Chapter 2
Basic Kinematics

In applied dynamics, we draw a distinction between free systems with elements that can move without restrictions and constrained systems, the elements of which are bound to each other or their surroundings by means of bearings. While, for example, satellite dynamics is largely concerned with free systems, machine dynamics deals almost solely with constrained systems. This chapter will provide a summary of the kinematic principles behind mass point systems, multibody systems, and continuous systems. From a kinematic standpoint, finite element systems are considered as continuous systems and will therefore not be treated separately. The kinematics of free and constrained systems will be presented both in a spatially fixed inertial frame as well as in a frame in relative motion.

2.1 Free Systems

Free mechanical systems have especially simple kinematics since their motion is not subject to constraints from any kind of bearings. Their mathematical description is first undertaken with reference to a spatially fixed frame, though often generalized coordinates are used in addition to Cartesian coordinates.

2.1.1 Kinematics of a Mass Point

The mass point is the simplest model in mechanics. However, a single free point has little engineering importance. Free mass point systems on the other hand can be encountered in flying elastic structures (e.g. in aerospace engineering) or in systems containing only coupling elements instead of bearings. Moreover, any elastic continuum can be seen as a free system of infinitely many mass points. In free systems, all points are kinematically equal. For this reason, we will first examine the single free point in some detail.

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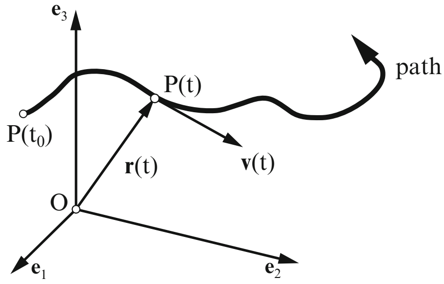


Fig. 2.1 Free motion of a mass point

The current position of a point in motion $P(t)$ at time t in space is clearly described with respect to the origin O of the fixed frame by means of the location vector $\boldsymbol{r}(t)$, see Fig. 2.1. In the course of time, the point in motion P changes its location, following the path marked by the position vector $\boldsymbol{r}(t)$. This motion is called displacement or translation.

Every location vector can be clearly resolved into its components in a Cartesian frame $\{O;\boldsymbol{e}_\alpha\}$, $\alpha = 1(1)3$, with the origin O and the basis vectors $\boldsymbol{e}_\alpha$. The current position is then defined for the location vector as

$$\boldsymbol{r}(t) = r_1(t)\boldsymbol{e}_1 + r_2(t)\boldsymbol{e}_2 + r_3(t)\boldsymbol{e}_3. \tag{2.1}$$

In a given frame, the location vector $\boldsymbol{r}(t)$ can thus be clearly represented, using (2.1), by the 3×1 vector of its coordinates

$$\boldsymbol{r}(t) = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}. \tag{2.2}$$

The coordinates are generally written without an argument, and no distinction is made between row and column vectors, see Sect. A.2 and (A.34).

A free point in space has three degrees of freedom. Three coordinates are required to describe these. In addition to the Cartesian coordinates $r_\alpha, \alpha = 1(1)3$, from (2.2), we can also make use of generalized coordinates $x_\gamma, \gamma = 1(1)3$, which are as a rule curvilinear. These generalized coordinates can then be merged into a 3×1 position vector

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}. \tag{2.3}$$

In general, there is a nonlinear correlation between the location vector $\boldsymbol{r}(t)$ and the position vector $\boldsymbol{x}(t)$,

$$\boldsymbol{r}(t) = \boldsymbol{r}(\boldsymbol{x}(t)) = \boldsymbol{r}(\boldsymbol{x}), \tag{2.4}$$

which in some cases simplifies the description of a point motion considerably. For example, circular motions can be represented more clearly by cylindrical coordinates than with Cartesian coordinates. Another example worth mentioning are the spatial central forces, which in spherical coordinates have only one nonzero coordinate. Some information on the notation used in this book can be found in Sect. A.1.

The velocity $\boldsymbol{v}(t)$ of point P is obtained by differentiation of (2.2) according to time, while its direction is determined by the tangent to the path, see Fig. 2.1. In a fixed frame (inertial frame), the 3×1 vector of the absolute velocity is thus written

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \left[\dot{r}_1 \; \dot{r}_2 \; \dot{r}_3 \right], \tag{2.5}$$

where $\dot{\boldsymbol{r}}$ is the derivative of $\boldsymbol{r}$ with respect to time t. The velocity can also be expressed in generalized coordinates. From (2.4) and (2.5), we obtain according to the chain rule

$$\boldsymbol{v}(t) = \boldsymbol{v}(\boldsymbol{x},\dot{\boldsymbol{x}}) = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt} = \boldsymbol{H}_T(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}}(t), \tag{2.6}$$

whereby we obtain the 3×3 Jacobian matrix of translation

$$\boldsymbol{H}_T(\boldsymbol{x}) = \begin{bmatrix} \dfrac{\partial r_1}{\partial x_1} & \dfrac{\partial r_1}{\partial x_2} & \dfrac{\partial r_1}{\partial x_3} \\ \dfrac{\partial r_2}{\partial x_1} & \dfrac{\partial r_2}{\partial x_2} & \dfrac{\partial r_2}{\partial x_3} \\ \dfrac{\partial r_3}{\partial x_1} & \dfrac{\partial r_3}{\partial x_2} & \dfrac{\partial r_3}{\partial x_3} \end{bmatrix}, \tag{2.7}$$

which establishes a relation between the location vector and the generalized coordinates. The velocity is therefore a linear function of the first time derivative $\dot{\boldsymbol{x}}(t)$ of the selected position vector.

The functional or Jacobian matrices are very important in applied dynamics. Their underlying mathematical principles are found in the differential and integral calculus of functions of several variables, see e.g. Bronstein and Semendjajew [12]. Since defining the Jacobian matrices element by element via scalar differential quotients is time-consuming, we shall resort to matrix notation. The 3×3 Jacobian matrix (2.7) in the form

$$\boldsymbol{H}_T(\boldsymbol{x}) = \frac{\partial \boldsymbol{r}(\boldsymbol{x})}{\partial \boldsymbol{x}} \tag{2.8}$$

thus follows from the 3×1 vector $\boldsymbol{r}(\boldsymbol{x})$ of the dependent variables and the 3×1 vector $\boldsymbol{x}$ of the independent variables, see (A.36). In this notation, the following relation is generally true for an $e \times 1$ vector $\boldsymbol{x}$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}} = \boldsymbol{E}, \tag{2.9}$$

where $\boldsymbol{E}$ is the $e \times e$ unit matrix. Also, for the $e \times 1$ vectors $\boldsymbol{r}$ and $\boldsymbol{x}$, we obtain a result corresponding to (2.9)

$$\frac{\partial \boldsymbol{r}(\boldsymbol{x}(\boldsymbol{r}))}{\partial \boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}} = \boldsymbol{E}. \tag{2.10}$$

In addition, the chain rule is written with an additional $f \times 1$ vector $\boldsymbol{y}$ as follows,

$$\frac{\partial \boldsymbol{r}(\boldsymbol{x}(\boldsymbol{y}))}{\partial \boldsymbol{y}} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}}, \tag{2.11}$$

yielding an $e \times f$ matrix. The computer-friendly notation introduced here will be used continually in the following. It also contains, for $e = 3$, the relations of the vector analysis.

The acceleration $\boldsymbol{a}(t)$ of point P is a measure of the change in time of its velocity and is determined by differentiation of (2.5) with respect to time. In a spatially fixed frame, the 3×1 vector of the absolute acceleration coordinates is thus defined by

$$\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t) = \left[\, \ddot{r}_1 \;\; \ddot{r}_2 \;\; \ddot{r}_3 \,\right]. \tag{2.12}$$

Acceleration can be expressed not only with Cartesian coordinates using (2.12), but also with generalized coordinates. With the product rule, (2.6) yields the relation

$$\begin{aligned} \boldsymbol{a}(t) = \boldsymbol{a}(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) &= \boldsymbol{H}_T(\boldsymbol{x}) \cdot \ddot{\boldsymbol{x}}(t) + \frac{d\boldsymbol{H}_T(\boldsymbol{x})}{dt} \cdot \dot{\boldsymbol{x}}(t) \\ &= \boldsymbol{H}_T(\boldsymbol{x}) \cdot \ddot{\boldsymbol{x}}(t) + \left(\frac{\partial \boldsymbol{H}_T(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \dot{\boldsymbol{x}}(t) \right) \cdot \dot{\boldsymbol{x}}(t). \end{aligned} \tag{2.13}$$

The acceleration is thus a linear function of the second derivative $\ddot{\boldsymbol{x}}(t)$ of the position vector. Moreover, it is also quadratically dependent on the first derivative $\dot{\boldsymbol{x}}(t)$ of the position vector.

All equations essential for the kinematics of a point have thereby been established.

Example 2.1 (Point Movement in Spherical Coordinates). For problems with centrally symmetric forces, the spherical coordinates ψ, ϑ, R, as shown in Fig. 2.2, are often to be recommended. The 3×1 position vector is then

$$\boldsymbol{x}(t) = \left[\, \psi \;\; \vartheta \;\; R \,\right]. \tag{2.14}$$

The 3×1 location vector then acquires the form

$$\boldsymbol{r}(t) = \begin{bmatrix} \cos\psi \sin\vartheta \\ \sin\psi \sin\vartheta \\ \cos\vartheta \end{bmatrix} R \tag{2.15}$$

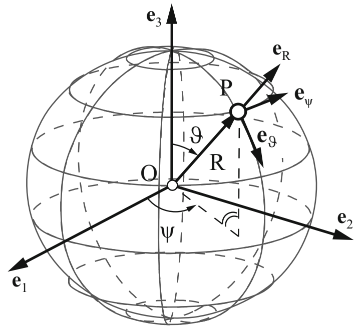


Fig. 2.2 Spherical coordinates

and the 3 × 3 Jacobian matrix of translation is written in accordance with (2.7) or (2.8)

$$\boldsymbol{H}_T(\boldsymbol{x}) = \begin{bmatrix} -R\sin\psi\sin\vartheta & R\cos\psi\cos\vartheta & \cos\psi\sin\vartheta \\ R\cos\psi\sin\vartheta & R\sin\psi\cos\vartheta & \sin\psi\sin\vartheta \\ 0 & -R\sin\vartheta & \cos\vartheta \end{bmatrix}. \tag{2.16}$$

With this, the 3 × 1 velocity vector is also determined with (2.6),

$$\boldsymbol{v}(\boldsymbol{x},\dot{\boldsymbol{x}}) = \begin{bmatrix} -R\dot{\psi}\sin\psi\sin\vartheta + R\dot{\vartheta}\cos\psi\cos\vartheta + \dot{R}\cos\psi\sin\vartheta \\ R\dot{\psi}\cos\psi\sin\vartheta + R\dot{\vartheta}\sin\psi\cos\vartheta + \dot{R}\sin\psi\sin\vartheta \\ -R\dot{\vartheta}\sin\vartheta + \dot{R}\cos\vartheta \end{bmatrix} \tag{2.17}$$

yielding the acceleration vector

$$\boldsymbol{a}(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) =$$
$$\begin{bmatrix} -R\ddot{\psi}\sin\psi\sin\vartheta + R\ddot{\vartheta}\cos\psi\cos\vartheta + \ddot{R}\cos\psi\sin\vartheta - R\dot{\psi}^2\cos\psi\sin\vartheta \\ -2R\dot{\psi}\dot{\vartheta}\sin\psi\cos\vartheta - 2\dot{R}\dot{\psi}\sin\psi\sin\vartheta - R\dot{\vartheta}^2\cos\psi\sin\vartheta + 2\dot{R}\dot{\vartheta}\cos\psi\cos\vartheta \\ R\ddot{\psi}\cos\psi\sin\vartheta + R\ddot{\vartheta}\sin\psi\cos\vartheta + \ddot{R}\sin\psi\sin\vartheta - R\dot{\psi}^2\sin\psi\sin\vartheta \\ +2R\dot{\psi}\dot{\vartheta}\cos\psi\cos\vartheta + 2\dot{R}\dot{\psi}\cos\psi\sin\vartheta - R\dot{\vartheta}^2\sin\psi\sin\vartheta + 2\dot{R}\dot{\vartheta}\sin\psi\cos\vartheta \\ -R\ddot{\vartheta}\sin\vartheta + \ddot{R}\cos\vartheta - R\dot{\vartheta}^2\cos\vartheta - 2\dot{R}\dot{\vartheta}\sin\vartheta \end{bmatrix}. \tag{2.18}$$

The acceleration vector is linearly dependent on the second derivatives and quadratically dependent on the first derivatives of the generalized coordinates.

End of Example 2.1.

By introducing generalized coordinates, the uniqueness of the kinematic description at singular points may be lost due to a loss of degrees of freedoms. Thus we should always require the full rank of the Jacobian matrix or

$$\det \boldsymbol{H}_T \neq 0. \tag{2.19}$$

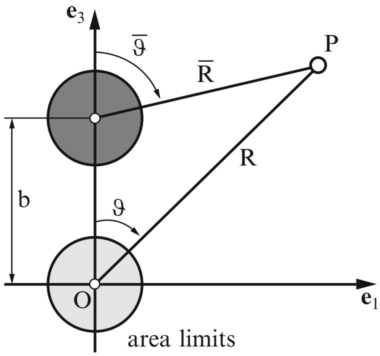


Fig. 2.3 Definition of complementary spherical coordinates

In Example 2.1, according to (2.16) there is a rank decrease of two in matrix $\boldsymbol{H}_T$ for $R = 0$, which certainly violates (2.19). The explanation for this is that the point P for $R \to 0$ can now only move in the direction

$$\boldsymbol{e}_R = \begin{bmatrix} \cos\psi \sin\vartheta & \sin\psi \sin\vartheta & \cos\vartheta \end{bmatrix}, \tag{2.20}$$

so it now only has one degree of freedom. This problem can be solved by the introducing complementary spherical coordinates $\overline{\psi}$, $\overline{\vartheta}$, see Fig. 2.3. By extending (2.15), we then obtain

$$\boldsymbol{r}(t) = \begin{bmatrix} R\cos\psi\sin\vartheta \\ R\sin\psi\sin\vartheta \\ R\cos\vartheta \end{bmatrix} = \begin{bmatrix} \overline{R}\cos\psi\sin\overline{\vartheta} \\ \overline{R}\sin\psi\sin\overline{\vartheta} \\ \overline{R}\cos\overline{\vartheta} + b \end{bmatrix}, \tag{2.21}$$

i.e. two different singular points $R = 0$ and $\overline{R} = 0$ appear, where $b > 0$ is an arbitrary distance. If we now limit, for example, the critical generalized coordinates by the areas shown in Fig. 2.3,

$$R \geq b/4, \qquad \overline{R} \geq b/4, \tag{2.22}$$

then with position vectors that are complementary to each other $\boldsymbol{x}(t)$ and $\overline{\boldsymbol{x}}(t)$, a unique description of position is always possible. If one of the limits (2.22) is violated, transition to the complementary spherical coordinates takes place and vice versa. Following (2.21), we have for this, for example, the following relation

$$\overline{R} = R\,\frac{\sin\vartheta}{\sin\overline{\vartheta}}, \qquad \cot\overline{\vartheta} = \cot\vartheta - \frac{b}{R\sin\vartheta}. \tag{2.23}$$

For many motions, the singular points are not critical. For example, the planets always move at a great distance from the singular point at the origin. Yet in the case of rotating rigid bodies we constantly find singular points, to which numerous

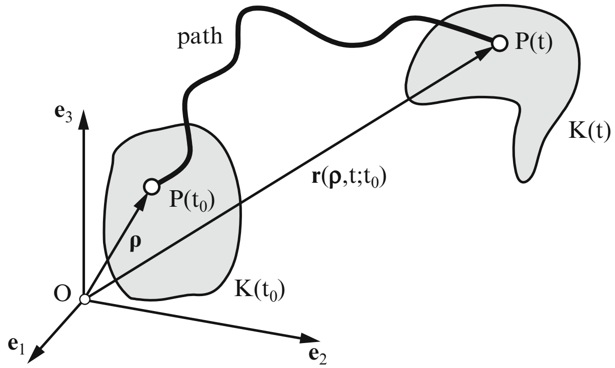


Fig. 2.4 Motion of a free body

papers have been dedicated in gyroscope theory. It is therefore worthwhile to deal with this problem already at the level of the point motion.

A free system of p mass points in space has $3p$ degrees of freedom. If we merge the $3p$ generalized coordinates of the total system into a $3p \times 1$ position vector $\boldsymbol{x}(t)$, the ith point is defined in accordance with (2.4) as

$$\boldsymbol{r}_i(t) = \boldsymbol{r}_i(\boldsymbol{x}), \qquad i = 1(1)p. \tag{2.24}$$

The relations (2.5), (2.6) and (2.12), (2.13) are also true for mass point systems. In particular, (2.8) turns into a $3 \times 3p$ Jacobian matrix $\boldsymbol{H}_{Ti}(\boldsymbol{x}), i = 1(1)p$.

2.1.2 Kinematics of a Rigid Body

The rigid body is a simple model of continuum mechanics. Like all continua, it consists of a coherent, compact quantity of mass points. However, in a rigid body, the distances between arbitrary mass points are constant. From the standpoint of continuum mechanics, a rigid body is thus free of strain. Yet it is also statically indeterminate, i.e. the forces and stresses arising in its interior cannot be calculated, see Sect. 5.4.2. Nonetheless, the rigid body is eminently suitable for the investigation of motions in many contexts within dynamics. This is especially the case for systems of rigid bodies, called multibody systems.

In order to describe free multibody systems kinematically, such as they appear in rotor dynamics for example, it is again sufficient to consider a single rigid body. In a free system, all rigid bodies are kinematically equal.

An arbitrary, rigid or nonrigid body K is described mathematically by its reference configuration, i.e. a constant and reversibly unique assignment of location vectors $\boldsymbol{\rho}$ to the mass points, see Fig. 2.4. If nothing else is stipulated, we use an

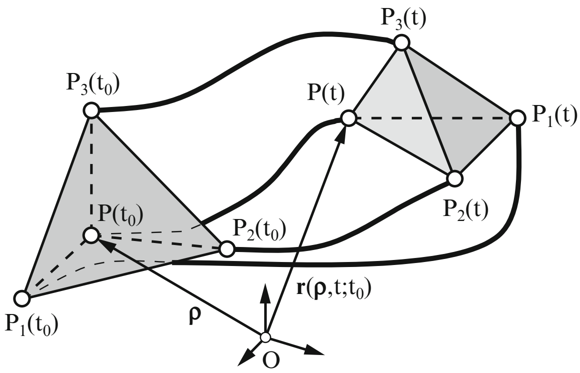


Fig. 2.5 Motion of a free tetrahedral element

inertial Cartesian frame $\{O, \boldsymbol{e}_\alpha\}, \alpha = 1(1)3$, and a nonpolar continuum. The current configuration of a body $K(t)$ in motion at time t in space,

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{\rho}, t; t_0), \tag{2.25}$$

is referred to the reference configuration of the body $K(t_0)$ at the reference time t_0,

$$\boldsymbol{\rho} = \boldsymbol{r}(\boldsymbol{\rho}, t_0; t_0). \tag{2.26}$$

On the other hand, the location vectors $\boldsymbol{\rho}$ are also determined by the inverse function of (2.25),

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\boldsymbol{r}, t; t_0), \tag{2.27}$$

yielding a unique assignment to the mass points. The fixed reference time t_0 is taken as the basis for Eqs. (2.25)–(2.27). Yet it is also possible to select the running time $t_0 = t$ as the reference time. We then obtain $\boldsymbol{\rho}(\boldsymbol{r}, t; t) = \boldsymbol{r}$, i.e. the mass point P designated by $\boldsymbol{\rho}$ coincides at the moment with the point in space described by $\boldsymbol{r}$. The running reference time t will prove useful for determining the current rotation velocity vector.

In the following, the variables $\boldsymbol{\rho}$, t and t_0 will only be written if needed. This does not affect the explicit dependence of the parameters considered in these variables. The coordinates of the vector $\boldsymbol{\rho}$ are also called material coordinates, while the coordinates of the vector $\boldsymbol{r}$ are designated as spatial coordinates.

The general motion of a nonrigid body K is composed of rotations and strains. This motion is called deformation. Since deformation changes from point to point within the body, it is properly characterized by the deformation gradient $\boldsymbol{F}(\boldsymbol{\rho}, t; t_0) = \partial \boldsymbol{r} / \partial \boldsymbol{\rho}$. The deformation gradient describes, for example, the motion of a tetrahedral element from a reference configuration into the current configuration, as shown in Fig. 2.5. A tetrahedral element comprises four infinitesimally neighboring mass points P, P_1, P_2, P_3. The line elements between point P and points P_1, P_2, P_3 are thus transformed from the respective reference configuration $d\boldsymbol{\rho}$ into the respective current configuration $d\boldsymbol{r}$. This transformation is effected by the deformation gradient $\boldsymbol{F}(\boldsymbol{\rho},t;t_0)$. From (2.25) we obtain

$$\boldsymbol{r}(\boldsymbol{\rho}+d\boldsymbol{\rho},t;t_0)-\boldsymbol{r}(\boldsymbol{\rho},t;t_0)=d\boldsymbol{r}=\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\rho}}\cdot d\boldsymbol{\rho}=\boldsymbol{F}(\boldsymbol{\rho},t;t_0)\cdot d\boldsymbol{\rho}. \tag{2.28}$$

Conversely, we obtain with (2.27) and (2.25)

$$d\boldsymbol{\rho}=\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{r}}\cdot d\boldsymbol{r}=\boldsymbol{F}^{-1}(\boldsymbol{\rho},t;t_0)\cdot d\boldsymbol{r}. \tag{2.29}$$

In order to avoid ambiguity in (2.25) and (2.27), the deformation gradient in (2.28) and (2.29) must be always regular, $\det\boldsymbol{F}\neq 0$. Due to (2.26), (2.28) also results in $\boldsymbol{F}(\boldsymbol{\rho},t_0;t_0)=\boldsymbol{E}$ and thus $\det\boldsymbol{F}(\boldsymbol{\rho},t_0;t_0)=+1$. Here $\boldsymbol{E}$ is again the 3×3 unit tensor.

Furthermore, if we take the consistency of the deformation into consideration, we obtain the condition

$$\det\boldsymbol{F}>0. \tag{2.30}$$

The current configuration of the tetrahedral element under consideration is determined by a total of 12 coordinates corresponding to the 12 degrees of freedom of the 4 mass points. These 12 coordinates can also be interpreted as the 3 displacement coordinates of the 3×1 location vector at point P and the 9 coordinates of the 3×3 tensor of the deformation gradient.

For a rigid body K, the pairwise distances of all mass points remain constant during deformation,

$$d\boldsymbol{r}\cdot d\boldsymbol{r}=\boldsymbol{F}\cdot d\boldsymbol{\rho}\cdot\boldsymbol{F}\cdot d\boldsymbol{\rho}=d\boldsymbol{\rho}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot d\boldsymbol{\rho}\overset{!}{=}d\boldsymbol{\rho}\cdot d\boldsymbol{\rho}. \tag{2.31}$$

We thus find for the deformation gradient of the rigid body

$$\boldsymbol{F}^T\cdot\boldsymbol{F}=\boldsymbol{E}. \tag{2.32}$$

Thus this deformation gradient $\boldsymbol{F}$ is independent of the location vector $\boldsymbol{\rho}$ of the mass points of the rigid body. It can therefore only be a function of time t. The deformation gradient thus corresponds to the 3×3 rotation tensor $\boldsymbol{S}(t;t_0)$ of the rigid body,

$$\boldsymbol{F}(\boldsymbol{\rho},t;t_0)=\boldsymbol{S}(t;t_0). \tag{2.33}$$

As a result of (2.30) and (2.32), the rotation tensor $\boldsymbol{S}(t;t_0)$ is actually an orthogonal tensor. In the following, the rotation tensor will always be applied to the reference configuration, so we need not write the reference time t_0.

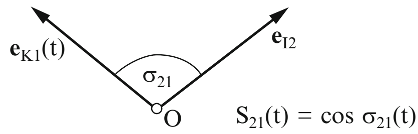


Fig. 2.6 Direction cosine

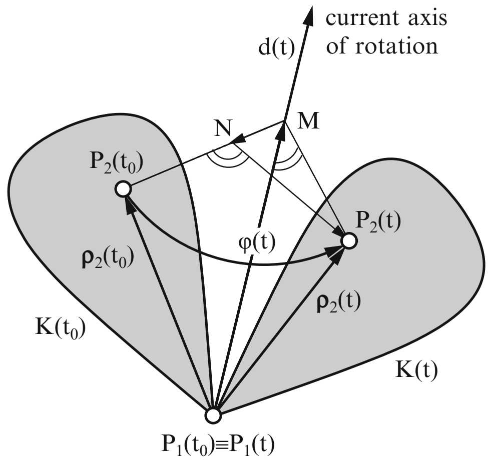


Fig. 2.7 Finite rotation of a rigid body

We will now introduce the properties of the rotation of a rigid body in detail. Various possible methods of description will be used in the process, either via the nine direction cosines, four rotation parameters or three rotation angles. In each case, there remain three generalized coordinates corresponding to the three degrees of freedom of the rotation of a rigid body.

Every Cartesian coordinate $S_{\alpha\beta}(t)$, $\alpha,\beta = 1(1)3$, of the rotation tensor (2.33) can be viewed as a direction cosine of the angle $\sigma_{\alpha\beta}(t)$ between the basis vector $\boldsymbol{e}_{I\alpha}$ of the spatially fixed inertial frame I and the basis vector $\boldsymbol{e}_{K\beta}(t)$ of the corresponding body-fixed frame K, see Fig. 2.6. The Cartesian body-fixed frame $\{P(t);\boldsymbol{e}_{K\beta}(t)\}$, $\beta = 1(1)3$ here coincides at time $t = t_0$ with the inertial frame,

$$\{P(t_0);\boldsymbol{e}_{K\beta}(t_0)\} = \{0;\boldsymbol{e}_{I\alpha}\}. \tag{2.34}$$

The nine direction cosines $S_{\alpha\beta}, \alpha, \beta = 1(1)3$, are subject to the six orthogonality constraints (2.32), so only three generalized coordinates remain.

The rotation tensor $\boldsymbol{S}$ according to (2.33) can also be expressed by the four rotation parameters, i.e. the three coordinates of the vector $\boldsymbol{d}$, normalized to length one, of the rotation axis and the scalar rotation angle $\varphi(t)$. The representation of a finite rotation by its rotation axis and a rotation angle is due to Euler. For this reason, we also denote the four rotation parameters as Euler parameters.

From Fig. 2.7 we conclude on the one hand

$$\boldsymbol{\rho}_2(t) = \boldsymbol{S}(t)\cdot\boldsymbol{\rho}_2(t_0), \tag{2.35}$$

while on the other hand we obtain from the vector polygon P_1MNP_2 the relation

$$\boldsymbol{\rho}_2(t) = \boldsymbol{dd}\cdot\boldsymbol{\rho}_2(t_0) + (\boldsymbol{\rho}_2(t_0) - \boldsymbol{dd}\cdot\boldsymbol{\rho}_2(t_0))\cos\varphi + \tilde{\boldsymbol{d}}\cdot\boldsymbol{\rho}_2(t_0)\sin\varphi. \tag{2.36}$$

Comparison of (2.35) and (2.36) directly yields

$$\boldsymbol{S}(t) = \boldsymbol{dd} + (\boldsymbol{E} - \boldsymbol{dd})\cos\varphi + \tilde{\boldsymbol{d}}\sin\varphi. \tag{2.37}$$

The skew-symmetric 3×3 tensor $\tilde{\boldsymbol{d}}$ of the 3×1 vector $\boldsymbol{d}$ and its dyadic product $\boldsymbol{dd}$ were introduced into (2.36), see Appendix A,

$$\boldsymbol{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \quad \tilde{\boldsymbol{d}} = -\tilde{\boldsymbol{d}}^T = \begin{bmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{bmatrix}, \quad \boldsymbol{dd} = \begin{bmatrix} d_1d_1 & d_1d_2 & d_1d_3 \\ d_2d_1 & d_2d_2 & d_2d_3 \\ d_3d_1 & d_3d_2 & d_3d_3 \end{bmatrix}. \tag{2.38}$$

The skew-symmetric tensor of a vector is denoted by the symbol $(\tilde{\ })$. It gives the cross or vector product

$$\tilde{\boldsymbol{a}}\cdot\boldsymbol{b} = \boldsymbol{a}\times\boldsymbol{b}. \tag{2.39}$$

Between the dyadic product $\boldsymbol{ab}$, the scalar product $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$, and the expanded vector product $\tilde{\boldsymbol{a}}\cdot\tilde{\boldsymbol{b}}$, there is also, using (A.30), the useful relation

$$\boldsymbol{ab} = (\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{E} + \tilde{\boldsymbol{b}}\cdot\tilde{\boldsymbol{a}}. \tag{2.40}$$

If we now consider that the vector $\boldsymbol{d}$ of the rotation axis is a unit vector,

$$\boldsymbol{d}\cdot\boldsymbol{d} = 1, \tag{2.41}$$

which corresponds exactly to a constraint between the four rotation parameters $d_\alpha, \alpha = 1(1)3$, and φ, we then obtain from (2.40) the relation

$$\boldsymbol{dd} = \boldsymbol{E} + \tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{d}}. \tag{2.42}$$

Then (2.37) can be written as

$$\boldsymbol{S}(t) = \boldsymbol{E} + \tilde{\boldsymbol{d}}\sin\varphi + \tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{d}}(1-\cos\varphi). \tag{2.43}$$

We can see that for $t = t_0$ the rotation tensor turns into the unit tensor $\boldsymbol{E}$ due to $\varphi(t_0) = 0$.

Closely related to the four rotation parameters are the four quaternions $q_n(t)$, $n = 0(1)3$, which we obtain following the transition to half the rotation angle

$$q_0 = \cos\frac{\varphi}{2}, \qquad \boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \boldsymbol{d}\sin\frac{\varphi}{2}. \tag{2.44}$$

Equation (2.43) thereby acquires the form

$$\boldsymbol{S}(t) = \boldsymbol{E} + 2q_0\tilde{\boldsymbol{q}} + 2\tilde{\boldsymbol{q}}\cdot\tilde{\boldsymbol{q}} \tag{2.45}$$

and the constraint (2.41) becomes

$$q_0^2 + \boldsymbol{q}\cdot\boldsymbol{q} = 1. \tag{2.46}$$

The three Rodrigues parameters $p_\alpha(t), \alpha = 1(1)3$, are obtained by normalizing the quaternions. They can be represented as a 3×1 vector $\boldsymbol{p}$,

$$\boldsymbol{p} = \boldsymbol{d}\tan\frac{\varphi}{2} = \frac{1}{q_0}\boldsymbol{q}. \tag{2.47}$$

The rotation tensor then has the form

$$\boldsymbol{S}(t) = \boldsymbol{E} + 2\frac{\tilde{\boldsymbol{p}} + \tilde{\boldsymbol{p}}\cdot\tilde{\boldsymbol{p}}}{1 + \boldsymbol{p}\cdot\boldsymbol{p}} \tag{2.48}$$

which is a nice, compact expression.

The four rotation parameters can also be determined conversely from the rotation tensor. To achieve this we can utilize, for example, the fact that a truly orthogonal 3×3 tensor has the eigenvalues $\lambda_1 = 1$, $\lambda_{2,3} = e^{\pm i\varphi}$. The eigenvector belonging to the real eigenvalue describes the rotation axis, while the argument φ of the imaginary eigenvalues gives the rotation angle. However, the rotation direction cannot be found by solving the eigenvalue problem. To do this, an additional comparison with the rotation tensor (2.43) is required. For $\varphi = 0, 2\pi, 4\pi, \ldots$, the rotation tensor has a triple eigenvalue $\lambda_{1,2,3} = 1$. Then each unit vector is also an eigenvector and consequently also a rotation axis.

Example 2.2 (Rotation Axis and Rotation Angle of a Rigid Body). Let a rotation tensor $\boldsymbol{S}(t)$ be given by

$$\boldsymbol{S}(t) = \begin{bmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{bmatrix}. \tag{2.49}$$

The eigenvalue problem

$$(\lambda\boldsymbol{E} - \boldsymbol{S})\cdot\boldsymbol{d} = \boldsymbol{0} \tag{2.50}$$

provides the characteristic equation

$$(\lambda - 1)(\lambda^2 - 2\lambda\cos\vartheta + 1) = 0 \tag{2.51}$$

with the eigenvalues

$$\lambda_1 = 1, \qquad \lambda_{2,3} = e^{\pm i\vartheta}. \tag{2.52}$$

The first normalized eigenvector is

$$\boldsymbol{d} = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}, \tag{2.53}$$

where the sign has been determined by inserting into (2.43) and comparing with (2.49). For the quaternions we obtain

$$\begin{aligned} q_0^2(t) &= \frac{1}{2}(1+\cos\vartheta) = \cos^2\frac{\vartheta}{2}, \qquad q_1^2(t) = 0, \\ q_2^2(t) &= \frac{1}{2}(1-\cos\vartheta) = \sin^2\frac{\vartheta}{2}, \qquad q_3^2(t) = 0. \end{aligned} \tag{2.54}$$

As a result of the quadratic quantities, here also the rotation direction must be determined by comparison with (2.49).

End of Example 2.2.

The four rotation parameters $d_\alpha(t), \alpha = 1(1)3$, and $\varphi(t)$ and the four quaternions $q_n(t), n = 0(1)3$, are subject to exactly one constraint, so here again only three generalized coordinates remain. The three Rodrigues parameters $p_\alpha(t)$ from (2.47) on the other hand can be used directly as generalized coordinates. However, their engineering applicability is limited by the infinite values of the tangent function for $\varphi = \pi/2, 3\pi/2, 5\pi/2, \ldots$.

Finally, the rotation tensor (2.33) can also be expressed by means of three rotation angles from elementary rotations. Elementary rotations exist when the rotation axis coincides with one of the coordinate axes. They are defined by the name of the rotation angle and the specification of the rotation axis. There are three elementary rotation matrices, these corresponding to the three basis vectors of a Cartesian frame.

In order to construct a unique rotation tensor, we now make use of the property that orthogonality is preserved in the multiplication of orthogonal tensors, and we restrict ourselves also to three independent angles as generalized coordinates

$$\boldsymbol{\alpha}_1(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \tag{2.55}$$

$$\boldsymbol{\beta}_2(t) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \tag{2.56}$$

$$\boldsymbol{\gamma}_3(t) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.57}$$

Among the numerous possible ways to describe finite rotations with three generalized coordinates, here we will mention the Euler angles

$$\boldsymbol{S}(t) = \boldsymbol{\psi}_3(t) \cdot \boldsymbol{\vartheta}_1(t) \cdot \boldsymbol{\varphi}_3(t) \tag{2.58}$$

and the Cardano angles

$$\boldsymbol{S}(t) = \boldsymbol{\alpha}_1(t) \cdot \boldsymbol{\beta}_2(t) \cdot \boldsymbol{\gamma}_3(t). \tag{2.59}$$

In the construction of rotation tensors from elementary rotations, it should also be kept in mind that the tensor product is not commutative. For this reason, a complete definition includes not only the angle name and rotation axis, but also the sequence of elementary rotations. If we now evaluate (2.59) with (2.55)–(2.57), we obtain

$$\boldsymbol{S}(t) = \begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{bmatrix}. \tag{2.60}$$

The Cardano angles can now be found conversely from the rotation tensor. For this purpose it is appropriate to use the sparsely populated coordinates, e.g.,

$$\sin\beta = S_{13}, \qquad \cos\alpha = \frac{S_{33}}{\cos\beta}, \qquad \cos\gamma = \frac{S_{11}}{\cos\beta}. \tag{2.61}$$

Singular rotation angles $\beta = \pi/2, 3\pi/2, 5\pi/2, \ldots$ exist here for $\cos\beta = 0$. They come from the fact that two elementary rotation axes coincide and thus one degree of freedom in the rotation is lost. This is especially clear if we consider, for example, the rotation tensor (2.60) in the area of a singularity, $\alpha = \Delta\alpha$, $\beta = \pi/2 + \Delta\beta$, $\gamma = \Delta\gamma$ with $\Delta\alpha, \Delta\beta, \Delta\gamma \ll 1$,

$$\boldsymbol{S}(t) = \begin{bmatrix} -\Delta\beta & 0 & 1 \\ (\Delta\alpha + \Delta\gamma) & 1 & 0 \\ -1 & (\Delta\alpha + \Delta\gamma) & -\Delta\beta \end{bmatrix}. \tag{2.62}$$

Only the sum of the angles $(\Delta\alpha + \Delta\gamma)$ and the single angle $\Delta\beta$ remain as generalized coordinates.

The singularities of rotation angles can be avoided by limiting the angle of the second elementary rotation and by introducing complementary rotation angles. If we limit, e.g., the second Cardano angle

$$-\pi/3 < \beta < \pi/3, \tag{2.63}$$

and if we add the complementary Cardano angle to (2.59)

$$\boldsymbol{S}(t) = \overline{\boldsymbol{\alpha}}_1(t)\cdot\overline{\boldsymbol{\beta}}_2(t)\cdot\overline{\boldsymbol{\gamma}}_1(t), \qquad |\overline{\beta}| > \pi/6, \tag{2.64}$$

then no singularity arises. For $\overline{\alpha} = \Delta\overline{\alpha}$, $\overline{\beta} = \frac{\pi}{2} + \Delta\overline{\beta}$, $\overline{\gamma} = \Delta\overline{\gamma}$ with $\Delta\overline{\alpha}$, $\Delta\overline{\beta}$, $\Delta\overline{\gamma} \ll 1$ we obtain from (2.64) the rotation tensor

$$\boldsymbol{S}(t) = \begin{bmatrix} -\Delta\overline{\beta} & \Delta\overline{\gamma} & 1 \\ \Delta\overline{\alpha} & 1 & \Delta\overline{\gamma} \\ -1 & \Delta\overline{\alpha} & -\Delta\overline{\beta} \end{bmatrix}. \tag{2.65}$$

With this we obtain three independent coordinates $\Delta\overline{\alpha}, \Delta\overline{\beta}, \Delta\overline{\gamma}$. At the boundaries (2.63) and (2.64), transformation of the angle takes place over the sparsely populated coordinates of both rotation tensors. The intersecting boundaries guarantee however a low number of shifts between (2.59) and (2.64). In particular, the relations

$$\beta = \arcsin(\sin\overline{\beta}\cos\overline{\gamma}), \tag{2.66}$$

$$\sin\alpha = \frac{1}{\cos\beta}(\cos\overline{\alpha}\sin\overline{\gamma} + \sin\overline{\alpha}\cos\overline{\beta}\cos\overline{\gamma}), \tag{2.67}$$

$$\cos\alpha = \frac{1}{\cos\beta}(-\sin\overline{\alpha}\sin\overline{\gamma} + \cos\overline{\alpha}\cos\overline{\beta}\cos\overline{\gamma}), \tag{2.68}$$

$$\sin\gamma = -\frac{1}{\cos\beta}(\sin\overline{\beta}\sin\overline{\gamma}), \tag{2.69}$$

$$\cos\gamma = \frac{1}{\cos\beta}\cos\overline{\beta} \tag{2.70}$$

apply and the complementary relations

$$\overline{\beta} = \arccos(\cos\beta\cos\gamma), \tag{2.71}$$

$$\sin\overline{\alpha} = \frac{1}{\sin\overline{\beta}}(\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma), \tag{2.72}$$

$$\cos\overline{\alpha} = \frac{1}{\sin\overline{\beta}}(-\sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma), \tag{2.73}$$

$$\sin\overline{\gamma} = -\frac{1}{\sin\overline{\beta}}(\cos\beta\sin\gamma), \tag{2.74}$$

$$\cos\overline{\gamma} = \frac{1}{\sin\overline{\beta}}\sin\beta. \tag{2.75}$$

The elementary rotations permit us to construct, by means of numerous combination possibilities, a suitable rotation tensor for every engineering problem. Especially

Table 2.1 Possible methods for describing the rotation of a rigid body

Coordinates of the rotation tensor	Relations of the coordinates	Generalized coordinates
9 direction cosines	6 constraints	e.g.
$\boldsymbol{S}(t)$	$\boldsymbol{S}\cdot\boldsymbol{S}^T=\boldsymbol{E}$	$S_{11}(t), S_{12}(t), S_{23}(t)$
4 rotation parameters	1 constraint	e.g.
$\boldsymbol{d}(t), \varphi(t)$	$\boldsymbol{d}\cdot\boldsymbol{d}=1$	$d_1(t), d_2(t), \varphi(t)$
4 quaternions	1 constraint	e.g.
$q_0(t)\ \boldsymbol{q}(t)$	$q_0^2+\boldsymbol{q}\cdot\boldsymbol{q}=1$	$q_0(t), q_1(t), q_2(t)$
3 Euler angles		
$\psi(t), \vartheta(t), \varphi(t)$	–	$\psi(t), \vartheta(t), \varphi(t)$
3 Cardano angles		
$\alpha(t), \beta(t), \gamma(t)$	–	$\alpha(t), \beta(t), \gamma(t)$

flight mechanics and theory of gyroscopes make extensive use of this, see e.g. Magnus [35] or Arnold and Maunder [2].

Possible methods of describing the rotation of a rigid body are summarized in Table 2.1. If we merge the remaining generalized coordinates of rotation again in a 3×1 position vector

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \tag{2.76}$$

then the following applies quite generally,

$$\boldsymbol{S}(t) = \boldsymbol{S}(\boldsymbol{x}(t)) = \boldsymbol{S}(\boldsymbol{x}), \tag{2.77}$$

independent of the particular choice of generalized coordinates.

A rigid body is in a state of general motion when its rotation is complemented with displacement. According to Fig. 2.8, the current configuration of the rigid body K is then

$$\boldsymbol{r}(\boldsymbol{\rho},t) = \boldsymbol{r}_1(t) + \boldsymbol{r}_P(\boldsymbol{\rho},t) = \boldsymbol{r}_1(t) + \boldsymbol{S}(t)\cdot\boldsymbol{\rho}. \tag{2.78}$$

This equation, first introduced in an intuitive manner, can also be found formally by integrating (2.28) with a fixed reference time t_0. In the case of the rigid body, this integration is possible in a closed form since the deformation gradient, in accordance with (2.33), does not depend on the mass coordinates.

In (2.78), $\boldsymbol{r}_1(t)$ is the 3×1 location vector of point $P_1(t)$. It describes the translation of the rigid body. The 3×3 rotation tensor $\boldsymbol{S}(t)$ denotes the rotation of the rigid body. Translation, as opposed to rotation, does not contribute to the deformation gradient, as follows from (2.78), (2.28). Furthermore, the following applies according to (2.29), (2.32) and (2.33) for the inverse deformation,

$$\boldsymbol{\rho} = \boldsymbol{S}^T(t)\cdot\boldsymbol{r}_P(\boldsymbol{\rho},t). \tag{2.79}$$

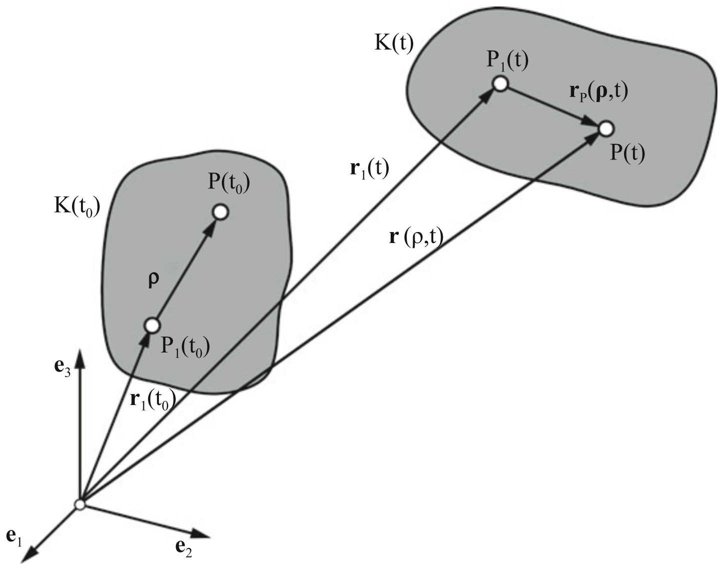


Fig. 2.8 Motion of a free rigid body

Inserted into (2.78) we obtain

$$\begin{aligned} \boldsymbol{r}(\boldsymbol{\rho},t) &= \boldsymbol{r}_1(t) + \boldsymbol{S}(t;t_0)\cdot\boldsymbol{S}^T(t;t_0)\cdot\boldsymbol{r}_P(\boldsymbol{\rho},t) \\ &= \boldsymbol{r}_1(t) + \boldsymbol{S}(t;t_0)\cdot\boldsymbol{S}(t_0;t)\cdot\boldsymbol{r}_P(\boldsymbol{\rho},t) = \boldsymbol{r}_1(t) + \boldsymbol{S}(t;t)\cdot\boldsymbol{r}_P(\boldsymbol{\rho},t) \\ &= \boldsymbol{r}_1(t) + \boldsymbol{r}_P(\boldsymbol{\rho},t). \end{aligned} \tag{2.80}$$

The instantaneous rotation tensor $\boldsymbol{S}(t,t) = \boldsymbol{E}$ is independent of the reference time t_0. It thus has the property of being a field in the sense of continuum mechanics.

A free rigid body has six degrees of freedom. For the three degrees of freedom of translation all the relations of point kinematics are valid, see Sect. 2.1.1. According to Table 2.1, the three degrees of freedom of rotation also require three generalized coordinates, so on the whole the 6×1 position vector

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix} \tag{2.81}$$

describes the general motion of a rigid body. The location vector and rotation tensor of the rigid body are thus written

$$\boldsymbol{r}(t) = \boldsymbol{r}(\boldsymbol{x}), \qquad \boldsymbol{S}(t) = \boldsymbol{S}(\boldsymbol{x}). \tag{2.82}$$

In special cases of pure translation or rotation (2.82) becomes (2.4) or (2.77), whereby the number of degrees of freedom is in both cases reduced to three.

The current velocity of a point on the rigid body K is obtained by the time derivative of (2.78) in the form

$$\boldsymbol{v}(\boldsymbol{\rho},t) = \frac{d}{dt}\boldsymbol{r}(\boldsymbol{\rho},t) = \dot{\boldsymbol{r}}_1(t) + \dot{\boldsymbol{S}}(t)\cdot\boldsymbol{\rho} \tag{2.83}$$

since $\boldsymbol{\rho}$ does not depend on time. If we also take (2.79) into consideration, we can also write

$$\boldsymbol{v}(\boldsymbol{\rho},t) = \dot{\boldsymbol{r}}_1(t) + \dot{\boldsymbol{S}}(t)\cdot\boldsymbol{S}^T(t)\cdot\boldsymbol{r}_P(\boldsymbol{\rho},t). \tag{2.84}$$

The first term on the right-hand side corresponds to the translation velocity of the reference point P_1 from (2.5). The second term clearly is based on rotation and should be inspected more closely here. From (2.84) we obtain, with a Taylor series expansion with respect to dt taking into account the orthogonality of $\boldsymbol{S}$, the result

$$\begin{aligned}\dot{\boldsymbol{S}}(t)\cdot\boldsymbol{S}^T(t) &= \frac{\boldsymbol{S}(t+dt;t_0)-\boldsymbol{S}(t;t_0)}{dt}\cdot\boldsymbol{S}^T(t;t_0)\\ &= \frac{\boldsymbol{S}(t+dt;t)-\boldsymbol{E}}{dt} = \tilde{\boldsymbol{d}}(t;t)\frac{d\varphi(t;t)}{dt} = \frac{d\tilde{\boldsymbol{s}}(t)}{dt} = \tilde{\omega}(t).\end{aligned} \tag{2.85}$$

Here $\tilde{\boldsymbol{d}}(t;t)$ denotes the rotation axis and $\dot{\varphi}(t;t)$ the velocity of the instantaneous rotation. Also, $d\tilde{\boldsymbol{s}}(t) = \tilde{\boldsymbol{d}}(t;t)d\varphi(t;t)$ is designated as the 3×3 tensor of infinitesimal instantaneous rotation and $\tilde{\boldsymbol{\omega}}(t)$ as the 3×3 tensor of rotation velocity. In accordance with (2.38), to this tensor is assigned the 3×1 vector $\boldsymbol{\omega}(t)$ of rotation velocity.

Infinitesimal rotation $d\boldsymbol{s}(t)$ thus has, in contrast to finite rotation, the property of being a vector. Moreover, it also no longer depends on the reference time and is thus a field quantity in the sense of continuum mechanics. Accordingly, (2.84) can also be written as

$$\boldsymbol{v}(\boldsymbol{\rho},t) = \boldsymbol{v}_1(t) + \tilde{\boldsymbol{\omega}}(t)\cdot\boldsymbol{r}_P(\boldsymbol{\rho},t) = \boldsymbol{v}_1(t) + \boldsymbol{\omega}(t)\times\boldsymbol{r}_P(\boldsymbol{\rho},t), \tag{2.86}$$

which corresponds to the known formula for the velocity field of a rigid body. The 3×1 vectors $\boldsymbol{v}_1(t)$ and $\boldsymbol{\omega}(t)$ clearly describe the state of velocity of the rigid body. They can also be merged into the 6×1 twist $(\boldsymbol{v}_1(t),\boldsymbol{\omega}(t))$, see Sect. 5.7.2.

In order to calculate the rotational velocity vector $\boldsymbol{\omega}(t)$, we thus have relation (2.85) available which means a formal time differentiation of the rotation tensor.

Example 2.3 (Rotational Velocity of a Rigid Body). With the rotation tensor (2.49) from Example 2.2 we get the rotational velocity tensor

$$\begin{aligned}\tilde{\boldsymbol{\omega}}(t) &= \dot{\vartheta}\begin{bmatrix} -\sin\vartheta & 0 & -\cos\vartheta \\ 0 & 0 & 0 \\ \cos\vartheta & 0 & -\sin\vartheta \end{bmatrix}\cdot\begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{bmatrix}\\ &= \dot{\vartheta}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\end{aligned} \tag{2.87}$$

and thereby the rotation velocity vector

$$\boldsymbol{\omega}(t) = \left[0 \; -\dot{\vartheta} \; 0\right]. \tag{2.88}$$

The rigid body is thus carrying out a planar rotation around the 2-axis with a negative direction of rotation, see also (2.53).

End of Example 2.3.

If we also apply (2.85) to (2.43), we then obtain after lengthy calculation the 3×1 rotation velocity vector as a function of the rotation parameters

$$\boldsymbol{\omega}(t) = \boldsymbol{d}\dot{\varphi} + \dot{\boldsymbol{d}}\sin\varphi + \tilde{\boldsymbol{d}} \cdot \dot{\boldsymbol{d}}(1-\cos\varphi). \tag{2.89}$$

It is obvious that the rotation velocity depends not only on the temporal change $\dot{\varphi}(t)$ of the rotation angle but also on the temporal change $\dot{\boldsymbol{d}}(t)$ of the direction of the rotation axis. This underlines it clearly that finite rotation and instantaneous rotation have different properties.

If we insert half the rotation angle into (2.89), with the quaternions (2.44) we obtain the simplified relation

$$\boldsymbol{\omega}(t) = 2(q_0\dot{\boldsymbol{q}} - \dot{q}_0\boldsymbol{q} + \tilde{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}}). \tag{2.90}$$

If we supplement (2.90) with the time derivative of (2.46)

$$q_0\dot{q}_0 + \boldsymbol{q} \cdot \dot{\boldsymbol{q}} = 0, \tag{2.91}$$

both equations can be merged into one 4×1 vector differential equation

$$\begin{bmatrix} 0 \\ ---- \\ \boldsymbol{\omega}(t) \end{bmatrix} = 2\boldsymbol{Q}(q_0, \boldsymbol{q}) \cdot \begin{bmatrix} \dot{q}_0 \\ -- \\ \dot{\boldsymbol{q}} \end{bmatrix} = 2\begin{bmatrix} q_0 & | & \boldsymbol{q} \\ -- & | & ------- \\ -\boldsymbol{q} & | & q_0\boldsymbol{E} + \tilde{\boldsymbol{q}} \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_0 \\ -- \\ \dot{\boldsymbol{q}} \end{bmatrix}, \tag{2.92}$$

which relates the rotation velocity with the quaternions. It should be noted in particular that the 4×4 coefficient matrix $\boldsymbol{Q}$ is orthogonal and thus nonsingular, so the inversion problem is easy to solve, see Table 2.2.

A further, intuitive way to calculate the rotation velocity vector is by using elementary rotations. For every elementary rotation there is one elementary rotational velocity. Following (2.57), we obtain

$$\boldsymbol{\omega}_{\alpha 1}(t) = \dot{\boldsymbol{\alpha}}_1(t) = [\dot{\alpha} \quad 0 \quad 0], \tag{2.93}$$

$$\boldsymbol{\omega}_{\beta 2}(t) = \dot{\boldsymbol{\beta}}_2(t) = [0 \quad \dot{\beta} \quad 0], \tag{2.94}$$

$$\boldsymbol{\omega}_{\gamma 3}(t) = \dot{\boldsymbol{\gamma}}_3(t) = [0 \quad 0 \quad \dot{\gamma}]. \tag{2.95}$$

Table 2.2 Kinematic differential equations

Sought coordinates	Rotation velocity in the spatially-fixed frame
9 direction cosines $\boldsymbol{S}$	$\dot{\boldsymbol{S}} = \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{S}$
4 quaternions $[q_0\ \boldsymbol{q}]$	$\begin{bmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{bmatrix} = \frac{1}{2}\boldsymbol{Q}^T(q_0, \boldsymbol{q}) \cdot [0 \mid \boldsymbol{\omega}]$ $\begin{bmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{bmatrix} = \frac{1}{2} \left[\begin{array}{c|c} 0 & -\boldsymbol{\omega} \\ \hline \boldsymbol{\omega} & \tilde{\boldsymbol{\omega}} \end{array}\right] \cdot \begin{bmatrix} q_0 \\ \boldsymbol{q} \end{bmatrix}$
3 Cardano angle $\alpha(t), \beta(t), \gamma(t)$	$\dot{\boldsymbol{x}} = \boldsymbol{H}_R^{-1}(\boldsymbol{x}) \cdot \boldsymbol{\omega}$ $\boldsymbol{H}_R^{-1} = \begin{bmatrix} 1 & \sin\alpha\tan\beta & -\cos\alpha\tan\beta \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\dfrac{\sin\alpha}{\cos\beta} & \dfrac{\cos\alpha}{\cos\beta} \end{bmatrix}$
	Rotation velocity in the body-fixed frame 1
9 direction cosines $\boldsymbol{S}$	$\dot{\boldsymbol{S}} = \boldsymbol{S} \cdot {}_1\tilde{\boldsymbol{\omega}}$
4 quaternions $[q_0\ \boldsymbol{q}]$	$\begin{bmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{bmatrix} = \frac{1}{2}{}_1\boldsymbol{Q}^T(q_0, \boldsymbol{q}) \cdot [0 \mid {}_1\boldsymbol{\omega}]$ $\begin{bmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}} \end{bmatrix} = \frac{1}{2} \left[\begin{array}{c|c} 0 & -{}_1\boldsymbol{\omega} \\ \hline {}_1\boldsymbol{\omega} & -{}_1\tilde{\boldsymbol{\omega}} \end{array}\right] \cdot \begin{bmatrix} q_0 \\ \boldsymbol{q} \end{bmatrix}$
3 Cardano angle $\alpha(t), \beta(t), \gamma(t)$	$\dot{\boldsymbol{x}} = {}_1\boldsymbol{H}_R^{-1}(\boldsymbol{x}) \cdot {}_1\boldsymbol{\omega}$ ${}_1\boldsymbol{H}_R^{-1} = \begin{bmatrix} \dfrac{\cos\gamma}{\cos\beta} & -\dfrac{\sin\gamma}{\cos\beta} & 0 \\ \sin\gamma & \cos\gamma & 0 \\ -\cos\gamma\tan\beta & \sin\gamma\tan\beta & 1 \end{bmatrix}$

These elementary rotation velocities can be added vectorially, whereby the sequence of rotations and the transformations of the coordinate axes through previous rotations must be accounted for. This yields for the Euler angle

$$\boldsymbol{\omega}(t) = \dot{\boldsymbol{\psi}}_3(t) + \boldsymbol{\psi}_3(t) \cdot \dot{\boldsymbol{\vartheta}}_1(t) + \boldsymbol{\psi}_3(t) \cdot \boldsymbol{\vartheta}_1(t) \cdot \dot{\boldsymbol{\phi}}_3(t) \tag{2.96}$$

and for the Cardano angle

$$\boldsymbol{\omega}(t) = \dot{\boldsymbol{\alpha}}_1(t) + \boldsymbol{\alpha}_1(t) \cdot \dot{\boldsymbol{\beta}}_2(t) + \boldsymbol{\alpha}_1(t) \cdot \boldsymbol{\beta}_2(t) \cdot \dot{\boldsymbol{\gamma}}_3(t). \tag{2.97}$$

We can also represent rotation velocity vectors in a body-fixed frame, e.g. with the Cardano angles as

$$\boldsymbol{\omega}(t) = \boldsymbol{\gamma}_3^T(t) \cdot \boldsymbol{\beta}_2^T(t) \cdot \dot{\boldsymbol{\alpha}}_1(t) + \boldsymbol{\gamma}_3^T(t) \cdot \dot{\boldsymbol{\beta}}_2(t) + \dot{\boldsymbol{\gamma}}_3(t). \tag{2.98}$$

Evaluation of (2.97) yields, with the Cardano angles as generalized coordinates,

$$\boldsymbol{x}(t) = \begin{bmatrix} \alpha \ \beta \ \gamma \end{bmatrix}, \tag{2.99}$$

the relation

$$\boldsymbol{\omega}(t) = \boldsymbol{H}_R(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}}(t) = \begin{bmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{bmatrix} \cdot \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}, \tag{2.100}$$

where the 3×3 Jacobian matrix $\boldsymbol{H}_R(\boldsymbol{x})$ of rotation has been introduced.

Calculation of the position or configuration from the rotation velocity is very important in dynamics. This can be done by integrating the corresponding differential equations. The rotation velocity is given in a body-fixed or spatially fixed frame, see Table 2.2. The kinematic differential equations of the direction cosines and quaternions are overdetermined. The constraints shown in Table 2.1 are included in differentiated form in the differential equations, although a first integral is known, namely the constraints themselves. This can lead to numerical difficulties, i.e. the constraints can be violated in case of longer integrations. It is therefore wise to provide a correction method that performs a normalization in accordance with (2.32) or (2.46) after each integration step. This normalization is undertaken automatically for the differential equation of the Cardano angle as well as for all other elementary rotations. However, then the functional matrix $\boldsymbol{H}_R$ is no longer regular in the singular configurations. Yet these singularities can be avoided by using complementary rotation angles, an adjustment which requires a higher level of programming effort.

Example 2.4 (Integration of the Direction Cosines). Let the rotation velocity vector (2.88) of the rotation be around the negative 2-axis and the initial condition be $\boldsymbol{S}(t = t_0) = \boldsymbol{S}_0$. According to Table 2.2, the differential equations of the direction cosines are then written $\dot{\boldsymbol{S}} = \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{S}$ or

$$\begin{array}{lll} \dot{S}_{11} = \omega_2 S_{31}, & \dot{S}_{12} = \omega_2 S_{32}, & \dot{S}_{13} = \omega_2 S_{33}, \\ \dot{S}_{21} = 0, & \dot{S}_{22} = 0, & \dot{S}_{23} = 0, \\ \dot{S}_{31} = -\omega_2 S_{11}, & \dot{S}_{32} = -\omega_2 S_{12}, & \dot{S}_{33} = -\omega_2 S_{13}. \end{array} \tag{2.101}$$

If we now consider that linear, time-variant differential equation systems of the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{2.102}$$

$$\begin{bmatrix} x_1(t = t_0) \\ x_2(t = t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{2.103}$$

have the general solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\left(\int_{t_0}^t \omega dt\right) & \sin\left(\int_{t_0}^t \omega dt\right) \\ -\sin\left(\int_{t_0}^t \omega dt\right) & \cos\left(\int_{t_0}^t \omega dt\right) \end{bmatrix} \cdot \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \tag{2.104}$$

then we obtain from (2.101) with $\omega_2 = -\dot{\vartheta}$,

$$\boldsymbol{S}(t) = \begin{bmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{bmatrix} \cdot \begin{bmatrix} S_{110} & S_{120} & S_{130} \\ S_{210} & S_{220} & S_{230} \\ S_{310} & S_{320} & S_{330} \end{bmatrix}. \tag{2.105}$$

If $\boldsymbol{S}_0 = \boldsymbol{E}$, we again obtain the rotation tensor (2.49).

If the differential equations (2.101) are solved not analytically but numerically, integration errors can destroy the orthogonality. For example, if we allow one integration error ε in a solution of the differential equation (2.101),

$$S_{11} = \cos\vartheta + \varepsilon, \tag{2.106}$$

then the corresponding condition of orthogonality is no longer satisfied. For $\boldsymbol{S}_0 = \boldsymbol{E}$ we obtain for example

$$S_{11}^2 + S_{21}^2 + S_{31}^2 = 1 + 2\varepsilon\cos\vartheta \neq 1. \tag{2.107}$$

A non-orthogonal rotation tensor corresponds however to the deformation gradient of a nonrigid body. For this reason, orthogonality must constantly be checked and enforced.

End of Example 2.4.

Example 2.5 (Integration of the Cardano Angle). The rotation shown in Example 2.4 will now be described with Cardano angles. In Table 2.2 we find

$$\dot{\alpha} = \omega_2 \sin\alpha \tan\beta, \qquad \dot{\beta} = \omega_2 \cos\alpha, \qquad \dot{\gamma} = -\omega_2 \frac{\sin\alpha}{\cos\beta}. \tag{2.108}$$

This nonlinear system of differential equations can now be solved in a closed form with the initial condition $\alpha_0 = 0$, $\beta_0 = 0$, $\gamma_0 = 0$

$$\alpha(t) = 0, \qquad \beta(t) = \int \omega_2 dt = -\vartheta, \qquad \gamma(t) = 0. \tag{2.109}$$

The rotation tensor (2.49) has again been determined, and orthogonality has by definition been preserved.

On the other hand, there is a singularity for $\alpha_0 = -\gamma_0 = 0$ and $\beta_0 = \pi/2$. This can be rectified by using complementary Cardano angles (2.64). According to (2.71), (2.72), and (2.74), the corresponding initial conditions are $\overline{\alpha}_0 = 0$,

$\overline{\beta}_0 = \pi/2$, and $\overline{\gamma}_0 = 0$ and the nonlinear time-variant system of differential equations has the form

$$\dot{\overline{\alpha}} = -\omega_2 \sin\overline{\alpha} \cot\overline{\beta}, \qquad \dot{\overline{\beta}} = \omega_2 \cos\overline{\alpha}, \qquad \dot{\overline{\gamma}} = \omega_2 \frac{\sin\overline{\alpha}}{\sin\overline{\beta}} \tag{2.110}$$

with the solution

$$\overline{\alpha}(t) = 0, \qquad \overline{\beta}(t) = \frac{\pi}{2} + \int \omega_2 dt, \qquad \overline{\gamma}(t) = 0. \tag{2.111}$$

We can see that the transition from Cardano angles to complementary Cardano angles was even possible in one singular position. But this should be avoided for numerical reasons.

End of Example 2.5.

This concludes our discussion of rotational velocity. According to (2.86), the velocity of a rigid body is given completely by the translational velocity $\boldsymbol{v}(t)$ of a mass point $P(t)$ and by the rotational velocity $\boldsymbol{\omega}(t)$ of the body, which is equal at every point. These velocities can however also be expressed in accordance with (2.6) and (2.100) by the generalized coordinates of the 6×1 position vector (2.81) as

$$\begin{aligned} \boldsymbol{v}(t) &= \boldsymbol{v}(\boldsymbol{x},\dot{\boldsymbol{x}}) = [\boldsymbol{H}_T(\boldsymbol{x}) \;\; \mathbf{0}] \cdot \dot{\boldsymbol{x}}(t) = \overline{\boldsymbol{H}}_T(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}}(t), \\ \boldsymbol{\omega}(t) &= \boldsymbol{\omega}(\boldsymbol{x},\dot{\boldsymbol{x}}) = [\mathbf{0} \;\; \boldsymbol{H}_R(\boldsymbol{x})] \cdot \dot{\boldsymbol{x}}(t) = \overline{\boldsymbol{H}}_R(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}}(t), \end{aligned} \tag{2.112}$$

whereby the 3×6 functional matrices can be obtained from (2.6) and (2.100) by extending the matrices by zero submatrices. Formally, it yields according to (2.7)

$$\overline{\boldsymbol{H}}_T(\boldsymbol{x}) = \frac{\partial \boldsymbol{r}(\boldsymbol{x})}{\partial \boldsymbol{x}}, \qquad \overline{\boldsymbol{H}}_R(\boldsymbol{x}) = \frac{\partial \boldsymbol{s}(\boldsymbol{x})}{\partial \boldsymbol{x}}. \tag{2.113}$$

It should be noted that, in the second formula of (2.113), the infinitesimal instantaneous rotation from (2.85) must be used. The transition from (2.82) to (2.113) is therefore somewhat laborious and must be undertaken using the skew-symmetric tensor of the infinitesimal instantaneous rotation tensor

$$\frac{\partial \tilde{s}_{\alpha\beta}}{\partial x_\delta} = \frac{\partial S_{\alpha\gamma}}{\partial x_\delta} S_{\beta\gamma}, \qquad \alpha,\beta,\gamma = 1(1)3, \quad \delta = 1(1)6. \tag{2.114}$$

Equation (2.114) is best analyzed with a formula manipulation program. However, the Jacobian matrix $\overline{\boldsymbol{H}}_R(\boldsymbol{x})$ in (2.113) can also be obtained descriptively with the help of elementary rotations from (2.100).

The current acceleration of the rigid body is given by a further time derivative from (2.83)

$$\boldsymbol{a}(\boldsymbol{\rho},t) = \frac{d}{dt}\boldsymbol{v}(\boldsymbol{\rho},t) = \ddot{\boldsymbol{r}}_1(t) + \ddot{\boldsymbol{S}}(t) \cdot \boldsymbol{\rho}. \tag{2.115}$$

If we again consider (2.79), we obtain

$$\boldsymbol{a}(\boldsymbol{\rho},t) = \ddot{\boldsymbol{r}}_1(t) + \ddot{\boldsymbol{S}}(t)\cdot\boldsymbol{S}^T(t)\cdot\boldsymbol{r}_P(\boldsymbol{\rho},t). \tag{2.116}$$

We again recognize the translational acceleration (2.12) as the first term, while the second term denotes the rotational acceleration, yielding

$$\begin{aligned}\ddot{\boldsymbol{S}}\cdot\boldsymbol{S}^T &= \ddot{\boldsymbol{S}}\cdot\boldsymbol{S}^T + \dot{\boldsymbol{S}}\cdot\dot{\boldsymbol{S}}^T - \dot{\boldsymbol{S}}\cdot\dot{\boldsymbol{S}}^T = \ddot{\boldsymbol{S}}\cdot\boldsymbol{S}^T + \dot{\boldsymbol{S}}\cdot\dot{\boldsymbol{S}}^T - \dot{\boldsymbol{S}}\cdot\boldsymbol{S}^T\cdot\boldsymbol{S}\cdot\dot{\boldsymbol{S}}^T \\ &= \dot{\tilde{\boldsymbol{\omega}}}(t) + \tilde{\boldsymbol{\omega}}(t)\cdot\tilde{\boldsymbol{\omega}}(t),\end{aligned} \tag{2.117}$$

where the definition (2.85) of angular velocity and its derivatives $\dot{\tilde{\boldsymbol{\omega}}} = \ddot{\boldsymbol{S}}\cdot\boldsymbol{S}^T + \dot{\boldsymbol{S}}\cdot\dot{\boldsymbol{S}}^T$ are applied. If we now introduce the 3×1 rotational acceleration vector

$$\boldsymbol{\alpha}(t) = \dot{\boldsymbol{\omega}}(t), \tag{2.118}$$

we then obtain

$$\boldsymbol{a}(\boldsymbol{\rho},t) = \ddot{\boldsymbol{r}}_1(t) + \left[\tilde{\boldsymbol{\alpha}}(t) + \tilde{\boldsymbol{\omega}}(t)\cdot\tilde{\boldsymbol{\omega}}(t)\right]\cdot\boldsymbol{r}_p(\boldsymbol{\rho},t). \tag{2.119}$$

The acceleration of a rigid body is thus given by the translational acceleration $\boldsymbol{a}_1(t)$ of the mass point P_1, its rotational acceleration $\boldsymbol{\alpha}(t)$ and the square of its rotational velocity $\boldsymbol{\omega}(t)$.

Just like in (2.13), the accelerations can be expressed with generalized coordinates

$$\boldsymbol{a}(t) = \boldsymbol{a}(\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \overline{\boldsymbol{H}}_T(\boldsymbol{x})\cdot\ddot{\boldsymbol{x}}(t) + \left(\frac{\partial\overline{\boldsymbol{H}}_T(\boldsymbol{x})}{\partial\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}(t)\right)\cdot\dot{\boldsymbol{x}}(t), \tag{2.120}$$

$$\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(\boldsymbol{x},\ddot{\boldsymbol{x}},\dot{\boldsymbol{x}}) = \overline{\boldsymbol{H}}_R(\boldsymbol{x})\cdot\ddot{\boldsymbol{x}}(t) + \left(\frac{\partial\overline{\boldsymbol{H}}_R(\boldsymbol{x})}{\partial\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}(t)\right)\cdot\dot{\boldsymbol{x}}(t). \tag{2.121}$$

We thus obtain, in accordance with (2.112), the 3×6 functional matrices $\overline{\boldsymbol{H}}_T$ and $\overline{\boldsymbol{H}}_R$ and their derivatives.

A free system of p rigid bodies has $6p$ degrees of freedom, which are described by the $6p\times 1$ position vector $\boldsymbol{x}(t)$ of the generalized coordinates of the overall system. Applying (2.82), for the ith body

$$\boldsymbol{r}_i(t) = \boldsymbol{r}_i(\boldsymbol{x}), \qquad \boldsymbol{S}_i(t) = \boldsymbol{S}_i(\boldsymbol{x}) \tag{2.122}$$

is true. We can likewise apply (2.112)–(2.114) and (2.121) to the ith body.

2.1.3 Kinematics of a Continuum

Like the rigid body, the continuum is a model of mechanics. The distances between the mass points of a continuum are not constant like those of a rigid body, however. A continuum in the deformation process is thus subject not only to a translation and a rotation but also to strain. However, strain is generally small in elastic materials, so we can usually work with linear relations. Fluids, which are subject to major strains, or plastic materials will not be dealt with in this book. Strains arising in a continuum also permit the calculation of internal forces and stresses, which are of crucial importance in strength tests. Nevertheless, the use of continuum models in dynamics is not always mandatory. Frequently, motions are calculated with a rigid body model and the strength tests – under consideration of inertia forces – are carried out using static methods. In order to describe free continua kinematically, it is again sufficient to consider a single body as was the case in the previous sections.

In order to describe the configuration of a continuum K mathematically, Fig. 2.4 and Eqs. (2.25)–(2.30) can be adopted without any alterations. The deformation gradient $\boldsymbol{F}(\boldsymbol{\rho},t)$ is now no longer orthogonal, however. But it can, like any second-order tensor, undergo a polar decomposition

$$\boldsymbol{F}(\boldsymbol{\rho},t) = \overline{\boldsymbol{S}}(\boldsymbol{\rho},t)\cdot\boldsymbol{U}(\boldsymbol{\rho},t), \tag{2.123}$$

where we see not only the location-dependent, proper orthogonal 3×3 rotation tensor

$$\overline{\boldsymbol{S}}^{T}(\boldsymbol{\rho},t) = \overline{\boldsymbol{S}}^{-1}(\boldsymbol{\rho},t) \tag{2.124}$$

but also the equally location-dependent, symmetric and positive definite 3×3 right stretch tensor

$$\boldsymbol{U}^{T}(\boldsymbol{\rho},t) = \boldsymbol{U}(\boldsymbol{\rho},t), \tag{2.125}$$

which is a measure of strain. Proof of the properties mentioned can be found e.g. in Lai, Rubin, Krempl [31] or Becker and Bürger [9] and will not be repeated here. From the 3×3 right stretch tensor we obtain the Green strain tensor

$$\boldsymbol{G} = \frac{1}{2}(\boldsymbol{U}\cdot\boldsymbol{U}-\boldsymbol{E}) = \frac{1}{2}(\boldsymbol{F}^{T}\cdot\boldsymbol{F}-\boldsymbol{E}), \tag{2.126}$$

which is also symmetric. With (2.123) and (2.125) we can also write the deformation gradient as

$$\boldsymbol{F} = \overline{\boldsymbol{S}}\cdot(\boldsymbol{E}+2\boldsymbol{G})^{\frac{1}{2}}. \tag{2.127}$$

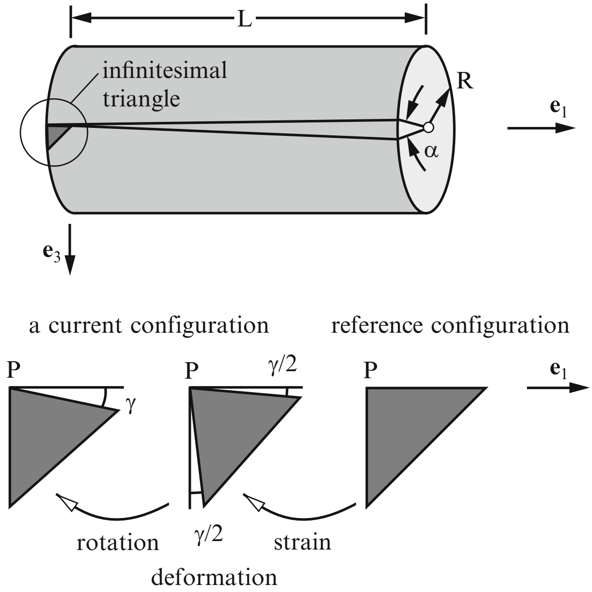


Fig. 2.9 Deformation of a round bar

Further information on the calculation of the roots of a matrix can be found in Zurmühl and Falk [69]. In the case of a rigid body, the Green strain tensor disappears, and from $\boldsymbol{U} = \boldsymbol{E}$ follows $\boldsymbol{G} = \boldsymbol{0}$, as a result of which (2.127) reverts to (2.33).

Example 2.6 (Strain of a Twisted Round Bar). The current configuration of a twisted round bar, Fig. 2.9, is described by point P with the 3×1 location vector

$$\boldsymbol{r}(\boldsymbol{\rho}, t) = \begin{bmatrix} \rho_1 \\ \rho_2 - \alpha\rho_3 \\ \rho_3 + \alpha\rho_2 \end{bmatrix}, \qquad \alpha(\rho_1, t) \ll 1, \tag{2.128}$$

where the 3×1 location vector $\boldsymbol{\rho}$ denotes the mass points in the reference configurations. The small angle α is a function of location and time, its location-dependence being restricted to the longitudinal direction of the bar.

According to (2.28), the deformation gradient is

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha'\rho_3 & 1 & -\alpha \\ \alpha'\rho_2 & \alpha & 1 \end{bmatrix}, \qquad \alpha' = \frac{\partial \alpha}{\partial \rho_1} \ll 1 \tag{2.129}$$

and the square of the right stretch tensor without taking quadratically small magnitudes into account, results in

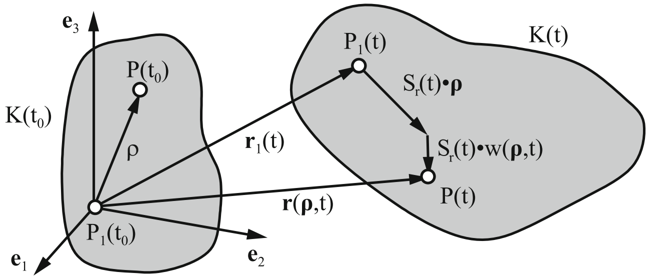


Fig. 2.10 Free motion of a linear-elastic body

$$\boldsymbol{U}\cdot\boldsymbol{U}=\boldsymbol{F}^T\cdot\boldsymbol{F}=\begin{bmatrix}1 & -\alpha'\rho_3 & \alpha'\rho_2\\ -\alpha'\rho_3 & 1 & 0\\ \alpha'\rho_2 & 0 & 1\end{bmatrix}. \tag{2.130}$$

If $\alpha \ll 1$, then $\boldsymbol{U}$ from (2.130) can be written in a closed form

$$\boldsymbol{U}=\begin{bmatrix}1 & -\frac{1}{2}\alpha'\rho_3 & \frac{1}{2}\alpha'\rho_2\\ -\frac{1}{2}\alpha'\rho_3 & 1 & 0\\ \frac{1}{2}\alpha'\rho_2 & 0 & 1\end{bmatrix}. \tag{2.131}$$

With (2.123) we now find the rotation tensor

$$\overline{\boldsymbol{S}}=\begin{bmatrix}1 & \frac{1}{2}\alpha'\rho_3 & -\frac{1}{2}\alpha'\rho_2\\ -\frac{1}{2}\alpha'\rho_3 & 1 & -\alpha\\ \frac{1}{2}\alpha'\rho_2 & \alpha & 1\end{bmatrix} \tag{2.132}$$

and with (2.126) we obtain for the linearized Green strain tensor

$$\boldsymbol{G}_{lin}=\frac{\alpha'}{2}\begin{bmatrix}0 & -\rho_3 & \rho_2\\ -\rho_3 & 0 & 0\\ \rho_2 & 0 & 0\end{bmatrix}. \tag{2.133}$$

The relation $R\alpha(\rho_1)=\gamma\rho_1$ applies for cases of static strain, see Fig. 2.9. If we consider the deformation of an infinitesimal triangle at the mass point $\boldsymbol{\rho}=[0\ R\ 0]$, we determine from Fig. 2.9 that in addition to strain, characterized by a pure change of angle, there is also a rotation of the infinitesimal triangle. This confirms the assertion of (2.127) concerning simultaneously possible rotations and strains in nonrigid bodies.

End of Example 2.6.

If we now note that elastic strain in elastic materials are usually small in relation to rigid body motion, then the above relations can generally be linearized, see Fig. 2.10.

Now the following is true for the current configuration with rigid body rotation $\boldsymbol{S}_r(t)$,

$$\boldsymbol{r}(\boldsymbol{\rho},t) = \boldsymbol{r}_1(t) + \boldsymbol{S}_r(t) \cdot \left[\boldsymbol{\rho} + \boldsymbol{w}(\boldsymbol{\rho},t) \right], \tag{2.134}$$

where the relative 3×1 displacement vector $\boldsymbol{w}(\boldsymbol{\rho},t)$ is small in relation to a characteristic length of the continuum. In addition, the following relation is valid in (2.134)

$$\boldsymbol{w}(\boldsymbol{0},t) = \boldsymbol{0}, \tag{2.135}$$

which specifies the location vector $\boldsymbol{r}_1(t)$ to the reference point P_1. With the associated 3×3 displacement gradient

$$\boldsymbol{F}_w(\boldsymbol{\rho},t) = \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{\rho}} \tag{2.136}$$

the linearized deformation gradient is

$$\boldsymbol{F}_{lin} = \overline{\boldsymbol{S}} \cdot (\boldsymbol{E} + \boldsymbol{G}_{lin}), \tag{2.137}$$

where the relations

$$\boldsymbol{G}_{lin} = \frac{1}{2}(\boldsymbol{F}_w + \boldsymbol{F}_w^T), \qquad \overline{\boldsymbol{S}} = \boldsymbol{S}_r(t) \cdot \boldsymbol{S}_w(\boldsymbol{\rho},t), \tag{2.138}$$

$$\boldsymbol{S}_w = \boldsymbol{E} + \frac{1}{2}(\boldsymbol{F}_w - \boldsymbol{F}_w^T), \qquad \boldsymbol{S}_w(\boldsymbol{0},t) = \boldsymbol{E}, \tag{2.139}$$

must be taken into account. In the linear case, we thus obtain the linear 3×3 Green strain tensor $\boldsymbol{G}_{lin}$ and the 3×3 tensor $\boldsymbol{S}_w$ of relative rotation by simple decomposition of the displacement gradient $\boldsymbol{F}_w$ into its symmetric and skew-symmetric components, see (2.138) and (2.139). It should also be mentioned that the tensor $\boldsymbol{S}_w(\boldsymbol{\rho},t)$ of relative motion, as opposed to the rotation tensor $\boldsymbol{S}_r(t)$, is dependent on location and time. According to (2.138), the total rotation $\overline{\boldsymbol{S}}(\boldsymbol{\rho},t)$ is composed of the rigid body rotation $\boldsymbol{S}_r(t)$ and the relative rotation $\boldsymbol{S}_w(\boldsymbol{\rho},t)$.

The linearized Green strain tensor has, for reasons of symmetry, only six essential elements

$$\boldsymbol{G}_{lin} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{31} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}, \tag{2.140}$$

which can also be merged into a 6×1 strain vector

$$\boldsymbol{e} = \left[\, \varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad \gamma_{12} \quad \gamma_{23} \quad \gamma_{31} \,\right]. \tag{2.141}$$

We call $\varepsilon_{\alpha\alpha}$, $\alpha = 1(1)3$ normal strains or elongations. The secondary diagonal elements $\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31}$ are called shear strains. Here, in the strain vector $\gamma_{12} = 2\varepsilon_{12}$, $\gamma_{23} = 2\varepsilon_{23}$, $\gamma_{31} = 2\varepsilon_{31}$ appear, which are referred to as slidings and describe the changes in one of the 90° angles in the reference configuration. The strains and shear stresses are not mutually independent however, since they are calculated from the three coordinates of the displacement vector $\boldsymbol{w}$. Yet the conditions of compatibility are now no longer denoted by algebraic equations but rather by differential equations. If we now introduce the 6×3 differential operator matrix,

$$\mathscr{V} = \begin{bmatrix} \partial/\partial\rho_1 & 0 & 0 \\ 0 & \partial/\partial\rho_2 & 0 \\ 0 & 0 & \partial/\partial\rho_3 \\ \partial/\partial\rho_2 & \partial/\partial\rho_1 & 0 \\ 0 & \partial/\partial\rho_3 & \partial/\partial\rho_2 \\ \partial/\partial\rho_3 & 0 & \partial/\partial\rho_1 \end{bmatrix}, \tag{2.142}$$

we can also calculate the strain vector directly from the displacement vector,

$$\boldsymbol{e} = \mathscr{V} \cdot \boldsymbol{w}. \tag{2.143}$$

The algorithms of matrix multiplication are applicable for the differential operator matrix $\mathscr{V}$, as shown in Sect. A.3.

As a consequence of linearization, the rotation tensor (2.139) has only three essential elements,

$$\boldsymbol{S}_w = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix}, \tag{2.144}$$

which correspond to the small Cardano angles α, β, γ. The essential elements of (2.144) can be merged in the 3×1 rotation vector

$$\boldsymbol{s} = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} \tag{2.145}$$

and with the 3×3 differential operator matrix of elastic strain,

$$\mathscr{D} = \frac{1}{2} \begin{bmatrix} 0 & -\partial/\partial\rho_3 & \partial/\partial\rho_2 \\ \partial/\partial\rho_3 & 0 & -\partial/\partial\rho_1 \\ -\partial/\partial\rho_2 & \partial/\partial\rho_1 & 0 \end{bmatrix}, \tag{2.146}$$

determined from the displacement vector,

$$\boldsymbol{s} = \mathscr{D} \cdot \boldsymbol{w}. \tag{2.147}$$

The rotation vector (2.145) plays an important role in the mechanics of polar continua, among which we can include the Bernoulli beam, too. A polar continuum is composed of mass points, which can execute both displacements and rotations and is also known as a Cosserat continuum.

A nonrigid continuum has an infinite number of degrees of freedom since it comprises infinitely many free mass points. This is also reflected in the fact that the deformation gradient is dependent not only on time but also on the location or material coordinates, respectively, of the mass points. A solution method often used in linear continuum mechanics exploits this fact in conjunction with the principles of separation and superposition,

$$\boldsymbol{w}(\boldsymbol{\rho},t) = \boldsymbol{A}(\boldsymbol{\rho}) \cdot \boldsymbol{x}(t), \tag{2.148}$$

where the $3 \times f$ matrix $\boldsymbol{A}(\boldsymbol{\rho})$ of the relative shape functions and the $f \times 1$ position vector $\boldsymbol{x}(t)$ of the generalized coordinates appear with $f \to \infty$. The approach (2.148), which does not always lead to the desired aim, thus in particular indicates the infinitely many degrees of freedom of the continuum. With (2.143), we also obtain for the strain vector

$$\boldsymbol{e}(\boldsymbol{\rho},t) = \boldsymbol{B}(\boldsymbol{\rho}) \cdot \boldsymbol{x}(t) \tag{2.149}$$

with the $6 \times f$ matrix $\boldsymbol{B}(\boldsymbol{\rho})$ of the strain functions,

$$\boldsymbol{B}(\boldsymbol{\rho}) = \mathscr{V} \cdot \boldsymbol{A}(\boldsymbol{\rho}). \tag{2.150}$$

For small elements of a continuum, it is also sufficient to choose with a finite number of generalized coordinates, such as are utilized in the finite element method, see Chap. 6. Furthermore, if we assume linear kinematics in the rigid body motion with reference to point P_1, we obtain from (2.134) and (2.148) for the displacement vector

$$\boldsymbol{r}(\boldsymbol{\rho},t) = \boldsymbol{\rho} + \boldsymbol{C}(\boldsymbol{\rho}) \cdot \boldsymbol{x}(t), \tag{2.151}$$

yielding the $3 \times f$ matrix $\boldsymbol{C}(\boldsymbol{\rho})$ of the absolute shape functions and a corresponding $f \times 1$ position vector $\boldsymbol{x}(t)$. Using the finite element method, the $f \times 1$ position vector $\boldsymbol{x}(t)$ is determined by means of the Cartesian coordinates of single mass points $P_j, j = 1, 2, 3, \ldots,$

$$\boldsymbol{x}(t) = \left[\boldsymbol{r}(\boldsymbol{\rho}_1,t)\ \boldsymbol{r}(\boldsymbol{\rho}_2,t)\ \boldsymbol{r}(\boldsymbol{\rho}_3,t) \ldots \right]. \tag{2.152}$$

In the case of continuous systems on the other hand, often the generalized coordinates belonging to the eigenforms are merged in the position vector, see Chap. 7.

The current velocity of a point in the continuum is determined by the time derivative of (2.25)

$$\boldsymbol{v}(\boldsymbol{\rho},t) = \frac{d}{dt}\boldsymbol{r}(\boldsymbol{\rho},t). \tag{2.153}$$

We can obtain additional information if deformation is noted in accordance with (2.28), (2.29),

$$\boldsymbol{v}(\boldsymbol{\rho}+d\boldsymbol{\rho}) = \boldsymbol{v}(\boldsymbol{\rho}) + \dot{\boldsymbol{F}}(\boldsymbol{\rho})\cdot\boldsymbol{F}^{-1}(\boldsymbol{\rho})\cdot d\boldsymbol{r}(\boldsymbol{\rho}). \tag{2.154}$$

This yields the tensor of the spatial velocity gradient,

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} = \frac{\partial\boldsymbol{v}(\boldsymbol{r})}{\partial\boldsymbol{r}}, \tag{2.155}$$

which can be decomposed into symmetric and skew-symmetric components

$$\boldsymbol{L} = \boldsymbol{D}+\boldsymbol{W}, \qquad \boldsymbol{D} = \frac{1}{2}(\boldsymbol{L}+\boldsymbol{L}^T), \qquad \boldsymbol{W} = \frac{1}{2}(\boldsymbol{L}-\boldsymbol{L}^T). \tag{2.156}$$

Here, $\boldsymbol{D}$ denotes the symmetric 3×3 strain velocity tensor, while $\boldsymbol{W}$ describes the skew-symmetric 3×3 rotation velocity tensor. Comparing (2.84) and (2.154), we can see clearly that, due to (2.33), the strain velocity tensor vanishes as expected in the case of a rigid body. In the linear case, we obtain from (2.137) to (2.139) with (2.155) and (2.156), ignoring quadratically small elements

$$\boldsymbol{D} = \overline{\boldsymbol{S}}\cdot\dot{\boldsymbol{G}}_w\cdot\overline{\boldsymbol{S}}^T, \qquad \boldsymbol{W} = \dot{\overline{\boldsymbol{S}}}\cdot\overline{\boldsymbol{S}}^T. \tag{2.157}$$

If we finally base our examination on the approach in (2.151), we then have

$$\boldsymbol{v}(\boldsymbol{\rho},t) = \boldsymbol{C}(\boldsymbol{\rho})\cdot\dot{\boldsymbol{x}}(t) \tag{2.158}$$

for the current velocity.

The current acceleration of a point in the continuum is taken from (2.153) via mass derivation of the velocity

$$\boldsymbol{a}(\boldsymbol{\rho},t) = \frac{d}{dt}\boldsymbol{v}(\boldsymbol{\rho},t) = \frac{\partial}{\partial t}\boldsymbol{v}(\boldsymbol{r},t) = \frac{\partial\boldsymbol{v}(\boldsymbol{r},t)}{\partial\boldsymbol{r}}\cdot\boldsymbol{v} + \frac{\partial\boldsymbol{v}(\boldsymbol{r},t)}{\partial t}. \tag{2.159}$$

In this case, first the inverse function (2.27) was utilized, followed by a separation of acceleration into a convective component (spatial velocity gradient) and a local component. Furthermore, from (2.158) we obtain for the linear kinematics

$$\boldsymbol{a}(\boldsymbol{\rho},t) = \boldsymbol{C}(\boldsymbol{\rho})\cdot\ddot{\boldsymbol{x}}(t). \tag{2.160}$$

The kinematics of the free continuum is now also concluded.

Example 2.7 (Velocity and Acceleration of a Round Bar). From the current configuration (2.128), we obtain the vectors from the time derivative, which will in the following be indicated by a point $(\dot{\cdot})$,

$$\boldsymbol{v}(\boldsymbol{\rho},t) = \begin{bmatrix} 0 \\ \dot{\alpha}\rho_3 \\ -\dot{\alpha}\rho_2 \end{bmatrix}, \qquad \dot{\alpha} = \frac{d\alpha(\rho_1,t)}{dt}, \tag{2.161}$$

$$\boldsymbol{a}(\boldsymbol{\rho},t) = \begin{bmatrix} 0 \\ \ddot{\alpha}\rho_3 \\ -\ddot{\alpha}\rho_2 \end{bmatrix}, \qquad \ddot{\alpha} = \frac{d^2\alpha(\rho_1,t)}{dt^2}. \tag{2.162}$$

If we now observe the inverse function of (2.128) or the material coordinates, respectively,

$$\boldsymbol{\rho}(\boldsymbol{r},t) = \begin{bmatrix} r_1 \\ r_2 - \alpha r_3 \\ r_3 + \alpha r_2 \end{bmatrix}, \tag{2.163}$$

we then see that (2.161) and (2.162) are also valid in spatial coordinates. Also, it can be shown quite generally that there are no differences between representations with material and spatial coordinates in linear kinematics.

End of Example 2.7.

2.2 Holonomic Systems

Constrained systems differ from free systems in that the freedom of motion of one or more position variables is limited by mechanical constraints. In engineering, holonomic constraints are realized by means of ideal, i.e. inflexible guides, joints, levers, bearings, rods, and other connections. The constraints between particular machine elements permit the engineer to arrive at a certain total motion in order to solve an engineering problem. On the other hand, constraints also serve to break down a complicated total motion into simple sub-motions, which can then be controlled independently of each other. For an industrial robot, see Fig. 2.11, each degree of freedom is normally assigned one rigid body and a drive motor.

When defining holonomic systems, it is sensible to allow free systems as a special case. This opens up more possibilities for the mathematical description of free systems. The representation of a free system in the form of a holonomic system means nothing more than an additional coordinate transformation. The number of degrees of freedom is unaffected by this, as is the mechanical issue of lacking constraints.

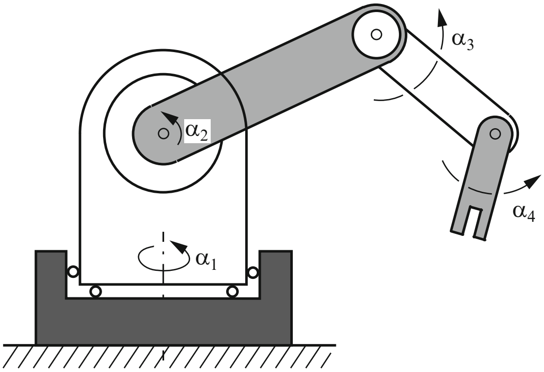


Fig. 2.11 Industrial articulated robot with 4 degrees of freedom

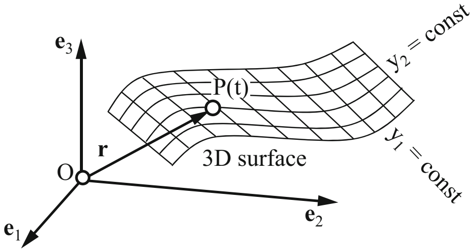


Fig. 2.12 Motion of a point on a three-dimensional surface

2.2.1 Mass Point Systems

Constraints will first be discussed again using the example of a single point. The motion of a mass point $P(t)$ can be restricted due to being confined to a surface or a path. Translational displacement on a surface that is changeable in time, see Fig. 2.12, can be represented uniquely by means of two generalized coordinates $y_1(t)$, $y_2(t)$ corresponding to the two degrees of freedom,

$$\boldsymbol{r}(t) = \boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{r}(y_1, y_2, t). \tag{2.164}$$

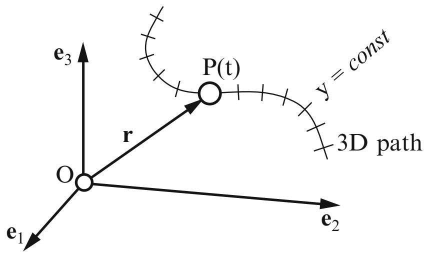
A surface in space is described by a scalar, algebraic and usually nonlinear equation,

$$\phi(\boldsymbol{x}, t) = 0, \tag{2.165}$$

where $\boldsymbol{x}(t)$ is the 3×1 position vector of the free point. This provides us with an implicit constraint onto a surface. With (2.164), we can also represent the constraint explicitly,

$$\boldsymbol{x} = \boldsymbol{x}(y_1, y_2, t). \tag{2.166}$$

Fig. 2.13 Motion of a point along a three-dimensional path



Both representations are equivalent. By means of (2.166), the reduction of the order of the position vector due to the constraint is obvious. Translation along a time-varying path, see Fig. 2.13, has only one degree of freedom with one generalized coordinate $y(t)$. The following applies

$$\boldsymbol{r}(t) = \boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{r}(y,t). \tag{2.167}$$

A path in space is given by two scalar equations,

$$\phi_1(\boldsymbol{x},t) = 0, \qquad \phi_2(\boldsymbol{x},t) = 0. \tag{2.168}$$

Both of these constraints read in explicit form

$$\boldsymbol{x} = \boldsymbol{x}(y,t). \tag{2.169}$$

The number of degrees of freedom of a constrained single point is uniquely determined by its number of constraints. For the point bound to the three-dimensional path, we obtain $f = 3 - 2 = 1$ degrees of freedom.

Example 2.8 (Pendulum). A pendulum of time-varying length $L(t)$ can move on a spherical surface with a varying radius. This introduces a constraint which can be written in Cartesian coordinates as

$$\phi = r_1^2 + r_2^2 + r_3^2 - L^2(t) = 0 \tag{2.170}$$

or using (2.14), (2.15), and Fig. 2.2 in spherical coordinates as

$$\phi = |\boldsymbol{r}| - L(t) = 0. \tag{2.171}$$

A constraint with the Cartesian coordinates r_1, r_2 as generalized coordinates is written in explicit form

$$\boldsymbol{r}(r_1,r_2,t) = \begin{bmatrix} r_1 \\ r_2 \\ \pm\sqrt{L^2(t) - r_1^2 - r_2^2} \end{bmatrix} \tag{2.172}$$

or with the spherical coordinates ψ, ϑ as generalized coordinates

$$\boldsymbol{r}(\psi,\vartheta,t) = \begin{bmatrix} \cos\psi\sin\vartheta \\ \sin\psi\sin\vartheta \\ \cos\vartheta \end{bmatrix} L(t). \tag{2.173}$$

Often, curvilinear coordinates serve better for the introduction of constraints than Cartesian coordinates.

End of Example 2.8.

The constraints limit motion not only of single points in space but in particular the freedom of movement between several mass points of a mass point system. The number of degrees of freedom in a system of p points with q constraints is

$$f = 3p - q. \tag{2.174}$$

The q constraints can be described implicitly by an algebraic, generally nonlinear $q \times 1$ vector equation

$$\boldsymbol{\phi}(\boldsymbol{x},t) = \mathbf{0} \tag{2.175}$$

or explicitly by the $3p \times 1$ vector equation

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{y},t), \tag{2.176}$$

using the $f \times 1$ position vector of the constrained mass point system

$$\boldsymbol{y}(t) = \begin{bmatrix} y_1 \ y_2 \ \dots \ y_f \end{bmatrix}. \tag{2.177}$$

Constraints of the form (2.175) or (2.176), which simultaneously restrict the position and velocity of the system, are called geometric constraints. Another type of constraint is the integrable kinematic constraint of the form

$$\boldsymbol{\phi}(\boldsymbol{x},\dot{\boldsymbol{x}},t) = \mathbf{0}, \tag{2.178}$$

which does indeed depend formally on the velocity parameters, but can be converted to the form (2.175) via integration. Holonomic constraints comprise geometric and integrable kinematic constraints and can always be written in the form (2.175).

Time-invariant constraints are called scleronomic constraints, while time-variant constraints are designated as rheonomic constraints. In addition to the bilateral constraints characterized by Eq. (2.175), there are also unilateral constraints that lead to inequalities. In the form (2.176), unilateral constraints lead to a variable number of degrees of freedom, e.g. such as those that arise in contact problems. An extensive treatment of this subject can be found in Pfeiffer and Glocker [42].

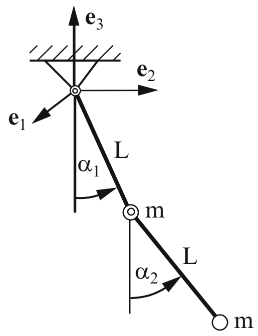


Fig. 2.14 Planar double pendulum

Example 2.9 (Planar Double Pendulum). The double pendulum, see Fig. 2.14, is a two-mass point system with four constraints (both points on the plane, both rod lengths are constant) and thus two degrees of freedom. The corresponding numbers are $p=2, q=4, f=3p-q=3\cdot 2-4=2$. For the Cartesian coordinates

$$\boldsymbol{x}(t)=\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{21} & r_{22} & r_{23} \end{bmatrix} \tag{2.179}$$

and the angular coordinates

$$\boldsymbol{y}(t)=\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \tag{2.180}$$

the scleronomic constraints are written in implicit form

$$\boldsymbol{\phi}=\begin{bmatrix} r_{11} \\ r_{12}^2+r_{13}^2-L^2 \\ r_{21} \\ (r_{22}-r_{12})^2+(r_{23}-r_{13})^2-L^2 \end{bmatrix}=\mathbf{0} \tag{2.181}$$

and in explicit form

$$\boldsymbol{x}=\begin{bmatrix} 0 \\ L\sin\alpha_1 \\ -L\cos\alpha_1 \\ 0 \\ L\sin\alpha_1+L\sin\alpha_2 \\ -L\cos\alpha_1-L\cos\alpha_2 \end{bmatrix}. \tag{2.182}$$

We can confirm the equivalence of both forms by inserting (2.182) into (2.181).

End of Example 2.9.

The translation of a holonomic mass point system is obtained from (2.24) and (2.176), yielding

$$\boldsymbol{r}_i(t)=\boldsymbol{r}_i(\boldsymbol{y},t), \qquad i=1(1)p. \tag{2.183}$$

For the velocity we obtain

$$\boldsymbol{v}_i(t) = \frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{y}} \cdot \dot{\boldsymbol{y}}(t) + \frac{\partial \boldsymbol{r}_i}{\partial t} = \boldsymbol{J}_{Ti}(\boldsymbol{y},t) \cdot \dot{\boldsymbol{y}}(t) + \bar{\boldsymbol{v}}_i(\boldsymbol{y},t), \qquad i = 1(1)p, \tag{2.184}$$

where, besides the $3 \times f$ Jacobian matrix $\boldsymbol{J}_{Ti}$ of translation, the local 3×1 velocity vector $\bar{\boldsymbol{v}}_i$ can appear in the case of rheonomic constraints. For the acceleration one obtains likewise

$$\begin{aligned} \boldsymbol{a}_i(t) &= \boldsymbol{J}_{Ti}(\boldsymbol{y},t) \cdot \ddot{\boldsymbol{y}}(t) + \dot{\boldsymbol{J}}_{Ti}(\boldsymbol{y},t) \cdot \dot{\boldsymbol{y}}(t) + \frac{d\bar{\boldsymbol{v}}_i}{dt} \\ &= \boldsymbol{J}_{Ti}(\boldsymbol{y},t) \cdot \ddot{\boldsymbol{y}}(t) + \bar{\boldsymbol{a}}_i(\boldsymbol{y},\dot{\boldsymbol{y}},t), \qquad i = 1(1)p. \end{aligned} \tag{2.185}$$

In the scleronomic case, the 3×1 acceleration vector $\bar{\boldsymbol{a}}_i$ is quadratically dependent on the first derivative of the position vector. With rheonomic constraints on the other hand, terms can also arise that, in purely mechanical systems, depend either linearly or not at all on the first derivative $\dot{\boldsymbol{y}}(t)$ of the position vector. These terms are calculated with the help of (2.185).

In addition to the real motions of a system, virtual motions are also important in dynamics. A virtual motion is an arbitrary, infinitesimal motion of the system which is compatible with scleronomic and rheonomic constraints (provided they are "frozen" at the given point in time). The symbol δ of virtual quantities possesses the properties of variations in mathematics. The following applies for holonomic constraints

$$\begin{aligned} &\delta \boldsymbol{r} \neq \boldsymbol{0} \quad \text{for movable bearings,} \\ &\delta \boldsymbol{r} = \boldsymbol{0} \quad \text{for firm restraints,} \\ &\delta t = 0. \end{aligned} \tag{2.186}$$

The virtual motion of a point is thus determined by the virtual displacement $\delta \boldsymbol{r}$, while time is not varied. With virtual motions, calculations are made the same way as with differentials

$$\delta(c\boldsymbol{r}) = c\delta\boldsymbol{r}, \qquad \delta(\boldsymbol{r}_1 + \boldsymbol{r}_2) = \delta\boldsymbol{r}_1 + \delta\boldsymbol{r}_2, \qquad \delta\boldsymbol{r}(\boldsymbol{y}) = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}} \cdot \delta\boldsymbol{y}. \tag{2.187}$$

The following applies in particular for the virtual motion of the ith point

$$\delta \boldsymbol{r}_i = \boldsymbol{J}_{Ti} \cdot \delta \boldsymbol{y}, \qquad i = 1(1)p. \tag{2.188}$$

The virtual change of position $\delta \boldsymbol{y}$ determines, using the Jacobian matrices $\boldsymbol{J}_{Ti}$, the entire virtual motion of the system.

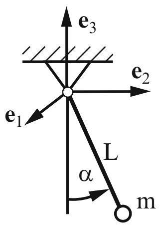


Fig. 2.15 Mathematical pendulum

According to the chain rule (2.11), there is a close connection between the Jacobian matrices $\boldsymbol{H}_{Ti}$ of the free system and $\boldsymbol{J}_{Ti}$ of the constrained system. With (2.8), (2.176) and (2.183), the following relation applies in particular,

$$\boldsymbol{J}_{Ti} = \frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{y}} = \frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}} = \boldsymbol{H}_{Ti}(\boldsymbol{y},t) \cdot \boldsymbol{I}(\boldsymbol{y},t) \tag{2.189}$$

with the $3p \times f$ matrix $\boldsymbol{I}(\boldsymbol{y},t)$. In this way, the practical calculation of the Jacobian matrices can often be simplified considerably.

Example 2.10 (Mathematical Pendulum). The mathematical pendulum is a planar pendulum with one degree of freedom, see Fig. 2.15. With spherical coordinates as generalized coordinates, see (2.14), the constraint equation is

$$\boldsymbol{x} = \left[\, \frac{\pi}{2} \;\; (\pi - \alpha) \;\; L \,\right]. \tag{2.190}$$

With this we obtain

$$\frac{\partial \boldsymbol{x}}{\partial \alpha} = \left[\, 0 \;\; -1 \;\; 0 \,\right]. \tag{2.191}$$

Taking (2.16) into account, from (2.189) we thus obtain the 3×1 Jacobian matrix

$$\boldsymbol{J}_T = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial \alpha} = \begin{bmatrix} 0 \\ L\cos\alpha \\ L\sin\alpha \end{bmatrix}. \tag{2.192}$$

This result can easily be checked by means of direct partial differentiation of the location vector (2.193), *End of Example 2.10*,

$$\boldsymbol{r}(\alpha) = \begin{bmatrix} 0 \\ L\sin\alpha \\ -L\cos\alpha \end{bmatrix}. \tag{2.193}$$

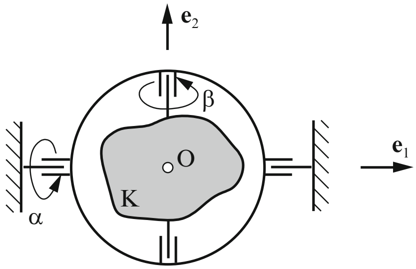


Fig. 2.16 Rotation of a rigid body in the Cardano joint

2.2.2 *Multibody Systems*

Just like the translation of a point, the rotation of a rigid body can also be restricted. The rotation of a rigid body K in a Cardano joint, see Fig. 2.16, is uniquely described by two degrees of freedom with the Cardano angles $\alpha(t), \beta(t)$ as generalized coordinates,

$$\boldsymbol{S}(t) = \boldsymbol{S}(\alpha, \beta). \tag{2.194}$$

The associated constraint is written implicitly with (2.99)

$$\phi(\boldsymbol{x}) = \gamma - \gamma_0 = 0 \tag{2.195}$$

and explicitly

$$\boldsymbol{x} = \boldsymbol{x}(\alpha, \beta) = \left[\, \alpha \;\; \beta \;\; \gamma_0 \,\right]. \tag{2.196}$$

We see that the relations (2.165) and (2.166) found for the translation of a point can be transferred directly to the rotation of a body.

The number of degrees of freedom in a system of p rigid bodies with q constraints is

$$f = 6p - q. \tag{2.197}$$

For the q constraints, (2.175)–(2.177) are valid again, whereby (2.176) represents, in the case of a multibody system, a $6p \times 1$ vector equation.

The position and orientation of a holonomic multibody system is described by

$$\boldsymbol{r}_i(t) = \boldsymbol{r}_i(\boldsymbol{y}, t), \qquad \boldsymbol{S}_i(t) = \boldsymbol{S}_i(\boldsymbol{y}, t), \qquad i = 1(1)p \tag{2.198}$$

in accordance with (2.122) and (2.176). Supplementing (2.184) and (2.185), the following applies for the rotation,

$$\boldsymbol{\omega}_i(t) = \frac{\partial \boldsymbol{s}_i}{\partial \boldsymbol{y}} \cdot \dot{\boldsymbol{y}}(t) + \frac{\partial \boldsymbol{s}_i}{\partial t} = \boldsymbol{J}_{Ri}(\boldsymbol{y},t) \cdot \dot{\boldsymbol{y}}(t) + \overline{\boldsymbol{\omega}}_i(\boldsymbol{y},t), \quad i = 1(1)p, \tag{2.199}$$

$$\begin{aligned}\boldsymbol{\alpha}_i(t) &= \boldsymbol{J}_{Ri}(\boldsymbol{y},t) \cdot \ddot{\boldsymbol{y}}(t) + \dot{\boldsymbol{J}}_{Ri}(\boldsymbol{y},t) \cdot \dot{\boldsymbol{y}} + \dot{\overline{\boldsymbol{\omega}}}_i(\boldsymbol{y},t) \\ &= \boldsymbol{J}_{Ri}(\boldsymbol{y},t) \cdot \ddot{\boldsymbol{y}}(t) + \overline{\boldsymbol{\alpha}}_i(\boldsymbol{y},\dot{\boldsymbol{y}},t), \qquad i = 1(1)p.\end{aligned} \tag{2.200}$$

Here, the instantaneous infinitesimal 3×1 rotation vector $\boldsymbol{s}_i$ from (2.85) has been used again, and like before the comments regarding (2.113) and (2.114) are applicable for the calculation of the $3 \times f$ Jacobian matrix $\boldsymbol{J}_{Ri}$ of rotation. Also, $\overline{\boldsymbol{\omega}}_i$ is the local 3×1 rotation velocity vector and $\overline{\boldsymbol{\alpha}}_i$ is a 3×1 local rotation acceleration vector defined by means of (2.185).

We obtain for the virtual motion of the multibody system

$$\delta \boldsymbol{r}_i = \boldsymbol{J}_{Ti} \cdot \delta \boldsymbol{y}, \qquad \delta \boldsymbol{s}_i = \boldsymbol{J}_{Ri} \cdot \delta \boldsymbol{y}, \qquad i = 1(1)p, \tag{2.201}$$

supplementing (2.188). Also, the following applies in accordance with (2.189),

$$\boldsymbol{J}_{Ri} = \boldsymbol{H}_{Ri}(\boldsymbol{y},t) \cdot \boldsymbol{I}(\boldsymbol{y},t), \tag{2.202}$$

a relation that is very valuable for calculating the Jacobian matrix of rotation.

Example 2.11 (Cardano Joint). The Cardano point, see Fig. 2.16, is a two-body system with ten constraints and two degrees of freedom, $p = 2, q = 10, f = 6p - q = 6 \cdot 2 - 10 = 2$. For the 12×1 position vector of the free system

$$\boldsymbol{x}(t) = [r_{11}\ r_{12}\ r_{13}\ r_{21}\ r_{22}\ r_{23}\ \alpha_1\ \beta_1\ \gamma_1\ \alpha_2\ \beta_2\ \gamma_2] \tag{2.203}$$

and the 2×1 position vector

$$\boldsymbol{y}(t) = \left[\, \alpha\ \beta \,\right] \tag{2.204}$$

the explicit constraints are

$$\boldsymbol{x} = \left[\, 0\ 0\ 0\ 0\ 0\ 0\ \alpha\ \beta\ 0\ \alpha\ 0\ 0 \,\right]. \tag{2.205}$$

Here we took into account the fact that the origin O of the frame is a fixed point of both bodies. Taking (2.100) and (2.202) into consideration, we obtain for the Jacobian matrices

$$\boldsymbol{J}_{T1} = \boldsymbol{J}_{T2} = \boldsymbol{0}, \quad \boldsymbol{J}_{R1} = \begin{bmatrix} 1 & 0 \\ 0 & \cos\alpha \\ 0 & \sin\alpha \end{bmatrix}, \quad \boldsymbol{J}_{R2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.206}$$

and the accelerations are

$$\boldsymbol{a}_1(t) = \boldsymbol{a}_2(t) = \boldsymbol{0}, \tag{2.207}$$

$$\boldsymbol{\alpha}_1(t) = \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta}\cos\alpha - \dot{\alpha}\dot{\beta}\sin\alpha \\ \ddot{\beta}\sin\alpha + \dot{\alpha}\dot{\beta}\cos\alpha \end{bmatrix}, \qquad \boldsymbol{\alpha}_2(t) = \begin{bmatrix} \ddot{\alpha} \\ 0 \\ 0 \end{bmatrix}. \tag{2.208}$$

In practice, we dispense with writing the $6p \times 1$ position vector $\boldsymbol{x}(t)$ of the free system in the case of large multibody systems, since this leads to long expressions, as (2.203) shows. The relations (2.198) are then evaluated directly with the $f \times 1$ position vector $\boldsymbol{y}(t)$.

End of Example 2.11.

By definition, holonomic systems also include free systems as a special case. Specifically, $q = 0, f = 6p$, $\boldsymbol{x} = \boldsymbol{y}$, $\boldsymbol{H}_{Ti} = \boldsymbol{J}_{Ti}$, $\boldsymbol{H}_{Ri} = \boldsymbol{J}_{Ri}$, $\boldsymbol{I} = \boldsymbol{E}$ then applies, i.e. the functional matrix $\boldsymbol{I}$ becomes the $6p \times 6p$ unit matrix.

2.2.3 Continuum

Constraints in a continuum are of a more theoretical nature since they cannot be influenced constructively. Nevertheless, we can easily model highly varying stiffness properties with good approximation by means of constraints. It is then possible to go from a general three-dimensional problem to a simpler two or one-dimensional task. One typical example for a holonomic constraint in a continuum is the Bernoulli hypothesis of beam bending, which requires planar cross-sectional surfaces – even under stress.

Deformation of a continuum is generally dependent on location, so the constraints must also be formulated locally. The constraints are then given as functions of the deformation gradient

$$\boldsymbol{\phi}(\boldsymbol{F}(\boldsymbol{\rho}, t)) = \boldsymbol{0}. \tag{2.209}$$

Rigidity is a constraint typical for a continuum. With (2.32), we can write

$$\boldsymbol{\phi} = \boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{E} = \boldsymbol{0}, \tag{2.210}$$

subjecting the nine coordinates of the deformation gradient to six constraints, so the three degrees of freedom of rotation remain. Besides the internal constraints given by (2.209), a continuum can also be bound to its surroundings. Additional external constraints then arise,

$$\boldsymbol{\phi}(\boldsymbol{r}(\boldsymbol{\rho}, t)) = \boldsymbol{0} \qquad \text{on } A^r, \tag{2.211}$$

which correspond to the boundary conditions on the surface A^r. Boundary conditions restrict deformation on a plane or line segment or at discrete single points on the surface.

Example 2.12 (Torsion of a Round Bar). A twisted round bar with the current configuration (2.128) represents a continuum characterized by six degrees of freedom of rigid body motion and infinitely many degrees of freedom of torsion. In particular, we obtain from (2.130) the relations

$$\phi_1 = U_{11}^2 - 1 = 0, \tag{2.212}$$

$$\phi_2 = U_{22}^2 - 1 = 0, \tag{2.213}$$

$$\phi_3 = U_{33}^2 - 1 = 0, \tag{2.214}$$

$$\phi_4 = U_{23}^2 = 0, \tag{2.215}$$

$$\phi_5 = \rho_2 U_{12}^2 + \rho_3 U_{13}^2 = 0. \tag{2.216}$$

These constraints express the fact that the cross-sectional areas remain flat and undistorted under stress. Also, the round bar can be attached at three points on its left end. The external constraints are then

$$\begin{aligned}
&r_1 - \rho_1 = 0,\ r_2 - \rho_2 = 0,\ r_3 - \rho_3 = 0, \qquad && \text{for } \boldsymbol{\rho} = [0\ 0\ \tfrac{R}{2}],\\
&r_1 - \rho_1 = 0,\ r_2 - \rho_2 = 0, && \text{for } \boldsymbol{\rho} = [0\ 0\ -\tfrac{R}{2}],\\
&r_1 - \rho_1 = 0, && \text{for } \boldsymbol{\rho} = [0\ \tfrac{R}{2}\ 0].
\end{aligned} \tag{2.217}$$

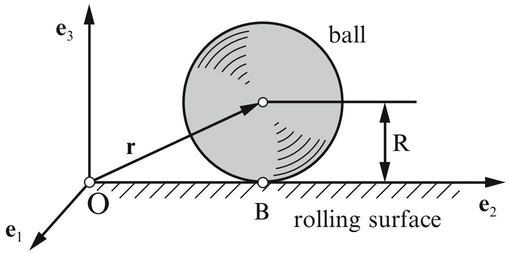
The number of degrees of freedom is $f \to \infty$ for the one-dimensional problem of torsion.

End of Example 2.12.

2.3 Nonholonomic Systems

While holonomic constraints limit the freedom of motion of the position variables, and thus simultaneously of the velocity variables, nonholonomic constraints lead only to a restriction of the velocity, not of the position. Nonholonomic constraints are relatively rare in technology. Linear nonholonomic constraints can be realized purely mechanically, e.g. via rolling rigid wheels, while nonlinear nonholonomic constraints require the use of control devices. However, with the help of nonholonomic constraints, generalized velocities can be employed for a simplified description of holonomic systems as well.

Fig. 2.17 Rolling ball



The number f of degrees of freedom of the position of a holonomic system is reduced by r nonholonomic constraints to the number g of degrees of freedom of velocity. Thus the following applies for a system of p rigid bodies,

$$g = f - r = 6p - q - r, \tag{2.218}$$

where (2.197) has been taken into account. The r nonholonomic constraints can be represented implicitly by the non-integrable $r \times 1$ vector equation

$$\boldsymbol{\psi}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t) = \mathbf{0} \tag{2.219}$$

or explicitly by the $f \times 1$ vector differential equation

$$\dot{\boldsymbol{y}} = \dot{\boldsymbol{y}}(\boldsymbol{y}, \boldsymbol{z}, t), \tag{2.220}$$

yielding the $g \times 1$ vector of the generalized velocity coordinates

$$\boldsymbol{z}(t) = \left[z_1 \; z_2 \; \ldots \; z_g \right]. \tag{2.221}$$

The nonholonomic constraints belong to the kinematic constraints, and they can be scleronomic or rheonomic. It is however an essential condition that (2.219) cannot be integrated. Otherwise the constraints will be holonomic, see (2.175).

Example 2.13 (Rolling Ball). A ball (radius R) rolling on a surface, Fig. 2.17, is a rigid body with a holonomic (motion on a plane) and two nonholonomic constraints (rolling without slipping), $p = 1$, $q = 1$, $r = 2$, $f = 5$, $g = 3$. With the generalized coordinates of the free ball

$$\boldsymbol{x}(t) = \left[r_1 \; r_2 \; r_3 \; \alpha \; \beta \; \gamma \right], \tag{2.222}$$

the generalized coordinates of the ball bound to the surface

$$\boldsymbol{y}(t) = \left[r_1 \; r_2 \; \alpha \; \beta \; \gamma, \right], \tag{2.223}$$

and the generalized velocities

$$\boldsymbol{z}(t) = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \tag{2.224}$$

the holonomic, scleronomic constraint is

$$\phi = r_3 - R = 0 \quad \text{or} \quad \boldsymbol{x} = \begin{bmatrix} r_1 & r_2 & R & \alpha & \beta & \gamma \end{bmatrix}. \tag{2.225}$$

The nonholonomic, scleronomic constraints are obtained implicitly from the rolling condition, yielding

$$\boldsymbol{\psi} = \begin{bmatrix} \dot{r}_1 - R(\dot{\beta}\cos\alpha - \dot{\gamma}\sin\alpha\cos\beta) \\ \dot{r}_2 + R(\dot{\alpha} + \dot{\gamma}\sin\beta) \end{bmatrix} = \boldsymbol{0} \tag{2.226}$$

and explicitly as

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \omega_2 R \\ -\omega_1 R \\ \omega_1 + \omega_2 \sin\alpha\tan\beta - \omega_3\cos\alpha\tan\beta \\ \omega_2\cos\alpha + \omega_3\sin\alpha \\ -\omega_2\frac{\sin\alpha}{\cos\beta} + \omega_3\frac{\cos\alpha}{\cos\beta} \end{bmatrix}. \tag{2.227}$$

The fact was taken into account that the absolute velocity of the point of contact B disappears, and (2.100) and Table 2.2 were consulted.

End of Example 2.13.

The configuration of a nonholonomic multibody system is given in unchanged form by (2.198). The state of velocity on the other hand is obtained from (2.184), (2.199), and (2.220), yielding

$$\boldsymbol{v}_i = \boldsymbol{v}_i(\boldsymbol{y},\boldsymbol{z},t), \qquad \boldsymbol{\omega}_i = \boldsymbol{\omega}_i(\boldsymbol{y},\boldsymbol{z},t), \qquad i = 1(1)p. \tag{2.228}$$

We thereby obtain for the acceleration

$$\boldsymbol{a}_i(t) = \frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{z}}\cdot\dot{\boldsymbol{z}}(t) + \frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{y}}\cdot\dot{\boldsymbol{y}}(t) + \frac{\partial \boldsymbol{v}_i}{\partial t} = \boldsymbol{L}_{Ti}(\boldsymbol{y},\boldsymbol{z},t)\cdot\dot{\boldsymbol{z}}(t) + \overline{\boldsymbol{a}}_i(\boldsymbol{y},\boldsymbol{z},t) \tag{2.229}$$

and also

$$\boldsymbol{\alpha}_i(t) = \frac{\partial \boldsymbol{\omega}_i}{\partial \boldsymbol{z}}\cdot\dot{\boldsymbol{z}}(t) + \frac{\partial \boldsymbol{\omega}_i}{\partial \boldsymbol{y}}\cdot\dot{\boldsymbol{y}}(t) + \frac{\partial \boldsymbol{\omega}_i}{\partial t} = \boldsymbol{L}_{Ri}(\boldsymbol{y},\boldsymbol{z},t)\cdot\dot{\boldsymbol{z}}(t) + \overline{\boldsymbol{\alpha}}_i(\boldsymbol{y},\boldsymbol{z},t). \tag{2.230}$$

For brevity's sake, the $3\times g$ Jacobian matrices $\boldsymbol{L}_{Ti}$ and $\boldsymbol{L}_{Ri}$ and the local 3×1 acceleration vectors $\overline{\boldsymbol{a}}_i$ and $\overline{\boldsymbol{\alpha}}_i$ have been introduced as in the holonomic case.

In analogy to the virtual motions of holonomic systems, we can also introduce the virtual velocity of nonholonomic systems. A virtual velocity is an arbitrary,

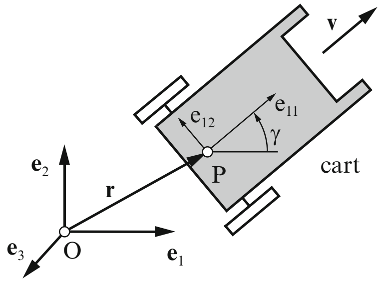


Fig. 2.18 Motion of a cart with rigid wheels

infinitesimal change in velocity which always agrees with the constraints. The symbol δ' of the virtual velocity has the properties

$$\delta'\boldsymbol{r}_i = \delta'\boldsymbol{s}_i = \boldsymbol{0}, \qquad \delta'\boldsymbol{v}_i \neq \boldsymbol{0}, \qquad \delta'\boldsymbol{\omega}_i \neq \boldsymbol{0}, \qquad \delta' t = 0. \tag{2.231}$$

Thus, when determining the virtual velocity, the position and time are not varied. In particular, the following applies for the virtual velocity of a multibody system,

$$\delta'\boldsymbol{v}_i = \boldsymbol{L}_{Ti} \cdot \delta'\boldsymbol{z}, \qquad \delta'\boldsymbol{\omega}_i = \boldsymbol{L}_{Ri} \cdot \delta'\boldsymbol{z}, \qquad i = 1(1)p. \tag{2.232}$$

The virtual change in velocity $\delta'\boldsymbol{z}$ determines, via the functional matrices $\boldsymbol{L}_{Ti}$, $\boldsymbol{L}_{Ri}$, the total virtual velocity of the system.

Also, there is a close connection between the different Jacobian matrices, as was already clarified by (2.202). The following is true,

$$\boldsymbol{L}_{Ti}(\boldsymbol{y},\boldsymbol{z},t) = \boldsymbol{J}_{Ti}(\boldsymbol{y},t) \cdot \boldsymbol{K}(\boldsymbol{y},\boldsymbol{z},t) \tag{2.233}$$

$$\boldsymbol{L}_{Ri}(\boldsymbol{y},\boldsymbol{z},t) = \boldsymbol{J}_{Ri}(\boldsymbol{y},t) \cdot \boldsymbol{K}(\boldsymbol{y},\boldsymbol{z},t) \tag{2.234}$$

with the $f \times g$ matrix

$$\boldsymbol{K}(\boldsymbol{y},\boldsymbol{z},t) = \frac{\partial \dot{\boldsymbol{y}}(\boldsymbol{y},\boldsymbol{z},t)}{\partial \boldsymbol{z}}. \tag{2.235}$$

With this we can often simplify our calculation of the Jacobian matrices.

Example 2.14 (Transport Cart). A transport cart with two independent, massless wheels, see Fig. 2.18, is characterized by the fact that the axial center P cannot move in the body-fixed 2-direction as a result of the static friction forces of the wheels. Assuming planar motion, we are concerned with a body with three holonomic constraints and one nonholonomic constraint, $p = 1, q = 3, r = 1, f = 3, g = 2$. With the 6×1 position vector of the free body

$$\boldsymbol{x}(t) = \begin{bmatrix} r_1 & r_2 & r_3 & \alpha & \beta & \gamma \end{bmatrix}, \tag{2.236}$$

the 3×1 position vector of the cart

$$\boldsymbol{y}(t) = \begin{bmatrix} r_1 & r_2 & \gamma \end{bmatrix} \tag{2.237}$$

and its 2×1 velocity vector

$$\boldsymbol{z}(t) = \left[v \; \dot{\gamma} \right], \tag{2.238}$$

the nonholonomic constraint is written in explicit form

$$\dot{\boldsymbol{y}} = \begin{bmatrix} v\cos\gamma \\ v\sin\gamma \\ \dot{\gamma} \end{bmatrix}. \tag{2.239}$$

For the 3×3 Jacobian matrices we obtain

$$\boldsymbol{J}_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{J}_R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.240}$$

the 3×2 functional matrix is written as

$$\boldsymbol{K}(\boldsymbol{y}) = \begin{bmatrix} \cos\gamma & 0 \\ \sin\gamma & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.241}$$

with which we can also determine using (2.233), (2.234), the 3×2 functional matrices $\boldsymbol{L}_T, \boldsymbol{L}_R$ where

$$\boldsymbol{L}_T = \begin{bmatrix} \cos\gamma & 0 \\ \sin\gamma & 0 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{L}_R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.242}$$

We also find

$$\overline{\boldsymbol{a}} = \begin{bmatrix} -v\dot{\gamma}\sin\gamma \\ v\dot{\gamma}\cos\gamma \\ 0 \end{bmatrix}, \qquad \overline{\boldsymbol{\alpha}} = \boldsymbol{0}. \tag{2.243}$$

Using (2.229), (2.230), this also yields the state of acceleration.

End of Example 2.14.

The nonholonomic constraints (2.219) and (2.220) are sometimes also designated as first-class nonholonomic constraints in order to differentiate them from second-class holonomic constraints, see e.g. Hamel [25]. Second-class nonholonomic constraints restrict accelerations, which is however only of theoretical interest.

Nonholonomic systems also include all holonomic systems as a special case. Since the concept of generalized velocities is lacking in the case of holonomic systems, this special case is not trivial, since the following is then true: $r = 0$, $g = f$, $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{y}, \boldsymbol{z})$, $\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{y}, \boldsymbol{z})$. It should also be mentioned that the case at hand (2.220) is always scleronomic and the $f \times f$ matrix $\boldsymbol{K}$ is usually regular and thus invertible. Generalized velocities offer especially for large holonomic multibody systems crucial advantages resulting from the separation of kinematics and dynamics.

Example 2.15 (Point Motion in Spherical Coordinates). The use of generalized velocities is already advantageous in the investigation of a simple point motion. The Jacobian matrix $\boldsymbol{H}_T$ can be reduced to a simpler functional matrix $\boldsymbol{L}_T$ by means of the appropriate choice of generalized velocities. With the generalized velocities

$$\boldsymbol{z}(t) = \left[\, r\dot{\psi} \; r\dot{\vartheta} \; \dot{r} \,\right] \tag{2.244}$$

we obtain

$$\dot{\boldsymbol{y}}(\boldsymbol{y},\boldsymbol{z}) = \left[\, \tfrac{1}{r}(r\dot{\psi}) \; \tfrac{1}{r}(r\dot{\vartheta}) \; \dot{r} \,\right] \tag{2.245}$$

and the 3×3 matrix

$$\boldsymbol{K}(\boldsymbol{y}) = \begin{bmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.246}$$

which, together with (2.16) and (2.233), leads to a dimensionless regular 3×3 matrix $\boldsymbol{L}_T$. This moves the singularity to (2.246) – it cannot be avoided even with generalized velocities.

End of Example 2.15.

2.4 Relative Motion of the Frame

The previous discussions were always based on a spatially fixed frame that is not in motion. This assumption was especially important in the calculation of velocity and acceleration, see e.g. (2.5) and (2.12). Yet for many engineering problems, it is useful to introduce a moving frame in addition to the fixed frame. The motion of the frame can either be predefined as a target motion, or it is obtained directly as a particular solution from the equations of motion. In the neighborhood of the target motion or of a particular, periodic solution, we can then often execute a linearization of the motion.

2.4.1 Moving Frame

In addition to the spatially fixed inertial frame $\{0_I; \boldsymbol{e}_{I\alpha}\}$, now a moving reference frame $\{0_R; \boldsymbol{e}_{R\alpha}\}$, $\alpha = 1(1)3$ is introduced. The motion of the frame R is described with respect to the frame I by the 3×1 vector $\boldsymbol{r}_R(t)$ and the 3×3 rotation tensor $\boldsymbol{S}_R(t)$. For the basis vectors, the transformation law is applicable,

$$\boldsymbol{e}_{I\alpha} = \boldsymbol{S}_R(t) \cdot \boldsymbol{e}_{R\alpha}(t), \qquad \alpha = 1(1)3, \tag{2.247}$$

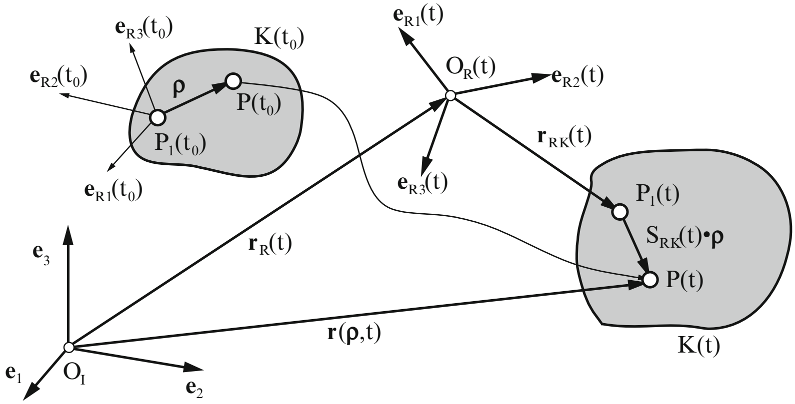


Fig. 2.19 Relative motion of a rigid body

which is analogously also valid for the coordinates of vectors and tensors. For the coordinates of a vector $\boldsymbol{a}$ or a tensor $\boldsymbol{A}$, we obtain the relation

$$ {}_I\boldsymbol{a} = \boldsymbol{S}_R \cdot {}_R\boldsymbol{a}, \qquad {}_I\boldsymbol{A} = \boldsymbol{S}_R \cdot {}_R\boldsymbol{A} \cdot \boldsymbol{S}_R^T. \tag{2.248} $$

In case it is required, the frame is displayed by the lower left index.

Using Fig. 2.19, we thus obtain the following for the current configuration of a rigid body K,

$$ {}_I\boldsymbol{r}(\boldsymbol{\rho},t) = {}_I\boldsymbol{r}_R(t) + \boldsymbol{S}_R(t) \cdot [{}_R\boldsymbol{r}_{R1}(t) + \boldsymbol{S}_{RK}(t) \cdot \boldsymbol{\rho}], \tag{2.249} $$

or completely written in the inertial frame I,

$$ \boldsymbol{r}(\rho,t) = \boldsymbol{r}_R(t) + \boldsymbol{r}_{R1}(t) + \boldsymbol{S}_R(t) \cdot \boldsymbol{S}_{RK}(t) \cdot \boldsymbol{\rho}. \tag{2.250} $$

By comparing with (2.78), we thus obtain for the absolute position of the rigid body, expressed in the parameters of relative motion,

$$ \boldsymbol{r}_1(t) = \boldsymbol{r}_R(t) + \boldsymbol{r}_{R1}(t), \tag{2.251} $$

$$ \boldsymbol{S}(t) = \boldsymbol{S}_R(t) \cdot \boldsymbol{S}_{RK}(t). \tag{2.252} $$

If we also take into account the inverse deformation

$$ \boldsymbol{\rho} = \boldsymbol{S}_{RK}^T(t) \cdot \boldsymbol{S}_R^T(t) \cdot \boldsymbol{r}_{RP}(\boldsymbol{\rho},t), \tag{2.253} $$

we then obtain, via the material derivative of (2.249), the absolute velocity

$$ \begin{aligned} {}_I\boldsymbol{v}(\boldsymbol{\rho},t) &= \frac{d}{dt}{}_I\boldsymbol{r}_R + \frac{d}{dt}\boldsymbol{S}_R \cdot [{}_R\boldsymbol{r}_{R1} + \boldsymbol{S}_{RK} \cdot \boldsymbol{\rho}] + \boldsymbol{S}_R \cdot [\frac{d}{dt}{}_R\boldsymbol{r}_{R1} + \frac{d}{dt}\boldsymbol{S}_{RK} \cdot \boldsymbol{\rho}] \\ &= {}_I\boldsymbol{r}_R^* + {}_I\tilde{\boldsymbol{\omega}}_R \cdot {}_I\boldsymbol{r}_{R1} + {}_I\dot{\boldsymbol{r}}_{R1} + ({}_I\tilde{\boldsymbol{\omega}}_R + {}_I\tilde{\boldsymbol{\omega}}_{RK}) \cdot {}_I\boldsymbol{r}_{RP}, \end{aligned} \tag{2.254} $$

where $(^*)$ signifies the derivative in the inertial frame and $(^\cdot)$ the derivative in the reference frame. Comparison with (2.86) yields the following for the laws of relative motion, see e.g. Magnus and Müller-Slany [36],

$$\boldsymbol{v}_1(t) = \boldsymbol{r}_R^*(t) + \tilde{\boldsymbol{\omega}}_R(t) \cdot \boldsymbol{r}_{R1}(t) + \dot{\boldsymbol{r}}_{RK}(t), \tag{2.255}$$

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_R(t) + \boldsymbol{\omega}_{RK}(t). \tag{2.256}$$

A corresponding calculation finally leads to the following for the absolute acceleration of the relative motion,

$$\boldsymbol{a}_1(t) = \boldsymbol{r}_R^{**} + (\dot{\tilde{\boldsymbol{\omega}}}_R + \tilde{\boldsymbol{\omega}}_R \cdot \tilde{\boldsymbol{\omega}}_R) \cdot \boldsymbol{r}_{R1} + 2\tilde{\boldsymbol{\omega}}_R \cdot \dot{\boldsymbol{r}}_{R1} + \ddot{\boldsymbol{r}}_{R1}, \tag{2.257}$$

$$\boldsymbol{\alpha}(t) = \dot{\boldsymbol{\omega}}_R + \tilde{\boldsymbol{\omega}}_R \cdot \boldsymbol{\omega}_{RK} + \dot{\boldsymbol{\omega}}_{RK}. \tag{2.258}$$

In (2.257), the first two terms denote the guidance acceleration, the third term the Coriolis acceleration, and the fourth term the relative acceleration.

The reference frame R can also be attached to the rigid body K. We then call it a body-fixed frame $\{O_1, \boldsymbol{e}_{11}\}$. In this special case, the following applies,

$$\boldsymbol{r}_{R1}(t) = \mathbf{0}, \qquad \boldsymbol{S}_{RK}(t) = \boldsymbol{E} \tag{2.259}$$

and (2.250) turns directly into (2.78). This means that the motion of a rigid body can also be interpreted as the motion of a Cartesian frame that is connected to the rigid body. If we restrict ourselves to rigid body mechanics from the outset, this gives us an easy access to the kinematics. However, describing rigid body motion with body-fixed frames complicates the continuum-mechanical approach that is privileged in this book.

Given a multibody system, a separate reference frame $\{O_{jR}; \boldsymbol{e}_{jR\alpha}\}$, $\alpha = 1(1)3$, $j = 1(1)n$ can be selected for each partial body K_i, $i = 1(1)p$. Then

$$\boldsymbol{r}_i(t) = \boldsymbol{r}_{jR}(t) + \boldsymbol{r}_{jRi}(t), \tag{2.260}$$

$$\boldsymbol{S}_i(t) = \boldsymbol{S}_{jR}(t) \cdot \boldsymbol{S}_{jRi}(t) \tag{2.261}$$

applies and the relations (2.255)–(2.258) must also be generalized accordingly.

2.4.2 Free and Holonomic Systems

Holonomic systems include the free systems, $q = 0, f = 6p, \boldsymbol{x} = \boldsymbol{y}, \boldsymbol{I} = \boldsymbol{E}$, as a special case. Mass point systems represent a subgroup of multibody systems with $f = 3p$. For this reason, it will be sufficient to deal only with holonomic multibody systems in this context.

The number of degrees of freedom of a system is not changed by the introduction of one or more reference frames. The degrees of freedom can however be distributed varyingly to the reference and relative motions. If the reference motion is predefined by pure time functions, the relative motion encompasses all degrees of freedom. If we choose body-fixed reference frames exclusively on the other hand, all degrees of freedom are found in the reference motion. In the common case of a mixed distribution of degrees of freedom therefore

$$\boldsymbol{r}_R = \boldsymbol{r}_R(\boldsymbol{y},t), \qquad \boldsymbol{S}_R = \boldsymbol{S}_R(\boldsymbol{y},t) \tag{2.262}$$

applies. Assuming that all vectors and tensors are represented in the reference frame, we then obtain in accordance with (2.184), (2.199) for the guidance velocities of the reference motion

$$\boldsymbol{r}_R^* = \boldsymbol{S}_R^T \cdot \left(\frac{\partial \boldsymbol{S}_r \cdot \boldsymbol{r}_R}{\partial \boldsymbol{y}} \cdot \dot{\boldsymbol{y}} + \frac{\partial \boldsymbol{S}_r \cdot \boldsymbol{r}_R}{\partial t} \right) = \boldsymbol{J}_{TR}(\boldsymbol{y},t) \cdot \dot{\boldsymbol{y}}(t) + \bar{\boldsymbol{v}}_R(\boldsymbol{y},t), \tag{2.263}$$

$$\boldsymbol{\omega}_R = \frac{\partial \boldsymbol{s}_R}{\partial \boldsymbol{y}} \cdot \dot{\boldsymbol{y}} + \frac{\partial \boldsymbol{s}_R}{\partial t} = \boldsymbol{J}_{RR}(\boldsymbol{y},t) \cdot \dot{\boldsymbol{y}}(t) + \bar{\boldsymbol{\omega}}_R(\boldsymbol{y},t) \tag{2.264}$$

with the $3 \times f$ Jacobian matrices $\boldsymbol{J}_{TR}$ and $\boldsymbol{J}_{RR}$ of the guidance motion,

$$\boldsymbol{r}_{Ri} = \boldsymbol{r}_{Ri}(\boldsymbol{y},t), \qquad \boldsymbol{S}_{Ri} = \boldsymbol{S}_{Ri}(\boldsymbol{y},t) \tag{2.265}$$

and for the relative velocities we obtain likewise

$$\dot{\boldsymbol{r}}_{Ri} = \frac{\partial \boldsymbol{r}_{Ri}}{\partial \boldsymbol{y}} \cdot \dot{\boldsymbol{y}} + \frac{\partial \boldsymbol{r}_{Ri}}{\partial t} = \boldsymbol{J}_{TRi}(\boldsymbol{y},t) \cdot \dot{\boldsymbol{y}}(t) + \bar{\boldsymbol{v}}_{Ri}(\boldsymbol{y},t), \tag{2.266}$$

$$\boldsymbol{\omega}_{Ri} = \frac{\partial \boldsymbol{s}_{Ri}}{\partial \boldsymbol{y}} \cdot \dot{\boldsymbol{y}} + \frac{\partial \boldsymbol{s}_{Ri}}{\partial t} = \boldsymbol{J}_{RRi}(\boldsymbol{y},t) \cdot \dot{\boldsymbol{y}}(t) + \bar{\boldsymbol{\omega}}_{Ri}(y,t). \tag{2.267}$$

Here, $\boldsymbol{J}_{TRi}$ and $\boldsymbol{J}_{RRi}$ are the $3 \times f$ Jacobian matrices of the relative motion. We then find the accelerations for the guidance and relative motions following (2.185) and (2.200).

The absolute velocities and accelerations are then obtained with (2.262)–(2.267) from (2.255) to (2.258). We hereby find for the Jacobian matrices the relation

$$\boldsymbol{J}_{Ti} = \boldsymbol{J}_{TR} + \boldsymbol{J}_{TRi} - \tilde{\boldsymbol{r}}_{Ri} \cdot \boldsymbol{J}_{RR}, \tag{2.268}$$

$$\boldsymbol{J}_{Ri} = \boldsymbol{J}_{RR} + \boldsymbol{J}_{RRi}. \tag{2.269}$$

We can see that a purely time-dependent guidance motion does not at all affect the Jacobian matrices of the multibody system at hand, $\boldsymbol{J}_{TR} = \boldsymbol{J}_{RR} = \boldsymbol{0}$.

Example 2.16 (Overturning Double Pendulum). Both bodies of the double pendulum in Fig. 2.20 have a high initial velocity. The initial conditions are $\alpha_{10} = \alpha_{20} = 0$,

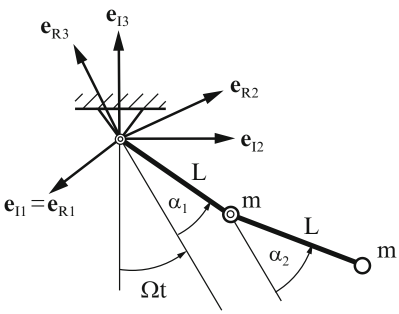


Fig. 2.20 Overturning double pendulum

$\dot{\alpha}_{10} = \dot{\alpha}_{20} = \Omega \gg \sqrt{g/L}$. In order to examine the motion, it is practical to use a reference frame that rotates with the rotation velocity Ω,

$$\boldsymbol{r}_R(t) = \mathbf{0}, \quad \boldsymbol{S}_R(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega t & -\sin\Omega t \\ 0 & \sin\Omega t & \cos\Omega t \end{bmatrix}, \tag{2.270}$$

$$\boldsymbol{r}_R^*(t) = \mathbf{0}, \quad \boldsymbol{\omega}_R = \overline{\boldsymbol{\omega}}_R = \begin{bmatrix} \Omega & 0 & 0 \end{bmatrix}, \quad \boldsymbol{J}_{TR} = \boldsymbol{J}_{RR} = \mathbf{0}. \tag{2.271}$$

Also, the following applies for the relative motion in the reference frame

$$\boldsymbol{r}_{R1} = \begin{bmatrix} 0 \\ \sin\alpha_1 \\ -\cos\alpha_1 \end{bmatrix} L, \qquad \boldsymbol{r}_{R2} = \begin{bmatrix} 0 \\ \sin\alpha_1 + \sin\alpha_2 \\ -\cos\alpha_1 - \cos\alpha_2 \end{bmatrix} L \tag{2.272}$$

with the Jacobian matrices

$$\boldsymbol{J}_{TR1} = \begin{bmatrix} 0 & 0 \\ \cos\alpha_1 & 0 \\ \sin\alpha_1 & 0 \end{bmatrix} L, \quad \boldsymbol{J}_{TR2} = \begin{bmatrix} 0 & 0 \\ \cos\alpha_1 & \cos\alpha_2 \\ \sin\alpha_1 & \sin\alpha_2 \end{bmatrix} L. \tag{2.273}$$

Observing (2.255), the absolute velocities in the reference frame read as

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ (\dot{\alpha}_1 + \Omega)\cos\alpha_1 \\ (\dot{\alpha}_1 + \Omega)\sin\alpha_1 \end{bmatrix} L, \tag{2.274}$$

$$\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ (\dot{\alpha}_1 + \Omega)\cos\alpha_1 + (\dot{\alpha}_2 + \Omega)\cos\alpha_2 \\ (\dot{\alpha}_2 + \Omega)\sin\alpha_1 + (\dot{\alpha}_2 + \Omega)\sin\alpha_2 \end{bmatrix} L. \tag{2.275}$$

By means of the moving reference frame R, the Jacobian matrices remain in the simple form (2.273), even when using relative coordinates. Further advantages will be seen in the linearization of the motion.

End of Example 2.16.

2.4.3 Nonholonomic Systems

Nonholonomic constraints in the explicit form (2.220) can be inserted into the expressions (2.263)–(2.267) of the guidance and relative velocities, so then they depend too on the $g \times 1$ vector of the generalized velocities. The corresponding accelerations are again obtained using (2.229) and (2.230). We finally arrive at the absolute velocities and accelerations with (2.255)–(2.258). For the Jacobian matrices, we obtain the relation

$$\boldsymbol{L}_{Ti} = \boldsymbol{L}_{TR} + \boldsymbol{L}_{TRi} - \tilde{\boldsymbol{r}}_{Ri} \cdot \boldsymbol{L}_{RR}, \tag{2.276}$$

$$\boldsymbol{L}_{Ri} = \boldsymbol{L}_{RR} + \boldsymbol{L}_{RRi}. \tag{2.277}$$

It is also true here that a purely time-dependent guidance motion does not influence the Jacobian matrices of the multibody system under consideration, $\boldsymbol{L}_{TR} = \boldsymbol{L}_{RR} = \boldsymbol{0}$.

2.5 Linearization of the Kinematics

We have already discussed the linearization of kinematic relations in our discussion of continua in Sect. 2.1.3. In this section, the linearization of the motion of point and multibody systems with respect to an arbitrary target motion will be considered. We will again ignore the distinction between free and holonomic systems in this context.

In engineering, a target motion $\boldsymbol{y}_S(t)$ is often defined by the task of a machine or device, whereby the actual motion $\boldsymbol{y}(t)$ deviates only slightly from it. If it is true that the velocities $\dot{\boldsymbol{y}}(t)$ and accelerations $\ddot{\boldsymbol{y}}(t)$ essentially also correspond to the target motion, then the following is valid for holonomic systems,

$$\boldsymbol{y}(t) = \boldsymbol{y}_S(t) + \boldsymbol{\eta}(t), \qquad |\boldsymbol{\eta}(t)| \ll a, \tag{2.278}$$

$$\dot{\boldsymbol{y}}(t) = \dot{\boldsymbol{y}}_S(t) + \dot{\boldsymbol{\eta}}(t), \qquad |\dot{\boldsymbol{\eta}}(t)| \ll b, \tag{2.279}$$

$$\ddot{\boldsymbol{y}}(t) = \ddot{\boldsymbol{y}}_S(t) + \ddot{\boldsymbol{\eta}}(t), \qquad |\ddot{\boldsymbol{\eta}}(t)| \ll c, \tag{2.280}$$

where $\boldsymbol{\eta}(t)$ is the $f \times 1$ position vector of the small deviations and a, b, c represent suitable reference values. With (2.278), we obtain from (2.198) after a Taylor series expansion,

$$\boldsymbol{r}_i(\boldsymbol{\eta}, t) = \boldsymbol{r}_{iS}(t) + \boldsymbol{J}_{TiS}(t) \cdot \boldsymbol{\eta} + \boldsymbol{r}_{i2}(\boldsymbol{\eta} \cdot \boldsymbol{\eta}, t) + \ldots, \tag{2.281}$$

$$\boldsymbol{S}_i(\boldsymbol{\eta}, t) = \boldsymbol{S}_{iS}(t) + \boldsymbol{S}_{i1}(\boldsymbol{\eta}, t) + \boldsymbol{S}_{i2}(\boldsymbol{\eta} \cdot \boldsymbol{\eta}, t) + \ldots, \tag{2.282}$$

where $\boldsymbol{r}_{iS}(t)$ and $\boldsymbol{S}_{iS}(t)$ denote the 3×1 position vector and the 3×3 rotation tensor of the target motion. Also, the following applies according to (2.184) for the linearized $3 \times f$ Jacobian matrix of translation

$$\boldsymbol{J}_{Ti}(\boldsymbol{\eta},t) = \boldsymbol{J}_{TiS}(t) + \boldsymbol{J}_{Ti1}(\boldsymbol{\eta},t) + \ldots \quad . \tag{2.283}$$

For the linear term of the series expansion of the Jacobian matrix it follows

$$\boldsymbol{J}_{Ti1}(\boldsymbol{\eta},t) = \frac{\partial \boldsymbol{r}_{i2}(\boldsymbol{\eta}\cdot\boldsymbol{\eta},t)}{\partial \boldsymbol{\eta}}. \tag{2.284}$$

If we now neglect the quadratic and higher components, we obtain for the velocity and acceleration of holonomic systems

$$\boldsymbol{v}_i(t) = \boldsymbol{J}_{TiS}(t)\cdot\dot{\boldsymbol{\eta}}(t) + \dot{\boldsymbol{J}}_{TiS}(t)\cdot\boldsymbol{\eta}(t) + \boldsymbol{v}_{iS}(t), \tag{2.285}$$

$$\boldsymbol{a}_i(t) = \boldsymbol{J}_{TiS}(t)\cdot\ddot{\boldsymbol{\eta}}(t) + 2\dot{\boldsymbol{J}}_{TiS}(t)\cdot\dot{\boldsymbol{\eta}}(t) + \ddot{\boldsymbol{J}}_{TiS}(t)\cdot\boldsymbol{\eta}(t) + \boldsymbol{a}_{iS}(t), \tag{2.286}$$

while for the virtual translational motion

$$\delta\boldsymbol{r}_i = \left[\boldsymbol{J}_{TiS}(t) + \boldsymbol{J}_{Ti1}(\boldsymbol{\eta},t)\right]\cdot\delta\boldsymbol{\eta} \tag{2.287}$$

is true.

Equation (2.283) applies analogously for the linearized $3 \times f$ Jacobian matrix of rotation. Calculation of the Jacobian matrices $\boldsymbol{J}_{RiS}(t)$ and $\boldsymbol{J}_{Ri1}(t)$ is much more complicated however. Taking the definition found in (2.113), (2.114) into account, we obtain

$$\frac{\partial \tilde{s}_{iS\alpha\beta}}{\partial \eta_\delta} = \frac{\partial S_{i1\alpha\gamma}}{\partial \eta_\delta} S_{iS\beta\gamma}, \tag{2.288}$$

$$\frac{\partial \tilde{s}_{i1\alpha\beta}}{\partial \eta_\delta} = \frac{\partial S_{i2\alpha\gamma}}{\partial \eta_\delta} S_{iS\beta\gamma} + \frac{\partial S_{i1\alpha\gamma}}{\partial \eta_\delta} S_{i1\beta\gamma}, \qquad \alpha,\beta,\gamma = 1(1)3, \delta = 1(1)f. \tag{2.289}$$

The rotation speed and rotation acceleration are thus

$$\begin{aligned}\boldsymbol{\omega}_i(t) =& \boldsymbol{J}_{RiS}(t)\cdot\dot{\boldsymbol{\eta}}(t) + \boldsymbol{J}'_{RiS}(t)\cdot\boldsymbol{\eta}(t) + \boldsymbol{\omega}_{iS}(t),\\ & \boldsymbol{J}'_{RiS}(t)\boldsymbol{\eta}(t) = \frac{\partial \boldsymbol{S}_{iS}}{\partial t}\cdot\boldsymbol{S}_{i1}^T + \frac{\partial \boldsymbol{S}_{i1}}{\partial t}\cdot\boldsymbol{S}_{iS}^T,\end{aligned} \tag{2.290}$$

$$\boldsymbol{\alpha}_i(t) = \boldsymbol{J}_{RiS}(t)\cdot\ddot{\boldsymbol{\eta}}(t) + (\dot{\boldsymbol{J}}_{RiS}(t) + \boldsymbol{J}'_{RiS}(t))\cdot\dot{\boldsymbol{\eta}}(t) + \dot{\boldsymbol{J}}'_{RiS}(t)\cdot\boldsymbol{\eta}(t) + \boldsymbol{\alpha}_{iS}(t), \tag{2.291}$$

while the virtual rotation corresponds to the relation (2.287).

We can see that the rotation, because of its nonlinearity, involves considerably more complexity in the linearization process than the translation. Here again, the relations (2.288) and (2.289) would only be applied in the context of a computer program. For smaller problems, it is advisable to proceed from the elementary rotations and to obtain the Jacobian matrices $\boldsymbol{J}_{RiS}(t)$ and $\boldsymbol{J}'_{RiS}(t)$ appearing in (2.290) intuitively using a geometric approach.

When linearizing, we must especially take heed that $\boldsymbol{\eta} \cdot \delta\boldsymbol{\eta}$ is not a quadratic term in the sense of a Taylor series expansion. This means that the series expansion in (2.281) is required up to the second term in order to determine the virtual motion. If the series expansion in (2.281) is already interrupted after the first term, completely false results could emerge when determining the generalized forces. This connection is commonly overlooked in the literature and in the development of program systems designed to investigate linear multibody systems. If we obtain the linearized accelerations (2.286), (2.291) not by total derivation of the linearized velocities (2.285), (2.290), but rather using general, nonlinear relations (2.185), (2.200), then even the third term in (2.281) must be taken into account for $\dot{\boldsymbol{y}}_S(t) = \boldsymbol{0}$. This method is therefore not recommended for setting up linear relations.

Nonholonomic systems can be linearized without difficulty. In addition to (2.281), (2.220) must then also undergo a series expansion, i.e. the target motion is determined by $\boldsymbol{y}_S(t)$ and $\boldsymbol{z}_S(t)$.

Furthermore, it is often useful to carry out a partial linearization. In this case, some of the position coordinates and/or some velocity coordinates are viewed as small due to the physics or the actual motion, while the rest may be large. We then of course do not obtain completely linear equations of motion, but the solution can nonetheless be substantially simplified.

Example 2.17 (Overturning Pendulum). Let the target motion of the double pendulum, see Fig. 2.20, be given by the motion of the reference frame. Then the following applies with respect to the inertial frame

$$\boldsymbol{y}_S(t) = \begin{bmatrix} \Omega t \\ \Omega t \end{bmatrix}, \qquad \boldsymbol{\eta}(t) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{2.292}$$

and it should be assumed here that, despite the large guidance motion $\boldsymbol{y}_S(t)$, only small deviations from this arise, i.e., $\alpha_1 \ll 1$, $\alpha_2 \ll 1$. The series expansion for the first location vector is written up to the second term

$$_I\boldsymbol{r}_1 = \begin{bmatrix} 0 \\ \sin\Omega t \\ -\cos\Omega t \end{bmatrix} L + \begin{bmatrix} 0 \\ \alpha_1 \cos\Omega t \\ \alpha_1 \sin\Omega t \end{bmatrix} L + \begin{bmatrix} 0 \\ -\frac{1}{2}\alpha_1^2 \sin\Omega t \\ \frac{1}{2}\alpha_1^2 \cos\Omega t \end{bmatrix} L \tag{2.293}$$

and the Jacobian matrix is obtained for the first mass point in the form

$$ {}_I\boldsymbol{J}_{T1S} = \begin{bmatrix} 0 & 0 \\ \cos\Omega t & 0 \\ \sin\Omega t & 0 \end{bmatrix} L, \tag{2.294} $$

$$ {}_I\dot{\boldsymbol{J}}_{T1s} = \begin{bmatrix} 0 & 0 \\ -\sin\Omega t & 0 \\ \cos\Omega t & 0 \end{bmatrix} L, \tag{2.295} $$

$$ {}_I\boldsymbol{v}_{is} = \begin{bmatrix} 0 \\ \Omega\cos\Omega t \\ \Omega\sin\Omega t \end{bmatrix}. \tag{2.296} $$

In accordance with (2.285), (2.286), now the velocity and acceleration of the first mass point are also determined. We then obtain for example

$$ {}_I\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ \dot\alpha_1\cos\Omega t - \alpha_1\Omega\sin\Omega t + \Omega\cos\Omega t \\ \dot\alpha_1\sin\Omega t + \alpha_1\Omega\cos\Omega t + \Omega\sin\Omega t \end{bmatrix} L. \tag{2.297} $$

If we observe the pendulum motion in the reference frame, then all expressions are simplified. Equations (2.293)–(2.297) yield with (2.270) Equations (2.298)–(2.299), *End of Example 2.17*,

$$ {}_R\boldsymbol{r}_1 = \begin{bmatrix} 0 \\ \alpha_1 \\ -1+\frac{1}{2}\alpha_1^2 \end{bmatrix} L, \qquad {}_R\boldsymbol{J}_{T1S} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} L \tag{2.298} $$

and

$$ {}_R\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ \dot\alpha_1 + \Omega \\ \alpha_1\Omega \end{bmatrix} L. \tag{2.299} $$

Kinematics is a very extensive branch of applied dynamics. Many important concepts and definitions have been introduced in this chapter, such as the point, rigid body, and continuum models, the motion types of translation, rotation, and strain, generalized coordinates and velocities, holonomic and nonholonomic constraints, motion relative to reference frames, and small, linearizable deviations from a target motion. All these basic concepts will be made use of again and again in the following chapters.