenses for a given sector unless the sector is the only consumer of each of its suppliers. This is again explained by the fact that the absolute value of the amounts of the discounts decreases.

Let $\Delta\Pi_i(t)$ be the change in profit for sector i in time period t when switching to the personalized prices while keeping the same price coefficients. This quantity is the difference between (2.8) and (2.33):

$$\begin{aligned}\Delta\Pi_i(t) = d_i(t)\Big(x_i^2(t) - \sum_{j\in N_i^-} y_{ij}^2(t+1)\Big)&\\ - \sum_{j\in N_i^+} d_j(t-1)(x_j(t-1) - y_{ji}(t))y_{ji}(t).&\end{aligned}$$

Here, the only thing that we know for sure is that the profit of a given sector increases if the sector is the only consumer of each of its suppliers and the sector's product is consumed by at least two sectors. On the other hand, the profit of a sector decreases if the sector has only one consumer and at least one of the sector's suppliers has more than one consumer.

2.6 Average Prices and Personalized Discounts

In this section, we consider another modification to the model from Sect. 2.2. Here, in addition to the pricing mechanism described in the previous sections, which is based on the "base price minus discount" principle, we introduce another mechanism that either adds surcharges on or offers additional discounts from product prices. This change in price is based on the order size, which again means personalized prices.

In order to avoid any confusion, we will refer to the prices in the original model as *average prices*. These prices are set as in the original model (i.e., using the base prices and discount coefficients). The dynamics of the price coefficients is as given by (2.23) and output plans are determined according to (2.11). This means that here, as in the original model, the parameters that determine the average prices are used as a basis when planning production. We will keep the notation $p_i(t)$ for the average price of sector i in time period t. Here, this price still depends on the volume of production, as in (2.7). Next, the initial values of the price coefficients are selected so that in balanced growth mode the average prices ensure that all the sectors have zero payment balances for all time periods.

The distinctive property of the model under consideration is that here *personalized discounts* are offered for products sold at the average prices. In concrete terms, we assume that the actual amount paid by sector j, $j \in N_i^-$, for the commodity produced by sector i in time period t is determined as follows:

$$p_i(t)y_{ij}(t+1) + \hat{d}_i(t)y_{ij}^q(t+1), \tag{2.40}$$

where the first summand equals the amount calculated using the average price and the second summand equals the amount of the additional discount. Here, the first amount depends on the order size linearly, whereas the second amount depends on the same parameter exponentially. In this formula, the coefficient $\hat{d}_i(t) < 0$ is set individually by each sector and, generally speaking, can be different in different time periods. On the other hand, the constant $q > 0$ is assumed to be the same for all the sectors in all time periods.

Remark 2.11. Formally, the value of q can be any positive number. However, when the size of every order in the system is significantly greater than 1, it makes more sense to assume that q belongs to the interval (0, 1) from an economic point of view.

Let $\Delta B_i(t)$ be the deviation (caused by the additional discounts) of the payment balance $B_i(t)$ from zero in time period t. It is easy to see that in the balanced growth mode we have

$$\Delta B_i(t) = \hat{d}_i(t) \sum_{j \in N_i^-} y_{ij}^q(t+1) - \sum_{j \in N_i^+} \hat{d}_j(t) y_{ji}^q(t+1).$$

The above formula can be rewritten as

$$\Delta B_i(t) = \hat{d}_i(t) \sum_{j \in N_i^-} a_{ij}^q x_j^q(t+1) - \sum_{j \in N_i^+} \hat{d}_j(t) a_{ji}^q x_i^q(t+1).$$

Obviously, if there exists a set of negative coefficients $\hat{d}_i(t)$ such that each $\Delta B_i(t)$ turns into zero, then the same set ensures that these indicators equal zero over the entire time-frame. This is because the system functions in balanced growth mode. For this reason, we can omit the index t when referring to the parameters $\hat{d}_i(t)$. For the same reason, the problem of the existence of a dynamics of the coefficients $d_i(t)$ such that we have zero payment balances for each sector in all time periods can now be reduced to the problem of the existence of a negative solution to the following system of equations:

$$\hat{d}_i \sum_{j \in N_i^-} a_{ij}^q x_j^q(0) - \left(\sum_{j \in N_i^+} \hat{d}_j a_{ji}^q \right) x_i^q(0) = 0, \tag{2.41}$$

where $x_i(0)$, $i = 1, \ldots, n$, are the components of the Frobenius vector of the matrix A.

Proposition 2.6. *Suppose the following conditions hold:*

(a) the system's technology matrix A is irreducible,
(b) the system functions in balanced growth mode,
(c) the amounts to be paid for products are set individually for each consumer, according to (2.40),
(d) the dynamics of the average prices ensures that the revenues of each sector equals its expenses in all time periods.

Then there exists a set of additional discounts $\hat{d}_i$, $i = 1, \ldots, n$, such that each sector's payment balance equals zero in all time periods.

Proof. Let w be the transpose of the row vector d. Then we can rewrite (2.41) as the matrix equation:

$$Gw = w,$$

where G is a square matrix of order n. The elements g_{ij} of G are defined using the elements of the technology matrix A and the components of the Frobenius vector $x(0)$ such that

$$g_{ij} = \frac{a^q_{ji} x^q_i(0)}{\sum\limits_{k=1}^{n} a^q_{ik} x^q_k(0)}.$$

Since A is irreducible, it cannot have zero rows; at the same time all the components of $x(0)$ are positive. Therefore the denominator is positive in the above equation. Denote by m_{ij} the $a^q_{ij} x^q_j(0)$. Then we can rewrite G as

$$G = \begin{bmatrix} \frac{m_{11}}{m_{11}+\cdots+m_{1n}} & \frac{m_{21}}{m_{11}+\cdots+m_{1n}} & \cdots & \frac{m_{n1}}{m_{11}+\cdots+m_{1n}} \\ \frac{m_{12}}{m_{21}+\cdots+m_{2n}} & \frac{m_{22}}{m_{21}+\cdots+m_{2n}} & \cdots & \frac{m_{n2}}{m_{21}+\cdots+m_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m_{1n}}{m_{n1}+\cdots+m_{nn}} & \frac{m_{2n}}{m_{n1}+\cdots+m_{nn}} & \cdots & \frac{m_{nn}}{m_{n1}+\cdots+m_{nn}} \end{bmatrix}. \tag{2.42}$$

It is readily seen that G is a (nonnegative) irreducible matrix. The equation $Gw = w$ has a nontrivial solution because the matrix $(G - I_n)$, where I_n is the identity matrix of order n, is singular. The last statement follows from the fact that the sum of the rows of the matrix

$$\tilde{M} = \begin{bmatrix} -m_{12}\cdots - m_{1n} & m_{21} & \dots & m_{n1} \\ m_{12} & -m_{21}\cdots - m_{2n} & \dots & m_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1n} & m_{2n} & \dots & -m_{n1}\cdots - m_{nn-1} \end{bmatrix} \tag{2.43}$$

equals the zero row vector and $\tilde{M}$ has the same determinant as the matrix $(G - I_n)$. Since the denominator of each fraction in (2.42) is strictly greater than zero, then we see that the solution set of the system of equations $Gw = w$ is the same as that of the system of equations $\tilde{M}w = 0$.

We now list the properties of $\tilde{M}$ that follow from the fact that A is irreducible. First, all the diagonal elements of $\tilde{M}$ are negative. Indeed, suppose to the contrary that there exists a zero diagonal element. Then the corresponding sector can function independently, i.e., it does not need the product of any other sector. Second, since for any column the sum of its elements equals zero, then we see that each column contains at least one positive element.

Based on the properties of $\tilde{M}$, we now show that by using the elimination method we can rewrite the system of equations $\tilde{M}w = 0$ as

$$\begin{aligned} &k_{21}w_1 - r_{22}w_2 = 0 \\ &k_{31}w_1 + k_{32}w_2 - r_{33}w_3 = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &k_{n1}w_1 + k_{n2}w_2 + k_{n3}w_3 + \dots - r_{nn}w_n = 0, \end{aligned} \tag{2.44}$$

where all the coefficients r_{ii}, $i = 2, \dots, n$, are positive and each row i, $i = 2, \dots, n$, contains at least one positive coefficient k_{ij}. Indeed, in the system of equations $\tilde{M}w = 0$ the coefficient at the variable w_n in the last equation is equal to

$$-m_{n1} - m_{n2} - \dots - m_{n(n-1)}.$$

Since A is irreducible, then we get that this sum is negative. If we take into account the form of columns of $\tilde{M}$, we see that we can eliminate the variable w_n from row i, $i = 2, \dots, n-1$, of the system $\tilde{M}w = 0$ as follows: Multiply row n by a scalar ω_i, $0 \leqslant \omega_i \leqslant 1$; then, add row i to the product.

If $\omega_i < 1$, then the number of positive coefficients of the remaining variables in row i does not decrease after the above operation. It is clear that we can have the case where $\omega_i = 1$ if and only if column n contains only two nonzero elements (corresponding to rows i and n); this means that $\omega_j = 0$, $j \neq i$. If we suppose that, after the rows are added together, the diagonal element at the position (i, i) equals zero, then column i must also contain only two nonzero elements, in the positions (i, i) and (n, i). But this means that sectors i and n are technologically isolated from the rest of the system, which contradict the condition that A is irreducible.

It is easy to see that if we eliminate w_n from rows $2, \dots, n-1$, then the values of all the coefficients do not decrease and the negative coefficients remain negative. Next, we eliminate w_{n-1} from rows $2, \dots, n-2$, etc.

Finally, we arrive at system of equations (2.44). Let $w_1 = 1$, $w_2 = k_{21}/r_{22}$, etc. As a result, we obtain a column vector all of whose components are positive.

Recall that if a matrix is irreducible, then we get that the matrix's strictly positive eigenvector corresponds only to its Frobenius eigenvalue. Therefore the Frobenius eigenvalue of the matrix G equals 1. □

Hence, the introduction of two types of discount allows us:

(a) to implement a personalized approach to each consumer,
(b) to ensure that each sector has a zero payment balance for all time periods.

2.7 Generalized Price Formula

The model of an economic system functioning in balance growth mode presented in Sect. 2.2 is based on the assumption that to determine the price of its product each sector of the system uses the linear dependencies given by (2.7). In this section, we consider a generalization of this approach. Here, the price of product i in time period t is determined as

$$p_i(t) = b_i(t) + d_i(t)x_i^q(t), \tag{2.45}$$

where $b_i(t) > 0$ and $d_i(t) < 0$, and q is a constant such that $0 < q < \infty$. Since the case where $q = 1$ is considered in Sect. 2.2, here we assume that $q \neq 1$. Thus, here the price of a product depends nonlinearly on the volume of production.

Using (2.45), we see that the profit of sector i in time period t is maximum when the volume of output is as follows:

$$x_i(t) = \left(-\frac{b_i(t) - \sum\limits_{j \in N_i^+} (b_j(t-1) + d_j(t-1)x_j^q(t-1))a_{ji}}{(1+q)d_i(t)} \right)^{1/q}.$$

Remark 2.12. Clearly, we assume that the value of the above root is positive. Such a root exists because the following holds: $b_i(t) > \sum\limits_{j \in N_i^+} p_j(t-1)a_{ji}$ (see Remark 2.8).

In balanced growth mode, the vector of outputs has the expansion rate equals ν [see (2.12)]. For this reason, to ensure the balanced growth mode it is sufficient that the base prices satisfy dynamic equations (2.14) and the discount coefficients satisfy the following equations:

$$d_i(t) = \frac{\nu^{-q}}{(1+q)} \sum_{j \in N_i^+} a_{ji} \left(\frac{x_j(0)}{x_i(0)} \right)^q d_j(t-1), \qquad t \geqslant 1. \tag{2.46}$$

It follows from these equations that the profit $\Pi_i(t)$ of sector i in the time period t is as stated below:

$$\Pi_i(t) = -q d_i(t) \left(x_i(t) \right)^{(1+q)}.$$

Let us now calculate the payment balance $B_i(t)$ (2.18) of sector i for time period t in balanced growth mode. Suppose that the initial row vector of base prices $b(0)$ is the left Frobenius vector of the matrix A and that the dynamics of the base prices is as given by (2.23). As we can see from Sect. 2.3, in this case the base prices contribute zero to the payment balance of each sector over the entire time-frame. Therefore the values of the payment balances depend exclusively on the amounts of the discounts. It is easy to see that the discount coefficients also contribute zero to the payment balance of each sector for all time periods if these coefficients satisfy the following dynamic equations:

$$d_i(t)x_i^q(0) = \nu \sum_{j \in N_i^+} d_j(t)x_j^q(0)a_{ji}, \qquad t \geqslant 1. \tag{2.47}$$

Comparing (2.46) with (2.47), we see that the dynamics of the variables $d_i(t)$ must be as follows:

$$d_i(t) = \frac{1}{(1+q)\nu^{(1+q)}} d_i(t-1). \tag{2.48}$$

If we reuse the arguments from the proof of Proposition 2.5, we conclude that for $t = 0$ system of equations (2.47) has a solution all of whose components are negative. This means that there exists an initial row vector of discount coefficients $d(0) < 0$ such that the discount coefficients (2.48) contribute zero to the payment balance of each sector over the entire economic time-frame.

Let us now apply exponential dependencies for calculating personalized prices from Sect. 2.5. The modified personalized price formula (2.32) is now as

$$p_{ij}(t) = b_i(t) + d_i(t) y_{ij}^{q(t)}(t+1), \qquad j \in N_i^-, \quad t \geqslant 1,$$

where $b_i(t) > 0$ and $d_i(t) < 0$ are fixed for given i and t. In addition, the parameter $q(t)$ (the same for all the sectors) can be different for different time periods; we require that $q(t) > 0, t \geqslant 0$.

Accordingly, the profit $\Pi_i(t)$ of sector i in time period t is now calculated as follows:

$$\begin{aligned} \Pi_i(t) = \sum_{j\in N_i^-} (b_i(t) + d_i(t) y_{ij}^{q(t)}(t+1)) y_{ij}(t+1) \\ - \sum_{j \in N_i^+} (b_j(t-1) + d_j(t-1) y_{ji}^{q(t)}(t)) y_{ji}(t). \end{aligned} \tag{2.49}$$

Using the reasoning and notation from Sect. 2.5, we now express the profit (2.49) as a function of one variable, namely $x_i(t)$) [see (2.34)]:

$$\begin{aligned} \Pi_i(t) = \sum_{j\in N_i^-} \left(b_i(t) + d_i(t) \left(\beta_{ij} x_i(t)\right)^{q(t)} \right) \beta_{ij} x_i(t) \\ - \sum_{j \in N_i^+} \left(b_j(t-1) + d_j(t-1) \left(a_{ji} x_i(t)\right)^{q(t)} \right) a_{ji} x_i(t). \end{aligned}$$

We assume that the dynamics of $d_i(t)$ and $q(t)$ is such that the following inequalities hold for all i and $t \geqslant 1$:

$$d_i(t) \sum_{j \in N_i^-} \beta_{ij}^{(1+q(t))} - \sum_{j\in N_i^+} d_j(t-1) \left(a_{ji}\right)^{(1+q(t))} < 0.$$

These inequalities ensure that the profit functions of all sectors have a maximum for all time periods. Besides that, we assume that the base prices satisfy (2.37) for all i

and t. In this case, the profit of sector i in time period t is maximum when $x_i(t)$ is as follows:

$$x_i(t) = \left(\frac{1}{1+q(t)} \frac{b_i(t) - \sum\limits_{j \in N_i^+} b_j(t-1)a_{ji}}{\sum\limits_{j \in N_i^+} d_j(t-1)a_{ji}^{(1+q(t))} - d_i(t) \sum\limits_{j \in N_i^-} \beta_{ij}^{(1+q(t))}} \right)^{1/q(t)}. \tag{2.50}$$

Here, we select a positive value of the root. It follows from (2.50) that in balanced growth mode the price coefficients must satisfy, for $t \geqslant 1$, the system of equations:

$$b_i(t) - \sum_{j \in N_i^+} b_j(t-1)a_{ji} - (1+q(t)) \\ \times \left(\sum_{j \in N_i^+} d_j(t-1)a_{ji}^{(1+q(t))} - d_i(t) \sum_{j \in N_i^-} \beta_{ij}^{(1+q(t))} \right) \left(\nu^t x_i(0)\right)^{q(t)} = 0.$$

Let us select a positive row vector $b(0)$ equals the left Frobenius vector of the matrix A. We assume that the vector of base prices is constant over the entire economic time-frame: $b(t) = b(0)$ for all $t \geqslant 1$. In this case, the numerator of the second fraction in (2.50) is constant and equals $(1-\nu^{-1})b_i(0)$, which is positive because $\lambda_A < 1$. Note that now the gross product of the system, if expressed in base prices, grows at an expansion rate equals ν.

The dynamic equations for the discount coefficients are here as

$$d_i(t) = \frac{1}{\sum\limits_{j \in N_i^-} \beta_{ij}^{(1+q(t))}} \left(\sum_{j \in N_i^+} d_j(t-1)a_{ji}^{(1+q(t))} - \frac{(1-\nu^{-1})\, b_i(0)}{(1+q(t))\,(\nu^t x_i(0))^{q(t)}} \right).$$

We can simplify the above formula significantly by selecting the dynamics of $q(t)$ appropriately. For example, if we assume that $q(t) = 1/t$, $t \geqslant 1$ then we have for sufficiently large t:

$$d_i(t) = \sum_{j \in N_i^+} d_j(t-1)a_{ji} - \frac{\nu-1}{\nu^2} b_i(0) + O(t), \tag{2.51}$$

where $O(t) \rightarrow 0$ as $t \rightarrow \infty$. In addition, if the matrix A is primitive then the sequence $\left\{\left(\lambda_A^{-1} d(\tilde{t})A\right)^t\right\}$, converges for any fixed $\tilde{t}$; in the general case, this sequence is bounded (see Ashmanov 1984; Horn and Johnson 1985). Since we assume that $\lambda_A < 1$, we see that the sequence $\left\{\left(d(\tilde{t})A\right)^t\right\}$ converges to the zero row vector. Therefore, if we apply induction to (2.51), we obtain a formula for calculating $d_i(t)$, where the initial conditions are limited to a value of the base price only:

$$d_i(t) = -\left(\frac{\nu - 1}{\nu^{(t+1)}} + \frac{\nu - 1}{\nu^t} + \cdots + \frac{\nu - 1}{\nu^2}\right) b_i(0) + \tilde{O}(t),$$

where $O(t) \to 0$ as $t \to \infty$. If we sum the geometric sequence in the right-hand side of this equality, we can see that the limit of the sequence $\{d_i(t)\}$ equals $-b_i(0)/\nu$. It is obvious that, as $t \to \infty$ the price of product i approaches the common value, namely $(\nu - 1)\nu^{-1}b_i(0)$, for all the consumers of i. It is clear that in this case the payment balance of each sector approaches zero.

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