Chapter 2
Distance

Metric spaces can be thought of as very basic spaces, with only a few axioms, where the ideas of *convergence* and *continuity* exist. The fundamental ingredient that is needed to make these concepts rigorous is that of a *distance*, also called a *metric*, which is a measure of how close elements are to each other.

Definition 2.1

A **distance** (or **metric**) on a **metric space** X is a function

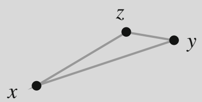
$$\begin{aligned} d: \quad X^2 &\to \mathbb{R}^+ \\ (x, y) &\mapsto d(x, y) \end{aligned}$$

such that the following properties (called *axioms*) hold for all $x, y, z \in X$,

(i) $d(x, y) \leqslant d(x, z) + d(z, y)$ (Triangle Inequality),

(ii) $d(y, x) = d(x, y)$, (Symmetry)

(iii) $d(x, y) = 0 \Leftrightarrow x = y$.



A metric space is not just a set, in which the elements have no relation to each other, but a set X equipped with a particular structure, its distance function d. One can emphasize this by denoting the metric space by the pair (X, d), although it is more convenient to denote different metric spaces by different symbols such as X, Y, etc.

In what follows, X will denote an abstract set with a distance, not necessarily $\mathbb{R}$ or $\mathbb{R}^N$, although these are of the most immediate interest. We still call its elements "points", whether they are in reality geometric points, sequences, or functions. What matters, as far as metric spaces are concerned, is not the internal structure of its points, but their outward relation to other points.

J. Muscat, *Functional Analysis*, DOI: 10.1007/978-3-319-06728-5_2,

Maurice Fréchet (1878–1973) studied under Hadamard (who had proved the prime number theorem and had succeeded Poincaré) and Borel at the University of Paris (École Normale Supérieure); his 1906 thesis developed "abstract analysis", an axiomatic approach to abstract functions that allows the Euclidean concepts of convergence and distance, as well as the usual algebraic operations, to be applied to functions. Many terms, such as metric space, completeness, compactness etc., are due to him.

Fig. 2.1 Fréchet

Although most distance functions treated in this book are of the type $d(x, y) = |x - y|$, as for $\mathbb{R}$, the point of studying metric spaces in more generality is not only that there are some exceptions that don't fit this type, but also to emphasize that addition/subtraction is not essential, as well as to prepare the groundwork for even more general spaces, called *topological spaces*, in which pure convergence is studied without reference to distances (but which are not covered in this book).

There are two additional axioms satisfied by some metric spaces that merit particular attention: *complete* metrics, which guarantee that their Cauchy sequences converge, and *separable* metric spaces whose elements can be handled by approximations. Both properties are possessed by *compact* metric spaces, which is what is often meant when the term "finite" is applied in a geometric sense. These are considered in later sections.

Easy Consequences

1. $d(x, z) \geqslant |d(x, y) - d(z, y)|$.
2. If $x_1, \ldots, x_n$ are points in X, then by induction on n,

$$d(x_1, x_n) \leqslant d(x_1, x_2) + \cdots + d(x_{n-1}, x_n).$$

Examples 2.2

1. The spaces $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ have the standard distance $d(a, b) := |a - b|$. Check that the three axioms for a distance are satisfied, making use of the in/equalities $|s + t| \leqslant |s| + |t|$, $|-s| = |s|$, and $|s| = 0 \Leftrightarrow s = 0$.
2. ► The vector spaces $\mathbb{R}^N$ and $\mathbb{C}^N$ have the standard *Euclidean* distance defined by $d(\boldsymbol{x}, \boldsymbol{y}) := \sqrt{\sum_{i=1}^N |a_i - b_i|^2}$ for $\boldsymbol{x} = (a_1, \ldots, a_N)$, $\boldsymbol{y} = (b_1, \ldots, b_N)$ (prove this for $N = 2$).
3. One can define distances on other more general spaces, e.g. we will later show that the space of real continuous functions f with domain $[0, 1]$ has a distance defined by $d(f, g) := \max_{x \in [0,1]} |f(x) - g(x)|$.
4. ◇ The space of 'shapes' in $\mathbb{R}^2$ (roughly speaking, subsets that have an area) have a metric $d(A, B)$ defined as the area of $(A \cup B) \setminus (A \cap B)$.

5. ▶ Any subset of a metric space is itself a metric space (with the 'inherited' or 'induced' distance). (The three axioms are such that they remain valid for points in a subset of a metric space.)

6. ▶ The product of two metric spaces, $X \times Y$, can be given several distances, none of which have a natural preference. Two of them are the following

$$D_1\left(\binom{x_1}{y_1}, \binom{x_2}{y_2}\right) := d_X(x_1, x_2) + d_Y(y_1, y_2),$$
$$D_\infty\left(\binom{x_1}{y_1}, \binom{x_2}{y_2}\right) := \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

   For convenience, we choose D_1 as our standard metric for $X \times Y$, except for $\mathbb{R}^N$ and $\mathbb{C}^N$, for which we take the Euclidean one.

   Proof for D_1: Positivity of D_1 and axiom (ii) are obvious. To prove axiom (iii), $D_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0$ implies $d_X(x_1, x_2) = 0 = d_Y(y_1, y_2)$, so $x_1 = x_2$, $y_1 = y_2$, and $\boldsymbol{x}_1 = \binom{x_1}{y_1} = \binom{x_2}{y_2} = \boldsymbol{x}_2$. As for the triangle inequality,

$$\begin{aligned} D_1(\boldsymbol{x}_1, \boldsymbol{x}_2) &= d_X(x_1, x_2) + d_Y(y_1, y_2) \\ &\leqslant d_X(x_1, x_3) + d_X(x_3, x_2) + d_Y(y_1, y_3) + d_Y(y_3, y_2) \\ &= D_1(\boldsymbol{x}_1, \boldsymbol{x}_3) + D_1(\boldsymbol{x}_3, \boldsymbol{x}_2). \end{aligned}$$

Exercises 2.3

1. Show that if $d(x, z) > d(z, y)$ then $x \neq y$.
2. Write in mathematical language,
   (a) The subsets A, B are close to within 2 distance units;
   (b) A and B are arbitrarily close.
3. The set of bytes, i.e., sequences of 0s and 1s (bits) of length 8 (or any length), has a "Hamming distance" defined as the number of bits where two bytes differ; e.g. the Hamming distance between 10010111 and 11001101 is 4.
4. Any non-empty set can be given a distance function. The simplest is the *discrete* metric $d(x, y) := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. Indeed, there are infinitely many other metrics on the same set (except when there is only one point!); for example, if d is a distance function then so are $2d$ and $d/(1 + d)$.
   (* Not every function of d will do though! The function d^2 is not generally a metric; what properties does $f\colon \operatorname{im} d \to \mathbb{R}^+$ need to have in order that $f \circ d$ also be a metric?)
5. A set may have several distances defined on it, but each has to be considered as a different metric space. For example, the set of positive natural numbers has a distance defined by $d(m, n) := |1/m - 1/n|$ (prove!); the metric space associated with it has very different properties from $\mathbb{N}$ with the standard Euclidean distance.

For example, in this space one can find distinct natural numbers that are arbitrarily close to each other.

6. Let $n = \pm 2^k 3^r \cdots$ be the prime decomposition of any $n \in \mathbb{Z}$ and define $|n|_2 := 1/2^k$, $|0|_2 := 0$. Show that $|\cdot|_2$ satisfies the same properties as the standard absolute value and hence that $d(m,n) := |m-n|_2$ is a distance on $\mathbb{Z}$ (called the 2-adic metric).
7. * Given the distances between n points in $\mathbb{R}^N$, can their positions be recovered? Can their relative positions be recovered?

2.1 Balls and Open Sets

The distance function provides an idea of the "surroundings" of a point. Given a point a and a number $r > 0$, we can distinguish between those points 'near' to it, satisfying $d(x,a) < r$, and those that are not.

Definition 2.4

An (open) **ball**, with *center a* and *radius r* > 0, is the set

$$B_r(a) := \{ x \in X : d(x,a) < r \}.$$

Despite the name, we should lay aside any preconception we may have of it being "round" or symmetric. We are now ready for our first, simple, proposition:

Proposition 2.5

Distinct points of a metric space can be separated by disjoint balls,

$$x \neq y \quad \Rightarrow \quad \exists r > 0 \quad B_r(x) \cap B_r(y) = \varnothing.$$

Proof If $x \neq y$ then $d(x,y) > 0$ by axiom (iii). Letting $r := d(x,y)/2$, then $B_r(x)$ is disjoint from $B_r(y)$ else we get a contradiction,

$$\begin{aligned} z \in B_r(x) \cap B_r(y) &\Rightarrow d(x,z) < r \text{ AND } d(y,z) < r \\ &\Rightarrow d(x,y) \leqslant d(x,z) + d(y,z) \\ &\qquad < 2r = d(x,y). \end{aligned}$$

□

Examples 2.6

1. In $\mathbb{R}$, every ball is an *open* interval

$$B_r(a) = \{ x \in \mathbb{R} : |x - a| < r \} = \,]a - r, a + r[.$$

Conversely, any open interval of the type $]a, b[$ is a ball in $\mathbb{R}$, namely $B_{|b-a|/2}(\frac{a+b}{2})$.

2. In $\mathbb{R}^2$, the ball $B_r(\boldsymbol{a})$ is the disk with center $\boldsymbol{a}$ and radius r without the circular perimeter.
3. In $\mathbb{Z}$, $B_{1/2}(m) = \{ n \in \mathbb{Z} : |n-m| < \frac{1}{2} \} = \{ m \}$ and $B_2(m) = \{ m-1, m, m+1 \}$.
4. It is clear that balls differ depending on the context of the metric space; thus $B_{1/2}(0) = \,]-\frac{1}{2}, \frac{1}{2}[$ in $\mathbb{R}$, but $B_{1/2}(0) = \{ 0 \}$ in $\mathbb{Z}$.

Open Sets

We can use balls to explore the relation between a point x and a given set A. As the radius of the ball $B_r(x)$ is increased, one is certain to include some points which are in A and some points which are not, unless $A = X$ or $A = \varnothing$. So it is more interesting to investigate what can happen when the radius is small. There are three possibilities as r is decreased: either $B_r(x)$ contains (i) *only* points of A, or (ii) *only* points in its complement A^{c}, or (iii) points of *both* A and A^{c}, no matter how small we take r.

Definition 2.7

A point x of a set A is called an **interior** point of A when it can be "surrounded completely" by points of A, i.e.,

$$\exists r > 0, \quad B_r(x) \subseteq A.$$

In this case, A is also said to be a *neighborhood* of x.

A point x (not in A) is an **exterior** point of A when

$$\exists r > 0, \quad B_r(x) \subseteq X \smallsetminus A.$$

All other points are called **boundary** points of A.

Accordingly, the set X is partitioned into three parts: its *interior* A°, its *exterior* $(\bar{A})^{\mathsf{c}}$, and its *boundary* ∂A. The set of interior and boundary points of A is called the *closure* of A and denoted by $\bar{A} := A^\circ \cup \partial A$.

A set A is **open** in X when all its points are interior points of it, i.e., $A = A^\circ$ (Fig. 2.2).

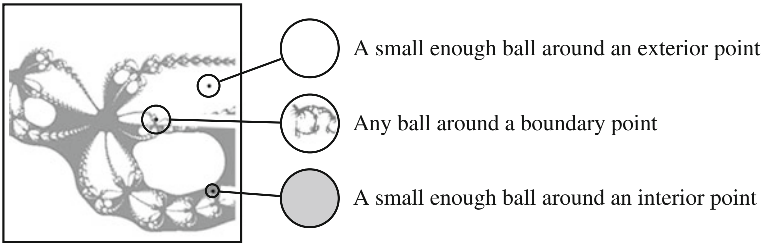


Fig. 2.2 The distinction between interior, boundary, and exterior points

Examples 2.8

1. In $\mathbb{R}$, the intervals $]a, b[$, $[a, b[$, $]a, b]$, and $[a, b]$ have the same interior $]a, b[$, exterior, and boundary $\{a, b\}$; their closure is $\overline{]a, b[} = [a, b]$.

   Proof For any $a < x < b$, let $0 < \epsilon < \min(x - a, b - x)$, then $a < x - \epsilon < x + \epsilon < b$, that is $B_\epsilon(x) \subset]a, b[$; this makes x an interior point of the interval.

   For $x < a$, there is an $\epsilon < a - x$ such that $x \in B_\epsilon(x) \subset]-\infty, a[\subset \mathbb{R} \smallsetminus [a, b]$. Similarly, any $x > b$ is an exterior point of the interval.

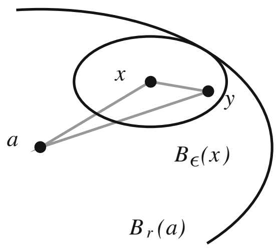
   For $x = a$, any small interval $B_\epsilon(a)$ contains points such as $a + \epsilon/2$, that are inside $B_\epsilon(a)$, and points outside it, such as $a - \epsilon/2$, making a (and similarly b) a boundary point.

2. ► The following sets are open in *any* metric space X:

   (a) $X \smallsetminus \{x\}$ for any point x. The reason is that any other point $y \neq x$ is separated from x by disjoint balls (our first proposition); this makes y an interior point of $X \smallsetminus \{x\}$.

   (b) The empty set is open by default, because it does not contain any point. The whole space X is also open because $B_r(x) \subseteq X$ for any $r > 0$ and $x \in X$.

   (c) Balls are open sets in any metric space.

   Proof Let $x \in B_r(a)$ be any point in the given ball, meaning $d(x, a) < r$. Let $\epsilon := r - d(x, a) > 0$; then $B_\epsilon(x) \subseteq B_r(a)$ since for any $y \in B_\epsilon(x)$,

   $$d(y, a) \leqslant d(y, x) + d(x, a) < \epsilon + d(x, a) = r.$$

3. ► The least upper bound of a set A in $\mathbb{R}$ is a boundary point of it.

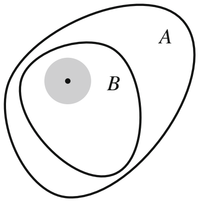
   Proof Let α be the least upper bound of A. For any $\epsilon > 0$, $\alpha + \epsilon/2$ is an upper bound of A but does not belong to it (else α would not be an upper bound).

Even if $\alpha \notin A$, then the interval $]\alpha - \epsilon/2, \alpha[$ cannot be devoid of elements of A, otherwise α would not be the *least* upper bound. So the neighborhood $B_\epsilon(\alpha)$ contains elements of both A and A^{c}.

Proposition 2.9

The set of interior points A° is the largest open set inside A.

Proof If $B \subseteq A$ then the interior points of B are obviously interior points of A, so $B^\circ \subseteq A^\circ$. In particular every open subset of A lies inside A° (because $B = B^\circ$), and every (open) ball in A lies in A°. This implies that if $B_r(x) \subseteq A$ then $B_r(x) \subseteq A^\circ$, so that every interior point of A is surrounded by other interior points, and A° is open.



□

Proposition 2.10

A set A is open $\Leftrightarrow$ A is the union of balls.

Proof Let A be an open set. Then every point of it is interior, and can be covered by a ball $B_{r(x)}(x) \subseteq A$. Taking the union of all the points of A gives

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} B_{r(x)}(x) \subseteq A,$$

forcing $A = \bigcup_{x \in A} B_{r(x)}(x)$, a union of balls.

Now let $A := \bigcup_i B_{r_i}(a_i)$ be a union of balls, and let x be any point in A. Then x is in at least one of these balls, say, $B_r(a)$. But balls are open and hence $x \in B_\epsilon(x) \subseteq B_r(a) \subseteq A$. Therefore A consists of interior points and so is open. □

The early years of research in metric spaces have shown that most of the basic theorems about metric spaces can be deduced from the following characteristic properties of open sets:

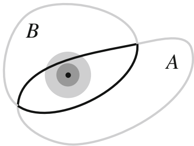
Theorem 2.11

Any union of open sets is open.
The finite intersection of open sets is open.

Proof (i) Consider the union of open sets, $\bigcup_i A_i$. Any $x \in \bigcup_i A_i$ must lie in at least one of the open sets, say A_j. Therefore,

$$x \in B_r(x) \subseteq A_j \subseteq \bigcup_i A_i$$

shows that it must be an interior point of the union.



(ii) It is enough, using induction (show!), to consider the intersection of two open sets $A \cap B$. Let $x \in A \cap B$, meaning $x \in A$ and $x \in B$, with both sets being open. Therefore there are open balls $B_{r_1}(x) \subseteq A$ and $B_{r_2}(x) \subseteq B$. The smaller of these two balls, with radius $r := \min(r_1, r_2)$, must lie in $A \cap B$,

$$x \in B_r(x) = B_{r_1}(x) \cap B_{r_2}(x) \subseteq A \cap B.$$

□

Examples 2.12

1. ▶ The exterior $(\bar{A})^{\mathsf{c}} = (A^{\mathsf{c}})^{\circ}$ of a subset A is open in X.
2. $A^{\circ} = A \smallsetminus \partial A$. So a set is open $\Leftrightarrow$ it does not contain any boundary points.
3. Let $Y \subseteq X$ inherit X's distance. Then A is open in Y if, and only if, $A = U \cap Y$ for some subset U open in X.

   Proof Care must be taken to distinguish balls in Y from those in X: $B_r^Y(x) = B_r^X(x) \cap Y$. If A is open in Y, then by Proposition 2.10,

   $$A = \bigcup_{a \in A} B_{r(a)}^Y(a) = \bigcup_{a \in A} B_{r(a)}^X(a) \cap Y = U \cap Y.$$

   For the converse, interior points of $U \subseteq X$ which happen to be in Y are interior points of A as a subset of Y,

   $$y \in B_r^X(y) \subseteq U \;\Rightarrow\; y \in B_r^X(y) \cap Y \subseteq U \cap Y = A.$$

Limit Points

It may happen that a point a of a set A is surrounded by points not in A, that is, there is a ball $B_r(a)$ which contains no points of A other than a itself. We call such points *isolated* points. The property that a point cannot be isolated from the rest of A is captured by the following definition:

Definition 2.13

A point b (not necessarily in A) is a **limit point** of a set A when every ball around it contains other points of A,

$$\forall \epsilon > 0,\ \exists a \neq b, \quad a \in A \cap B_\epsilon(b).$$

Thus every point of $\bar{A}$ is either a limit point or an isolated point of A.

Exercises 2.14

1. In $\mathbb{R}$, the set $\{a\}$ has no interior points, a single boundary point a, and all other points are exterior. It is not an open set in $\mathbb{R}$. There are ever smaller open sets that contain a, but there is no smallest one.
2. In $\mathbb{R}$, $\overline{\{1/n : n \in \mathbb{N}\}} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$.
3. The set $\mathbb{Q}$, and also its complement, the set of irrational numbers $\mathbb{Q}^{\mathsf{c}}$, do not have interior (or exterior) points in $\mathbb{R}$. Every real number is a boundary point of $\mathbb{Q}$.
   Similarly every complex number is a boundary point of $\mathbb{Q} + i\mathbb{Q}$.
4. The set $\{m\}$ in $\mathbb{Z}$ does not have any boundary points; it is an open set in $\mathbb{Z}$ ($B_{1/2}(m) = \{m\}$).

   ► Notice that whether a point is in the interior (or boundary, or exterior) of a set depends on the metric space under consideration. For example, $\{m\}$ is open in $\mathbb{N}$ but not open in $\mathbb{R}$; the interval $]a, b[$ is open in $\mathbb{R}$, but not open when considered as a subset of the x-axis in $\mathbb{R}^2$. We thus need to specify that a set A is open *in* X.
5. Describe the interior, boundary and exterior of the sets

$$\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leqslant 1\}, \qquad \{(x, y) \in \mathbb{R}^2 : \tfrac{1}{2} < \max(|x|, |y|) \leqslant 1\}.$$

6. Of the proper intervals in $\mathbb{R}$, only $]a, b[$, $]a, \infty[$, and $]-\infty, a[$ are open.
7. In $\mathbb{R}^2$, the half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ and the rectangles $]a, b[\times]c, d[:= \{(x, y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$ are open sets.
8. ► A^{c} has the same boundary as A; its interior is the exterior of A, that is, $(\bar{A})^{\mathsf{c}} = (A^{\mathsf{c}})^{\circ}$ (and $\bar{A} = A^{\mathsf{c}\circ\mathsf{c}}$); so $\partial A = \bar{A} \cap \overline{A^{\mathsf{c}}}$.
9. Find an open subset of $\mathbb{R}$, apart from $\mathbb{R}$ itself, without an exterior.
   So, the exterior of the exterior of A need not be the interior of A. Similarly, the boundary of $\bar{A}$ or A° need not equal the boundary of A.
10. ► *An infinite intersection of open sets need not be open.* For example, in $\mathbb{R}$, the open intervals $]-1/n, 1/n[$ are nested one inside another. Their intersection is the non-open set $\{0\}$ (prove!). Find another example in $\mathbb{R}^2$.

11. Deduce from the theorem that if every $\{x\}$ is open in X, then *every* subset of X is open in X. This 'extreme' property is satisfied by $\mathbb{N}$, and also by any discrete metric space.

12. Any point x with $d(x, a) > r$ is in the exterior of the open ball $B_r(a)$. But the boundary of $B_r(a)$ need not be the set $\{x : d(x, a) = r\}$. Find a counterexample in $\mathbb{Z}$.

13. * Every open set in $\mathbb{R}$ is a countable disjoint union of open intervals. (Hint: An open set in $\mathbb{R}$ is the disjoint union of open intervals; take a rational interior point for each.)
    In contrast to this simple case, the open sets in $\mathbb{R}^2$, say, can be much more complicated—there is no simple characterization of them, apart from the definition.

14. Can a set *not* have limit points? Can an infinite set not have limit points?

15. In $\mathbb{R}$, the set of integers $\mathbb{Z}$ has no limit points, but all real numbers are limit points of $\mathbb{Q}$.

16. (a) 1 is an interior isolated point of $\{1, 2\}$ in $\mathbb{Z}$;
    (b) 1 is a boundary isolated point of $\{1, 2\}$ in $\mathbb{R}$;
    (c) 1 is an interior limit point of $[0, 2]$ in $\mathbb{R}$;
    (d) 1 is a boundary limit point of $[0, 1]$ in $\mathbb{R}$.

17. In $\mathbb{R}$ and $\mathbb{Q}$, an isolated point of a subset must be a boundary point, or, equivalently, an interior point is a limit point.

2.2 Closed Sets

Definition 2.15

A set F is **closed** in a space X when $X \setminus F$ is open in X.

Proposition 2.16

A set F is closed $\Leftrightarrow$ F contains its boundary $\Leftrightarrow$ $\bar{F} = F$.

Proof We have already seen that the boundary of a set F and of its complement F^{c} are the same (because the interior of F^{c} is the exterior of F). So F is closed, and F^{c} open, precisely when this common boundary does not belong to F^{c}, but belongs instead to $F^{\mathsf{cc}} = F$. □

Examples 2.17

1. In $\mathbb{R}$, the set $[a, b]$ is closed, since $\mathbb{R} \setminus [a, b] =]-\infty, a[\cup]b, \infty[$ is the union of two open sets, hence itself open. Similarly $[a, \infty[$ and $]-\infty, a]$ are closed in $\mathbb{R}$.
2. $\mathbb{N}$ and $\mathbb{Z}$ are closed in $\mathbb{R}$, but $\mathbb{Q}$ is not.
3. ▶ In any metric space X, the following sets are closed in X (by inspecting their complements):
   (a) the singleton sets $\{ x \}$,
   (b) the 'closed balls' $B_r[a] := \{ x \in X : d(x, a) \leqslant r \}$,
   (c) X and $\varnothing$,
   (d) the boundary of any set (the complement of ∂A is $A^\circ \cup (A^{\mathsf{c}})^\circ$).
4. ▶ The complement of an open set is closed. More generally, if U is an open set and F a closed set in X, then $U \setminus F$ is open and $F \setminus U$ is closed. The reasons are that $U \setminus F = U \cap F^{\mathsf{c}}$ and $(F \setminus U)^{\mathsf{c}} = F^{\mathsf{c}} \cup U$.

Closed sets are complements of open ones, and their properties reflect this:

Proposition 2.18

The finite union of closed sets is closed.
Any intersection of closed sets is closed.

Proof These are the complementary results for open sets (Theorem 2.11). For F, G closed sets in X, F^{c}, G^{c} are open, so the result follows from

$$(F \cup G)^{\mathsf{c}} = F^{\mathsf{c}} \cap G^{\mathsf{c}}, \quad \Big(\bigcap_i F_i\Big)^{\mathsf{c}} = \bigcup_i F_i^{\mathsf{c}},$$

and the definition that the complement of a closed set is open. □

Theorem 2.19 Kuratowski's closure 'operator'

$\bar{A}$ **is the smallest closed set containing** A**, called the closure of** A**.**

$$A \subseteq B \ \Rightarrow\ \bar{A} \subseteq \bar{B}; \quad \bar{\bar{A}} = \bar{A}; \quad \overline{A \cup B} = \bar{A} \cup \bar{B}.$$

Proof The complement of $\bar{A}$ is the exterior of A, which is an open set, so $\bar{A}$ is closed. This implies $\bar{\bar{A}} = \bar{A}$ Proposition 2.16.

If $A \subseteq B$, then an exterior point of B is obviously an exterior point of A, that is $(\bar{B})^c \subseteq (\bar{A})^c$; so $\bar{A} \subseteq \bar{B}$. It follows that if F is any closed set that contains A, then $\bar{A} \subseteq \bar{F} = F$, and this shows that $\bar{A}$ is the smallest closed set containing A. (Alternatively, Proposition 2.9 can be used: how?)

Of course, $\bar{A} \subseteq \overline{A \cup B}$ follows from $A \subseteq A \cup B$; combined with $\bar{B} \subseteq \overline{A \cup B}$, it gives $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$. Moreover, $\bar{A} \cup \bar{B}$ is a closed set which contains $A \cup B$, and so must contain its closure $\overline{A \cup B}$. □

Exercises 2.20

1. It is easy to find sets in $\mathbb{R}$ which are neither open nor closed (so contain only part of their boundary). Can you find any that are both open and closed?
   The terms "open" and "closed" are misnomers, but they have stuck in the literature, being derived from the earlier use of "open/closed intervals".
2. The set $\{ x \in \mathbb{Q} : x^2 < 2 \}$ is closed, and open, in $\mathbb{Q}$.
3. In any metric space, a finite collection of points $\{ a_1, \ldots, a_N \}$ is a closed set.
4. The following sets are closed in $\mathbb{R}$: $[0, 1] \cup \{ 5 \}$, $\bigcup_{n=0}^{\infty}[n, n + \frac{1}{2}]$.
5. The infinite union of closed sets may, but need not, be closed. For example, the set $\bigcup_{n=1}^{\infty}\{ 1/n \}$ is not closed in $\mathbb{R}$; which boundary point is not contained in it?
6. Find two disjoint closed sets (in $\mathbb{R}^2$ or $\mathbb{Q}$, say) that are arbitrarily close to each other.
7. Start with the closed interval $[0, 1]$; remove the open middle interval $]\frac{1}{3}, \frac{2}{3}[$ to get two closed intervals $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Remove the middle interval of each of these intervals to obtain four closed intervals $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. If we continue this process indefinitely we end up with the *Cantor set*. Show it is a closed set.

8. Denote the decimal expansion of any number in [0,1] by $0.n_1n_2n_3\ldots$. Show that

$$\{ x \in [0, 1] : x = 0.n_1n_2n_3\ldots \Rightarrow \frac{n_1 + \cdots + n_k}{k} \leqslant 5 \quad \forall k \}$$

   is closed in $\mathbb{R}$.
9. ► One can define the "distance" between a *point* and a *subset* of a metric space by $d(x, A) := \inf_{a \in A} d(x, a)$. Then $x \in \bar{A}$ exactly when $d(x, A) = 0$.
10. Let x be an exterior point of A, and let $y \in \bar{A}$ have the least distance between x and $\bar{A}$. Do you think that y is unique? or that it must be on the boundary of A? Prove or disprove. For starters, take the metric space to be $\mathbb{R}^2$.

11. Show equality need not hold in $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. Indeed two disjoint sets may 'touch' at a common boundary point.
12. Show the complementary results of the theorem: $A^\circ \cap B^\circ = (A \cap B)^\circ$, $A^{\circ\circ} = A^\circ$.
13. If $A \subseteq \bar{B}$, does it follow that $A^\circ \subseteq B$?

Dense Subsets

We often need to approximate an element $x \in X$ to within some small distance ϵ by an element from some special subset $A \subseteq X$. The elements of A may be simpler to describe, or more practical to work with, or may have nicer theoretical qualities. For example, computers cannot handle arbitrary real numbers and must approximate them by rational ones; polynomials are easier to work with than general continuous functions. The property that elements of a set A can be used to approximate elements of X to within any ϵ, namely,

$$\forall x \in X,\ \forall \epsilon > 0,\ \exists a \in A,\ d(x, a) < \epsilon,$$

is equivalent to saying that any ball $B_\epsilon(x)$ contains elements of A, in other words A has no exterior points.

Definition 2.21

A set A is **dense** in X when $\bar{A} = X$ (so $\bar{A}$ contains all balls).
A set A is **nowhere dense** in X when $\bar{A}$ contains no balls.

Exercises 2.22

1. ► $\mathbb{Q}$ is dense in $\mathbb{R}$. (This is equivalent to the Archimedean property of $\mathbb{R}$.) More generally, a set A is dense in $\mathbb{R}$ when for any two distinct real numbers $x < y$, there is an element $a \in A$ between them $x < a < y$.
2. The intersection of two open dense sets is again open and dense.
3. A finite union of straight lines in $\mathbb{R}^2$ is nowhere dense. $\mathbb{Z}$ and the Cantor set are nowhere dense in $\mathbb{R}$.
4. Nowhere dense sets have no interior points.
5. A is nowhere dense in X $\Leftrightarrow$ $X \setminus \bar{A}$ is dense in X $\Leftrightarrow$ $\bar{A}$ is the boundary of an open set.
6. * What are the nowhere dense sets in $\mathbb{R}$? (Hint: Exercise 2.14(13))

Remarks 2.23

1. If $d(x, y) = 0$ does not guarantee $x = y$, but d satisfies the other two axioms, then it is called a *pseudo-distance*. In this case, let us say that points x and y are *indistinguishable* when $d(x, y) = 0$ ($\Leftrightarrow \forall z,\ d(x, z) = d(y, z)$). This is an equivalence relation, which induces a partition of the space into equivalence classes $[x]$. The function $D([x], [y]) := d(x, y)$ is then a legitimate well-defined metric.

   In a similar vein, if d satisfies the triangle inequality, but is not symmetric, then $D(x, y) := d(x, y) + d(y, x)$ is symmetric and still satisfies the triangle inequality.

   Positivity of d follows from axioms (i) and (ii), $d(x, y) \geqslant |d(x, z) - d(y, z)| \geqslant 0$.

2. The axioms for a distance can be re-phrased as axioms for balls:

   (a) $B_0(x) = \varnothing$, $\bigcap_{r>0} B_r(x) = \{ x \}$, $\bigcup_{r>0} B_r(x) = X$,
   (b) $\{ y : x \in B_r(y) \} = B_r(x)$,
   (c) $B_s \circ B_r(x) \subseteq B_{r+s}(x)$, i.e., if $y \in B_s(z)$ where $z \in B_r(x)$ then $y \in B_{r+s}(x)$.

3. The concept of open sets is more basic than that of distance. One can give a set X a collection of open sets satisfying the properties listed in Theorem 2.11 (taken as axioms), and study them without any reference to distances. It is then called a *topological space*; most theorems about metric spaces have generalizations that hold for topological spaces. There are some important topological spaces that are not metric spaces, e.g. the arbitrary product of metric spaces $\prod_i X_i$, and spaces of functions $X^Y := \{ f : Y \to X \}$.

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