Chapter 2
Stability of Dynamical Systems: Linear Approach

2.1 Introduction

Our understanding of the stability of a particular operating mode of a dynamical system is formed intuitively as we build up our experience and understanding of everyday life and nature. The first steps of a small child give him or her very real representations of the stability of walking, although these representations may not yet enter consciousness. Looking at the famous painting entitled *Young Acrobat on a Ball* by P. Picasso, we have a distinct feeling that the girl's equilibrium is not quite stable. As adults, we can already discuss the stability of a ship on a stormy sea, the stability of economic trends in relation to the actions of managers and politicians, the stability of our nervous system with regard to stressful perturbation, etc. In each case, we talk about different properties that are specific to the considered systems. However, if we think about it carefully, we can find something in common, inherent in any system. The common feature is that, when we talk about stability, we understand the way the dynamical system reacts to a small perturbation of its state. If arbitrarily small changes in the system state begin to grow in time, the system is unstable. Otherwise, small perturbations decay with time and the system is stable.

It is extremely important from a practical point of view to be able to analyse the stability of the operating modes of dynamical systems. Stability of such systems as a car, an aircraft, or an ocean liner to perturbations is certainly a vital factor in the truest sense of the word, since such perturbations are always going to be present in one form or another.

These arguments are qualitative and can be made precise only if we manage to translate them into the formal language of mathematics. The fundamentals of the rigorous mathematical theory of stability were laid down in the works of the prominent Russian mathematician A.M. Lyapunov 100 years ago, while the

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development of the qualitative theory and bifurcation theory of dynamical systems is associated with Russian scientists A.A. Andronov, V.I. Arnold, and their pupils.

In this chapter we shall define the stability of a dynamical system and, with the help of some simple and clear examples, attempt to illustrate its content and also some methods for solving stability problems.

2.2 Definition of Stability

There are in fact many different definitions of stability, among which the following are the most frequently encountered: stability according to Poisson, stability according to Lyapunov, and asymptotic stability. Let a DS be described by the system of ordinary differential equations (1.5) or by (1.6) in the vector form. We are interested in the stability of a trajectory $\mathbf{x}^0(t)$.

Stability according to Poisson means that, after a while, the phase trajectory returns to an arbitrarily small neighbourhood of the initial point $\mathbf{x}_0^0 = \mathbf{x}^0(t_0)$. Moreover, if the system is reversible, return occurs both forward and backward in time. The time interval after which the trajectory returns to a neighborhood of the point $\mathbf{x}_0^0$ with given radius ε is called the *Poincaré recurrence time*. Recurrence times may correspond to the period or quasiperiod of a regular motion and represent a random sequence in the regime of dynamical chaos (Fig. 2.1).

Stability according to Poisson is an important but weak form of stability. We can say nothing about the behavior of neighboring trajectories, initially close to $\mathbf{x}^0(t)$. In practical problems we are often interested in another property of stability, associated with a small perturbation of a given trajectory. Depending on the temporal dynamics of the perturbation, we distinguish stability according to Lyapunov and asymptotic stability.

The trajectory $\mathbf{x}^0(t)$ is said to be *stable according to Lyapunov* if, for any arbitrarily small $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that, for any trajectory $\mathbf{x}(t)$ for which $\|\mathbf{x}(t_0) - \mathbf{x}^0(t_0)\| < \delta$, the inequality $\|\mathbf{x}(t) - \mathbf{x}^0(t)\| < \varepsilon$ is satisfied for all $t > t_0$. The symbol $\|\ldots\|$ denotes the vector norm in $\mathbb{R}^N$. Thus, a small initial perturbation does not grow in time for a phase trajectory that is stable according to Lyapunov. If the small perturbation δ vanishes as time goes by, i.e., $\|\mathbf{x}(t) - \mathbf{x}^0(t)\| \to 0$ as $t \to \infty$, the trajectory possesses a stronger stability property, namely, *asymptotic stability*. Any asymptotically stable phase trajectory is stable according to Lyapunov. The opposite is not generally true.

The stability properties of phase trajectories belonging to limit sets, e.g., attractors, are of special importance for understanding the system dynamics. In many cases, a change in the kind of stability of one or another limit set can change the operating mode of the system.

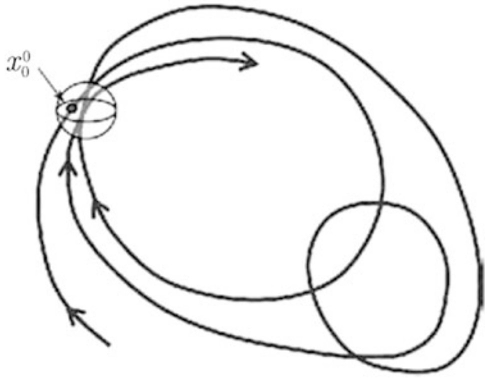

Fig. 2.1 Poisson-stable non-closed trajectory

2.3 Linear Analysis of Stability

2.3.1 Stability of Solutions of a First-Order Differential Equation

Any dynamical system (physical, chemical, mechanical, etc.) is associated in our minds with an evolution in time. Anticipating objections, we note that an equilibrium state, i.e., a stationary state, in which the rate of the process under study is equal to zero, can also be treated as a limiting case of the temporal evolution of the system. Consider a simple model of a DS described by the single first-order ordinary differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \dot{x} = F(x) \ , \tag{2.1}$$

where $x(t)$ is the state variable and F is a function characterizing the evolution law. The state space of such a system is a set of real numbers $\mathbb{R}^1$. If the initial condition $x(t_0)$ is given, there is a unique solution of (2.1) that defines the state $x(t)$ at any time t.

Because the problem of Lyapunov stability and asymptotic stability involves analysis of the way the system reacts to a small perturbation, it can be studied in a linear approximation. Let us explain this. Suppose $x^0(t)$ is a particular solution of (2.1) whose stability we would like to investigate. We introduce a variable $y(t)$ which specifies a small deviation from $x^0(t)$, i.e.,

$$y(t) = x(t) - x^0(t) \ . \tag{2.2}$$

Here $x(t)$ is a perturbed solution.

Our task is to study the time evolution of the small perturbation $y(t)$ which obeys (2.1). We expand the function F in a series in the neighborhood of $x^0(t)$:

$$F(x_0 + y) = \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=x^0(t)} y(t) + \frac{1}{2}\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right|_{x=x^0(t)} y^2(t) + \dots \,. \tag{2.3}$$

The derivatives of F must be calculated at points corresponding to the particular solution. We now rewrite (2.1) for the perturbation $y(t)$ using (2.3), whence

$$\dot{y}(t) = \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=x^0(t)} y(t) + \Phi(y) \,, \tag{2.4}$$

where

$$\Phi(y) = \frac{1}{2}\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right|_{x=x^0(t)} y^2(t) + \dots \,. \tag{2.5}$$

The terms in $\Phi(y)$ include all terms going as y^n ($n \geq 2$), i.e., they account for all the nonlinear components. By definition, the variable $y(t)$ is a small deviation from the particular solution. Therefore, the nonlinear terms in (2.4) can be neglected in a first approximation. The evolution of the small perturbation can thus be described by the linear equation

$$\dot{y} = A(t)y \,, \quad \text{where } A(t) = \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=x^0(t)} . \tag{2.6}$$

Consider now the following example. Let a dynamical system be described by

$$\dot{x} = a - bx^2 \,, \quad a > 0 \,,\ b > 0 \,. \tag{2.7}$$

We find stationary states x^0 of the system and analyze their stability. Since there are no temporal changes in a stationary state, $\mathrm{d}x/\mathrm{d}t|_{x^0} = 0$ and we obtain

$$x^0_{1,2} = \pm\sqrt{\frac{a}{b}} \,. \tag{2.8}$$

We now apply Eq. (2.6) for the perturbation to the first stationary state x^0_1, yielding

$$\dot{y} = -(2bx^0_1)y = (-2\sqrt{ab})y = sy \,, \qquad s = \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x^0_1} = -2\sqrt{ab} \,. \tag{2.9}$$

Equation (2.9) has solution $y = \exp(st)$. The perturbation y decays exponentially in time because s is negative. This means that the state x^0_1 is stable. Since the second state x_2^0 differs from the first only by its sign, the solution of (2.9) increases in time. Hence, the stationary state x_2^0 is unstable.

A sufficiently simple idea for predicting stability in the linear approximation has proved to be very fruitful. Equation (2.6) for the perturbation can also be generalized to N state variables.

2.3.2 Stability of a Dynamical System in $\mathbb{R}^N$

Consider a DS given by a vector differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \ , \tag{2.10}$$

where $\mathbf{x} \in \mathbb{R}^N$. We analyze the stability of a particular solution $\mathbf{x}^0(t)$. Whereas the one-dimensional equation (2.1) describes the evolution exclusively in the neighborhood of equilibria, solutions of (2.10) can include equilibrium points and periodic, quasiperiodic, and chaotic orbits.

We introduce a perturbation vector $\mathbf{y} = \mathbf{x}(t) - \mathbf{x}^0(t)$, assuming that its length $\|\mathbf{y}\|$ is small. For $\mathbf{y}$ we may write

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{x}^0 + \mathbf{y}) - \mathbf{F}(\mathbf{x}^0) \ . \tag{2.11}$$

Expanding $\mathbf{F}(\mathbf{x}^0 + \mathbf{y})$ in a series in the vicinity of $\mathbf{x}^0$ and taking into account the fact that the perturbation is small in norm, we arrive at the following equations, linearized with respect to $\mathbf{y}$:

$$\dot{\mathbf{y}} = \hat{A}(t)\mathbf{y} \ , \tag{2.12}$$

where $\hat{A}(t)$ is a matrix with elements

$$a_{jk}(t) = \left.\frac{\partial f_j}{\partial x_k}\right|_{\mathbf{x}(t)=\mathbf{x}^0(t)} , \qquad j,k = 1,2,\ldots,N \ , \tag{2.13}$$

called the *linearization matrix* of the system in the vicinity of the solution $\mathbf{x}^0(t)$, and f_j are the components of the vector function $\mathbf{F}$. As the elements of the matrix $\hat{A}$ depend on a point on the studied trajectory, they generally vary in time. The matrix is characterized by eigenvalues $s_i(t)$ which are also time dependent. The eigenvalues are roots of the characteristic equation

$$\det\left[\hat{A}(t) - s\hat{E}\right] = 0 \ , \tag{2.14}$$

where $\hat{E}$ is the unit matrix. The N eigenvalues (counting multiple eigenvalues) are associated with N linearly independent eigenvectors $\mathbf{e}_i(t)$ which change direction as one moves along the trajectory $\mathbf{x}^0(t)$. These eigenvectors satisfy

$$\hat{A}(t)\mathbf{e}_i(t) = s_i(t)\mathbf{e}_i(t) , \quad i = 1, 2, \ldots, N . \tag{2.15}$$

The increase or decrease of the perturbation $\mathbf{y}(t)$ is determined by the sign of the real part of $s_i(t)$. As one moves along the trajectory $\mathbf{x}^0(t)$, it may be that the perturbation grows at some points of the given trajectory and decreases at others. The problem is to specify those characteristics of the perturbation behavior that would define it as a whole along the given trajectory, and the relevant tool for this is the *Lyapunov characteristic exponent*.

According to Lyapunov's theorem, if the matrix $\hat{A}(t)$ is bounded, then for each nontrivial solution $\mathbf{y}(t)$ of the system (2.12), there is a finite *Lyapunov characteristic exponent*, i.e., the real number defined by

$$\lambda[\mathbf{y}(t)] = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|\mathbf{y}(t)\| , \tag{2.16}$$

where the bar indicates the upper limit and $\| \ \|$ denotes the vector norm. Linearly independent solutions are characterized, in general, by different Lyapunov exponents. For N linearly independent solutions $\mathbf{y}^i(t)$, $i = 1, 2, \ldots, N$ making up a fundamental matrix of solutions $\hat{Y}(t)$ of the system (2.12), there are N Lyapunov characteristic exponents:

$$\lambda_i = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \left\|\mathbf{y}^i(t)\right\| , \quad i = 1, \ldots, N . \tag{2.17}$$

Arranged in decreasing order, the real numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ form the *Lyapunov characteristic exponent spectrum (LCE spectrum)*. λ_1 is called the maximal Lyapunov exponent. For certain sufficiently general conditions, the LCE spectrum does not depend on the choice of the fundamental matrix of solutions and completely defines the local stability properties of trajectory $\mathbf{x}^0(t)$. Each exponent in the LCE spectrum determines the rate of exponential contraction or stretching of a perturbation component in the direction of a relevant eigenvector of the fundamental matrix $\hat{Y}(t)$, on average, along the trajectory.

If $\hat{A}(t)$ is a bounded real matrix, the Lyapunov inequality is satisfied:

$$\sum_{i=1}^{N} \lambda_i \geq \overline{\lim_{t\to\infty}} \frac{1}{t - t_0} \int_{t_0}^{t} \mathrm{Tr}\, \hat{A}(t')\mathrm{d}t' , \tag{2.18}$$

where $\mathrm{Tr}\, \hat{A}(t)$ is the trace of the matrix $\hat{A}(t)$. Equality holds in (2.18) for *systems that are said to be tame according to Lyapunov*. According to the Ostrogradsky–Liouville formula, the trace of the linearization matrix determines the evolution of a small volume element of the phase space along the trajectory $\mathbf{x}^0(t)$:

$$V(t) = V(t_0) \exp\left[\int_{t_0}^{t} \mathrm{Tr}\, \hat{A}(t')\mathrm{d}t'\right] . \tag{2.19}$$

It is obvious that $\mathrm{Tr}\,\hat{A}(t) = \mathrm{div}\,\mathbf{F}(\mathbf{x}(t))$, where $\mathbf{F}(\mathbf{x}(t))$ is a phase velocity field. Accordingly, the mean divergence along the trajectory $\mathbf{x}^0(t)$ satisfies the inequality

$$\left\langle \mathrm{div}\mathbf{F}(\mathbf{x}(t)) \right\rangle \le \sum_{i=1}^{N} \lambda_i \ . \tag{2.20}$$

If the sum of Lyapunov exponents is negative, the divergence of $\mathbf{F}$ is on average negative and the phase volume vanishes with time. This indicates the presence of dissipation in the system.

If the trajectory $\mathbf{x}^0(t)$ is stable according to Lyapunov, then an arbitrary initial perturbation $\mathbf{y}(t_0)$ does not grow, on average, along the trajectory. A necessary and sufficient condition for this is that the LCE spectrum should not contain positive exponents.

If an arbitrary bounded trajectory $\mathbf{x}^0(t)$ belongs to a limit set of the autonomous system (2.10) which is not an equilibrium or a saddle separatrix, then at least one of the Lyapunov exponents is always equal to zero. Indeed, the small perturbation remains on average unchanged along the direction tangent to the trajectory.

A phase volume element must be contracted for phase trajectories located near the attractor. In this case the dissipative dynamical system has a negative average divergence $\mathbf{F}(\mathbf{x}(t))$ and the sum of the Lyapunov exponents satisfies the inequality

$$\sum_{i=1}^{N} \lambda_i < 0 \ . \tag{2.21}$$

Stability of Equilibrium States in $\mathbb{R}^N$

If the particular solution $\mathbf{x}^0(t)$ of a system (2.10) is an equilibrium point, i.e., $\mathbf{F}(\mathbf{x}^0) = 0$, the linearization matrix $\hat{A}$ is considered at only one point of phase space, so it is a matrix with constant elements a_{ij}. The eigenvectors and eigenvalues of the matrix $\hat{A}$ are constant in time and the Lyapunov exponents coincide with the real parts of the eigenvalues, i.e., $\lambda_i = \mathrm{Re}\,s_i$. The signature of the LCE spectrum indicates whether the equilibrium is stable or not. To analyze the behavior of phase trajectories in a local neighborhood of an equilibrium, one also needs to know the imaginary parts of the linearization matrix eigenvalues. In a phase plane, $N = 2$, the equilibrium is characterized by the two eigenvalues of the matrix $\hat{A}$, namely, s_1 and s_2. The following cases can be realized in the phase plane:

1. s_1 and s_2 are real negative numbers, in which case the equilibrium is a stable node.
2. s_1 and s_2 are real positive numbers, in which case the equilibrium is an unstable node.
3. s_1 and s_2 are real numbers but with different signs, in which case the equilibrium is a saddle.

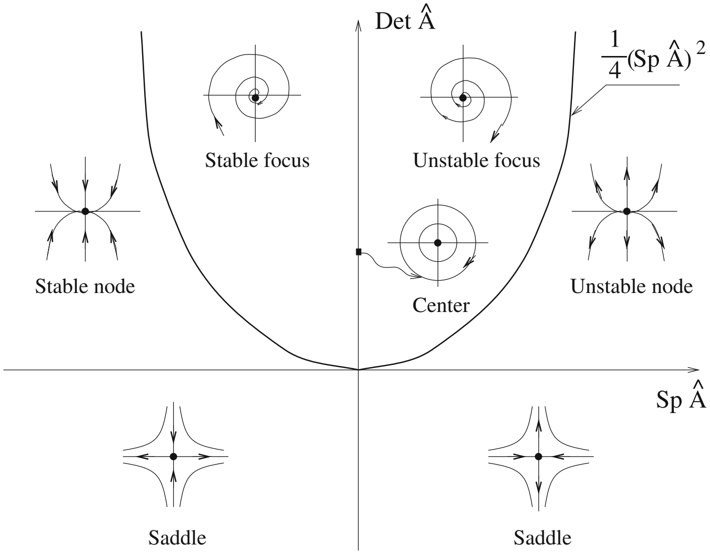

Fig. 2.2 Equilibria in the plane. Phase portraits are shown in transformed coordinates

4. s_1 and s_2 are complex conjugates with $\mathrm{Re}\, s_{1,2} < 0$ and the equilibrium is a stable focus.
5. s_1 and s_2 are complex conjugates with $\mathrm{Re}\, s_{1,2} > 0$ and the equilibrium is an unstable focus.
6. s_1 and s_2 are pure imaginary, so can be written $s_{1,2} = \pm \mathrm{i}\omega$, in which case the equilibrium is a center.

Figure 2.2 shows the equilibria realized in the plane for different values of the determinant and trace of the matrix $\hat{A}$, i.e., det $\hat{A} = s_1 s_2$ and Tr $\hat{A} = s_1 + s_2$.

Besides the aforementioned states, other kinds of equilibria are possible in a phase space with dimension $N \geq 3$, e.g., an equilibrium state called a saddle-focus which is unstable according to Lyapunov. Figure 2.3 shows two possible types of saddle-focus in a three-dimensional phase space. These are distinguished by the dimensions of their stable and unstable manifolds.

To identify which type of limit set the equilibrium corresponds to, it is enough to know the Lyapunov exponents. The equilibrium is considered to be an attractor if it is asymptotically stable in all directions and its LCE spectrum consists only of negative exponents (stable node and stable focus). If the equilibrium is unstable in all directions, it is a repeller (unstable node and unstable focus). If the LCE spectrum includes both positive and negative exponents, the equilibrium is of saddle type (simple saddle or a saddle-focus). In addition, the exponents $\lambda_i \geq 0$ ($\lambda_i \leq 0$) determine the dimension of the unstable (stable) manifold.

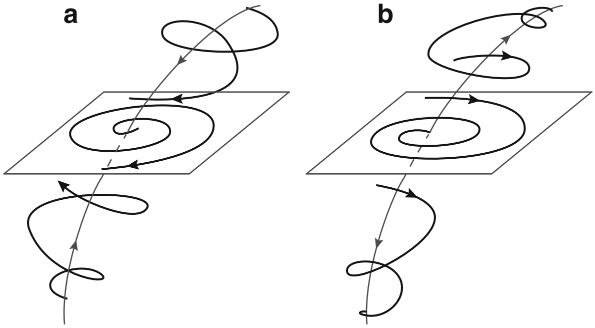

Fig. 2.3 Saddle-foci in three-dimensional phase space: (**a**) s_1 is real and negative, while $s_{2,3}$ are complex conjugate with $\mathrm{Re}\, s_{2,3} > 0$. (**b**) s_1 is real and positive, while $s_{2,3}$ are complex conjugates with $\mathrm{Re}\, s_{2,3} < 0$

Stability of Periodic Solutions

Any periodic solution $\mathbf{x}^0(t)$ of the system (2.10) satisfies the condition

$$\mathbf{x}^0(t) \equiv \mathbf{x}^0(t+T) \,, \tag{2.22}$$

where T is the period of the solution. The linearization matrix $\hat{A}(t)$ calculated at points of the trajectory corresponding to the periodic solution $\mathbf{x}^0(t)$ is also periodic:

$$\hat{A}(t) = \hat{A}(t+T) \,. \tag{2.23}$$

In this case, Eq. (2.12) for perturbations is linear with periodic coefficients. The stability of a periodic solution can be estimated once it is known how a small perturbation $\mathbf{y}(t_0)$ evolves over the period T. Its evolution can be represented by

$$\mathbf{y}(t_0+T) = \hat{M}_T \mathbf{y}(t_0) \,, \tag{2.24}$$

where $\hat{M}_T$ is the *monodromy matrix*. It is independent of time. The eigenvalues of the monodromy matrix, i.e., the roots of the characteristic equation

$$\det\left[\hat{M}_T - \mu \hat{E}\right] = 0 \,, \tag{2.25}$$

are called *multipliers* of the periodic solution $\mathbf{x}^0(t)$ and define its stability. Indeed, the monodromy operator (2.24) acts as follows. The initial perturbation of a periodic solution, considered via its projections onto the eigenvectors of the matrix $\hat{M}_T$, is multiplied by an appropriate multiplier μ_i over the period T. Thus, a necessary and sufficient requirement for the periodic solution $\mathbf{x}^0(t)$ to be stable according to Lyapunov is that its multipliers should satisfy $|\mu_i| \le 1$, $i = 1, 2, \ldots, N$. At least one of the multipliers is equal to $+1$. Since they are the eigenvalues of the monodromy matrix, the multipliers obey the relations

$$\sum_{i=1}^{N} \mu_i = \mathrm{Tr}\, \hat{M}_T \ , \qquad \prod_{i=1}^{N} \mu_i = \det \hat{M}_T \ . \tag{2.26}$$

They are related to the Lyapunov exponents of the periodic solution by

$$\lambda_i = \frac{1}{T} \ln |\mu_i| \ . \tag{2.27}$$

One of the LCE spectrum exponents of a limit cycle is always zero and corresponds to a unit multiplier. The limit cycle is an attractor if all the other exponents are negative. If the LCE spectrum includes exponents of different sign, the limit cycle is a saddle. The dimension of its unstable manifold is equal to the number of non-negative exponents in the LCE spectrum, and the dimension of its stable manifold is equal to the number of exponents for which $\lambda_i \leq 0$. If, besides the zero exponent, all the other exponents satisfy $\lambda_i > 0$, then the limit cycle is absolutely unstable (a repeller).

Stability of Quasiperiodic and Chaotic Solutions

Let a particular solution $\mathbf{x}^0(t)$ of the system (2.10) correspond to quasiperiodic oscillations with k independent frequencies ω_j, $j = 1, 2, \ldots, k$. Then

$$\begin{aligned}\mathbf{x}^0(t) &= \mathbf{x}^0\big(\varphi_1(t), \varphi_2(t), \ldots, \varphi_k(t)\big) \\ &= \mathbf{x}^0\big(\varphi_1(t) + 2\pi m, \varphi_2(t) + 2\pi m, \ldots, \varphi_k(t) + 2\pi m\big) \ ,\end{aligned} \tag{2.28}$$

where m is an arbitrary integer and $\varphi_j(t) = \omega_j t$, $j = 1, 2, \ldots, k$. The stability of the quasiperiodic solution is characterized by the LCE spectrum. The linearization matrix $\hat{A}(t)$ is quasiperiodic, and the Lyapunov exponents are strictly defined only in the limit as $t \to \infty$. In the case of *ergodic quasiperiodic oscillations*, the periodicity of the solution with respect to each of the arguments φ_j results in the LCE spectrum containing k zero exponents. If all other exponents are negative, the toroidal k-dimensional hypersurface (which we shall refer to as the k-dimensional torus for simplicity) on which the relevant quasiperiodic trajectory lies is an attractor. When all other exponents are positive, the k-dimensional torus is a repeller. The torus is said to be a saddle[1] if the LCE spectrum of quasiperiodic trajectories on the torus has, besides zero exponents, both positive and negative ones.

A chaotic trajectory which belongs to a chaotic attractor is always unstable in at least one direction. The LCE spectrum of a chaotic solution always has at least one positive Lyapunov exponent. There is no contradiction between instability of

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[1]This situation should be distinguished from the case of chaos on a k-dimensional torus, which is observed for $k \geq 3$.

phase trajectories and the attracting nature of the limit set to which they belong. Phase trajectories starting from close initial points in the basin of attraction tend to the attractor but they are separated on it. Hence, chaotic trajectories are unstable according to Lyapunov, but stable according to Poisson.

2.4 Stability of Phase Trajectories in Discrete-Time Systems

Let a discrete time system be described by the return map

$$\mathbf{x}_{n+1} = \mathbf{P}(\mathbf{x}_n) \ , \tag{2.29}$$

where $\mathbf{x} \in \mathbb{R}^N$ is the state vector, n is a discrete time variable, and $\mathbf{P}(\mathbf{x})$ is a vector function with components P_j, $j = 1, 2, \ldots, N$. Let us analyze the stability of an arbitrary solution $\mathbf{x}_n^0$. Introducing a small perturbation $\mathbf{y}_n = \mathbf{x}_n - \mathbf{x}_n^0$ and linearizing the map in the vicinity of the solution $\mathbf{x}_n^0$, we deduce the linear equation for the perturbation:

$$\mathbf{y}_{n+1} = \hat{M}(n)\mathbf{y}_n \ , \tag{2.30}$$

where $\hat{M}(n)$ is the linearization matrix with elements

$$m_{jk} = \left.\frac{\partial P_j(\mathbf{x})}{\partial x_k}\right|_{\mathbf{x}\in\mathbf{x}_n^0} . \tag{2.31}$$

It follows from (2.30) that the initial perturbation evolves according to the law

$$\mathbf{y}_{n+1} = \hat{M}(n)\hat{M}(n-1)\ldots\hat{M}(1)\mathbf{y}_1 \ . \tag{2.32}$$

By analogy with differential systems, we consider the Lyapunov exponents of the solution $\mathbf{x}_n^0$:

$$\lambda_i = \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \left\| \mathbf{y}_n^i \right\| \ , \tag{2.33}$$

where $\mathbf{y}_n^i$, $i = 1, \ldots, N$ are linearly independent solutions of the system (2.30).

The stability of fixed points and cycles of the map is characterized by multipliers. The sequence of states $\mathbf{x}_1^0, \mathbf{x}_2^0, \ldots, \mathbf{x}_l^0$ is called a *period-l cycle* of the map, or simply an *l-cycle*, if the following condition is satisfied:

$$\mathbf{x}_1^0 = \mathbf{P}^l(\mathbf{x}_1^0) \ . \tag{2.34}$$

If $l = 1$, i.e.,

$$\mathbf{x}^0 = \mathbf{P}(\mathbf{x}^0) \ , \tag{2.35}$$

the state $\mathbf{x}^0$ is a *fixed point* or *period-1 cycle*. The linearization matrix $\hat{M}$ along the periodic solution $\mathbf{x}_n^0$ is periodic, i.e., $\hat{M}(n+l) = \hat{M}(n)$. The perturbation component $\mathbf{y}_1^i$ transforms as follows over the period l:

$$\mathbf{y}^i l + 1 = \hat{M}(l)\hat{M}(l-1)\ldots\hat{M}(1)\mathbf{y}_1^i = \hat{M}_l \mathbf{y}_1^i \,. \tag{2.36}$$

The matrix $\hat{M}_l$ does not depend on the initial point and is an analogue of the monodromy matrix in a differential system. The eigenvalues μ_i^l of the matrix $\hat{M}_l$ are called *multipliers of the l-cycle* of the map. They characterize how projections of the perturbation vector onto the eigenvectors of the linearization matrix $\hat{M}_l$ change over the period l. The multipliers μ_i^l are related to the Lyapunov exponents by

$$\lambda_i = \frac{1}{l}\ln|\mu_i^l| \,. \tag{2.37}$$

The l-cycle of the map is asymptotically stable if its multipliers satisfy $|\mu_i^l| < 1$, $i = 1, 2, \ldots, N$. Thus, the LCE spectrum involves only negative numbers.

If the map has the phase space dimension $(N-1)$ and is the Poincaré map of some N-dimensional continuous-time system, then it has the following property: the eigenvalues μ_i^l, $i = 1, 2, \ldots, (N-1)$, of the matrix $\hat{M}_l$ for the l-cycle, supplemented by the unit multiplier $\mu_N^l = 1$, are strictly equal to the eigenvalues of the monodromy matrix of the corresponding limit cycle in this continuous-time system. On this basis, the stability of periodic oscillations in differential systems can be described quantitatively by the multipliers of the relevant cycle in the Poincaré map.

2.5 Summary

In this chapter we have given a brief and simplified description of the basic ideas and methods of the theory of stability. The main focus has been on linear analysis of the stability of trajectories. The theory of stability is essential for nonlinear dynamics. By studying the stability of trajectories, one can determine the character of the system's limit sets and obtain a qualitative phase portrait. In addition, when the system parameters are varied, the resulting change in the stability of trajectories belonging to a particular limit set can be used to diagnose bifurcations. The most typical bifurcations of dynamical systems will be discussed in the next chapter. We note that, even though the linear analysis of stability is very important, it is not sufficient to provide a complete picture of the system behavior or describe the possible bifurcations in the system. The stability of dynamical systems is described in more detail in [1–17].

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