Chapter 2
Interaction of Electrons with Laser Fields

To describe the emission by electrons, it is necessary to first analyze their dynamics. There are several theories available, as to how this task is to be accomplished. In this chapter we are going to outline two fundamentally different approaches. We will start by presenting the classical framework, in which an electron of charge e is viewed as a point source of an electromagnetic field. The emission of such a classical current in discussed in Sect. 2.1.

As discussed in Chap. 1, in the past century it emerged a quantum theory of matter. In this theory, due to the intrinsic uncertainty of conjugate observables, in particular position and momentum, no point particles can exist. Much rather, quantum electrodynamics describes particles as exited states of quantized fields and describes a scattering as the probability of a quantum state, formed in the far past, to go over into another quantum state, observed in the far future. A short introduction into the schemes and techniques of the according scattering theory of quantum electrodynamics is given in Sect. 2.2.

2.1 Classical Electrodynamics

The basics of classical electrodynamics are the famous Maxwell equations, unifying the electric and magnetic interaction. In covariant form they state that a spatial charge current $\boldsymbol{j}$, which is combined with its charge density ρ to a four dimensional current $j^\mu = (c\rho, \boldsymbol{j})$, will generate electromagnetic fields according to [1]

$$\partial_\mu F^{\mu\nu}(x) = \frac{4\pi}{c} j^\nu(x), \tag{2.1}$$

where $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ is the antisymmetric field strength tensor derived from the four potential $A^\mu(x) = (\phi(x), \boldsymbol{A}(x))$. This potential, however, is not a measurable quantity as it is not uniquely defined by Eq. (2.1). In fact, applying a gauge transformation to the four potential, the field strength tensor, entering the

K. F. Mackenroth, *Quantum Radiation in Ultra-Intense Laser Pulses*,
Springer Theses, DOI: 10.1007/978-3-319-07740-6_2,

Maxwell equations, remains unchanged. The same current $j^\mu(x)$ thus generates a whole equivalence class of vector potentials, all connected by gauge freedom. The electromagnetic quantities, uniquely defined by Eq. (2.1) are the three dimensional vector fields of the electric and magnetic field $\boldsymbol{\mathcal{E}}(x)$ and $\boldsymbol{\mathcal{B}}(x)$, respectively, which are derived from the four potential via the relations

$$\boldsymbol{\mathcal{E}}(x) = -\nabla\phi(x) - \frac{\partial \boldsymbol{A}(x)}{\partial t} \tag{2.2a}$$

$$\boldsymbol{\mathcal{B}}(x) = \nabla \times \boldsymbol{A}(x). \tag{2.2b}$$

For describing electromagnetic fields propagating in free space ($j^\nu \equiv 0$) one usually adopts the Lorenz gauge $\partial_\mu A^\mu(x) = \partial_t\phi(x) + \nabla\boldsymbol{A}(x) = 0$, whence Eq. (2.1) turns into a wave equation of the form

$$\Box A^\mu(x) = 0, \tag{2.3}$$

where the D'Alembert operator $\Box = \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$ is introduced. As we wish to describe physical fields propagating in free space, the employed solutions have to satisfy Eq. (2.3). An important class of such solutions is given by plane waves, which are defined by a wave vector $k_L^\mu = \omega_L(1, \boldsymbol{n}_L)$, with the central angular frequency of the electric field ω_L, and depend on the spatial coordinates only via the so-called invariant phase $\eta = x_\mu k_L^\mu$ [1]. Much of the discussion presented in this work will be based on linearly polarized plane wave solutions of Eq. (2.3), for which we will use the notation $A_L^\mu(\eta) = A_L\epsilon_L^\mu\psi_{\mathcal{A}}(\eta)$, with the constant (positive) amplitude $A_L = -m\xi/e$, the wave's polarization four vector ϵ_L^μ and the shape function $\psi_{\mathcal{A}}(\eta)$ encoding the temporal structure of the field. It is customary to consider plane waves, propagating in free space, in a reference system, where the static potential vanishes ($\phi \equiv 0$) and all physical fields are derived from the vector potential $\boldsymbol{A}_L(\eta)$. This corresponds to a purely spatial polarization vector $\epsilon_L^\mu = (0, \boldsymbol{\epsilon}_L)$. For a plane wave field the Lorenz gauge condition reduces to $A_L k_L = 0$, whence with the above choices we find

$$\boldsymbol{\epsilon}_L\boldsymbol{k}_L = 0. \tag{2.4}$$

A plane wave thus is always polarized perpendicularly to its propagation direction. The shape function is an essential ingredient in obtaining specific results. It is customary to model it by a function of the form $\psi_{\mathcal{A}}(\eta) = g(\eta)\sin(\eta + \eta_0)$, where $g(\eta)$ is the so-called *pulse envelope*, which has a peak value of unity. The validity of this approach has been proven on the basis that the central carrying frequency is uniquely defined by the envelope. In particular, it needs to be independent of the quantity η_0, which quantifies a relative phase shift between the carrier wave and the envelope and is hence labeled *carrier-envelope phase (CEP)* [2]. Due to the importance of the shape function, we wish to explicitly present two possible choices here. Here we are going to present the shape functions as functions of the invariant laser phase η, but also point out that by virtue of the relation $\eta = k_L^+ x^-$ any function $F(\eta)$ is easily translated to a function of the light cone coordinate x^- according to

$$F(x^-) := F(k_L^+ x^-). \tag{2.5}$$

This equivalence is frequently used for the shape function and the corresponding expression for the four potential $A_L^\mu(x^-) = A_L \epsilon_L^\mu \psi_{\mathcal{A}}(x^-)$ in the course of this thesis, but for the sake of notational simplicity we are not going to introduce a separate symbol for these two, rigorously speaking different, functions of Eq. (2.5). To model a few-cycle pulse we use

$$\psi_{\mathcal{A}}(\eta) = \begin{cases} \sin^4\left(\frac{\eta}{2n_C}\right)\sin(\eta+\eta_0) & \text{if } \eta \in [0, 2\pi\, n_C] \\ 0 & \text{else,} \end{cases} \tag{2.6}$$

where n_C is the number of cycles contained in the laser pulse. The favors of this choice are its simple analytic structure alongside its smooth rise and fall of the electric field, derived from Eq. (2.2a), which is well suited to model a laser pulse [3]. One drawback is that the average of the envelope function (2.6) is constantly smaller than one, irrespective of the number of cycles contained in the laser pulse. This can be seen by the computation

$$\langle g \rangle = \frac{\int\limits_0^{2\pi\, n_C} d\eta\ \sin^4\left(\frac{\eta}{2n_C}\right)}{2\pi\, n_C} = \frac{\int\limits_0^{2\pi} d\eta'\ \sin^4\left(\frac{\eta'}{2}\right)}{2\pi} = \frac{3}{8}. \tag{2.7}$$

Despite the good applicability of Eq. (2.6) to model few-cycle pulses, due to the sketched drawback, it is problematic, in case one wants to model a long laser pulse, or particularly recover the monochromatic limit. This limit namely is recovered, if the laser pulse can approximately be described by the sole oscillation frequency. The amplitude variation over the whole pulse must accordingly be negligible, which in turn, for a shape function normalized to unity, implies that Eq. (2.7) has to equal unity. To recover the monochromatic limit or model a longer laser pulse we thus employ the following, however more complicated envelope function

$$g^{\text{long}}(\eta) = \begin{cases} \frac{\eta}{2\pi\, n_{\text{switch}}} & \text{if } \eta \in [0, 2\pi\, n_{\text{switch}}] \\ 1 & \text{if } \eta \in [2\pi\, n_{\text{switch}}, 2\pi\,(n_{\text{switch}} + n_{\text{flat}})] \\ \frac{(2n_{\text{switch}} + n_{\text{flat}}) - \eta/2\pi}{n_{\text{switch}}} & \text{if } \eta \in [2\pi\,(n_{\text{switch}} + n_{\text{flat}}), 2\pi\,(2n_{\text{switch}} + n_{\text{flat}})] \\ 0 & \text{else.} \end{cases} \tag{2.8}$$

To obtain a proper shape function $\psi_{\mathcal{A}}(\eta)$, one has to multiply this envelope function with the oscillating carrier function $\sin(\eta+\eta_0)$, analogous to Eq. (2.6). That Eq. (2.8) indeed allows to take the monochromatic limit, is seen in analogy to Eq. (2.7)

$$\langle g^{\text{long}} \rangle = \frac{\int_0^{2\pi n_C} d\eta \, g^{\text{long}}(\eta)}{2\pi \, (2n_{\text{switch}} + n_{\text{flat}})} = \frac{n_{\text{flat}} + n_{\text{switch}}}{2n_{\text{switch}} + n_{\text{flat}}} \xrightarrow{n_{\text{flat}} \to \infty} 1. \tag{2.9}$$

In the light of the previous computations please note that, due to the relation $\boldsymbol{\mathcal{E}}_L(\eta) = -\partial_t \boldsymbol{A}_L(\eta)$, the value of the four potential can be viewed as

$$\boldsymbol{A}_L(\eta) = -\int_{-\infty}^{\eta} d\eta' \, \frac{dt}{d\eta'} \boldsymbol{\mathcal{E}}_L(\eta'), \tag{2.10}$$

where any integration constant can be chosen as zero by gauge freedom. The above expression, however, for $\eta \to \infty$ is proportional to the zero-frequency, i.e. constant field, Fourier component of the laser's electric field. Since such a constant field mode, however, does not propagate, it is essential that for any choice of $\psi_{\mathcal{A}}(\eta)$ it holds $A_L(\eta \to \infty) = 0$. Consequently, even though $|A_L(\eta \to \infty)| > 0$ would correspond to a physically reasonable electric field vanishing at infinity, this possibility is ruled out.

From here on the following discussion is again valid for arbitrary electromagnetic fields fulfilling Eq. (2.3). To determine the dynamics of an electron moving inside such an electromagnetic field, one has to solve its equation of motion. Any external electromagnetic field described by its field strength tensor $F_L^{\mu\nu}(x)$, obtained via Eq. (2.3), exerts a force on the electron according to the Lorentz force equation [1]

$$\frac{dp^{\mu}(s)}{ds} = \frac{e}{m} F_L^{\mu\nu}(x) p_{\nu}(s). \tag{2.11}$$

In this expression s is the proper time of the electron and $p^{\mu}(s)$ its kinetic momentum. Integrating Eq. (2.11) together with the electron's initial momentum $\boldsymbol{p}_i$ and position $\boldsymbol{x}_i$ then yields the classical trajectory of the electron subjected to the electromagnetic field in question. To be able to show a specific example, however, we first have to specify the coordinate frame, in which we wish to investigate the interaction. We introduce the reference frame, in which we will observe the interaction of a laser pulse with an electron throughout this thesis in Fig. 2.1. We are going to consider the laser's propagation and polarization axes to be the z- and x-axis, respectively. This corresponds to the representations $\boldsymbol{k}_L = (0, 0, 1)$ and $\boldsymbol{\epsilon}_L = (1, 0, 0)$. The coordinate frame will be chosen such that the electron is initially counterpropagating to the laser pulse ($\boldsymbol{p}_i = \varepsilon_i(0, 0, -\beta_i)$), where β_i is the electron's initial velocity. The observation direction of the electron radiation, observed in this reference frame will be denoted by a direction vector $\boldsymbol{n}_1 = (\sin(\vartheta_1)\cos(\varphi_1), \sin(\vartheta_1)\sin(\varphi_1), \cos(\vartheta_1))$, where ϑ_1 is the angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_L$ and φ_1 is the angle between $\boldsymbol{n}_1$ and the x-z plane. Exemplary electron trajectories, obtained via numerical integration of Eq. (2.11) and observed in a reference frame according to Fig. 2.1, are shown in Fig. 2.2. The trajectory inside a circularly polarized laser wave in Fig. 2.2b is shown for mere comparison,

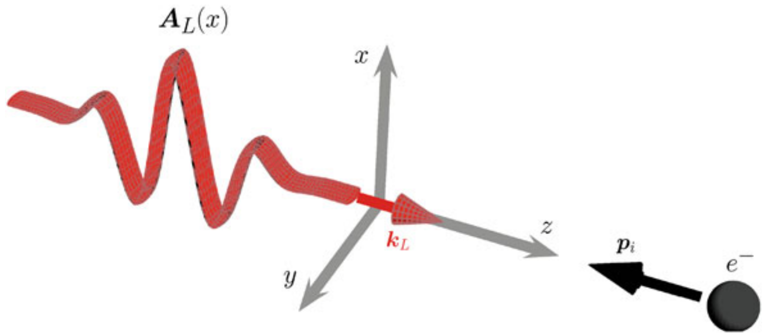

Fig. 2.1 Generic choice for the reference frame the scattering will be observed in

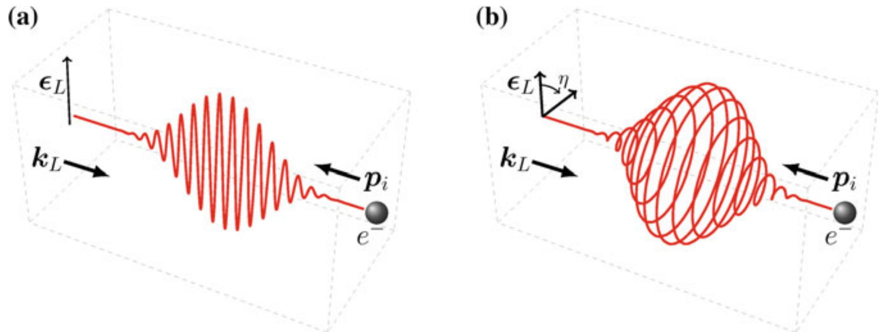

Fig. 2.2 Classical trajectories of an electron colliding head on with a laser pulse. **a** Exemplary electron trajectory for propagation in a linearly polarized laser beam. **b** Exemplary electron trajectory for propagation in a circularly polarized laser beam

as we are going to consider exclusively linearly polarized laser pulses in this work. As seen in Fig. 2.2a, in this case the classical trajectory is confined to the $\boldsymbol{k}_L$-$\boldsymbol{\epsilon}_L$-plane. It is then sufficient to give the two dimensional electron trajectory within this plane to fully describe the electron's classical dynamics. From here on we will adopt this simplified visualization scheme in the course of this work. Please note that by employing Eq. (2.11) in this work, we are going to neglect any influence of the electron's self-field on its own dynamics. Such radiation reaction, however, was shown to possibly significantly influence the dynamics [4], and is a matter of intense scientific discourse [5–7].

Solving Eq. (2.11) results in a given trajectory for the electron $r^\mu(t) = (t, \boldsymbol{r}(t))$ with a velocity $u^\mu(t) = \partial_s r^\mu(t) = \gamma(t)\beta^\mu(t)$, where $\gamma(t) = \varepsilon(t)/m$ is the electron's relativistic factor, s its proper time and $\beta^\mu(t) = (1, \boldsymbol{\beta}(t))$. From this quantity one defines the classical charge current of an electron $j^\mu(x) = e\beta^\mu(t)\delta(\boldsymbol{x}-\boldsymbol{r}(t))$. The electron's emission can then be computed by means of the Lienard Wiechert potentials [1, 8]. According to this formalism the energy emitted by an accelerated point-like electron into the observation direction $\boldsymbol{n}$ per unit frequency and solid angle element is given by

$$\frac{dE}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2}\left|\int_{-\infty}^{\infty} dt\, \boldsymbol{n} \times (\boldsymbol{n} \times \beta(t))\; \mathrm{e}^{\mathrm{i}\omega\,(t-\boldsymbol{nr}(t))}\right|^2 . \tag{2.12}$$

The integrand is simplified through

$$\boldsymbol{n} \times (\boldsymbol{n} \times \beta(t)) = \boldsymbol{n}\,(\boldsymbol{n}\beta(t)) - \beta(t). \tag{2.13}$$

To simplify the term containing the factor $\boldsymbol{n}\beta(t)$ one writes

$$\int_{-\infty}^{\infty} dt\;(1-\boldsymbol{n}\beta(t))\;\mathrm{e}^{\mathrm{i}\omega\,(t-\boldsymbol{nr}(t))} = \int_{-\infty}^{\infty} dt\left(\frac{d}{dt}\frac{\mathrm{e}^{\mathrm{i}\omega\int_{-\infty}^{t}dt'\;(1-\boldsymbol{n}\beta(t'))}}{\mathrm{i}\omega}\right) = 0. \tag{2.14}$$

The transformed integral has to vanish since its integrand is the total differential of an expression, which does not contribute at $t = \pm\infty$. Hence, for Eq. (2.12) one obtains

$$\frac{dE}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2}\left|\int_{-\infty}^{\infty} dt\;(\boldsymbol{n}-\beta(t))\mathrm{e}^{\mathrm{i}\omega\,(t-\boldsymbol{nr}(t))}\right|^2 . \tag{2.15}$$

Taking the square in this expression results in

$$\begin{aligned}\frac{dE}{d\omega d\Omega} &= \frac{e^2\omega^2}{4\pi^2}\int_{-\infty}^{\infty} dtdt'\;(1-\boldsymbol{n}\boldsymbol{\beta}(t)-\boldsymbol{n}\boldsymbol{\beta}(t')+\boldsymbol{\beta}(t)\boldsymbol{\beta}(t'))\\ &\qquad\times\,\mathrm{e}^{\mathrm{i}\omega\,(t-\boldsymbol{nr}(t))}\mathrm{e}^{\mathrm{i}\omega\,(t'-\boldsymbol{nr}(t'))}\\ &= \frac{e^2\omega^2}{4\pi^2}\int_{-\infty}^{\infty} dtdt'\;(\boldsymbol{\beta}(t)\boldsymbol{\beta}(t')-1)\mathrm{e}^{\mathrm{i}\omega\,(t-\boldsymbol{nr}(t))}\mathrm{e}^{\mathrm{i}\omega\,(t'-\boldsymbol{nr}(t'))},\end{aligned} \tag{2.16}$$

where, for obtaining the second line, Eq. (2.14) is used. Now, Eq. (2.16) is easily written in a covariant form, recalling the four velocity $u^\mu(t) = p^\mu(t)/m$. Plugging this into Eq. (2.16) we find the covariant expression [9]

$$\frac{dE}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2}\left|\int_{-\infty}^{\infty} dt\;\frac{p^\mu(t)}{\varepsilon(t)}\mathrm{e}^{\mathrm{i}kr(t)}\right|^2 , \tag{2.17}$$

where we defined $k^\mu = \omega(1, \boldsymbol{n})$ in the exponential. Please note that the appearance of $\varepsilon(t)$ does not make this expression change under Lorentz transformation since it is integrated over the time t, which compensates the transformation of the energy.

We finally wish to discuss some qualitative features of the angular and frequency distribution that is expected in the radiation from an electron scattered from a laser field (see [1, 8]). The former distribution is dominated by the properties of the Lorentz transformations, as we wish to briefly sketch. Consider an electron which propagates with the instantaneous velocity four vector $\beta^\mu = (1, 0, 0, \beta^z)$, observed in a laboratory frame according to Fig. 2.1. In a reference frame, copropagating with the electron at β^z, the radiation emitted by the electron will feature a wave vector $k'^\mu = \omega'(1, \sin(\vartheta'), 0, \cos(\vartheta'))$, where ϑ' is the angle between $\boldsymbol{k}'$ and the negative z-axis measured in the copropagating reference frame. Additionally the perpendicular axes are chosen such that the emission is confined to the x-z-plane. Transforming this four vector into the laboratory frame, where the emitted radiation is observed, one obtains

$$k^\mu = \omega \begin{pmatrix} 1 \\ \sin(\vartheta) \\ 0 \\ \cos(\vartheta) \end{pmatrix} = \frac{\omega'}{m} \begin{pmatrix} \varepsilon(1 + \beta^z \cos(\vartheta')) \\ m \sin(\vartheta') \\ 0 \\ \varepsilon(\beta^z + \cos(\vartheta')) \end{pmatrix}, \tag{2.18}$$

where ϵ is the electron's energy, measured in the laboratory frame. The angle between the wave vector and the electron's propagation direction in the laboratory frame thus becomes

$$\sin(\vartheta) = \frac{m}{\varepsilon} \frac{\sin(\vartheta')}{(1 + \beta^z \cos(\vartheta'))} \sim \frac{m}{\varepsilon} \tag{2.19}$$

One concludes that the emission of an electron in highly relativistic motion ($\varepsilon \gg m$) is confined to a narrow cone of opening angle $\Delta\vartheta \sim {}^m/_\varepsilon$ around its velocity vector at the time of emission. Any observer detecting the emission from the electron will then detect only a short burst of radiation, whenever the electron's velocity points into his observation direction. Over such short times the change of the electron's propagation direction can be approximated by a circular orbit with the instantaneous radius of curvature ρ. The time in the highly relativistic electron's rest frame, over which its emission cone will accordingly illuminate a detector, then scales as

$$\Delta s \sim \frac{m\rho}{\varepsilon}. \tag{2.20}$$

The transformation of this illumination time to the observation time t, measured in the laboratory in which the radiation is observed, requires the Lorentz transformation factor

$$\frac{dt}{ds} = 1 - \boldsymbol{n\beta} \sim \left(\frac{m}{\varepsilon}\right)^2, \tag{2.21}$$

where $\boldsymbol{n}$ again is a unit vector, pointing along the direction of observation. A laboratory detector will thus detect a radiation flash of the approximate duration

$$\Delta t \sim \rho \left(\frac{m}{\varepsilon}\right)^3, \tag{2.22}$$

whenever the electron points in its direction. According to the general theory of Fourier transformation, a field flash of such short duration has to contain frequencies up to

$$\omega_c \sim \frac{1}{\rho} \left(\frac{\varepsilon}{m}\right)^3. \tag{2.23}$$

One thus expects the radiation of an electron scattered by an intense laser pulse to scale as the cube of its instantaneous energy.

2.1.1 Electron Radiation in a Plane Wave

The classical equation of motion was solved analytically for the momentum in the case of an electron moving in a plane wave laser field exactly by solving the Hamilton-Jacobi equation [10–12] as well as by direct integration [13, 14]. In the former case the phase dependent position of the electron is found directly as derivative of the action (or alternatively principal function) of an electron entering a plane wave field $A_L^\mu(\eta)$ with a momentum $p_i = p(\eta \to -\infty)$ [1]

$$S_{p_i}(x) = -p_i x - \int_0^\eta \mathrm{d}\phi \left(e\frac{p_i A_L(\phi)}{p_i k_L} - \frac{e^2 A_L^2(\phi)}{2(p_i k_L)^2} \right). \tag{2.24}$$

In the latter approach it is obtained by another integration of the equation of motion (2.11) and $u^\mu(t) = {dr^\mu(t)}/{ds}$. An advantage of the latter computation is that its results are given in an explicitly covariant manner which is why we sketch this way of solving the classical equations of motion. As a first step we note that for a plane wave electromagnetic field the field strength tensor satisfies $F_L^{\mu\nu}(\eta) = k_L^\mu \partial_\eta A^\nu(\eta) - k_L^\nu \partial_\eta A^\mu(\eta)$. From the classical equation of motion (2.11) one concludes

$$\frac{dp^\mu(s)}{ds} = \frac{e}{m} \left(k_L^\mu (A_L p(s)) - A_L^\mu (p(s) k_L) \right) \partial_\eta \psi_{\mathcal{A}}(\eta). \tag{2.25}$$

Multiplying Eq. (2.25) with the constant plane wave's wave vector we conclude

$$\frac{d(p(\eta) k_L)}{ds} = 0, \tag{2.26}$$

where we utilized the gauge condition $k_L A_L = 0$ and whence we conclude that $p(s)k_L = p_i k_L$ is a constant of motion and can thus be set to its initial value. Furthermore, for the laser's invariant phase one finds

$$\frac{d\eta}{ds} = k_L^\mu \frac{dx_\mu}{ds} = \frac{p_i k_L}{m}. \tag{2.27}$$

It is then advantageous to parameterize the electron's kinetic momentum by η and one can change the variable in Eq. (2.25) according to

$$\begin{aligned} \frac{dp^\mu(\eta)}{d\eta} &= \frac{ds}{d\eta}\frac{dp^\mu(s)}{ds} \\ &= \frac{e}{p_i k_L}\left(k_L^\mu(A_L p(\eta)) - A_L^\mu(p_i k_L)\right)\partial_\eta \psi_{\mathcal{A}}(\eta). \end{aligned} \tag{2.28}$$

Multiplying Eq. (2.28) with the constant plane wave's amplitude vector A_L^μ we find

$$\begin{aligned} \frac{d(A_L p(\eta))}{d\eta} &= -eA_L^\mu \partial_\eta A_L^\mu(\eta) \\ \Rightarrow (A_L p(\eta)) &= (A_L p_i) - eA_L^2 \psi_{\mathcal{A}}(\eta). \end{aligned} \tag{2.29}$$

Inserting now Eq. (2.26) and (2.29) into Eq. (2.28) we find

$$\frac{dp^\mu(\eta)}{d\eta} = e\left(k_L^\mu \frac{(A_L p_i) - eA_L^2\psi_{\mathcal{A}}(\eta)}{p_i k_L} - A_L^\mu\right)\partial_\eta \psi_{\mathcal{A}}(\eta), \tag{2.30}$$

which is readily integrated to give the covariant form of an electron's momentum in the presence of a plane wave laser field as

$$p^\mu(\eta) = p_i^\mu - eA^\mu(\eta) + k_L^\mu\left[e\frac{p_i A_L(\eta)}{p_i k_L} - \frac{e^2 A_L^2(\eta)}{2(p_i k_L)}\right] \tag{2.31a}$$

$$r^\mu(\eta) = r_i + \int_{-\infty}^{\eta} \mathrm{d}\eta' \, \frac{p_i^\mu - eA^\mu(\eta')}{p_i k_L} + k^\mu \int_{-\infty}^{\eta} \mathrm{d}\eta' \, \frac{e(A_L(\eta')p_i) - \frac{e^2 A_L^2}{2}}{(p_i k_L)^2}. \tag{2.31b}$$

The latter line is obtained by a direct integration of the former analogous to Eq. (2.28). The results of [11, 12], in contrast to Eqs. (2.31a) and (2.31b), are given in a chosen reference frame but their generalization to a covariant form is straightforward and the results agree with Eqs. (2.31a) and (2.31b), as it must be.

To find now the classical emission formula for an electron moving in a plane wave field, these solutions are inserted into Eq. (2.17). To simplify the exponential phase, one writes it in the form

$$k_1 r(t) = \int_{-\infty}^{t} dt' k_1^\mu \frac{u_\mu}{\gamma(t')} = \int_{-\infty}^{\eta} d\eta' \frac{dt'}{d\eta'} k_1^\mu \frac{p_\mu}{m\gamma(t')}. \tag{2.32}$$

Now one may use the relation between the invariant phase of the incident laser pulse and the laboratory time

$$\frac{d\eta}{dt} = k_L^\mu \frac{dr_\mu(t)}{dt} = k_L^\mu \frac{p_\mu(t)}{m\gamma(t)}. \tag{2.33}$$

The numerator in this expression is given by the constant of motion $k_L^+ p_i^-$. Inserting furthermore Eqs. (2.31a) and (2.31b) we find

$$\begin{aligned} k_1 r(t) &= \frac{k_1^\mu}{p_i k_L} \int_{-\infty}^{\eta} d\eta' p_\mu(\eta') \\ &= \int_{-\infty}^{\eta} d\eta' \frac{k_1 p_i}{p_i k_L} - e\frac{k_1 A_L}{p_i k_L} \psi_{\mathcal{A}}(\eta') - \frac{e^2 A_L^2 (k_1 k_L)}{2(p_i k_L)^2} \psi_{\mathcal{A}}^2(\eta'). \end{aligned} \tag{2.34}$$

Finally, since it is customary to formulate all quantities in the interaction with a plane wave as functions of the wave's invariant phase η or equally the light-cone coordinate x^-, by virtue of Eq. (2.33) we reformulate Eq. (2.17) to yield

$$\frac{dE}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 p_i^{-2}} \left| \int_{-\infty}^{\infty} dx^- p^\mu(x^-) \mathrm{e}^{\mathrm{i}k_1 r(x^-)} \right|^2. \tag{2.35}$$

2.1.2 *Interaction with a Monochromatic Plane Laser Wave*

If the temporal duration of a laser pulse τ_L is much larger than its cycle period ω_L^{-1}, its spectrum will be very narrowly confined around ω_L. It is then a good approximation to model the spectrum as monochromatic, i.e. by a δ-spike in frequency space. The temporal structure of this type of fields is strictly periodic and allows for significant simplifications in the calculations. For instance, the trajectory of an electron inside a monochromatic plane wave field, reduces to a simple form. Observing it in a reference frame, in which the electron is on average at rest, it moves on a trajectory with a well-known *figure-8* shape [11, 12]. Due to this strictly monochromatic motion, the computation of the energy spectra can be largely simplified and central features of a QED computation of the scattering from a monochromatic laser field are already found in this classical analysis.

The following discussion largely follows [15] and summarizes the results of that work. For reasons of convenience and without loss of generality, however, we will analyze the interaction in a reference frame where the laser wave propagates along the positive z-direction ($\boldsymbol{k}_L = \omega_L(0, 0, 1)$) and the electron's energy and velocity in the absence of the field reduce to the free values ε_i and $\boldsymbol{\beta}_i = \beta_i(0, 0, -1)$, respectively. It was shown that in such a reference frame for a monochromatic laser field of the form $A_L^\mu(\eta) = A_L \epsilon_L^\mu \cos(\eta)$ Eq. (2.17) can be written as

$$\frac{dE^2}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2}\left[(1-n_x^2)K_x^2 - 2n_x n_y K_x K_y + (1-n_z^2)K_z^2\right] \tag{2.36}$$

where n_i are the components of the vector $\boldsymbol{n}$ pointing in the observation direction and it was defined the vector function

$$\boldsymbol{K} = \int_{-\infty}^{\infty} d\eta \frac{d\boldsymbol{r}(\eta)}{d\eta} \exp\left[\mathrm{i}\frac{\omega}{\omega_L}\left(\eta + \omega_L(z - \boldsymbol{n}\boldsymbol{r}(\eta))\right)\right]. \tag{2.37}$$

For the assumed monochromatic potential the trajectory, given by the spatial components of Eq. (2.31b), is found to be

$$\boldsymbol{r}(\eta) = \boldsymbol{a}\eta + \boldsymbol{b}\sin(\eta) + \boldsymbol{c}\sin(2\eta), \tag{2.38}$$

where the following constant vectors are defined

$$\boldsymbol{a} = \frac{1}{\omega_L}\left[\left(\frac{m\xi}{2\varepsilon_i}\right)^2 \frac{\boldsymbol{n}_L}{(1-\boldsymbol{n}_L\boldsymbol{\beta}_i)^2} + \frac{\boldsymbol{\beta}_i}{(1-\boldsymbol{n}_L\boldsymbol{\beta}_i)}\right] \tag{2.39a}$$

$$\boldsymbol{b} = \boldsymbol{\epsilon}_L \frac{\frac{m\xi}{\varepsilon_i}}{\omega_L(1-\boldsymbol{n}_L\boldsymbol{\beta}_i)} \tag{2.39b}$$

$$\boldsymbol{c} = \boldsymbol{k}_L \frac{\left(\frac{m\xi}{2\varepsilon_i}\right)^2}{2(\omega_L(1-\boldsymbol{n}_L\boldsymbol{\beta}_i))^2}. \tag{2.39c}$$

The exponential factors in Eq. (2.37) containing trigonometric functions can then be transformed to Bessel functions of integer order J_n via their generating function

$$\mathrm{e}^{\mathrm{i}u\sin(\eta)} = \sum_{n=-\infty}^{\infty} J_n(u)\mathrm{e}^{\mathrm{i}n\eta}. \tag{2.40}$$

This relation has been widely employed in problems involving intense monochromatic laser fields [11] and also takes an important role in respective QED analyses to assign photon numbers to an actually unquantized laser field [16].

The resulting expression of the integrals $\boldsymbol{K}$ then allows to perform the integration in η, yielding δ-functions of the form

$$\boldsymbol{K} \propto \sum_{n=-\infty}^{\infty} \ldots \delta\left(\omega - n\frac{\omega_L}{1 - \boldsymbol{a}(\boldsymbol{n} - \boldsymbol{n}_L)}\right). \tag{2.41}$$

The explicit form of the integrals is rather involved [15] but is not needed here. Taking the square of this δ-function, as is required for evaluating Eq. (2.36), by usual methods employed in S-Matrix calculations one obtains an expression for the overall emitted power in the form

$$\frac{dP}{d\Omega} \propto \sum_{n=0}^{\infty} \frac{dP^{(n)}}{d\Omega}. \tag{2.42}$$

The n$^{\text{th}}$ term in this series represents the power emitted into the n$^{\text{th}}$ harmonic, whose frequencies according to the conservation law of Eq. (2.41) are equidistantly distributed and given by

$$\omega^n = n\frac{\omega_L}{1 - \omega_L \boldsymbol{a}(\boldsymbol{n} - \boldsymbol{n}_L)}. \tag{2.43}$$

This result is not equivalent to Eq. (1.8), because of the radiation pressure of the laser field, which may reduce the Doppler shift. In fact, we note that in the limit $\xi \rightarrow 0$ Eq. (2.43) for $n = 1$ goes over to the expression

$$\omega = \omega_L \frac{(1 - \boldsymbol{n}_L\boldsymbol{\beta}_i)}{(1 - \boldsymbol{n}_L\boldsymbol{\beta}_i) - \boldsymbol{\beta}_i(\boldsymbol{n} - \boldsymbol{n}_L)} = \omega_L \frac{1 - \boldsymbol{n}_L\boldsymbol{\beta}_i}{1 - \boldsymbol{n}\boldsymbol{\beta}_i}, \tag{2.44}$$

which in fact is equivalent to the ordinary Doppler shift of Eq. (1.8). This result of the scattered harmonic frequencies alongside the utilization of the generating function of the Bessel functions according to Eq. (2.40) will enable us to compare the presented classical calculations to the QED analyses for monochromatic laser waves (see Eq. (2.40)).

2.1.3 Interaction with a Focused Laser

The assumption of plane wave, though often justified to a good extent, is never complete [17]. This is easily seen by recalling that any function $f(t - x^{\|})$, where $x^{\|}$ is the propagation direction of the laser, is a plane wave solution of the wave equation Eq. (2.3). This solution, however, does not feature a dependence on the two transversal coordinates $\boldsymbol{x}^{\perp}$. Consequently, at every phase value $\eta = k_L x$ the field is constant in a plane stretching infinitely in the $\boldsymbol{x}^{\perp}$-plane (hence the label *plane wave*).

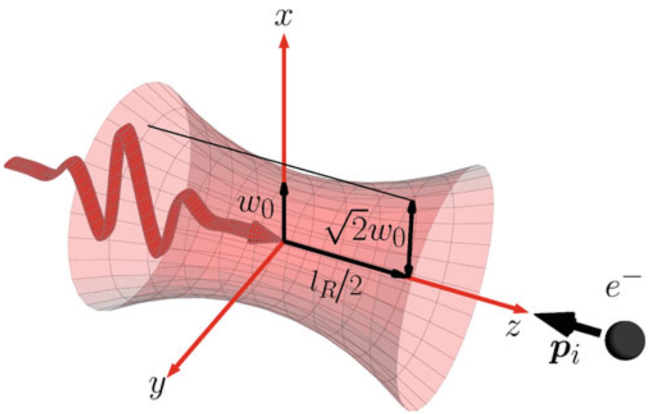

Fig. 2.3 Gaussian beam focus in a coordinate system according to Fig. 2.1

Though this is of course, rigorously speaking, unphysical, in many cases it is still a valid approximation, as we will see. To treat a laser beam consistently, however, it would much rather be necessary to include the natural transversal extent of the beam. The standard formalism of describing the field distribution of a monochromatic laser wave focused to a perpendicular spot size w_0 (see Fig. 2.3) is due to Davis [18]. Once analytic expressions for the laser fields are found, it is a straightforward task in classical electrodynamics to obtain the electron's trajectory from Eq. (2.11) and thus via Eq. (2.17) its emission pattern. We wish to briefly sketch the concept of this treatment. Assume the laser pulse's vector potential to be given by

$$A_L^\mu(x) = A_L \epsilon_L^\mu \Psi_L(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i}k_L x} + \varepsilon_\phi^\mu \phi(x). \tag{2.45}$$

In this expression the space dependent factor $\Psi_L(\boldsymbol{r})$ is introduced to describe the spatial focusing of the laser pulse, $\epsilon_L^\mu = (0, \boldsymbol{\epsilon}_L)$ is the well known polarization vector of the laser pulse and $\varepsilon_\phi = (1, 0, 0, 0)$ denotes that the second term introduces a nontrivial scalar potential into $A_L^\mu(x)$. One can no longer assume the vector potential to be purely spatially polarized, since the Lorenz gauge condition could not be fulfilled in that case. To circumvent this difficulty it is customary to incorporate a nonzero scalar potential, as indicated in Eq. (2.45). The introduced scalar potential is connected with the spatial components of $A_L^\mu(x)$ via the Lorenz gauge condition

$$\partial_t \phi(x) = \nabla \Big(A_L \boldsymbol{\epsilon}_L \Psi_L(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i}k_L x} \Big). \tag{2.46}$$

Thus, the four potential still obeys the Lorenz gauge condition $\partial_\mu A_L^\mu(x) = 0$. Due to Eq. (2.46) the problem is fully determined if we find a solution of the wave equation for the spatial vector potential $\boldsymbol{A}_L(x)$. We will thus restrict the following discussion to this quantity. The electric and magnetic fields, derived from Eq. (2.1), then of course also are no longer linearly polarized, but will exhibit longitudinal field components, typical of a focused beam. Inserting Eq. (2.45), the Lorenz gauge wave equation $\Box \boldsymbol{A}_L(x) = 0$ reduces to

$$\nabla^2 \Psi_L(\boldsymbol{x}) - 2\mathrm{i}\omega_L \frac{\partial \Psi_L(\boldsymbol{x})}{\partial x^{\parallel}} = 0. \tag{2.47}$$

Solving Eq. (2.47) exactly for $\Psi_L(\boldsymbol{x})$ would mean to find the class of electromagnetic potentials, that vary in time like $\mathrm{e}^{-\mathrm{i}\omega_L t}$ and fulfill the wave Eq. (2.3). Unfortunately such a complete solution has not been reported up to today, whence a perturbative ansatz for Eq. (2.47) is called for. Such an ansatz is found from the consideration that a laser beam cannot be focussed to spot sizes w_0 smaller than its central wavelength $\lambda_L = 2\pi/\omega_L < w_0$. A dimensionless and always small parameter, characterizing the focusing of the laser beam, is then given by

$$s_L = \frac{1}{\omega_L w_0} = \frac{\lambda_L}{2\pi w_0}. \tag{2.48}$$

In addition to this perpendicular confinement, a laser focus also exhibits a characteristic longitudinal spreading length $l_R = w_0/s_L = \omega_L w_0^2$, often called *Rayleigh length*. At a distance $z = l_R/2$ from the focal plane the laser's intensity has dropped to half its value at $z = 0$, whence l_R is often referred to as the longitudinal extent of the focal spot. It is then useful to transform Eq. (2.47) to the dimensionless variables $\boldsymbol{\rho} = (\rho^{\parallel}, \rho_1^{\perp}, \rho_2^{\perp})$ with $\rho^{\parallel} = x^{\parallel}/l_R$ and $\boldsymbol{\rho}^{\perp} = \boldsymbol{x}^{\perp}/w_0$, resulting in

$$\left(\frac{\partial^2}{\partial^2 \rho_1^{\perp}} + \frac{\partial^2}{\partial^2 \rho_2^{\perp}}\right)\Psi_L(\boldsymbol{\rho}) + s_L^2 \frac{\partial^2 \Psi_L}{\partial {\rho^{\parallel}}^2} - 2\mathrm{i}\frac{\partial \Psi_L(\boldsymbol{\rho})}{\partial \rho^{\parallel}} = 0. \tag{2.49}$$

From this equation we can guess the correct way of a perturbative ansatz for the solution. In fact, assuming that the focus parameter would vanish ($s_L = 0$), Eq. (2.49) is solved by the function [18]

$$\Psi_L^0(\boldsymbol{\rho}) = \left(\frac{w_0^2}{w^2(\rho^{\parallel})} + \mathrm{i}\frac{l_R}{2R(\rho^{\parallel})}\right)\exp\left[-\left(\frac{w_0^2}{w^2(\rho^{\parallel})} + \mathrm{i}\frac{l_R}{2R(\rho^{\parallel})}\right)\boldsymbol{\rho}_{\perp}^2\right], \tag{2.50}$$

where the following definitions are used

$$w(\rho^{\parallel}) = w_0\sqrt{1 + 4{\rho^{\parallel}}^2} \tag{2.51a}$$

$$R(\rho^{\parallel}) = l_R \rho^{\parallel}\left(1 + \frac{1}{4{\rho^{\parallel}}^2}\right). \tag{2.51b}$$

From Eq. (2.50) we read off that $w(\rho^{\parallel})$ gives the perpendicular extent of the laser focus in dependence of the longitudinal position (note that at $|\boldsymbol{\rho}^{\perp}| = w(\rho^{\parallel})$ the function $\Psi_L^0(\boldsymbol{\rho})$ is always damped by at least a factor e^{-1}). The factor $R(\rho^{\parallel})$ gives the radius of curvature of the non-plane wavefront going through the laser-axis at $x^{\parallel} = l_R \rho^{\parallel}$. The meaning of the quantity l_R is also apparent. Its half (corresponding to $\rho^{\parallel} = 1/2$) indicates the distance from the origin $\rho^{\parallel} = 0$ along the laser's propagation direction, after which the focus' perpendicular extent has increased to $w(\rho^{\parallel} = 1/2) = \sqrt{2} w_0$, as indicated in Fig. 2.3. In this sense l_R can be interpreted as a measure of the overall longitudinal extent of the laser focus. Having now found a lowest order perturbative solution of Eq. (2.49) we can readily guess the proper form of a perturbative expansion of the complete focusing function to be [18]

$$\Psi_L(\boldsymbol{\rho}) = \sum_{n=0}^{\infty} s_L^{2n} \Psi_L^{2n}(\boldsymbol{\rho}). \tag{2.52}$$

Inserting this ansatz into Eq. (2.49) we find as determining equation for the second term in this series

$$\left(\frac{\partial^2}{\partial^2 \rho_1^{\perp}} + \frac{\partial^2}{\partial^2 \rho_2^{\perp}} - 2\mathrm{i} \frac{\partial}{\partial \rho^{\parallel}} \right) \Psi_L^2(\boldsymbol{\rho}) = -\frac{\partial^2 \Psi_L^0(\boldsymbol{\rho})}{\partial {\rho^{\parallel}}^2}, \tag{2.53}$$

and all higher orders accordingly. From the thusly found vector potential Eq. (2.45) it is simple to derive the electromagnetic field strength tensor $F_L^{\mu\nu}(x)$ and hence, via e.g. numerical integration of Eq. (2.11), an electron's trajectory in the focused laser beam (compare Fig. 2.2). The procedure just outlined, however, is not applicable for pulsed laser fields, since it is found assuming a time variation of the form $\mathrm{e}^{-\mathrm{i}\omega_L t}$ for all times $t \in [-\infty, \infty]$ with a fixed laser frequency ω_L. To describe a focused laser pulse the ansatz for the spatial vector potential of Eq. (2.45) can be modified according to [17]

$$\boldsymbol{A}_L(x) = g(\eta) \boldsymbol{\epsilon}_L \Psi_L(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i} k_L x}, \tag{2.54}$$

where the function $g(\eta)$ is used to introduce an arbitrary temporal shaping of the laser pulse, depending only on the invariant phase $\eta = \omega_L(t - x^{\parallel})$. The determining equation for the focusing function then turns into

$$\nabla^2 \Psi_L(\boldsymbol{x}) - 2\mathrm{i}\omega_L \frac{\partial \Psi_L(\boldsymbol{x})}{\partial x^{\parallel}} \left(1 - \mathrm{i} \frac{\partial_\eta g(\eta)}{g(\eta)} \right) = 0. \tag{2.55}$$

It is then usual to assume, that the envelope function's derivative is small compared to its function value $\partial_\eta g(\eta) \ll g(\eta)$, which is a valid approximation for a long laser pulse. This is also called *slowly varying envelope approximation* [19, 20]. For the case of few-cycle laser pulses, however, the approximation of a slowly varying envelope is not a good one. In fact, for few-cycle laser pulses, which are well described by the model Eq. (2.6) we find the assumedly small fraction to be $\partial_\eta g(\eta)/g(\eta) = -(2\cot(\eta/2n_C))/n_C \propto n_C^{-1}$. For $n_C \sim 1$, corresponding to a few-cycle laser pulse, this quantity is non-negligible, and for $\eta \to 2\pi n_C$ it even diverges. Consequently, for few-cycle pulses a new approximation scheme is called for, to describe a focused laser pulse.

To properly account for spatial focusing of a few-cycle laser pulse we propose a scheme which is based on the idea to impose a spatial focusing on a previously plane wave laser potential. Before doing so, however, we suggest to decompose the plane wave potential into its Fourier components and then consider the focusing of each separate (monochromatic) mode to the same focal spot size. In spirit of this idea, we note that any shape function which vanishes outside of a finite interval $[0, \tau_L]$ has a finite Fourier sum according to

$$\psi_{\mathcal{A}}(\eta) = \sum_{n=-N}^{N} c_n \mathrm{e}^{\mathrm{i}n\,\omega_0(t-x^{\parallel})} \tag{2.56}$$

with the fundamental frequency $\omega_0 = {}^{2\pi\omega_L}/_{\tau_L}$. The factors c_n are the usual Fourier series coefficients

$$c_n = \frac{1}{\tau_L}\int_0^{\tau_L} d\eta\, \psi_{\mathcal{A}}(\eta)\mathrm{e}^{-\mathrm{i}n\omega_0\eta/\omega_L}. \tag{2.57}$$

From here on we will outline the proposed method for the specific model of a few-cycle pulse Eq. (2.6) and we note that in this case it is $\tau_L = 2\pi n_C$ and we thus find the simple relation $\omega_0 = {}^{\omega_L}/_{n_C}$. Employing Eq. (2.56) instead of Eq. (2.54), the ansatz for the vector potential becomes

$$\boldsymbol{A}_L(x) = \boldsymbol{\epsilon}_L \sum_{n=-N}^{N} c_n \Psi_{L,n}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}n\,\omega_0(t-x^{\parallel})}. \tag{2.58}$$

Plugging this expression into the wave equation we arrive at a differential equation for the determination of the $\Psi_{L,n}(\boldsymbol{x})$ analogous to Eq. (2.47)

$$\sum_{n=-N}^{N} c_n \left(\nabla^2\Psi_{L,n}(\boldsymbol{x}) - 2\mathrm{i}\left(n\frac{\omega_L}{n_C}\right)\frac{\partial\Psi_{L,n}(\boldsymbol{x})}{\partial x^{\parallel}}\right) = 0. \tag{2.59}$$

Since the separate frequency components represented by the single terms of this series do not mix, a solution of the above equation can be readily written down in terms of solutions of Eq. (2.47). We accordingly find that the focusing functions for the Fourier modes of the vector potential Eq. (2.58) are given by

$$\Psi^0_{L,n}(\boldsymbol{\rho_n}) = \left(\frac{w_0^2}{w_n^2(\rho_n^{\parallel})} + \mathrm{i}\frac{l_{R,n}}{2R_n(\rho_n^{\parallel})}\right)\exp\left[-\left(\frac{w_0^2}{w_n^2(\rho_n^{\parallel})} + \mathrm{i}\frac{l_{R,n}}{2R_n(\rho_n^{\parallel})}\right)\boldsymbol{\rho}_{\perp}^2\right]. \tag{2.60}$$

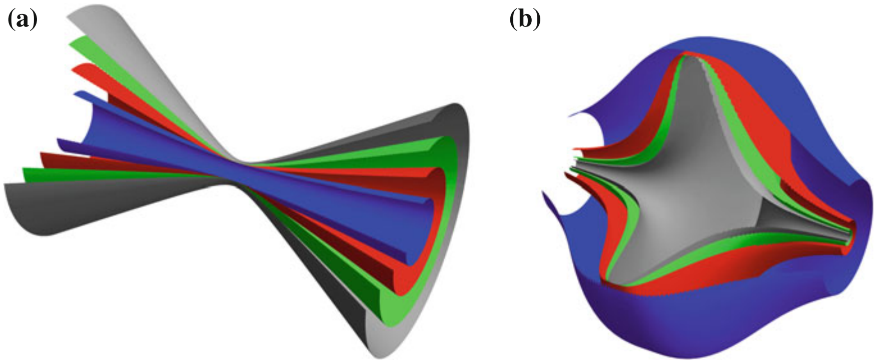

Fig. 2.4 Fourier frequency component foci of a pulse derived from inserting Eq. (2.6) into Eq. (2.56) for $n_C = 2$. The only non-vanishing Fourier components are $n = 1$ (*blue*), $n = 2$ (*red*), $n = 3$ (*green*), $n = 4$ (*gray*). **a** Surfaces of constant $w(\rho^{\parallel})$, all coinciding with w_0 at $\rho^{\parallel} = 0$. **b** Surfaces of constant intensity, all featuring the same peak intensity

It is now essential that for all n the same w_0 enters the equation, corresponding to a focusing to the same spot size. The definitions of the remaining variables are analogous to the presented monochromatic analysis

$$\begin{aligned} l_{R,n} &= n\frac{\omega_L}{n_C}w_0^2 \\ \rho_n^{\parallel} &= \frac{x^{\parallel}}{l_{R,n}} \\ w_n(\rho_n^{\parallel}) &= w_0\sqrt{1+4{\rho_n^{\parallel}}^2} \\ R_n(\rho_n^{\parallel}) &= l_{R,n}\rho_n^{\parallel}\left(1+\frac{1}{4{\rho_n^{\parallel}}^2}\right). \end{aligned}$$

The computation of higher order terms is carried out according to Eq. (2.53). Essentially the found behaviour is explained by the observation that the focusing to a Gaussian beam is a linear operation on the laser field and doesn't mix its frequency components. Accordingly one can picture the focused laser beam as a superposition of frequency components focused to the same focal spot as sketched in Fig. 2.4. We wish to point out that Eqs. (2.58) and (2.60) are exact in the temporal focusing of the plane wave field. There was no need for a slowly varying envelope approximation $\partial_\eta g(\eta)/g(\eta) \ll 1$. In such a short laser pulse, focussed to small spot sizes, the classical electron trajectory may be significantly changed, as can be traced in Fig. 2.5, where we show the trajectory only within the $\boldsymbol{\epsilon}_L$-$\boldsymbol{k}_L$ plane, as was motivated below Fig. 2.2. From that figure we also conclude that the plane wave approximation gives useful results for the trajectory already for a focusing as small as $w_0 \geq 2\lambda_L$.

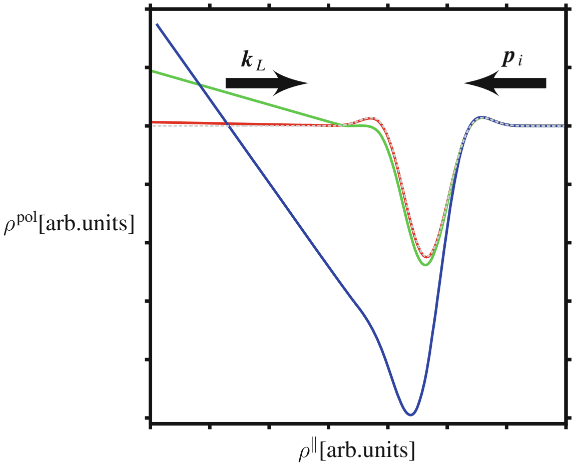

Fig. 2.5 Trajectories of an electron colliding head on with a laser pulse with shape derived from Eq. (2.6) with $n_C = 2$ and central wavelength $\lambda_L = 800\,\mathrm{nm}$ for a focusing of $w_0 = 2\lambda_L$ (*red*), $w_0 = \lambda_L$ (*green*) and $w_0 = \lambda_L/2$ (*blue*, corresponds to a focal spot size of λ_L). For comparison the plane wave trajectory is shown in *gray dashes*

2.2 Quantum Electrodynamics

It is in order to shortly review the fundamental concepts of QED. The basic concept of this highly successful theory is, that all electrically charged elementary particles, as well as the gauge particles mediating the electromagnetic interaction between them, have to be interpreted as excitations of quantum fields. The fundamental quantity of the accordingly required field theory for the coupling of a charged spinor field $\Psi(x)$ with an electromagnetic potential $A^\mu(x)$ is the Lagrangian of quantum electrodynamics [21, 22]. The Lagrangian is not unique, since due to gauge invariance several expressions will lead to equivalent equations of motion of the described fields. One may take advantage of this freedom and employ an expression of the QED Lagrangian, which circumvents several problems such as the vanishing of the conjugate momentum of the photon field [23, 24]. To enable a canonical quantization scheme for the photon field, we will employ the so-called Fermi-Lagrangian [25] for its description, turning the QED Lagrangian into the expression

$$\mathcal{L}_{\mathrm{QED}} = \overline{\Psi}(x)\left(\mathrm{i}\not{\mathcal{D}} - m\right)\Psi(x) - \frac{1}{8\pi}\partial_\nu A_\mu(x)\partial^\mu A^\nu(x). \tag{2.61}$$

Here $\mathcal{D}^\mu(x) = \partial^\mu + \mathrm{i}eA^\mu(x)$ is the so-called gauge covariant derivative [22]. The equations of motion for the involved fields $\Phi(x) \in \left[\Psi(x), \overline{\Psi}(x), A^\mu(x)\right]$ are obtained as extremal points of the variation of the action with respect to the corresponding fields, as is usual in Lagrangian field theory, resulting in the Euler-Lagrange equations

$$\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi(x))}\right) - \frac{\partial\mathcal{L}}{\partial\phi(x)} = 0. \tag{2.62}$$

This procedure results in the Dirac equation and its conjugate for the spinor field, its conjugate and the photon field, respectively [26]

$$(\mathrm{i}\not{D}(x) - m)\Psi(x) = 0 \tag{2.63a}$$

$$\overline{\Psi}(x)(\mathrm{i}\not{D}(x) + m) = 0 \tag{2.63b}$$

$$\Box A^{\mu}(x) = 4\pi e j^{\mu}_{\text{Dirac}}(x) = 4\pi e\overline{\Psi}(x)\gamma^{\mu}\Psi(x). \tag{2.63c}$$

Given in this form the fields $\Psi(x), \overline{\Psi}(x), A^{\mu}(x)$ are classical unquantized quantities. Since one is now interested in a relativistic theory of interacting particles, one naturally has to consider a multi-particle theory, since particles may be created or annihilated taking or providing the amount of energy corresponding to their rest masses, respectively [22]. The fields will then be represented by operators, acting on a space of physical quantum states, representing the number of existing particles. To quantize the theory one introduces conjugate momenta for the fields according to

$$\pi_{\Psi}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_t\Psi(x))} = \mathrm{i}\overline{\Psi}(x)\gamma^0 = \mathrm{i}\Psi^{\dagger}(x) \tag{2.64a}$$

$$\pi_{\overline{\Psi}}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_t\overline{\Psi}(x))} = 0 \tag{2.64b}$$

$$\pi^{\mu}_{A}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_t A_{\mu}(x))} = -\frac{1}{4\pi}\partial_t A^{\mu}(x). \tag{2.64c}$$

The problem that in these equations the conjugate momentum of the conjugate spinor field $\overline{\Psi}$ vanishes, can be resolved by symmetrizing the Lagrangian (2.61) and does not cause any particular difficulties [21]. The Hamiltonian of the theory is formed in the standard way

$$\begin{aligned} H(t) &= \int \mathrm{d}\boldsymbol{x}\ \left(\pi_{\Psi}(x)\partial_t\Psi(x) + \pi^{\mu}_{A}(x)\partial_t A_{\mu}(x) - \mathcal{L}(\Psi(x), \overline{\Psi}(x), A^{\mu}(x))\right) \\ &= H_{\text{Dirac}}(t) + H_{\text{Maxwell}}(t) + H_{\text{int}}(t), \qquad (2.65) \\ H_{\text{Dirac}}(t) &= \int d\boldsymbol{x}\ \overline{\Psi}(x)(-\mathrm{i}\boldsymbol{\gamma}\nabla + m)\Psi(x) \\ H_{\text{Maxwell}}(t) &= \frac{1}{8\pi}\int d\boldsymbol{x}\ \left(-\pi^{\mu}_{A}(x)\pi_{A,\mu}(x) + \partial_{\nu}A_{\mu}(x)\partial^{\mu}A^{\nu}(x)\right) \\ H_{\text{int}}(t) &= e\int d\boldsymbol{x}\ j^{\mu}_{\text{Dirac}}(x)A_{\mu}(x). \end{aligned}$$

To now quantize the theory one interprets the above fields, as well as the derived Hamiltonian, as Schrödinger picture operators, indicated by a superscript S, and subjects them to the anti-commutation and commutation relations for the fermionic and bosonic fields, respectively

$$\left[\hat{\Psi}_r^S(\boldsymbol{x}),\, \hat{\Psi}_q^{S\dagger}(\boldsymbol{x}')\right]_+ = \delta_{rq}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \tag{2.66a}$$

$$\left[\hat{A}^{S\mu}(\boldsymbol{x}),\, \hat{\pi}_A^{S\nu}(\boldsymbol{x}')\right]_- = \mathrm{i}g^{\mu\nu}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \tag{2.66b}$$

$$\left[\hat{\Psi}_r^S(\boldsymbol{x}),\, \hat{\Psi}_q(\boldsymbol{x}')\right]_+ = \left[\hat{\Psi}_r^{S\dagger}(\boldsymbol{x}),\, \hat{\Psi}_q^{S\dagger}(\boldsymbol{x}')\right]_+ = 0$$

$$\left[\hat{A}^{S\mu}(\boldsymbol{x}), \hat{A}^{S\nu}(\boldsymbol{x}')\right]_- = \left[\hat{\pi}_A^{S\mu}(\boldsymbol{x}),\, \hat{\pi}_A^{S\nu}(\boldsymbol{x}')\right]_- = 0.$$

where we have written the spin indices of the spinor fields $\hat{\Psi}^S$, $\hat{\overline{\Psi}}^S$ explicitly. In the Schrödinger picture operators do not carry a dynamic time dependency, whence the time dependency in Eq. (2.65) can be only due to an explicitly time dependent field, as e.g. an external electromagnetic current. The dynamical evolution of quantum systems is governed by the evolution of the quantum states, which obey the Schrödinger equation

$$\mathrm{i}\frac{d}{dt}\big|\Psi, A;\,(t)\rangle^S = \hat{H}^S\big|\Psi, A;\,(t)\rangle^S, \tag{2.67}$$

where $\hat{H}^S$ is the Schrödinger picture Hamiltonian from Eq. (2.65), with the entering fields understood as operators. For the sake of simplicity we do not consider an explicit time dependence of the Hamiltonian. Eq. (2.67) can formally be integrated by defining a time evolution operator

$$\big|\Psi, A;\,(t)\rangle^S = \hat{U}^S(t, t_0)\big|\Psi, A;\,(t_0)\rangle^S \tag{2.68}$$

$$\hat{U}^S(t, t_0) = \hat{\mathcal{T}}\exp\left[-\mathrm{i}\int_{t_0}^{t} dt\hat{H}^S\right], \tag{2.69}$$

where $\hat{\mathcal{T}}$ denotes time ordering [21, 25]. Even though we do not consider an explicitly time dependent Hamiltonian at this point, we introduce this general notion already for later convenience. For Eq. (2.69), however, no exact solution is known. It is thus customary to transform the quantum states and operators to the interaction picture of quantum dynamics. To this end one splits up the full Hamiltonian into its free and its interaction part

$$\hat{H}^S = \hat{H}_0^S + \hat{H}_{\mathrm{int}}^S \tag{2.70a}$$

$$\hat{H}_0^S = \hat{H}_{\mathrm{Dirac}}^S + \hat{H}_{\mathrm{Maxwell}}^S. \tag{2.70b}$$

The transition from the Schrödinger to the interaction picture is accomplished by the unitary transformation

$$\hat{\mathcal{O}}^I(t) = \hat{U}_0^{S\dagger}(t,t_0)\hat{\mathcal{O}}^S\hat{U}_0^S(t,t_0) \tag{2.71a}$$

$$\left|\Psi, A; (t)\right\rangle^I = \hat{U}_0^{S\dagger}(t,t_0)\left|\Psi, A; (t)\right\rangle^S \tag{2.71b}$$

$$\hat{U}_0^S(t,t_0) = \hat{\mathcal{T}}\exp\left[-\mathrm{i}\int_{t_0}^{t} dt\hat{H}_0^S\right], \tag{2.71c}$$

where the superscript I denotes states and operators in the interaction picture. In this picture both states and operators carry a dynamic time evolution, however, with different governing equations. For the evolution of states in the interaction picture, we find a Schrödinger-type, whereas the time dependence of the operators is determined by a Heisenberg-type equation

$$\mathrm{i}\frac{d}{dt}\hat{\mathcal{O}}^I(t) = \left[\hat{\mathcal{O}}^I, \hat{H}_0^I\right] \tag{2.72a}$$

$$\mathrm{i}\frac{d}{dt}\left|\Psi, A; (t)\right\rangle^I = \hat{H}_{\mathrm{int}}^I\left|\Psi, A; (t)\right\rangle^I. \tag{2.72b}$$

We note that due to the transformation law (2.71a) it holds $\hat{H}_0^I \equiv \hat{H}_0^S$. Evaluating Eq. (2.72a) for the field operators $\hat{\Psi}(x)$ and $\hat{A}^\mu(x)$ we recover wave equations formally equivalent to the classical equations of motion for the field modes Eqs. (2.63). To describe all degrees of freedom of the theory, we have to expand the full field operator in a complete basis of solutions of the wave equation with operator-valued coefficients. We have to pay attention that the wave Eqs. (2.63) allow positive and negative energy solutions for the photon and the spinor fields (see Appendix B). Both contributions have to be included in a complete basis, whence the expanded field operators read

$$\hat{A}^\mu(x) = \int d\boldsymbol{k}\left[\hat{a}_{\boldsymbol{k}}A_{\boldsymbol{k}}^\mu(x) + \hat{a}_{\boldsymbol{k}}^\dagger A_{\boldsymbol{k}}^{*\mu}(x)\right] \tag{2.73a}$$

$$\hat{\Psi}(x) = \int d\boldsymbol{p}\left[\hat{c}_{\boldsymbol{p}}\Psi_p(x) + \hat{d}_{\boldsymbol{p}}^\dagger\Psi_{-p}(x)\right] \tag{2.73b}$$

$$\hat{\overline{\Psi}}(x) = \int d\boldsymbol{p}\left[\hat{c}_{\boldsymbol{p}}^\dagger\overline{\Psi}_p(x) + \hat{d}_{\boldsymbol{p}}\overline{\Psi}_{-p}(x)\right]. \tag{2.73c}$$

In this expression $A_{\boldsymbol{k}}^\mu(x)$ are the solutions of the photon field wave equation to the spatial wave vector $\boldsymbol{k}$ and $\hat{a}_{\boldsymbol{k}}$, $\hat{a}_{\boldsymbol{k}}^\dagger$ are creation and annihilation operators of this field mode. In the fermionic field operators $\Psi_{\pm p}(x)$ and $\overline{\Psi}_{\pm p}(x)$ are positive and negative energy solutions of the free wave equation for the fermionic fields to the spatial momenta $\pm\boldsymbol{p}$ and the coefficients $\hat{c}_{\boldsymbol{p}}$ and $\hat{c}_{\boldsymbol{p}}^\dagger$ ($\hat{d}_{\boldsymbol{p}}$, $\hat{d}_{\boldsymbol{p}}^\dagger$) are the according electronic (positronic) creation and annihilation operators, respectively.

The Schrödinger-type Eq. (2.72b), governing the time evolution of quantum states in the interaction picture, can be formally integrated out, in analogy to Eq. (2.68), yielding an explicit time evolution of the form

$$\big|\Psi, A; (t)\rangle^I = \hat{U}^I(t, t_0)\big|\Psi, A; (t_0)\rangle^I \tag{2.74}$$
$$\hat{U}^I(t, t_0) = \hat{\mathcal{T}} \exp\left[-\mathrm{i}\int_{t_0}^{t} dt \hat{H}^I_{\text{int}}\right],$$

Although also for Eq. (2.74) no closed analytical solution has been found so far, this type of time evolution is well suited for a perturbative approach, due to the smallness of the coupling constant $|e| \approx 137^{-1/2}$ at low energies (recall $H_{\text{int}} \propto e$). In fact, it is possible to truncate the exponential series, as which the operator $\hat{U}^I(t, t_0)$ is defined, at a desired order of accuracy of perturbation theory. Every interaction between the fermionic and photon fields is then treated as a small perturbation of the free theories. The result of this procedure are the well-known Feynman diagrams of free QED (see Sect. 2.2.2 and [21, 22]). As a fixed basis for the space of states evolving according to Eq. (2.74), one chooses the Fock representation, with each state $\big|\ldots n_{\boldsymbol{k}} \ldots; \ldots n_{\boldsymbol{p}} \ldots\rangle$ corresponding to a given number of photons $n_{\boldsymbol{k}}$ in the momentum mode $\boldsymbol{k}$ and a number of massive fermions $n_{\boldsymbol{p}}$ in the mode $\boldsymbol{p}$. The ground state of this basis is then the vacuum state $\big|0\rangle$ defined by the action of the field annihilators on it

$$\hat{a}_{\boldsymbol{k}}\big|0\rangle = \hat{c}_{\boldsymbol{p}}\langle|0\rangle = \hat{d}_{\boldsymbol{p}}\big|0\rangle = 0. \tag{2.75}$$

A particular Fock state representing l $\boldsymbol{k}$-mode photons and $m(n)$ electrons with momentum $\boldsymbol{p}_{e^-}$ (positrons with momentum $\boldsymbol{p}_{e^+}$), is then formed from this vacuum state by the action of field construction operators on it

$$\big|l^{\gamma}_{\boldsymbol{k}} \ldots; \ldots m_{\boldsymbol{p}_{e^-}} \ldots; \ldots n_{\boldsymbol{p}_{e^+}}\rangle \propto \left(\hat{a}^{\dagger}_{\boldsymbol{k}}\right)^l \ldots \left(\hat{c}^{\dagger}_{\boldsymbol{p}_{e^-}}\right)^m \ldots \left(\hat{d}^{\dagger}_{\boldsymbol{p}_{e^+}}\right)^n \ldots \big|0\rangle. \tag{2.76}$$

2.2.1 Quantization in the Presence of a Strong External Field

A perturbative expansion, as outlined in the last section is no longer possible, if the coupling between the spinor and the electromagnetic fields, mediated by the coupling term $-e\int dt\, j_\mu A^\mu$ in the Lagrange function (2.61), can no longer be treated as a small perturbation. As was pointed out in Chap. 1, the expansion parameter of the perturbation series of the interaction between an electron and several photons from a strong laser field is the intensity parameter ξ. Hence, a laser field exceeding $\xi \gtrsim 1$ cannot be accounted for perturbatively. Similarly, there are other strong fields imaginable, that do not lend themselves to a perturbative treatment. In this case the spinor field has to be quantized in the presence of the electromagnetic background field. This task is canonically reached by investigating the quantum dynamics in the so-called Furry picture of quantum dynamics [21, 24, 27]. We will see, however, that large bits of the simpler discussion of the free theory still maintain their validity with slightly changed definitions

The essential concept of the Furry picture is to employ a split up of the QED-Hamiltonian differing from Eq. (2.70). To this end, one takes advantage of the physical fact, that electromagnetic fields, which are sufficiently strong to render the perturbative approach disfavorable, usually fulfill two assumptions: All photons in the field stem from one coherent source, realized by e.g. an atomic nucleus or a laser field. Secondly due to the tremendous photon-flux densities present in high-intensity laser fields, these fields can be treated as unquantized, neglecting the single photons' quantum dynamics. To obtain the Furry picture one can then split up the electromagnetic potential entering the Hamiltonian Eq. (2.65) into two separate components

$$A^\mu(x) = A^\mu_{\text{ext}}(x) + A^\mu_{\text{rad}}(x), \tag{2.77}$$

where $A^\mu_{\text{ext}}(x)$ is the explicitly time dependent strong *external* electromagnetic potential. To treat the potential $A^\mu_{\text{ext}}(x)$ as an unquantized field, one must not raise it to an operator level and impose no commutation relations on its components. Hence any contribution to the Hamiltonian operator, depending solely on $A^\mu_{\text{ext}}(x)$ can be omitted. The second contribution $A^\mu_{\text{rad}}(x)$ is the total of all remaining electromagnetic field modes, not belonging to the strong external field. In particular all single emitted (or absorbed) photons are excitations of this field term, hence the index referring to *radiation*. Inserting this expansion into Eq. (2.65), we find that the QED Hamiltonian takes the form

$$H(t) = H_{\text{Dirac}}(t) + H^{\text{ext}}_{\text{int}}(t) + H^{\text{rad}}_{\text{Maxwell}}(t) + H^{\text{rad}}_{\text{int}}(t). \tag{2.78}$$

The term $H_{\text{Dirac}}(t)$ is the same as in Eq. (2.65), the terms $H^{\text{ext,rad}}_{\text{Maxwell}}(t)$ are derived from that equation by replacing $A^\mu \to A^\mu_{\text{ext,rad}}$. In Eq. (2.78) terms coupling the two four potentials A^μ_{ext} to A^μ_{rad} were already dropped, since they do not influence the equations of motion of these fields. In the Furry picture the above Hamiltonian is split up according to

$$H(t) = H^{\text{Furry}}_0(t) + H^{\text{Furry}}_{\text{int}}(t) \tag{2.79}$$

$$\begin{aligned} H^{\text{Furry}}_0(t) &= H_{\text{Dirac}} + H^{\text{ext}}_{\text{int}}(t) + H^{\text{rad}}_{\text{Maxwell}} \\ H^{\text{Furry}}_{\text{int}} &= H^{\text{rad}}_{\text{int}}, \end{aligned} \tag{2.80}$$

where the explicit time dependence of the free Hamiltonian of the Furry picture $H^{\text{Furry}}_0(t)$ is indicated. The transition from the Schrödinger to the Furry picture is accomplished by the unitary transformation

$$\hat{\mathcal{O}}^F(t) = \left(\hat{U}^{\text{Furry}}_0(t, t_0)\right)^\dagger \hat{\mathcal{O}}^S \hat{U}^{\text{Furry}}_0(t, t_0) \tag{2.81a}$$

$$\left|\Psi, A; (t)\right\rangle^F = \left(\hat{U}_0^{\text{Furry}}(t, t_0)\right)^\dagger \left|\Psi, A; (t)\right\rangle \tag{2.81b}$$

$$\hat{U}_0^{\text{Furry}}(t, t_0) = \hat{\mathcal{T}} \exp\left[-\mathrm{i} \int_{t_0}^{t} dt \hat{H}_0^{\text{Furry}}(t)\right]. \tag{2.81c}$$

Due to the explicit time dependence of $\hat{H}_0^{\text{Furry}}(t)$, caused by the external electromagnetic current, the time ordering is indispensable in the definition of the time evolution operator of the Furry picture $\hat{U}_0^{\text{Furry}}(t, t_0)$. We can then largely adopt the discussion subsequent to Eq. (2.70), albeit, respecting the changed definitions of the free and interaction Hamiltonian. In particular, the quantization schemes of the fermionic fields and the field A^μ_{rad} are analogous to the discussion of the previous section. The intricacies of the fact that now the free Hamilton operator $\hat{H}_0^{\text{Furry}}(t)$ is explicitly time dependent can be found e.g. in [24]. We do not repeat those discussions but directly turn to discussing the dynamic evolutions of the quantized fermionic field operators. Analogous to Eq. (2.72) we find the dynamic evolution of operators and states in the Furry picture to be governed by the equations

$$\mathrm{i}\frac{d}{dt}\hat{\mathcal{O}}^F(t) = \left[\hat{O}^F, \hat{H}_0^{\text{Furry}}\right] \tag{2.82a}$$

$$\mathrm{i}\frac{d}{dt}\left|\Psi, A; (t)\right\rangle^F = \hat{H}^F_{\text{int}}\left|\Psi, A; (t)\right\rangle^F. \tag{2.82b}$$

Please note that the interaction Hamiltonian in the Furry picture is given by the expression

$$\hat{H}^F_{\text{int}} = \left(\hat{U}_0^{\text{Furry}}(t, t_0)\right)^\dagger \hat{H}^{\text{Furry}}_{\text{int}} \hat{U}_0^{\text{Furry}}(t, t_0) = e \int d\boldsymbol{x}\, \hat{\overline{\Psi}}^F \hat{\not{A}}^F_{\text{rad}} \hat{\Psi}^F. \tag{2.83}$$

Solving the operator equation Eq. (2.82a), we find the wave equation

$$\left(\mathrm{i}\not{\partial} - e\hat{\not{A}}^F_{\text{ext}} - m\right) \hat{\Psi}^F(x) = 0, \tag{2.84}$$

which is just the Dirac equation in the presence of the assumedly strong external potential A^μ_{ext}. We find that the operators of the fermionic fields are given by an expression analogous to Eq. (2.73)

$$\hat{\Psi}_{A_{\text{ext}}}(x) = \int d\boldsymbol{p} \left[\hat{c}_{\boldsymbol{p}} \Psi_{p,A_{\text{ext}}}(x) + \hat{d}^\dagger_{\boldsymbol{p}} \Psi_{-p,A_{\text{ext}}}(x)\right] \tag{2.85a}$$

$$\hat{\overline{\Psi}}_{A_{\text{ext}}}(x) = \int d\boldsymbol{p} \left[\hat{c}^\dagger_{\boldsymbol{p}} \overline{\Psi}_{p,A_{\text{ext}}}(x) + \hat{d}_{\boldsymbol{p}} \overline{\Psi}_{-p,A_{\text{ext}}}(x)\right], \tag{2.85b}$$

where only in this case the wave functions $\Psi_{\pm p,A_{\rm ext}}(x)$ need to fulfill the Dirac equation in the presence of $A_{\rm ext}$, analogous to Eq. (2.84) and the coefficients $\hat{c}_{\boldsymbol{p}}$ and $\hat{c}_{\boldsymbol{p}}^{\dagger}$ $(\hat{d}_{\boldsymbol{p}}, \hat{d}_{\boldsymbol{p}}^{\dagger})$ are the creation and annihilation operators of these field modes.

Summarizing the above discussion we state that, to obtain the Furry picture, the potential term $A^{\mu}_{\rm ext}$ is attributed to the free Hamiltonian H_0 of the interaction picture. Hence it becomes clear, why this contribution to the electromagnetic potential is labeled *external*: It enters the equation of motion of the spinor field as an additional term, independent of the interaction of the spinors with the radiation field. Furthermore, since $A^{\mu}_{\rm ext}$ is not written as an operator term, it is obvious that in the Furry picture the external potential is not quantized, but treated as a classical current. The dynamic evolution of the Furry picture states, given by Eq. (2.82b), is determined by the radiation field modes. The interaction with these - assumedly weak - field modes, however, is again of order α_{QED} and thus accessible to a perturbative expansion as we wish to outline in the following section.

2.2.2 The S-Matrix Expansion

A key role in the investigation of QED is taken by scattering experiments. A typical experiment of this kind would be the respective scattering of an electron and an intense laser pulse. To describe such scenarios theoretically, one relies on the *S -Matrix formalism*, where the S can be understood to mean *scattering*. The idea underlying this formalism is, that the interaction is confined to a small region in space and time, as is reasonable for a realistic laboratory experiment. The particles entering and leaving the interaction region can then be considered to origin from and propagate to an infinitely remote past and future, respectively, without further interaction. In the Furry picture the evolution equation for the states (Eq. (2.82b)) can be formally integrated in analogy to Eq. (2.74) resulting in an equation of the form

$$\big|\Psi, A; (t)\rangle^F = \hat{U}^F(t, t_0)\big|\Psi, A; (t_0)\rangle^F \tag{2.86a}$$

$$\hat{U}^F(t, t_0) = \hat{\mathcal{T}} \exp\left[-\mathrm{i}\int_{t_0}^{t} dt \hat{H}^F_{\rm int}\right]. \tag{2.86b}$$

A given initial state, formed at a time t_i and subsequently evolved by $\hat{U}^F$ can then be projected onto a complete basis of states, formed at a time t_f, after the scattering took place [21]

$$\begin{aligned}&\sum_f |\Psi_f, A_f; (t_f)\rangle^{F\,F}\langle\Psi_f, A_f; (t_f)|\hat{U}^F(t_f, t_i)|\Psi, A_i; (t_i)\rangle^F\\ &\quad= \sum_f |\Psi_f, A_f; (t_f)\rangle^F \hat{U}^F_{fi}(t_f, t_i),\end{aligned} \tag{2.87}$$

where we defined the evolution operator matrix elements

$$\hat{U}^F_{fi}(t_f, t_i) = {}^F\langle \Psi_f, A_f; (t_f)|\hat{U}^F(t_f, t_i)|\Psi, A; (t_i)\rangle^F. \tag{2.88}$$

The subscripts refer to the initial and final state, respectively. To capture an experimental scattering scenario as described above, one then has to consider states formed at $t_i \to -\infty$ and observed at $t_f \to \infty$. The scattering matrix in the Furry picture is thus recovered by the limit

$$\begin{aligned} S^F_{fi} &= U^F_{fi}(t_f \to \infty, t_i \to -\infty) \\ &= {}^F\langle \Psi_f, A_f; (t_f \to \infty)|\hat{S}^F|\Psi_i, A_i; (t_i \to -\infty)\rangle^F, \end{aligned} \tag{2.89}$$

in analogy to the ordinary QED result, obtained in the interaction picture [21]. The entries of the infinitely dimensional S-Matrix can then be understood as amplitudes of a given initial state to evolve into a specific final state. The scattering operator in the Furry picture is then in accordance with Eq. (2.74)

$$\hat{S}^F = \hat{\mathcal{T}} \exp\left[-\mathrm{i} \int_{-\infty}^{\infty} dt H^F_{\text{int}}(t)\right] = \hat{\mathcal{T}} \exp\left[-\mathrm{i}e \int d^4x\, \hat{\bar{\Psi}}^F \hat{\not{A}}^F_{\text{rad}}\, \hat{\Psi}^F\right], \tag{2.90}$$

The scattering operator in the Furry picture thus only describes the interaction with the quantized radiation field. From Eq. (2.90) one then constructs the usual perturbation series in orders of the exponential operator function. This perturbation series can then also be represented by Feynman graphs. However, all electron states in the Furry picture perturbation series are formed in the presence of the strong electromagnetic potential $A^\mu_{\text{ext}}(x)$. This particularity is conventionally indicated in the Feynman graph representation of the Furry picture, by drawing double lines for the electron states and propagators. This convention is adopted in the thesis and we will explicitly call attention to Feynman graphs which are drawn with single lines for the electron states, which are then ordinary perturbative QED graphs.

2.2.2.1 The Volkov Solution

According to the theory outlined in the previous section, we can take arbitrary external fields analytically into account in QED, as long as we can provide an exact solution of the Dirac equation (2.63a) in the given electromagnetic field. Unfortunately there exists only a fairly limited class of electromagnetic fields, the Dirac equation has been solved for analytically so far. One such known solution is given for the case of the external potential being a plane wave [21, 28]. For many applications, treating a laser field as a plane wave is a sufficiently good approximation. Furthermore, the plane wave solution is the leading order approximation to the electron state function, even for the external field being a focused Gaussian beam (see Appendix C). To obtain a solution of the Dirac equation one multiplies Eq. (2.63a) from the left with the operator $\left(\hat{\not{p}} - e\not{A}_L + m\right)$ to arrive at the second order differential equation

$$\left[\left(\hat{\not{p}} - e\not{A}_L\right)^2 - m^2 - \frac{i}{2} e F_{\mu\nu}\sigma^{\mu\nu}\right]\Psi(x) = 0, \tag{2.91}$$

where we used the antisymmetric tensor of the Dirac matrices defined in Appendix B. Eq. (2.91) is usually the starting point of the derivation. Recalling the Lorenz gauge condition we have $\partial_\mu A_L^\mu = 0$ and $\not{p}\not{p} = p^2$, the square operator term is evaluated to

$$\left(\hat{\not{p}} - e\not{A}_L\right)^2 = -\partial_\mu\partial^\mu - 2\mathrm{i}e\left(A_L^\mu\partial_\mu\right) + e^2 A_L^2, \tag{2.92}$$

where the four dimensional unit matrix is not written explicitly. Due to the known plane wave solutions of the free Dirac equation (see Appendix B) it is sensible to expect the change in the wave function due to the external field to be summable in a prefactor, leading to the ansatz

$$\Psi_p(x) = \mathrm{e}^{-\mathrm{i}px} F_p(\eta). \tag{2.93}$$

Inserting this expression into Eq. (2.91) and expanding the square operator term as shown above one arrives at the equation

$$2\mathrm{i}(pk_L)F_p'(\eta) + \left[-2e\left(pA_L\right) + e^2A_L^2 - \mathrm{i}e\not{k}_L\left(\partial_\eta \not{A}(\eta)\right)\right]F_p(\eta) = 0. \tag{2.94}$$

This equation, however, is a simple first order differential equation for the prefactor $F_p(\eta)$ which is readily integrated to yield the expression

$$\Psi_p(x) = \mathrm{e}^{-\mathrm{i}px}\exp\left[-\mathrm{i}\int_0^\eta d\phi\left(\frac{e(pA_L(\phi))}{pk_L} - \frac{e^2A_L^2(\phi)}{2(pk_L)}\right) + e\frac{\not{k}_L\not{A}_L\,A(\eta)}{2(pk_L)}\right]\frac{u_p}{\sqrt{2\varepsilon V}}. \tag{2.95}$$

with a yet arbitrary spinor u_p and normalization factor $(2\varepsilon V)^{-1/2}$. The exponential series involving Dirac matrices is seen to vanish after its linear term, due to the relation $(\not{k}_L\not{A}_L)^2 = 0$. To eliminate all solutions of the second order Eq. (2.91), which are not solutions of the original first order Dirac Eq. (2.63a), we demand the solution to fulfill this first order equation at any point in space. For an arbitrarily small damping in the field $A_L(\eta)$ this request at $|\boldsymbol{r}| \to \infty$ goes over into

$$\left(\hat{\not{p}} - m\right)u_p = 0, \tag{2.96}$$

whence we conclude that the request of the constant spinor in Eq. (2.95) to be a solution of the free Dirac equation (see Appendix B) is already sufficient to ensure that the following wave functions are solutions of the Dirac equation in the presence of a plane wave

$$\Psi_p(x) = \left[1 + e\frac{\not{k}_L \not{A}_L(\eta)}{2(pk_L)}\right] \frac{u_p}{\sqrt{2\varepsilon V}} \mathrm{e}^{\mathrm{i}S_p(\eta)}. \tag{2.97}$$

This wave function is called *Volkov function* after D. Volkov who first published its derivation. The exponential phase given by

$$S_p(\eta) = -px - g_p(\eta) \tag{2.98}$$
$$g_p(\eta) = \int\limits_0^\eta \mathrm{d}\phi \left(\frac{e(pA_L(\phi))}{pk_L} - \frac{e^2 A_L^2(\phi)}{2(pk_L)}\right).$$

It is noteworthy that Eq. (2.98) is equivalent to the action of a classical electron in a plane wave fields given in Eq. (2.24) rendering the Volkov solution explicitly quasiclassical. It is further customary to assign a separate symbol to the combined matrix and exponential prefactors according to [29]

$$E_p(x) = \left[1 + e\frac{\not{k}_L \not{A}_L(\eta)}{2(pk_L)}\right] \mathrm{e}^{\mathrm{i}S_p(\eta)}, \tag{2.99}$$

which is called the *Ritus matrix*. To obtain the Dirac current associated with the Volkov functions we need the Dirac conjugate of Eq. (2.97)

$$\overline{\Psi}_p(x) = \frac{\bar{u}_p}{\sqrt{2\varepsilon V}} \overline{E}_p(x), \tag{2.100}$$

with the conjugate of the Ritus matrix defined in accordance with Eq. (2.99)

$$\overline{E}_p(x) = \left[1 + e\frac{\not{A}_L(\eta)\not{k}_L}{2(pk_L)}\right] \mathrm{e}^{-\mathrm{i}S_p(x)}. \tag{2.101}$$

In Eq. (2.102) we summarize the action of the Dirac operator on the Ritus matrices, the resulting explicit commutation relation with the contraction of the momentum with the Dirac matrices as well as the fact that the Ritus matrix and its Dirac conjugate are their respective inverse

$$(\mathrm{i}\not{\partial} - e\not{A}_L)\, E_p(x) = E_p(x)\not{p} \tag{2.102a}$$
$$\overline{E}_p(x)\, (\mathrm{i}\not{\partial} - e\not{A}_L) = -\not{p}\, \overline{E}_p(x) \tag{2.102b}$$
$$\left[\overline{E}(p,x), \not{p}\right] = \left[\not{p}, E(p,x)\right] = e\not{A}(x) - e\frac{A(x)p}{k_L p}\not{k}_L \tag{2.102c}$$

$$E_p(x)\overline{E}_p(x) = \mathbb{1}^4. \tag{2.102d}$$

We stress that the Ritus matrices depend nontrivially only on the laser phase η or equivalently the x^--coordinate. For later convenience we write the part of $E_p(x)$, depending solely on x^- explicitly

$$E_p(x^-) = \left[1 + e\frac{\not{k}_L \not{A}_L(x^-)}{2(pk_L)}\right] \mathrm{e}^{-\mathrm{i}(p^+x^- + g_p(x^-))}, \tag{2.103a}$$

$$\overline{E}_p(x^-) = \left[1 + e\frac{\not{A}_L(x^-)\not{k}_L}{2(pk_L)}\right] \mathrm{e}^{\mathrm{i}(p^+x^- + g_p(x^-))}, \tag{2.103b}$$

where we recall the discussion in connection to Eq. (2.5). The Volkov current is accordingly found to be

$$j_p^\mu(\eta) = \overline{\Psi}_p \gamma^\mu \Psi_p = \frac{1}{\varepsilon V}\left(p^\mu - eA_L^\mu(\eta) + k_L^\mu\left[e\frac{pA_L(\eta)}{pk_L} - \frac{e^2A_L^2(\eta)}{2(pk_L)}\right]\right). \tag{2.104}$$

We note that the expression in round brackets is equivalent to the classical electron momentum in a plane wave from Eq. (2.31a). If we consider the external plane wave field to be confined in time such that $A_L(\eta \to \pm\infty) \to 0$, we infer that Volkov solutions, formed in the far past and future, are normalized to one particle per volume V. Hence, we can employ the usual state statistics for computing scattering probabilities for a pulsed field entering the Volkov solutions. Finally, owing to some discussion that was had on the topic, we state that the Volkov solutions are orthogonal and normalized according to [1, 29–31]

$$\frac{V}{(2\pi)^3}\int d\boldsymbol{x}\, \overline{\Psi}_{p'}(x)\gamma^0\Psi_p(x) = \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}). \tag{2.105}$$

It has recently been proven that the Volkov solutions fulfill the completeness relation [32]

$$\frac{V}{(2\pi)^3}\int d\boldsymbol{p}\, \overline{\Psi}_p(x)\gamma^0\Psi_p(y) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}). \tag{2.106}$$

For the Ritus matrices Eqs. (2.99) and (2.101) this implies the orthogonality and completeness relations

$$\int \frac{d^4p}{(2\pi)^4}\overline{E}_p(x)E_p(y) = \delta^{(4)}(x-y), \int \frac{d^4x}{(2\pi)^4}\overline{E}_{p'}(x)E_p(x) = \delta^{(4)}(p'-p). \tag{2.107}$$

Having established the mentioned properties of the Volkov solutions we may feel free to use them as a basis for building a SF-QED field theory.

2.2.2.2 The Volkov Propagator

Irrespective of the picture in which the quantum dynamics are described, in addition to wave functions for the incoming and outgoing particles, in QED one is in need of the two-point Green's function or the propagator of the involved quantum fields. By virtue of the above given argument, that in tree level QED electromagnetic potentials do not interact, the dressed photon Green's function equals its free counterpart. The Green's function of a charged spinor field, on the other hand, is altered by including an external potential. In fact, the defining equation of the Green's function of the Dirac equation to an external plane wave potential $A_L(\eta)$ is [21]

$$(\mathrm{i}\not{\partial} - e\not{A} - m)\, G(x, y) = \delta^{(4)}(x - y). \tag{2.108}$$

The Green's function solving this equation can be expressed as [29, 33]

$$G(x, y) = \lim_{\epsilon\to 0} \int \frac{d^4 p}{(2\pi)^4} E_p(x) \frac{\not{p} + m}{p^2 - m^2 + \mathrm{i}\epsilon} \overline{E}_p(y), \tag{2.109}$$

as can be checked by employing Eqs. (2.102a) and (2.102b). The pole prescription of Eq. (2.109) is the Feynman prescription of the free propagator [22]

$$G^{\text{free}}(x, y) = \lim_{\epsilon\to 0} \int \frac{d^4 p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}p(x-y)}, \tag{2.110}$$

to which the given dressed propagator reduces in the limit $A_L \to 0$. This prescription for bypassing the poles of Eq. (2.109) ensures that for $x^0 < y^0$, corresponding to propagation forward in time, only positive energy solutions ($\varepsilon > 0$) of Eq. (2.63a) enter Eq. (2.109), whereas in the opposite case $x^0 > y^0$, corresponding to propagation backwards in time, the negative energy components ($\varepsilon < 0$) are described.

Separating these two cases ab initio, we can split up the time ordered product $\hat{\mathcal{T}}\{\dots\}$ in the definition of the dressed propagator [21] into its respective forms

$$G(x, y) = -\mathrm{i}\langle 0|\hat{\mathcal{T}}\left\{\hat{\Psi}(y)\hat{\overline{\Psi}}(x)\right\}|0\rangle = \begin{cases} -\mathrm{i}\langle 0|\hat{\overline{\Psi}}(x)\hat{\Psi}(y)|0\rangle & \text{if } x_0 > y_0 \\ \mathrm{i}\langle 0|\hat{\Psi}(y)\hat{\overline{\Psi}}(x)|0\rangle & \text{if } x_0 < y_0 \end{cases}, \tag{2.111}$$

using the Dirac field operators Eqs. (2.73b) and (2.73c) expanded in the basis of Volkov functions. By virtue of the commutation relations Eq. (2.66a) one then finds a result equivalent to Eq. (2.109), as it has to be.

In a monochromatic external plane wave field the dressed electron propagator of SF-QED features infinitely many singularities depending on the number of photons absorbed from the external field [34, 35]. These resonances were addressed in numerous work employing the dressed electron propagator in monochromatic fields to compute numerous physical quantities such as lepton pair creation [36–39], lepton-lepton scattering [40, 41], resonant lepton-photon scattering (e.g. bremsstrahlung if the external photon is a nuclear Coulomb field photon) [42, 43] and, on a more fundamental level, those resonances were also investigated in the study of the electron self-energy [44]. These poles are located at the dressed mass m^*, which is discussed in the following section. In contrast to those previous works, there has been some effort to study SF-QED processes involving the dressed electron propagator in pulsed plane wave fields [45, 46]. There the authors, however, considered the strongly restrictive condition of temporally only mildly focussed ($\omega\tau \gg 1$) and low intense ($\xi \ll 1$) fields.

2.2.3 Interaction with a Monochromatic Laser Wave

In this chapter we outline the quantum analysis of the scattering of an electron from a monochromatic laser wave, in analogy to the discussion of Sect. 2.1.2. As discussed there, modeling the laser field as monochromatic allows for great simplifications in the calculations. In fact, most of the theoretical works on nonlinear Compton scattering, performed before this thesis was started, considered a monochromatic laser wave [29, 47–51]. There had been some work on electron scattering from a laser pulse of duration τ_L and frequency ω [19], but there the authors considered a pulse explicitly violating Eq. (1.9), i.e. a pulse containing many cycles of the carrier field. In this case, as one can see from Eq. (2.31a, 2.31b), the classical electron trajectory is strictly monotonic. Comparable to the Fourier decomposition of the radiation formula Eq. (2.17) it is possible to expand the Volkov states, and equally the dressed propagator in a Fourier series. To this end we consider a monochromatic laser wave of the form $\psi_{\mathcal{A}}(\eta) = \sin(\eta)$ (the discussion for $\psi_{\mathcal{A}}(\eta) = \cos(\eta)$ is analogous), whence the Volkov solutions becomes analytically integrable, yielding

$$\Psi_p(x) = \left[1 + e\frac{\not{k}_L \not{A}_L(\eta)}{2(pk_L)}\right] \frac{u_p}{\sqrt{2\varepsilon V}} \mathrm{e}^{-\mathrm{i}S_p^{\mathrm{m.c.}}(\eta)}, \tag{2.112}$$

with the monochromatic exponential $S_p^{\mathrm{m.c.}}(\eta) = \alpha\cos(\eta) + \beta\sin(2\eta) + qx$. In this expression we defined the quantities

$$q^\mu = p^\mu + \frac{m^2\xi^2}{4(pk_L)} k_L^\mu \tag{2.113a}$$

$$\alpha = -e\frac{pA_L}{pk_L} \tag{2.113b}$$

$$\beta = -\frac{e^2A_L^2}{8pk_L}. \tag{2.113c}$$

The so-called dressed momentum q^μ contains an additional momentum component along the laser's wave vector k_L, which arises due to the non-vanishing average over the square contribution of the laser wave in the classical action and is thus attributed to the wiggling motion of the electron in the laser wave. The square of the dressed momentum yields the dressed mass m^* at which the poles of the electron propagator in a monochromatic laser wave are located

$$q^2 = m^2 \left(1 + \frac{m^2\xi^2}{2}\right) =: m^{*2}. \tag{2.114}$$

This dressed mass exceeds a free electron's rest mass by an intensity dependent term $m^{*2} - m^2 = m^2\xi^2/2$. This mass increase is caused by the periodic wiggling motion the electron undergoes inside the strong laser wave providing it with additional energy, which translates to an increased mass. We will find that when considering a laser pulse, the divergences from Eq. (2.109) are naturally regularized and the dressed mass loses its unambiguous meaning. Though this observation does not render the concept of a dressed mass obsolete, it is clear that it will definitely need further investigation (see Sect. 3.5 and [52]). The expression for the Volkov states (2.112) allows for the deployment of the generating function of Bessel functions analogous to Eq. (2.40) from classical electrodynamics. Utilizing the relation

$$\sin^i(\eta)\mathrm{e}^{-\mathrm{i}(\alpha\sin(\eta)+\beta\sin(2\eta))} = \sum_{n=-\infty}^{\infty} C_{i,n}\mathrm{e}^{-\mathrm{i}n\eta} \tag{2.115}$$

where the coefficients are defined according to [29]

$$C_{i,n} = \frac{1}{2\pi}\int d\eta' \sin(\eta')\mathrm{e}^{\mathrm{i}\left(\alpha\sin(\eta')+\beta\sin(2\eta')-n\eta'\right)} \tag{2.116}$$

allows then for a Fourier expansion of Eq. (2.112) resulting in

$$\Psi_p(x) = \sum_{n=-\infty}^{\infty}\left[C_{0,n} + e\frac{\not{k}_L\not{A}_L}{2(pk_L)}C_{1,n}\right]\frac{u_p}{\sqrt{2q^0V}}\mathrm{e}^{-\mathrm{i}(q+nk_L)x}. \tag{2.117}$$

The replacement $\varepsilon \to q^0$ in the normalization of the monochromatic wave function is due to the special form of an infinitely stretched external laser field. The wave functions are required to be normalized to one particle per normalization volume and as in the monochromatic case one cannot perform the limit $A_L^\mu(\eta \to \pm\infty) = 0$, one has to average this quantity over one laser cycle, yielding

$$\bar{j}_p^\mu(\eta) = \frac{1}{q^0V}\left(p^\mu - k_L^\mu\frac{m^2\xi^2}{4(pk_L)}\right) = \frac{q^\mu}{q^0V}, \tag{2.118}$$

which again corresponds to one particle per volume V. Equation (2.117) is now well suited for computing scattering matrix elements, since the exponential factors will always cancel the four dimensional space-time integration, to give a momentum conserving δ-function of the form $\delta^{(4)}(\sum_{\text{in}} q - \sum_{\text{out}} q)$, i.e., where in the sums for spinor particles there have to be inserted the dressed momenta q, whereas uncharged particles, such as photons, enter with their ordinary free momentum. Illustrating this concept at the exemplary process of an electron of initial momentum p_i being scattered by a monochromatic laser wave, described by $A_L(\eta)$ into a final momentum state q_f upon emission of a single photon with wave vector k_1, the scattering matrix amplitude is given by [29]

$$S_{fi} = -\mathrm{i}\frac{\sqrt{2\pi}e}{\sqrt{\omega_1 V}} \int \mathrm{d}^4x\, \overline{\Psi}_{p_f}(x) \not{\epsilon}_1^* \mathrm{e}^{\mathrm{i}k_1 x} \Psi_{p_i}(x), \tag{2.119}$$

where for the emitted photon the free wave function

$$A_1^\mu = \frac{\sqrt{4\pi}}{\sqrt{2\omega_1 V}} \epsilon_1^\mu \mathrm{e}^{\mathrm{i}k_1 x} \tag{2.120}$$

solving Eq. (2.1) is employed with the polarization index not written explicitly. Expanding this expression now into a Fourier series analogously to Eq. (2.117) one obtains the expression

$$S_{fi} = -\mathrm{i}\frac{\sqrt{\pi}e(2\pi)^4}{\sqrt{2\omega_1 q_i^0 q_f^0 V^3}} \sum_{n=-\infty}^{\infty} \bar{u}_{p_f} M_n u_{p_i} \delta^{(4)}(q_i + nk_L - k_1 - q_f), \tag{2.121}$$

with the reduced matrix elements given by

$$M_n = \left[\not{\epsilon}_1^* C_{0,n} + e\left(\frac{\not{A}_L \not{k}_L}{2(p_f k_L)} + \frac{\not{k}_L \not{A}_L}{2(p_i k_L)} \right) C_{1,n} - \frac{e^2 A_L^2 (k_L \epsilon_1^*)}{2(p_i k_L)(p_f k_L)} \not{k}_L C_{2,n} \right]. \tag{2.122}$$

The scattering matrix element is then easily translated into an emission probability per unit time by taking its modulus square, summing and averaging over all outgoing and incoming particles' spins and polarizations, respectively, and multiplying the result by the phase space of the final state's particles. The result of this procedure is given by

$$\begin{aligned} \frac{1}{T} dW^{\text{m.c.}} &= \frac{1}{2} \sum_{\{\sigma,\lambda\}} \left|S_{fi}\right|^2 \frac{d\boldsymbol{k}_1 V}{(2\pi)^3} \frac{d\boldsymbol{p}_f V}{(2\pi)^3} \\ &= \frac{e^2}{16\pi\omega_1 q_i^0 q_f^0} \sum_{\{\sigma,\lambda\}} \sum_{n=-\infty}^{\infty} \left| \bar{u}_{p_f} M_n u_{p_i} \right|^2 d\boldsymbol{k}_1 d\boldsymbol{p}_f \delta^{(4)}(q_i + nk_L - k_1 - q_f), \end{aligned} \tag{2.123}$$

where the square of the four dimensional δ-function yielded the customary factor $(VT)/(2\pi)^4$, $\sum_{\{\sigma,\lambda\}}$ means the summation over all incoming and outgoing polarization and spin quantum numbers and the additional factor $1/2$ turns the sum over the initial state's electron spins into an average. In Eq. (2.123) the three spatial δ-functions fix the final electron's spatial momentum, fixing final electron's energy to the value $q_f^{0\,2} = q_i^{0\,2} + n\omega_L^2 + \omega_1^2 + 2\left(\boldsymbol{q}_i\boldsymbol{k}_L - \boldsymbol{q}_i\boldsymbol{k}_1 - n\boldsymbol{k}_L\boldsymbol{k}_1\right)$. This procedure introduces a conversion factor $\left|\partial q_f/\partial p_f\right|^{-1} = \left|\partial q_f^{\parallel}/\partial p_f^{\parallel}\right|^{-1}$ into Eq. (2.123). The fourth δ-function is customarily used to fix the outgoing photon's energy to the harmonic frequencies

$$\omega_{1,n}^{\text{m.c.}} = \frac{n(p_i k_L)}{(q_i + nk_L)n_1}, \tag{2.124}$$

with an integer $n \in [0, \infty]$. In the classical limit $k_1 k_L \ll p_i k_L$ this expression goes over to the classical harmonic formula (2.43), whence one concludes that the ordinary Doppler shift can be recovered only in the limit $\xi \ll 1$. Using the fourth δ-function in Eq. (2.123) to fix ω_1 to the values of Eq. (2.124), introduces an additional factor of the form [29, 53]

$$\left|\frac{d(q_f^0 + \omega_1)}{d\omega_1}\right| = \frac{(q_i + nk_L)n_1}{q_f^0}. \tag{2.125}$$

As a result of the described steps we obtain the final expression for the single photon emission probability of an electron scattered from a monochromatic laser field per unit time and solid angle

$$\frac{1}{T}\frac{dW^{\text{m.c.}}}{d\Omega_1} = \frac{e^2\omega_1}{16\pi q_i^0 q_f^0 \left|\frac{d(q_f^0+\omega_1)}{d\omega_1}\right|} \sum_{\{\sigma,\lambda\}} \sum_{n=-\infty}^{\infty} \left|\bar{u}_{p_f} M_n u_{p_i}\right|^2. \tag{2.126}$$

In the same fashion one can easily obtain expressions for higher order SF-QED processes in a monochromatic laser wave.

2.2.4 Interaction with a Laser Pulse

Concerning temporal compression, there has been an increasingly fast growing number of works analyzing QED amplitudes of electrons interacting with plane wave laser fields of arbitrary strength. These works, which are applicable to the realm of pulse durations distinguished by Eq. (1.9), were performed for single photon emission [20, 51, 54–58] and recently also for two photon emission [59, 60]. This family of calculations, applicable to the rapidly evolving regime of few-cycle laser pulses, is a research field of swiftly increasing interest and importance.

It is this research field the present thesis is dedicated to.

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http://www.springer.com/978-3-319-07739-0

Quantum Radiation in Ultra-Intense Laser Pulses
Mackenroth, K.F.
2014, XIV, 175 p. 41 illus., 17 illus. in color., Hardcover
ISBN: 978-3-319-07739-0