Chapter 2
Mathematical and Statistical Properties of Decomposition Techniques. The Splines Method

2.1 Introduction

As discussed in Chap. 1, the theoretical Divisia index is calculated upon the basis of the continuous time paths of the observed variables. However, only a finite number of discrete observations is available in practice. As seen in the above chapter, two basic strategies may be applied in order to alleviate that discrepancy between continuous theory and discrete data. Most authors have opted for discretizing theory. This is the spirit of methods such as general PDM1 and PDM2 discussed in Chap. 1. Alternatively, a few contributions in the literature have taken the opposite direction, i.e., generating approximate continuous time paths that more properly adapt to the theory. This is the basic idea of some techniques as the path-based method (Fernández Vázquez and Fernández González 2008).

In this chapter we propose a new continuous time decomposition method that is based on spline interpolation. Our analysis relies on classical results of the mathematical theories of spline interpolation and approximation of functions (e.g., Powell 1981, Chap. 23), as well as on certain stochastic analogues of these results. We begin by studying some mathematical properties of additive and multiplicative decompositions. For the sake of brevity we shall focus on the specific problem of decomposing the variation of a ratio (namely, the energy intensity ratio) in two factors (respectively, structural and intensity effects), although our theoretical results are valid in more general cases, as to be detailed below.

Our proposal may be regarded as nonparametric, since no functional form is assumed for the (deterministic or stochastic) paths to be reconstructed. We shall only impose the requirement of convergence of the approximate, spline-interpolation-based decompositions to their theoretical, continuous time analogues.

Our analysis will follow this sequence: first, upon the basis of a finite set of discrete observations of the relevant variables (namely, production levels and

P. Fernández González et al., *The Driving Forces of Change in Environmental Indicators*, Lecture Notes in Energy 25, DOI: 10.1007/978-3-319-07506-8_2,

energy consumptions), a reconstruction of their continuous time trajectories is generated.[1]

Secondly, approximations to the quantities of interest (namely, the intensity change for the period under study and its components) in Divisia-based decomposition analysis are constructed by plugging the interpolated trajectories into the relevant path integrals. Then, convergence—as the sampling is performed on an increasingly finer time mesh—towards the exact (continuous time) decomposition of energy intensity is derived. Finally, the analysis is extended to the stochastic field, by the expedient of assuming that the time paths are generated by continuous time stochastic processes having appropriate regularity properties.

2.2 Path Reconstruction Through Interpolation. The Splines Method

We shall consider classical polynomial splines, with a single variable t (the time index), where $-\infty \leq a \leq t \leq b \leq \infty$ for arbitrary values a and b. Following the usual definition (e.g., Powell 1981, Chap. 3, page 29), a piecewise polynomial function $Q(t)$ is called a spline of degree K in $[a, b]$ if $Q(t)$ is a polynomial of degree K in each section and has continuous derivatives up to order $K-1$ in $[a, b]$. Formally we will say that $Q(t)$ belongs to the function space $C^{K-1}[a, b]$ of all functions with continuous derivatives up to order $K-1$ in $[a, b]$. In particular, we are interested in the approximation capabilities of splines on interval [0,1]. In that specific case, every spline $Q(t)$ of degree K is characterized by a set $N_n = \{t_1, \ldots, t_n\}$ of points called *knots*, such that $-\infty \leq a < 0 = t_1 < t_2 < \cdots < t_n = 1 < b \leq \infty$ and the spline may be expressed in the following form:

$$Q(t) = \sum_{j=0}^{K} c_j t^j + \sum_{i=2}^{n-1} d_i (t - t_i)_+^K \tag{2.1}$$

where c_j and d_j are constants and $()_+^K$ denotes the truncated power of degree K, i.e., $(z)_+^K = [\max(0, z)]^K$.

Splines are very flexible structures that allow function interpolation in a way that preserves a number of interesting features, such as monotonicity and convexity of the interpolated functions (e.g., DeVore and Lorentz 1993, Chap. 13).

[1] Path reconstruction can be accomplished through a number of interpolation techniques. Two basic requirements are derived from the characteristics of the above theoretical decomposition: (a) the interpolants must be able to approximate the relevant *time paths and their derivatives* up to first order, and (b) the proposed methods should lead to exact decompositions of the variation in energy intensity. These two requirements, as well as computational simplicity, are fulfilled by the splines method.

Within the many spline-based approximation and interpolation techniques, those based on so-called *natural splines* have particularly interesting capabilities. Let $W^m[a,b]$ be the Sobolev space consisting of all functions having continuous derivatives in $[a,b]$ up to order $m-1$ (i.e., the set functions belonging to the function space $C^{m-1}[a,b]$ and having square integrable mth order derivatives in $[a,b]$, with $m = 1, 2, \ldots$; e.g., Adams 1975, Chap. 3). Natural splines were proposed by Schoenberg (1964) as a solution to the following variational problem (we slightly adapt Schoenberg's general formulation to our problem): find a function $\hat{f}$ belonging to $W^m[a,b]$ which solves the following minimization problem

$$\min_{\hat{f}\in W^m[a,b]} \int_a^b \left(D^m\hat{f}(t)\right)^2 \mathrm{d}t \tag{2.2}$$

subject to the set of conditions $\hat{f}(t_i) = f(t_i)$, where $i = 1,\ldots,n$ and $-\infty \le a < 0 = t_1 < t_2 < \cdots < t_n = 1 < b \le \infty$. Classical results (e.g., Powell 1981, Chap. 23, Theorems 23.1–23.2) show that, provided that the number of observations is $n \ge m$, the above variational problem has a unique solution which is a natural spline of order m, i.e., a spline of degree $2m-1$ and continuous derivatives up to order $2m-2$.

An especially interesting feature of natural splines of order m is that they are able to uniformly approximate arbitrary smooth functions and their derivatives up to order $m-1$ on compact sets. Proposition 1 below formally states this property (again this is a particularization, adapted to our setting, of a general property of natural splines; e.g., Wahba 1990; also see Schultz 1973): we shall focus on the problem of approximating in interval $0 \le t \le 1$ an arbitrary function f belonging to Sobolev space $W^m[a,b]$, with $-\infty \le a < 0 \le t \le 1 < b \le \infty$, upon the basis of a sample of n observations (i.e., n points in the graph of the function).[2] In the remainder of the chapter we will denote by $D^\alpha f$ the αth order derivative of f, with $D^0 f = f$.

Proposition 1 (Wahba 1990, pp. viii–ix) *Let f be a function belonging to $W^m[a,b]$. Then, for some constant $L < \infty$, the natural spline of interpolation $\hat{f}$ (with order m and degree $2m-1$) that interpolates f at $N_n = \{t_i = (i-1)/(n-1), i = 1,\ldots,n\}$, satisfies*

$$\max_{0\le\alpha\le m-1}\ \sup_{0\le t\le 1}\left|D^\alpha\hat{f}(t) - D^\alpha f(t)\right| \le L(n-1)^{-(m-\alpha)} \tag{2.3}$$

[2] For notational simplicity we will consider data evenly spaced in time, although the results of this chapter are valid for unevenly spaced observations, under the condition that the maximum distance between any two consecutive observations converges to zero as n goes to infinity.

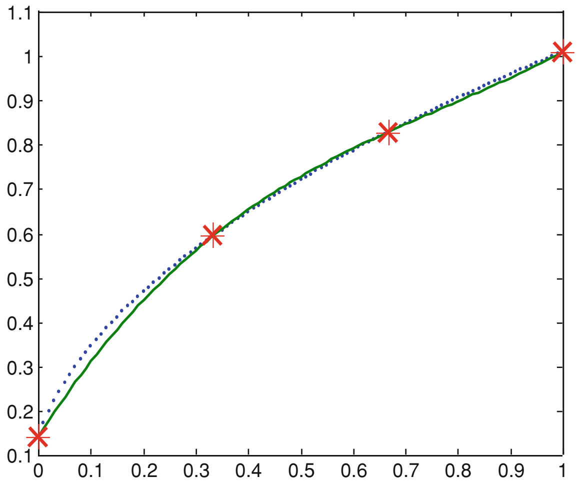

Fig. 2.1 Function $f(t) = (t + 0.02)^{1/2}$ (*dotted line*) and spline interpolant (*solid line*) with $n = 4$ observations (represented as *asterisks*)

Proposition 1 ensures that, under very general conditions, if the function to be reconstructed is smooth enough, the natural spline interpolant and its derivatives up to order $m - 1$ converge uniformly to f and its respective derivatives, as the graph of the function is more densely sampled. Our derivations in this chapter rely heavily on this property.

As a simple illustration, Fig. 2.1 displays a reconstruction of $f(t) = (t + 0.02)^{1/2}$, $0 \leq t \leq 1$, on the basis of $D_n = \{(t_i, f(t_i)), t_i = (i - 1)/(n - 1), i = 1, 2, \ldots, n\}$, using only $n = 4$ observations.

As shown in Fig. 2.1, the deviation between the function and its natural (cubic) spline interpolant is barely noticeable.

2.3 Mathematical Properties. Convergence

The above smooth approximation properties of splines can be applied to path approximation in the general Divisia problem. As to be shown below, decompositions based on spline interpolation converge to the theoretical (continuous time) solution of the Divisia problem. We shall focus on the representative case of the decomposition of the variation of energy intensity in a given period. For simplicity we consider the unit interval, $0 \leq t \leq 1$, with t being the time index (any other finite interval may be chosen instead).

We shall assume that the economy is composed of r sectors ($j = 1, \ldots, r$), and the following notation will be used:

$e_j(t)$: Instantaneous energy consumption in sector j evaluated at time t,

$y_j(t)$: Instantaneous production of sector j at time t,

$e(t) = \sum_{j=1}^{r} e_j(t)$: Total energy consumption at t,

$y(t) = \sum_{j=1}^{r} y_j(t)$: Total production at t.

2.3.1 Additive Decomposition

Aggregate energy intensity at time t is defined as usual:

$$I(t) = \frac{e(t)}{y(t)} \tag{2.4}$$

As $e(t) = \sum_{j=1}^{r} e_j(t)$, the following decomposition is readily obtained:

$$I(t) = \sum_{j=1}^{r} \frac{e_j(t)}{y(t)} = \sum_{j=1}^{r} \frac{e_j(t)}{y_j(t)} \frac{y_j(t)}{y(t)} = \sum_{j=1}^{r} I_j(t) \cdot S_j(t) \tag{2.5}$$

where $I_j(t) = e_j(t)/y_j(t)$ is the energy intensity in sector j and $S_j(t) = y_j(t)/y(t)$ is the share of sector j in the total production at time t.

Under Assumptions 1 and 2 below, the intensity function $I(t)$ has a continuous first order derivative in $[0, 1]$ admitting the following decomposition:

$$D^1 I(t) = \sum_{j=1}^{r} D^1 I_j(t) \cdot S_j(t) + \sum_{j=1}^{r} I_j(t) \cdot D^1 S_j(t) \tag{2.6}$$

The above expression directly results in the following additive decomposition of the intensity variation:

$$I_1 - I_0 = TE = IE + SE \tag{2.7}$$

where $I_0 = I(0)$, $I_1 = I(1)$, TE is the total effect (or intensity change), $IE = \int_0^1 \left(\sum_{j=1}^{r} D^1 I_j(t)\, S_j(t) \right) \mathrm{d}t$ is the intensity effect, and $SE = \int_0^1 \left(\sum_{j=1}^{r} I_j(t) D^1 S_j(t) \right) \mathrm{d}t$ is the structural effect, all of them referred to the accumulation period between $t = 0$ and $t = 1$.

More generally, the variation of total intensity between 0 and t, denoted by $TE(t)$, with $0 \le t \le 1$, may be decomposed as $TE(t) = IE(t) + SE(t)$, with $IE(t) = \int_0^t \sum_{j=1}^{r} D^1 I_j(u) S_j(u)\, \mathrm{d}u$ being the intensity effect accumulated up to t and

$SE(t) = \int_0^t \sum_{j=1}^r D^1 S_j(u) I_j(u)\, du$ being the structural effect accumulated along the same period, so that evidently $TE(1) = I_1 - I_0$.

We shall consider natural cubic spline interpolants, denoted by $\hat{e}_j(t)$ and $\hat{y}_j(t)$, respectively, for the time paths $e_j(t)$ and $y_j(t)$, $j = 1, \ldots, r$, $0 \leq t \leq 1$. Spline interpolants for total energy consumption $e(t)$ and total production $y(t)$ are readily obtained upon the basis of $e_j(t)$ and $y_j(t)$, respectively, as $\hat{e}(t) = \sum_{j=1}^r \hat{e}_j(t)$ and $\hat{y}(t) = \sum_{j=1}^r \hat{y}_j(t)$, which evidently coincide with the natural spline interpolants for $e(t)$ and $y(t)$, and under the conditions of Proposition 1 below will converge uniformly to $e(t)$ and $y(t)$, respectively, and their derivatives up to order 1. Plug-in interpolants for the intensities and production shares are defined directly on the basis of $\hat{e}_j(t)$ and $\hat{y}_j(t)$. In particular:

$\hat{I}_j(t) = \hat{e}_j(t)/\hat{y}_j(t)$: interpolant for energy intensity in sector j,

$\hat{I}(t) = \hat{e}(t)/\hat{y}(t)$: interpolant for aggregate energy intensity,

$\hat{S}_j(t) = \hat{y}_j(t)/\hat{y}(t)$: interpolant for the production share of sector j.

Evidently, the above three interpolants are no longer splines, although they inherit most approximation capabilities of splines $\hat{e}_j(t)$ and $\hat{y}_j(t)$, which suffices for our purposes.

The above structures interpolate the discrete set of observations of the relevant variables (consumptions, productions, intensities, shares) and, as to be shown below, they also provide suitable approximations to the continuous time paths of these variables and their first derivatives.

We shall consider the following plug-in approximants to functions $IE(t)$, $SE(t)$ and $TE(t)$, respectively, $\hat{I}E(t) = \int_0^t \sum_{j=1}^r D^1 \hat{I}_j(u) \hat{S}_j(u) \mathrm{d}u$, $\hat{S}E(t) = \int_0^t \sum_{j=1}^r \hat{I}_j(u) D^1 \hat{S}_j(u) \mathrm{d}u$ and $\hat{T}E(t) = \hat{I}E(t) + \hat{S}E(t)$, with $0 \leq t \leq 1$. Evidently, $\hat{I}E = \hat{I}E(1)$, $\hat{S}E = \hat{S}E(1)$ and $\hat{T}E = \hat{T}E(1)$ will be, respectively, plug-in approximants for IE, SE and TE, corresponding to the whole accumulation period.

To ensure that the above interpolants satisfy Proposition 1 we will impose the following regularity conditions on the paths we want to reconstruct:

Assumption 1 For each $j = 1, \ldots, r$: (i) sector consumption $e_j(t)$ has continuous derivatives up to order 1 in [0,1], with $\int_0^1 \left(D^2 e_j(t)\right)^2 \mathrm{d}t \leq c < \infty$, and (ii) $e_j(t) \geq m > 0$.

Assumption 2 For each $j = 1, \ldots, r$: (i) sector production $y_j(t)$ has continuous derivatives up to order 1 in [0,1], with $\int_0^1 \left(D^2 y_j(t)\right)^2 \mathrm{d}t \leq c < \infty$, and (ii) $y_j(t) \geq m > 0$.

The following proposition shows that the above approximations converge to the theoretical, continuous time effects as the number of observations (n) and its denseness in $[0,1]$ increase.

Proposition 2 (Convergence of the additive decomposition) *Let* $\hat{T}E_n = \int_0^1 \sum_{j=1}^{r} \left[D^1 \hat{I}_j(t)\hat{S}_j(t) + \hat{I}_j(t) D^1 \hat{S}_j(t)\right] \mathrm{d}t$ *be the plug-in approximant to TE generated through natural spline interpolation (of order 1 and degree 3) of the time paths, applied to the set of observations* $D_n = \{(t_i, e_1(t_i), \ldots, e_r(t_i), y_1(t_i), \ldots, y_r(t_i)),\ t_i = (i-1)/(n-1);\ i = 1, \ldots, n\}$. *Then the following holds under Assumptions 1 and 2*:

(a) $\hat{T}E_n \to TE$ *as* $n \to \infty$, *and in particular*

(b) $\left|\hat{T}E_n - TE\right| \leq L_1 (n-1)^{-1}$ *for some* $L_1 < \infty$ *and all n large enough.* □

An analogue result holds for the plug-in approximations for *IE* and *SE*, and for those of functions $\hat{I}E(t)$, $\hat{S}E(t)$ and $\hat{T}E(t)$ themselves (see the proof of Proposition 2 in Appendix I below).

2.3.2 Multiplicative Decomposition

The above ideas are readily extended to the multiplicative case, where the logarithmic total effect is defined as follows:

$$LTE = LTE(1) = ln(I_1/I_0) = \int_0^1 D^1 \ln I(t)\, \mathrm{d}t = \int_0^1 \sum_{j=1}^{r} \frac{D^1 I_j(t) S_j(t)}{I(t)}\, \mathrm{d}t + \int_0^1 \sum_{j=1}^{r} \frac{I_j(t) D^1 S_j(t)}{I(t)}\, \mathrm{d}t \tag{2.8}$$

The intensity effect accumulated until t is $R_{\mathrm{int}}(t) = \exp(LIE(t))$, with $LIE(t) = \int_0^t \sum_{j=1}^{r} \frac{D^1 I_j(u) S_j(u)}{I(u)}\, \mathrm{d}u$ being the logarithmic intensity effect for the $[0,t]$ period. Similarly the structural effect accumulated up to t is $R_{\mathrm{str}}(t) = \exp(LSE(t))$, where $LSE(t) = \int_0^t \sum_{j=1}^{r} \frac{I_j(t)\, D^1 S_j(t)}{I(t)}\, \mathrm{d}t$ is the logarithmic structural effect.

The magnitude to decompose is the total effect accumulated at t, which is simply $R(t) = R_{\mathrm{int}}(t)\, R_{\mathrm{str}}(t)$, with $R(1) = I_1/I_0$ being the ratio of intensities at $t = 1$ and $t = 0$.

We will approximate *LTE* by its spline-based analogue, namely,

$$\hat{LTE}_n = \int_0^1 \sum_{j=1}^{r} \frac{D^1\hat{I}_j(t)\,\hat{S}_j(t)}{\hat{I}(t)}\,\mathrm{d}t + \int_0^1 \sum_{j=1}^{r} \frac{\hat{I}_j(t)\,D^1\hat{S}_j(t)}{\hat{I}(t)}\,\mathrm{d}t \tag{2.9}$$

Analogously to the additive case, the following convergence result holds for (2.9):

Proposition 3 (Convergence of the multiplicative decomposition) *Let $\hat{LTE}_n$ be the plug-in approximant to LTE generated through natural interpolation splines (of order 1 and degree 3) by using the observation set $D_n = \{(t_i, e_1(t_i), \ldots, e_r(t_i), y_1(t_i), \ldots, y_r(t_i)),\ t_i = (i-1)/(n-1);\ i = 1, \ldots, n\}$. Then the following holds under Assumptions 1 and 2*:

(a) *$\hat{LTE}_n \to LTE$ as $n \to \infty$, and in particular*

(b) *$\left|\hat{LTE}_n - LTE\right| \le L_1(n-1)^{-1}$ for some $L_1 < \infty$ and all n large enough.*

Therefore, the logarithmic total effect (and also the intensity and structural ones) are uniformly approximated by their natural spline analogues.

2.4 Stochastic Analysis

In this section we will analyze the behaviour of $\hat{TE}_n$ and $\hat{LTE}_n$ in a probabilistic setting where the data used for path reconstruction are generated by a sufficiently regular stochastic process. We will show that, under general conditions, these two approximants are random variables and converge with probability 1 (and therefore in distribution) to *TE* and *LTE* (which are also random variables), respectively, as $n \to \infty$.

Let $(\Omega, \mathbf{A}, P)$ be a complete probability space. The set of time paths in $[0, 1]$ will be given by vector function $Z = (z_1, \ldots, z_{2r})$, where $z_j = e_j,\ z_{r+j} = y_j,\ j = 1, \ldots, r$ are the trajectories of energy consumption and production for each of the r sectors. Thus, for each (t, ω) with $t \in [0, 1]$ and $\omega \in \Omega$, we will have $Z(t, \omega) = (e_1(t, \omega), \ldots, e_r(t, \omega), y_1(t, \omega), \ldots, y_r(t, \omega))$, which is a vector of observations at time t. By allowing t to range between 0 and 1, a path vector is obtained. Regarding the set of trajectories, we assume as in the previous section that, for all fixed $\omega \in \Omega$, each component of $Z = Z(t, \omega)$ has continuous derivatives up to order 1 in $[0, 1]$. Thus, we will say that Z is defined on function space $S = \prod_{i=1}^{2r} C^1[0, 1]$, endowed with the metric induced by the norm $||Z|| = \max_{j=1,\ldots,2r} \max_{0 \le \alpha \le 1} \max_{0 \le t \le 1} \left|D^\alpha z_j(t)\right|$. Equipped with this norm, S is a complete separable metric space.[3]

[3] See Dudley (1973) for sets of conditions ensuring differentiability, continuity and Lipschitz properties for stochastic processes, both Gaussian and non-Gaussian.

We shall denote by $\mathbf{B}(S)$ the class of Borel sets in S, i.e., the smallest σ-algebra containing all open subsets (in the sense of the metric induced by the norm $||.||$) of S. The ordered pair $(S, \mathbf{B}(S))$ is a probabilizable space. We assume that the stochastic process Z is a random element of S, that is, a measurable mapping of Ω into S. Therefore, for each event B in $\mathbf{B}(S)$, an event A in $\mathbf{A}$ exists such that $Z(A) = B$. Note that the norm $||.||$ is a $\mathbf{B}(S)$-measurable function. The mapping $P'(B) = P(Z^{-1}(B))$, $B \in \mathbf{B}(S)$, defines a probability measure P' induced by Z, and a final probability space $(S, \mathbf{B}(S), P')$. (For simplicity we will also use the same symbol P to denote P', with the proper interpretation to be deduced from the context in each case.)

For each $m > 0$ we will denote by A_m the set of paths $Z = \left(z_1, \ldots, z_{2r}\right) \in S$ such that $\min_{j=1,\ldots,2r} \min_{0 \le t \le 1} D^1 z_j(t) \ge m$. We will impose the following condition, which is a stochastic analogue of Assumptions 1 and 2 above:

Assumption 1′ For every $\omega \in \Omega$: (i) each component of $Z(.,\omega)$ belongs to $W^2[a,b]$, and (ii) for some $m > 0$ not depending on $\omega \in \Omega$, it holds $Z(.,\omega) \in A_m$.[4]

Under Assumption 1′, the total effects TE and LTE, can be expressed as $TE = g(Z)$ and $LTE = h(Z)$, respectively, where $g(.)$ and $h(.)$ are functionals of Z with expressions given by Propositions 2 and 3, respectively. It is readily verified that both g and h are continuous in A_m, that is, for every $Z, Z' \in A_m$, $||Z' - Z|| \to 0$ implies $|g(Z') - g(Z)| \to 0$ and $|h(Z') - h(Z)| \to 0$. (This may be readily deduced by the same procedure used in the proof of Propositions 2 and 3 and Lemmas A.1–A.3 in Appendix I). As Z is $\mathbf{B}(S)$-measurable, continuity of $g(.)$ and $h(.)$ implies that both $TE = g(Z)$ and $LTE = h(Z)$ are random variables (*e.g.,* Billingsley, 1968, Appendix II, p. 222), i.e. measurable functions with respect to the σ-algebra $\mathbf{B}(\Re)$, defined on the real line $\Re$.

We now consider the probabilistic behaviour of $\hat{T}E_n$ and $\hat{L}TE_n$, constructed as described in the previous section. It is easily derived that both $\hat{T}E_n$ and $\hat{L}TE_n$ are random variables under Assumption 1′. For each $\omega \in \Omega$ the sample realizations $\hat{T}E_n(\omega)$ and $\hat{L}TE_n(\omega)$ of $\hat{T}E_n$ and $\hat{L}TE_n$ are standard Riemann integrals so extension to the stochastic field is straightforward.[5]

Proposition 4 *Under Assumption 1′:*

(a) For all n large enough $\hat{T}E_n$ and $\hat{L}TE_n$ *are* $\mathbf{B}(\Re)$*-measurable.*

(b) As $n \to \infty$, $\hat{T}E_n \to TE$ *and* $\hat{L}TE_n \to LTE$ *with probability 1 (and therefore in distribution).*

[4] The results in these section are also obtained if Assumption 1′ holds almost surely, i.e., for each $\omega \in \Omega$ excepting a set $N \in \mathbf{A}$ with $P(N) = 0$.

[5] The proof of Proposition 4 in Appendix I relies on a version of the Continuous Mapping Theorem that requires all integrals to be defined pointwise, i.e., as standard Riemann integrals for almost each $\omega \in \Omega$. Therefore, use of more general (mean square) stochastic integrals is not sufficient for our purposes. In addition, mean square integration requires more restrictive conditions, such as finite variances (e.g., Tanaka 1996, Chap. 3, page 71), not imposed in Assumption 1' above.

2.5 Concluding Remarks

In this chapter we have established—using as an illustration the problem of decomposing the change in energy intensity—that the exact decompositions based on spline interpolation of the time paths converge to the values derived from the theory of continuous time Divisia indices. These theoretical quantities may be seen as (deterministic and stochastic, respectively) limits of sequences of spline-based approximations. To obtain these results we have relied on some mathematical properties of classical cubic spline interpolation.

The splines method provides a workable alternative to mainstream techniques developed so far in the IDA literature. It incorporates a number of advantages, including (i) its being nonparametric in nature, and (ii) its *objective* (or fully automatic) character, as it does not depend on any parameters to be subjectively chosen by the researcher. In addition, (iii) it is an exhaustive method (in the sense of having zero residual) that naturally verifies the circular property of index numbers, thus allowing so-called time series decompositions. Finally, (iv) the splines method may be applied in multilevel decompositions, provided that time series of sufficient length are available for all the quantities involved in the decomposition analysis.

The results of this chapter can be readily extended to a number of closely related problems. For instance, in Chap. 3 below the splines method is applied to a decomposition of energy intensity under a different number of factors than considered in this chapter, and in Chap. 4 it is used to decompose the variation of an absolute magnitude (namely, greenhouse gas emissions). More generally, the splines method may be applied to the (respectively, additive or multiplicative) decomposition of the variation of the product of any finite number of components or time paths (e.g., Fernández Vázquez and Fernández González 2008), and more generally to decomposing the variation of a differentiable functional of a vector of smooth time paths. The arguments developed in this chapter for the case of energy intensity directly extend to those more general settings.

Appendix I: Mathematical Proofs of Chapter 2

The following two lemmas are required for the proof of Proposition 2.

Lemma A.1 *Under Assumptions 1 and 2, there exists a constant* $B < \infty$ *such that the following holds for* $j = 1, \ldots, r$ *and* $0 \leq \alpha \leq 1$:

(a) $\max_{t \in [0,1]} \left| D^{\alpha}\hat{e}_j(t) - D^{\alpha}e_j(t) \right| \leq B(n-1)^{-(2-\alpha)}$, *and*

(b) $\max_{t \in [0,1]} \left| D^{\alpha}\hat{y}_j(t) - D^{\alpha}y_j(t) \right| \leq B(n-1)^{-(2-\alpha)}$.

Proof The Lemma is a direct consequence of Proposition 1.

Lemma A.2 *Under Assumptions 1 and 2, there exist constants* $B_k < \infty, k = 1, \ldots, 4$ *such that, for each* $j = 1, \ldots, r$, *the following holds for* $0 \le t \le 1$ and $n > 1 + (2B/m)^{1/2}$, *where B is as in Lemma A.1 above and does not depend on t*:
(a) $\left|\hat{I}_j(t) - I_j(t)\right| \le B_1(n-1)^{-2}$,
(b) $\left|\hat{S}_j(t) - S_j(t)\right| \le B_2(n-1)^{-2}$,
(c) $\left|D^1\hat{I}_j(t) - D^1 I_j(t)\right| \le B_3(n-1)^{-1}$,
(d) $\left|D^1\hat{S}_j(t) - D^1 S_j(t)\right| \le B_4(n-1)^{-1}$.

Proof In the proof of this Lemma (and throughout the rest of the Appendix) we will rely on the fact that any continuous function in $[0,1]$ is bounded in that interval. This implies that, under Assumptions 1(i) and 2(i), there exists a constant $M < \infty$ such that $\max_{0 \le \alpha \le 1} \max_{0 \le t \le 1} \left|D^\alpha e_j(t)\right| \le M$ and $\max_{0 \le \alpha \le 1} \max_{0 \le t \le 1} \left|D^\alpha y_j(t)\right| \le M$ for each $j = 1, \ldots, r$. The same is true for the aggregate consumption and production functions (respectively, $e(t)$ and $y(t)$).

Let us select an arbitrary point $t \in [0,1]$. Regarding statement (a), we have: $\left|\hat{I}_j(t) - I_j(t)\right| = \left|\frac{\hat{e}_j(t)}{\hat{y}_j(t)} - \frac{e_j(t)}{y_j(t)}\right| \le A_I + A_{II}$, where $A_I = \left|\frac{\hat{e}_j(t)}{\hat{y}_j(t)} - \frac{e_j(t)}{\hat{y}_j(t)}\right|$ and $A_{II} = \left|\frac{e_j(t)}{\hat{y}_j(t)} - \frac{e_j(t)}{y_j(t)}\right|$. Let $A_{III} = \hat{y}_j(t) - y_j(t)$. Lemma A.1 ensures both $\left|\hat{e}_j(t) - e_j(t)\right| \le B(n-1)^{-2}$ and $\left|\hat{y}_j(t) - y_j(t)\right| \le B(n-1)^{-2}$. Thus, arbitrarily small $\left|\hat{e}_j(t) - e_j(t)\right|$ and $|A_{III}|$ can be obtained for large n. In particular, if $n > 1 + (2B/m)^{1/2}$ we have $A_{III} < m/2$. Therefore, Assumption 2(ii) implies that, for any $n > 1 + (2B/m)^{1/2}$, it holds $\hat{y}_j(t) = y_j(t) + A_{III} \ge m - m/2 = m/2 > 0$, so $A_I = \left|\hat{y}_j(t)\right|^{-1}\left|\hat{e}_j(t) - e_j(t)\right| \le \frac{B}{m/2}(n-1)^{-2}$.

As for A_{II} we have:

$$A_{II} = \left|e_j(t)\right| \cdot \left|\frac{y_j(t) - \hat{y}_j(t)}{y_j(t)\hat{y}_j(t)}\right|$$

As $\left|e_j(t)\right| \le M < \infty$ by continuity in $[0,1]$, Lemma A.1(b) ensures that, for $n > 1 + (2B/m)^{1/2}$, it holds $A_{II} \le \frac{M}{m^2/2} B(n-1)^{-2}$.

Therefore, $A_I + A_{II} \le B_1(n-1)^{-2}$ for all n large enough and some finite B_1. As t was arbitrary, uniform convergence is obtained, which completes the proof of statement (a).

Regarding (b) we have, for any $t \in [0,1]$, $\left|\hat{S}_j(t) - S_j(t)\right| = \left|\frac{\hat{y}_j(t)}{\hat{y}(t)} - \frac{y_j(t)}{y(t)}\right| \le A_I + A_{II}$, where $A_I = \frac{\left|\hat{y}_j(t) - y_j(t)\right|}{\hat{y}(t)}$ and $A_{II} = \left|y_j(t)\frac{y(t) - \hat{y}(t)}{y(t)\hat{y}(t)}\right|$.

For $n > 1 + (2B/m)^{1/2}$ it holds $\hat{y}(t) = \sum_{j=1}^r \hat{y}_j(t) = \sum_{j=1}^r y_j(t) + \sum_{j=1}^r \left(\hat{y}_j(t) - y_j(t)\right) \ge rm - rm/2 = rm/2$.

Thus $A_I \leq \frac{2}{rm} B(n-1)^{-2}$ and $A_{II} \leq \frac{2M}{r^2 m^2} B(n-1)^{-2}$, so for $n > 1 + (2B/m)^{1/2}$ we have $\left|\hat{S}_j(t) - S_j(t)\right| \leq B_2(n-1)^{-2}$. Again, as t is arbitrary, convergence is uniform.

Regarding (c), select an arbitrary t in $[0, 1]$ and apply the quotient rule for derivatives. We have $\left|D^1 \hat{I}_j(t) - D^1 I_j(t)\right| = |A_I - A_{II}|$, where

$$A_I = \frac{D^1 \hat{e}_j(t)\hat{y}_j(t) - \hat{e}_j(t) D^1 \hat{y}_j(t)}{\left(\hat{y}_j(t)\right)^2}$$

and

$A_{II} = \frac{D^1 e_j(t) y_j(t) - e_j(t) D^1 y_j(t)}{\left(y_j(t)\right)^2}$.

By the triangle inequality

$|A_I - A_{II}| \leq |A_I - A_{III}| + |A_{III} - A_{II}|$, where $A_{III} = \frac{D^1 e_j(t) y_j(t) - e_j(t) D^1 y_j(t)}{\left(\hat{y}_j(t)\right)^2}$.

It is directly obtained that, for $n > 1 + (2B/m)^{1/2}$, there exists $B_5 < \infty$ (not depending on n) such that $|A_I - A_{II}| \leq B_5(n-1)^{-1}$.

Analogously, continuity (and so boundedness in $[0, 1]$) of the first derivatives of y_j and e_j implies that, for some $M < \infty$, $\left|D^1 e_j(t) y_j(t) - e_j(t) D^1 y_j(t)\right| \leq \left|D^1 e_j(t)\right| \cdot \left|y_j(t)\right| + \left|e_j(t)\right| \cdot \left|D^1 y_j(t)\right| \leq 2M^2$.

So it follows, for $n > 1 + (2B/m)^{1/2}$ and some $B_6 < \infty$ (not depending on n), $|A_{II} - A_{III}| \leq \frac{4M^2}{m^2} B_6 (n-1)^{-2}$.

Therefore, $|A_I - A_{II}| \leq |A_I - A_{III}| + |A_{III} - A_{II}| \leq B_3(n-1)^{-1}$ for $n > 1 + (2B/m)^{1/2}$ and some B_3 finite, with convergence being uniform.

As for (d), a similar procedure is applied. We have $\left|D^1 \hat{S}_j(t) - D^1 S_j(t)\right| \leq |A_I - A_{III}| + |A_{III} - A_{II}|$, where $A_I = \frac{D^1 \hat{y}_j(t)\hat{y}(t) - \hat{y}_j(t) D^1 \hat{y}(t)}{(\hat{y}(t))^2}$, $A_{II} = \frac{D^1 y_j(t) y(t) - y_j(t) D^1 y(t)}{(y(t))^2}$ and $A_{III} = \frac{D^1 y_j(t) y(t) - y_j(t) D^1 y(t)}{(\hat{y}(t))^2}$.

Again we obtain that, for $n > 1 + (2B/m)^{1/2}$, there exists $B_6 < \infty$ (not depending on n) so that $|A_I - A_{II}| \leq B_6(n-1)^{-1}$.

As $\left|D^1 y_j(t) y(t) - y_j(t) D^1 y(t)\right| \leq \left|D^1 y_j(t)\right| \cdot |y(t)| + \left|y_j(t)\right| \cdot \left|D^1 y(t)\right| \leq rM^2$ for some $M < \infty$, and since $\left|(y(t))^2 - (\hat{y}(t))^2\right| = |y(t) + \hat{y}(t)| \cdot |y(t) - \hat{y}(t)| \leq \left(2M + B(n-1)^{-2}\right) B(n-1)^{-2}$, we obtain for $n > 1 + (2B/m)^{1/2}$:

$$\begin{aligned} |A_{III} - A_{II}| &\leq \frac{\left|D^1 y_j(t) y(t) - y_j(t) D^1 y(t)\right| \cdot \left|(y(t))^2 - (\hat{y}(t))^2\right|}{(y(t))^2 (\hat{y}(t))^2} \\ &\leq \frac{4rM^2(2M+1)}{r^4 m^4} \left(2M + B(n-1)^{-2}\right) B(n-1)^{-2} \end{aligned}$$

Therefore, for each $n > 1 + (2B/m)^{1/2}$ and $0 \le t \le 1$, there exists $B_4 < \infty$ (not depending on n or t), such that $|A_I - A_{II}| \le |A_I - A_{III}| + |A_{III} - A_{II}| \le B_3(n-1)^{-1}$.

Lemma A.3 *Let* $D^1 I(t) = \sum_{j=1}^{r} \left(D^1 I_j(t) S_j(t) + I_j(t) D^1 S_j(t)\right)$ *and* $D^1 \hat{I}(t) = \sum_{j=1}^{r} \left(D^1 \hat{I}_j(t) \hat{S}_j(t) + \hat{I}_j(t) D^1 \hat{S}_j(t)\right)$. *Under Assumptions 1 and 2, there exists a constant* $B_5 < \infty$ *such that, for each* $0 \le t \le 1$ *and* $n > 1 + (2B/m)^{1/2}$, *with B being as in Lemma A.1, it holds* $\left|D^1 \hat{I}_j(t) - D^1 I_j(t)\right| \le B_5(n-1)^{-1}$.

Proof It straightforwardly derives from Lemma A.2, which establishes that each component of $D^1 \hat{I}(t)$ converges uniformly to its analogue in $D^1 I(t)$. In order to obtain uniform convergence we will rely on the fact that all the components appearing in $D^1 I(t)$ are uniformly bounded in $0 \le t \le 1$. In particular: $\left|I_j(t)\right| = \left|\frac{e_j(t)}{y_j(t)}\right| \le \frac{M}{m}$, $\left|S_j(t)\right| = \left|\frac{y_j(t)}{y(t)}\right| \le \frac{M}{rm}$, $\left|D^1 I_j(t)\right| = \frac{\left|D^1 e_j(t) y_j(t) - e_j(t) D^1 y_j(t)\right|}{\left(y_j(t)\right)^2} \le \frac{2M^2}{m^2}$ and $\left|D^1 S_j(t)\right| = \frac{\left|D^1 y_j(t) y(t) - y_j(t) D^1 y(t)\right|}{(y(t))^2} \le \frac{2M^2}{r^2 m^2}$.

Similar uniform bounds are obtained for the components of $D^1 \hat{I}(t)$, since according to Lemma A.2, as $n > 1 + (2B/m)^{1/2}$ we have, for each $0 \le t \le 1$,
$\left|\hat{I}_j(t)\right| \le \left|I_j(t)\right| + \left|\hat{I}_j(t) - I_j(t)\right| \le \frac{M}{m} + B_1(n-1)^{-2}$,
$\left|\hat{S}_j(t)\right| \le \left|S_j(t)\right| + \left|\hat{S}_j(t) - S_j(t)\right| \le \frac{M}{rm} + B_2(n-1)^{-2}$,
$\left|D^1 \hat{I}_j(t)\right| \le \left|D^1 I_j(t)\right| + \left|D^1 \hat{I}_j(t) - D^1 I_j(t)\right| \le \frac{2M^2}{m^2} + B_3(n-1)^{-1}$
and
$\left|D^1 \hat{S}_j(t)\right| \le \left|D^1 S_j(t)\right| + \left|D^1 \hat{S}_j(t) - D^1 S_j(t)\right| \le \frac{2M^2}{r^2 m^2} + B_4(n-1)^{-1}$.

By using the decomposition $D^1 \hat{I}(t) - D^1 I(t) = A_I + A_{II}$, where

$$A_I = \sum_{j=1}^{r} \left(D^1 \hat{I}_j(t) \hat{S}_j(t) + \hat{I}_j(t) D^1 \hat{S}_j(t)\right) - \sum_{j=1}^{r} \left(D^1 \hat{I}_j(t) S_j(t) + \hat{I}_j(t) D^1 S_j(t)\right)$$

and

$$A_{II} = \sum_{j=1}^{r} \left(D^1 \hat{I}_j(t) S_j(t) + \hat{I}_j(t) D^1 S_j(t)\right) - \sum_{j=1}^{r} \left(D^1 I_j(t) S_j(t) + I_j(t) D^1 S_j(t)\right).$$

and applying the above bounds, it is readily obtained that there exists some $B_5 < \infty$ such that, for each t in $[0,1]$ and $n > 1 + (2B/m)^{1/2}$, it holds $\left|D^1 \hat{I}(t) - D^1 I(t)\right| \le |A_I| + |A_{II}| \le B_5(n-1)^{-1}$.

Proof of Proposition 2 It is a direct consequence of Lemma A.3. It is readily checked that $D^1 I(t)$ is continuous in $[0,1]$, so it is also bounded in that interval. In particular, for each t in $[0,1]$, it holds $\left|D^1 I(t)\right| = \frac{\left|D^1 e(t) y(t) - e(t) D^1 y(t)\right|}{(y(t))^2} \le \frac{2M^2}{m^2}$. An analogous result is obtained for $D^1 \hat{I}(t)$ when $n > 1 + (2B/m)^{1/2}$, as $\hat{y}(t) \ge rm/2 > 0$ in that case, so $D^1 \hat{I}(t)$ is a ratio of continuous functions, with strictly positive denominator (having a positive lower bound not depending on n).

Applying Lemma 3, we obtain

$$\left|\hat{T}E_n - TE\right| = \left|\int_0^1 D^1\hat{I}(t)\mathrm{d}t - \int_0^1 D^1 I(t)\mathrm{d}t\right| \le \int_0^1 \left|D^1\hat{I}(t) - D^1 I(t)\right|\mathrm{d}t \le B_5(n-1)^{-1}\int_0^1 \mathrm{d}t = B_5(n-1)^{-1}$$

for $n > 1 + (2B/m)^{1/2}$, which proves part (*b*), and therefore part (*a*). □

Proof of Proposition 3

It is a consequence of Lemma A.3. Let us select an arbitrary point t in $[0,1]$ and apply the decomposition $\left|\frac{D^1\hat{I}(t)}{\hat{I}(t)} - \frac{D^1 I(t)}{I(t)}\right| \le A_I + A_{II}$, where $A_I = \frac{\left|D^1\hat{I}(t) - D^1 I(t)\right|}{\hat{I}(t)}$ and $A_{II} = \left|\frac{D^1 I(t)}{\hat{I}(t)} - \frac{D^1 I(t)}{I(t)}\right|$.

The inequality $\left|D^1\hat{I}(t) - D^1 I(t)\right| \le B_5(n-1)^{-1}$ was obtained in the proof of Lemma A.3 above, for each t in $[0,1]$ and $n > 1 + (2B/m)^{1/2}$. In addition, $\frac{m}{M} \le \frac{1}{I(t)} = \frac{y(t)}{e(t)} \le \frac{M}{m}$.

It is readily shown, by a procedure analogous to that used in Lemma A.2 (*a*), that $\left|\hat{I}(t) - I(t)\right| \le B_6(n-1)^{-2}$ for each t in $[0,1]$ and $n > 1 + (2B/m)^{1/2}$, with $B_6 < \infty$ not depending on t.

Since $\frac{1}{\hat{I}(t)} = \frac{1}{I(t)} + \frac{I(t) - \hat{I}(t)}{I(t)\hat{I}(t)}$, and as $\left|\hat{I}(t) - I(t)\right| \le B_6(n-1)^{-2} \le \frac{m}{2M}$ for all large enough n (for this, it suffices to select $n \ge \sqrt{B_6 \frac{2M}{m} - 1}$), we arrive at $\hat{I}(t) = I(t) + \left(\hat{I}(t) - I(t)\right) \ge \frac{m}{M} - \frac{m}{2M} = \frac{m}{2M}$.

Therefore, it eventually holds:

$$\left|\frac{1}{\hat{I}(t)}\right| = \left|\frac{1}{I(t)} + \frac{I(t) - \hat{I}(t)}{I(t)\hat{I}(t)}\right| \le \frac{M}{m} + \frac{2M^2}{m^2} B_6(n-1)^{-2}$$

So, for large enough n, we obtain $A_I = \frac{\left|D^1\hat{I}(t) - D^1 I(t)\right|}{\hat{I}(t)} \le \left(1 + \frac{M}{m}\right) B_5(n-1)^{-1}$.

As for A_{II}, we have $A_{II} = \frac{\left|D^1 I(t)\right| \cdot \left|I(t) - \hat{I}(t)\right|}{I(t)\hat{I}(t)}$. As shown in the proof of Proposition 2, it holds $\left|D^1 I(t)\right| \le \frac{2M^2}{m^2}$ for all large enough n, and by applying the bounds obtained for A_I above the following inequality is readily obtained:

$$A_{II} = \frac{\left|D^1 I(t)\right| \cdot \left|I(t) - \hat{I}(t)\right|}{I(t)\hat{I}(t)} \le \frac{\frac{2M^2}{m^2} \times M}{m}\left(1 + \frac{M}{m}\right) B_6(n-1)^{-2}$$

Therefore, there exists $B_7 < \infty$ such that, for each t in $[0,1]$, $\left|\frac{D^1\hat{I}(t)}{\hat{I}(t)} - \frac{D^1 I(t)}{I(t)}\right| \le B_7(n-1)^{-1}$.

By the same procedure as in Proposition 2 it is shown that, for each t in $[0,1]$, it holds $I(t) > 0$, and the same is true for $\hat{I}(t)$ for large enough n. This ensures that

$$\left|\hat{LTE}_n - LTE\right| = \left|\int_0^1 \left(D^1 \ln \hat{I}(t) - D^1 \ln I(t)\right) \mathrm{d}t\right| \leq \int_0^1 \left|D^1 \ln \hat{I}(t) - D^1 \ln I(t)\right| \mathrm{d}t$$

$$\leq B_7 (n-1)^{-1} \int_0^1 \mathrm{d}t = B_5 (n-1)^{-1}$$

for sufficiently large n, which completes the proof of both parts of the proposition.

Proof of Proposition 4

First, we will prove (a), that for each n large enough and each (fixed) set of knots $N_n = \{t_i = (i-1)/(n-1),\ i = 1, \ldots, n\}$, the vector of interpolated time paths, $\hat{Z}_n = \left(\hat{z}_{1,n}, \ldots, \hat{z}_{2r,n}\right)$, is $\mathbf{B}(S)$-measurable. We shall use symbol $(.)_n$ to denote the operator that associates with each path in $C^1[0,1]$ its natural spline interpolant, with n knots located at N_n. Thus, $\hat{z}_{j,n} = \left(z_j\right)_n$ is the natural spline that interpolates path z_j at N_n.

It suffices to show that $(.)_n$ is a continuous mapping of $C^1[0,1]$ into $C^1[0,1]$, which implies that it is also continuous as a vector function of S into S, and therefore $\hat{Z}_n = (Z)_n$ is $\mathbf{B}(S)$-measurable. Select two arbitrary functions $f, f' \in C^1[0,1]$, with that function space endowed with the norm $||.||$ particularized to the case of a single trajectory. We will show that, given n fixed (and therefore, a fixed set of n knots), $||f - f'|| \to 0$ implies $\left|\left|\hat{f} - \hat{f}'\right|\right| \to 0$. This stems, as we shall see, from the fact that $(.)_n$ is a continuous linear operator.

Indeed it can be shown that, provided that $n \geq 2$, the set of natural splines (having degree 3, continuous derivatives up to order 1 and knots at N_n) is a vector space of dimension n. This implies that the natural spline interpolant $\hat{f}_n(t)$ for $f \in C^1[0,1]$ is unique (Powell 1981, Chap. 23, Theorem 23.1.) and has the following expression:

$$\hat{f}_n(t) = \sum_{i=1}^{n} \hat{\beta}_{i,n} \varphi_{i,n}(t)$$

where $\left(\varphi_{1,n}, \ldots, \varphi_{n,n}\right)$ is a vector of n linearly independent functions in $C^1[0,1]$. The coefficient vector $\hat{\beta}_n = \left(\hat{\beta}_{1,n}, \ldots, \hat{\beta}_{n,n}\right)^T$ is obtained by imposing the n conditions of interpolation at N_n, i.e., $f(t_i) = \hat{f}(t_i),\ i = 1, \ldots, n$. This is equivalent to solving the system of linear equations $f_n = \Phi_n \hat{\beta}_n$, where $f_n = (f(t_1), \ldots, f(t_n))^T$ and $\Phi_n = \left[c_{i,k}\right],\ i, k = 1, \ldots, n,$ is a squared matrix whose elements are $c_{i,k} = \varphi_{i,n}(t_k)$. Uniqueness of the solution of this problem implies that matrix Φ_n is nonsingular, so $\hat{\beta}_n = \Phi_n^{-1} f_n$.

Now consider another function $f' \in C^1[0,1]$, and let $\hat{f}'_n(t) = \sum_{i=1}^{n} \hat{\beta}'_{i,n}\varphi_{i,n}(t)$ be its natural spline interpolant at N_n, with $\hat{\beta}'_n = \left(\hat{\beta}'_{1,n}, \ldots, \hat{\beta}'_{n,n}\right)^T$. For any integer $0 \le \alpha \le 1$, select an arbitrary point $0 \le t \le 1$. We have

$$\left|D^\alpha f'_n(t) - D^\alpha f_n(t)\right| = \left|\sum_{i=1}^{n} D^\alpha \varphi_{i,n}(t)\left(\beta'_{i,n} - \beta_{i,n}\right)\right| \le \sum_{i=1}^{n} \left|D^\alpha \varphi_{i,n}(t)\right| \cdot \left|\beta'_{i,n} - \beta_{i,n}\right| \le$$

$$\max_{0\le\alpha\le1} \max_{i=1,\ldots,n} \left|D^\alpha \varphi_{i,n}(t_i)\right| \cdot \sum_{i=1}^{n} \left|\hat{\beta}'_{i,n} - \hat{\beta}_{i,n}\right| \le \max_{0\le\alpha\le1} \max_{i=1,\ldots,n} \max_{t\in[0,1]} \left|D^\alpha \varphi_{i,n}(t)\right| n\, n^{-1} \sum_{i=1}^{n} \left|\hat{\beta}'_{i,n} - \hat{\beta}_{i,n}\right| \le$$

$$\max_{0\le\alpha\le1} \max_{i=1,\ldots,n} \max_{t\in[0,1]} \left|D^\alpha \varphi_{i,n}(t)\right| n \sqrt{n^{-1} \sum_{i=1}^{n} \left(\hat{\beta}'_{i,n} - \hat{\beta}_{i,n}\right)^2}$$

As $\hat{\beta}'_n - \hat{\beta}_n = \Phi_n^{-1} e_n$, with $e_n = f'_n - f_n$, we have $\sqrt{\left(\hat{\beta}'_n - \hat{\beta}_n\right)^T \left(\hat{\beta}'_n - \hat{\beta}_n\right)} = \sqrt{e_n^T \left(\Phi_n^{-1}\right)^T \Phi_n^{-1} e_n}$.

Since matrix $\left(\Phi_n^{-1}\right)^T \Phi_n^{-1}$ is symmetric and positive definite, it admits the orthogonal decomposition $\left(\Phi_n^{-1}\right)^T \Phi_n^{-1} = P_n^T \Lambda_n P_n$, where $P_n^T P_n = I_n$, with I_n being the unit matrix of order n and Λ_n being a diagonal matrix with the eigenvalues of $\left(\Phi_n^{-1}\right)^T \Phi_n^{-1}$ on its main diagonal. It is readily obtained that $\left[e_n^T \left(\Phi_n^{-1}\right)^T \Phi_n^{-1} e_n\right]^{1/2} = \left[e_n^T P_n^T \Lambda_n P_n e_n\right]^{1/2} \le \lambda_1^{1/2} \left[e_n^T e_n\right]^{1/2}$, where λ_1 is the largest eigenvalue of $\left(\Phi_n^{-1}\right)^T \Phi_n^{-1}$.

Therefore, we obtain

$$\sqrt{\left(\hat{\beta}_n - \hat{\beta}_n\right)^T \left(\hat{\beta}_n - \hat{\beta}_n\right)} = \sqrt{e_n^T \left(\phi_n^{-1}\right)^T \phi_n^{-1} e_n} \le \sqrt{\lambda_1}\sqrt{e_n^T e_n} = \sqrt{\lambda_1}\sqrt{\sum_{i=1}^{n} \left(f'(t_i) - f(t_i)\right)^2}$$

$$\le \sqrt{\lambda_1 n} \max_{i=1,\ldots,n} \left|f'(t_i) - f(t_i)\right| \le \sqrt{\lambda_1 n} \cdot \|f' - f\|$$

and it holds $\left|D^\alpha \hat{f}'_n(t) - D^\alpha \hat{f}_n(t)\right| \le \max_{0\le\alpha\le1} \max_{i=1,\ldots,n} \max_{t\in[0,1]} \left|D^\alpha \varphi_{i,n}(t)\right| n\sqrt{\lambda_1}\ \|f' - f\|$. Since both α and t are arbitrary the above bound is uniform, so $\left\|\hat{f}'_n - \hat{f}_n\right\| \to 0$ as $\|f' - f\| \to 0$, which ensures continuity—in terms of the distance induced by the norm $\|f\| = \max_{0\le\alpha\le1} \max_{0\le t\le1} \left|D^\alpha f(t)\right|$ in $C^1[0,1]$)—of the natural spline interpolant $\hat{f}_n$. This evidently implies that the mapping from S into S defined by the natural spline interpolation operator with knots at N_n, applied element by element of $Z = (z_1, \ldots, z_{2r})$, i.e., $\hat{Z}_n = (Z)_n = \left((z_1)_n, \ldots, (z_{2r})_n\right)$, is continuous with respect to the metric induced by the norm $\|Z\| = \max_{j=1,\ldots,2r} \max_{0\le\alpha\le1} \max_{0\le t\le1} \left|D^\alpha z_j(t)\right|$. (For brevity we use the same symbol, $\|.\|$, for the norms of

$C^1[0,1]$ and S; the proper interpretation will be clear in each case depending on the context.)

Since we have assumed that Z—the vector of time paths—is $\mathbf{B}(S)$-measurable, the approximant vector $\hat{Z}_n$, generated by natural spline interpolation, is also $\mathbf{B}(S)$-measurable, as it is obtained by a continuous (and so, measurable) transformation of Z.

Part (b) of the statement is obtained directly. For $n \geq 2$ the natural spline interpolant $\hat{Z}_n$ is unique and $\mathbf{B}(S)$-measurable, as established in part (a). Select an arbitrary point $\omega \in \Omega$. By Assumption $1'$.(i) each component of the observed path vector $Z(.,\omega)$ belongs to W^2[0,1], and Lemma A.1 establishes, given $\omega \in \Omega$, uniform convergence with respect to t, i.e., $\left|\left|\hat{Z}_n(.,\omega) - Z(.,\omega)\right|\right| \to 0$ as $n \to \infty$. Since Assumption $1'$.(ii) imposes that, for some $m > 0$ and each $\omega \in \Omega$, it holds $\min_{0 \leq t \leq 1} D^1 z_j(t,\omega) \geq m$, it is then obtained that, for each j and all large enough n (possibly depending on ω), it holds (uniformly in t) $D^1\hat{z}_j(t,\omega) = D^1 z_j(t,\omega) + \left(D^1\hat{z}_j(t,\omega) - D^1 z_j(t,\omega)\right) \geq m - m/2 = m' > 0$. This is a consequence of $\min_{j=1,\ldots,2r} \min_{0 \leq t \leq 1} D^1 z_j(t,\omega) \geq m$, which is ensured by Assumption $1'$.(ii) and the fact that $max_{0 \leq t \leq 1}\left|D^1\hat{z}_j(t,\omega) - D^1 z_j(t,\omega)\right| \to 0$ as $n \to \infty$ by Lemma A.1. Therefore, for each $\omega \in \Omega$ and n large enough, it holds $\min_{j=1,\ldots,2r} \min_{0 \leq t \leq 1} D^1\hat{z}_j(t,\omega) \geq m' > 0$, that is, $\hat{Z}_n(.,\omega) \in A_{m'}$.

It is easily checked that, for any $m' > 0$, the mappings $g(Z)$ and $h(Z)$ defining, respectively, *TE* and *LTE*, are continuous (and thus $\mathbf{B}(S)$-measurable) in $A_{m'}$, which in turn is a closed subset of S with nonempty interior, made up of all the vector functions Z in S with coordinates belonging to $W^2[0,1]$ and having $\min_{j=1,\ldots,2r} \min_{0 \leq t \leq 1} D^1\hat{z}_j(t,\omega) \geq m'$ for some $m' > 0$ fixed a priori and not depending on ω.

So, for each $\omega \in \Omega$ and n large enough, the sample realization of the interpolant for Z also has first derivative that is (uniformly in [0,1]) bigger than some $m' > 0$, i.e., $\hat{Z}_n(\omega) \in A_{m'}$. Therefore, with probability 1, it holds $\hat{Z}_n \in A_{m'}$ as $n \to \infty$, and $\hat{TE}_n = g\left(\hat{Z}_n\right)$ and $\hat{LTE}_n = h\left(\hat{Z}_n\right)$ are continuous (and so $\mathbf{B}(\Re)$-measurable) functions of $\hat{Z}_n$ for each n large enough.

Once we have established that, for large enough n, $\hat{TE}_n$ and $\hat{LTE}_n$ are random variables, convergence with probability 1 to *TE* y *LTE,* respectively, is derived by the same procedure as in Propositions 2 y 3, applied to an arbitrary realization $\omega \in \Omega$. For each $\omega \in \Omega$, the integrals appearing in the definitions of *TE* and *LTE* are classical Riemann integrals, so the proofs of Lemmas A.1–A.3 and Propositions 2 and 3 readily extend (for each fixed $\omega \in \Omega$) to the random case, with the only inconsequential issue that the Lipschitz bounds $(B_1, \ldots, B_6)$ will generally range with each realization ω of the random experiment.

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http://www.springer.com/978-3-319-07505-1

The Driving Forces of Change in Environmental Indicators
An Analysis Based on Divisia Index Decomposition
Techniques
Fernández González, P.; Landajo, M.; Presno, M.J.
2014, XII, 80 p. 16 illus., 14 illus. in color., Hardcover
ISBN: 978-3-319-07505-1