Black Hole Formation from a Complete Past for the Einstein–Vlasov System

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Abstract A natural question in general relativity is to find initial data for the Einstein equations whose past evolution is regular and whose future evolution contains a black hole. In [1] initial data of this kind is constructed for the spherically symmetric Einstein–Vlasov system. One consequence of the result is that there exists a class of initial data for which the ratio of the Hawking mass $\mathring{m} = \mathring{m}(r)$ and the area radius r is arbitrarily small everywhere, such that a black hole forms in the evolution. This result is analogous to the result [2] for a scalar field. Another consequence is that there exist black hole initial data such that the solutions exist for all Schwarzschild time $t \in (-\infty, \infty)$. In the present article we review the results in [1].

1 Introduction

In the study of gravitational collapse it is important to identify physically admissible initial data, and it is natural to require that the past evolution of the data is regular. However, in numerical relativity it is often the case that the given initial data, which form black holes to the future, also result in a singular past due to topological reasons. Moreover, most of the existing mathematical results which ensure a regular past also ensure a regular future which rules out the study of the formation of black holes. The exceptions being the classical result for dust [3], and the recent result [4] for a scalar field. In the latter work, which in part rests on the studies [2, 5], initial data whose past evolution is regular and whose future evolution forms a black hole is constructed. Neither dust nor a scalar field are realistic matter models in the sense that they are used by astrophysicists. Dust is a perfect fluid where the pressure is assumed to be zero, and a scalar field is merely a toy model. Thus, there is so far

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no example of a solution to the Einstein-matter system for a realistic matter model possessing a regular past and a singular future.

Here we consider Vlasov matter, or collisionless matter, governed by the Vlasov equation, cf. [6] for an introduction. Although this is a simple matter model, it has rich dynamics and many features that are desirable of a realistic matter model. For instance, there is a large number of stable and unstable spherically symmetric and axially symmetric stationary solutions, there is numerical support that time periodic solutions exist, it behaves as Type I matter in critical collapse, and it is used by astrophysicists, cf. [7]. The following theorem is the main result in [1].

Theorem 1 *There exists a class of initial data $\mathcal{J}$ for the spherically symmetric Einstein–Vlasov system with the property that black holes form in the future time direction and in the past time direction spacetime is causally geodesically complete.*

The following corollary is analogous to a result in [2] for a scalar field. Let $\mathring{m}$ be the initial Hawking mass. We then have

Corollary 1 *Given $\epsilon > 0$, there exists a class $\mathcal{J}_r$ of initial data for the spherically symmetric Einstein–Vlasov system which satisfy*

$$\sup_r \frac{\mathring{m}(r)}{r} \leq \epsilon,$$

for which black holes form in the evolution.

Another consequence of our result is that there exists a class of black hole initial data such that the corresponding solutions exist for all Schwarzschild time $t \in (-\infty, \infty)$, cf. Corollary 2 in [1].

Theorem 1 relies in part on the previous studies [8–10]. In [9] global existence in a maximal time gauge is shown for a particular class of initial data where the particles are moving rapidly outwards. One of the restrictions imposed on the initial data is that

$$\sup_r \frac{2\mathring{m}(r)}{r} < k_0, \tag{1}$$

where the constant k_0 is roughly $1/10$. The situation considered in [10] is in a sense the reverse since the initial data is such that the particles move rapidly inwards and the quantity $\sup_r 2\mathring{m}/r$ is required to be close to one. The main result of [10] is that data of this kind guarantee the formation of black holes in the evolution. Particles that move inward in the future time direction move outward in the past time direction. It is thus natural to try to combine these two results with the goal of constructing solutions with a regular past and a singular future. The conditions on the ratio $2\mathring{m}/r$ are clearly very different in [9] compared to [10], and moreover, the Cauchy hypersurfaces are different since a maximal time gauge and a polar time gauge are imposed in the respective cases. The reason a maximal time gauge is used in [9] is due to the difficulties related to the so called pointwise terms which appear in the characteristic equations in a polar time gauge. In [8] the problem of global existence for general initial data is investigated under conditional assumptions on the solutions. The analysis along characteristics is applied to a modified quantity for which the problem with the pointwise terms do not appear.

The proof of Theorem 1 is obtained by combining the strategies in [8] and [9], and a sketch of proof is given in Sect. 3. The spherically symmetric Einstein–Vlasov system is introduced in Sect. 2.

2 The Einstein–Vlasov System

For an introduction to the Einstein–Vlasov system and kinetic theory we refer to [6, 11]. In Schwarzschild coordinates the spherically symmetric metric takes the form

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2}$$

The Einstein equations read

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2\rho, \tag{3}$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p, \tag{4}$$

$$\lambda_t = -4\pi r e^{\lambda+\mu} j, \tag{5}$$

$$e^{-2\lambda}(\mu_{rr} + (\mu_r - \lambda_r)(\mu_r + \frac{1}{r})) - e^{-2\mu}(\lambda_{tt} + \lambda_t(\lambda_t - \mu_t)) = 8\pi p_T. \tag{6}$$

The indices t and r denote partial derivatives. The Vlasov equation for the density function $f = f(t, r, w, L)$ is given by

$$\partial_t f + e^{\mu-\lambda}\frac{w}{E}\partial_r f - (\lambda_t w + e^{\mu-\lambda}\mu_r E - e^{\mu-\lambda}\frac{L}{r^3 E})\partial_w f = 0, \tag{7}$$

where

$$E = E(r, w, L) = \sqrt{1 + w^2 + L/r^2}. \tag{8}$$

Here $w \in (-\infty, \infty)$ can be thought of as the radial component of the momentum variables, and $L \in [0, \infty)$ is the square of the angular momentum. The matter quantities are defined by

$$\rho(t, r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty} Ef(t, r, w, L)\, dwdL, \tag{9}$$

$$p(t,r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{E} f(t,r,w,L)\, dwdL, \tag{10}$$

$$j(t,r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t,r,w,L)\, dwdL, \tag{11}$$

$$p_T(t,r) = \frac{\pi}{2r^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{L}{E} f(t,r,w,L)\, dwdL. \tag{12}$$

Here ρ, p, j and p_T are the energy density, the radial pressure, the current and the tangential pressure respectively. The following boundary conditions are imposed to ensure asymptotic flatness

$$\lim_{r\to\infty} \lambda(t,r) = \lim_{r\to\infty} \mu(t,r) = 0, \tag{13}$$

and if a regular centre is required we set

$$\lambda(t,0) = 0. \tag{14}$$

As initial data it is sufficient to prescribe a density function $\mathring{f} = \mathring{f}(r,w,L) \geq 0$ such that

$$\int_0^r 4\pi\eta^2 \mathring{\rho}(\eta)\, d\eta < \frac{r}{2}. \tag{15}$$

Here we denote by $\mathring{\rho}$ the energy density induced by the initial distribution function $\mathring{f}$. This condition ensures that no trapped surfaces are present initially. We now introduce a couple of notations. From (4) and (13) we have

$$\mu(t,r) = -\int_r^{\infty} \frac{m(t,\eta)}{\eta^2} e^{2\lambda} - \int_r^{\infty} 4\pi\eta p e^{2\lambda}\, d\eta =: \hat{\mu} + \check{\mu}. \tag{16}$$

Moreover, the Hawking mass $m = m(t,r)$ is given by

$$m(t,r) = 4\pi \int_0^r \eta^2 \rho(t,\eta) d\eta. \tag{17}$$

Finally, we note that in [6, 12] local existence theorems are proved for compact and non-compact initial data respectively and it will be used below that solutions exist on some time interval $[0,T[$, which is assumed to be maximal.

3 Global Existence for Outgoing Matter

The aim in this section is to consider initial data of the type constructed in [10], which guarantee the formation of black holes to the future, and show that global existence holds to the past for such data. We remark that the time direction is reversed in this section so that the particles move outwards initially and the global existence to the past refers to the time interval $[0, \infty[$. Furthermore, in [1] two different classes of initial data are given adapted to the two corollaries of Theorem 1 mentioned above. Here we only consider one of these classes of data.

Let $0 < r_0' < r_0 < r_1$ be given and put $M = r_1/2$. Let $\mathring{f}_s$ be data of a steady state supported in $[r_0', r_0]$ and let

$$M_{\text{in}} := \int_{r_0'}^{r_0} 4\pi r^2 \mathring{\rho}(r) dr. \tag{18}$$

The results in [13, 14] guarantee that there are such steady states and moreover that

$$\sup_{0 \le r \le r_0} \frac{2\mathring{m}(r)}{r} < \frac{8}{9},$$

and in particular $2M_{\text{in}}/r_0 < 8/9$ so that $M > M_{\text{in}}$. Let $M_{\text{out}} := M - M_{\text{in}}$. Let $R_1 > r_1$ be such that

$$R_1 - r_1 < \frac{r_1 - r_0}{6}, \tag{19}$$

and define

$$R_0 := \frac{1}{2}(r_1 + R_1).$$

Let $L_+ > 0$ and let $W_* > 0$ be such that

$$|W_*| \ge 1 + \frac{\sqrt{L_+}}{R_0}. \tag{20}$$

Let $W_- > 0$ satisfy

$$|W_-| e^{\frac{-5M}{2R_0(1-\frac{2M}{R_0})}} (1 - \frac{2M}{R_0})^{3/2} \ge 3|W_*|. \tag{21}$$

We can now specify the initial data. Let $\mathring{f} = \mathring{f}_s + \mathring{f}_m$ be initial data of ADM mass M, and such that

$$\text{supp}\, \mathring{f}_m \subset [R_0, R_1] \times [W_-, \infty[\times [0, L_+],$$

and

$$\int_{r_0}^{\infty} 4\pi r^2 \mathring{\rho}(r)dr = \int_{R_0}^{R_1} 4\pi r^2 \mathring{\rho}_m(r)dr = M_{\text{out}}. \tag{22}$$

In view of [10] the initial data $\mathring{f}$ guarantee the formation of black holes. Hence, Theorem 1 follows from the following global existence theorem.

Theorem 2 *Assume that r_0', r_0, M_{in}, M, L_+, R_0, R_1, W_*, W_- and $\mathring{f}$ are given as above, and consider a solution f of the system (3)-(6), launched by $\mathring{f}$, on its maximal existence interval $[0, T[$. Then $T = \infty$, and there is a $\kappa_* > 0$ such that*

$$\text{supp } f_m(t) \subset [R_0 + |t\,\kappa_*|, \infty[\times[W_*, \infty[\times[0, L_+], \tag{23}$$

and the resulting spacetime is future causally geodesically complete.

Sketch of proof: We focus here on the main idea of the proof and refer to [1] for the complete argument. It is shown in [1] that f, λ and μ_r remain time independent for $r \le r_0$ and therefore the present arguments only concern the outer matter given by f_m. The steady state is needed to guarantee the formation of black holes.

Let $[0, t_1[$ be the maximal time interval such that for $t \in [0, t_1[$ and $(r, w, L) \in \text{supp } f_m(t)$, $w > W_*$. By continuity $t_1 > 0$. Suppose that $t_1 \in]0, T[$, then we must have $w = W_*$ for some $w \in \text{supp } f_m(t_1)$, but we will show that $w > W_*$ for all $w \in \text{supp } f_m(t_1)$. Thus $t_1 = T$ and since the matter stays strictly away from $r = 0$ it follows that $T = \infty$ in view of [8].

Consider a characteristic $(R(s), W(s), L)$ with $R(0) \in [R_0, R_1]$ and define

$$G(s) := E(R(s), W(s), L) + W(s).$$

Below we suppress the arguments but it should be clear that $R = R(s)$, $\mu_r = \mu_r(s, R(s))$ etc. The main idea of the proof is to consider the evolution of the quantity

$$G(t)e^{\hat{\mu}(t,R(t))}(1 - 2M/R(t))$$

along the characteristic $(R(s), W(s), L)$. The following inequality is then obtained in [1]:

$$\frac{d}{ds}\left(Ge^{\hat{\mu}}(1 - \frac{2M}{R})\right) \ge -\left[\lambda_t \frac{W}{E} + \mu_r e^{\mu-\lambda} - \hat{\mu}_t\right] Ge^{\hat{\mu}} \left(1 - \frac{2M}{R}\right). \tag{24}$$

This implies that

$$G(t_1)e^{\hat{\mu}(t_1,R(t_1))}\left(1 - \frac{2M}{R(t_1)}\right) \ge e^{-\int_0^{t_1}\left[\lambda_t(s,R(s))\frac{W}{E} + \check{\mu}_r(s,R(s))e^{(\mu-\lambda)(s,R(s))} - \hat{\mu}_t(s,R(s))\right]ds}$$
$$\times\, G(0)e^{\hat{\mu}(0,R(0))}(1 - \frac{2M}{R(0)}). \tag{25}$$

Let γ be the curve

$$\gamma := \{(t, r) : 0 \le t \le t_1,\ r = R(t)\}.$$

The time integral in (25) can be written as

$$\int_\gamma e^{(-\mu+\lambda)(t,r)}\lambda_t(t, r)\, dr + \left(e^{(\mu-\lambda)(t,r)}\check{\mu}_r(t, r) - \hat{\mu}_t(t, r)\right) dt. \tag{26}$$

An application of Green's formula in the plane to this curve integral, making crucial use of the second order Einstein equation (6) and the Vlasov equation, leads in [1] to the inequality

$$\int_\gamma e^{-\mu+\lambda}\lambda_t\, dr + (e^{\mu-\lambda}\check{\mu}_r - \hat{\mu}_t)\, ds \le \frac{5M}{2R_0\left(1 - \frac{2M}{R_0}\right)}.$$

Inserting this into the main inequality (25) we get

$$G(t_1)e^{\hat{\mu}(t_1,R(t_1))}\left(1 - \frac{2M}{R(t_1)}\right) \ge e^{\frac{-5M}{2R_0(1-\frac{2M}{R_0})}} G(0)e^{\hat{\mu}(0,R(0))}\left(1 - \frac{2M}{R(0)}\right).$$

Noticing that $\hat{\mu}$ is monotone in r and nonpositive, and that $R(0) \ge R_0$, we find that

$$\begin{aligned} G(t_1) &\ge e^{\frac{-5M}{2R_0(1-\frac{2M}{R_0})}} G(0)e^{\hat{\mu}(0,R_0)}(1 - \frac{2M}{R_0}) \\ &\ge e^{\frac{-5M}{2R_0(1-\frac{2M}{R_0})}} G(0)\sqrt{\frac{R_0 - 2M}{R_0}}(1 - \frac{2M}{R_0}). \end{aligned} \tag{27}$$

Here we made use of the estimate

$$\hat{\mu}(t, R_0) \ge -\int_{R_0}^\infty \frac{M\, d\eta}{\eta^2(1 - \frac{2M}{\eta})} = \frac{1}{2}\log\left(1 - \frac{2M}{R_0}\right). \tag{28}$$

We have that $G(0) > W(0) \ge W_-$, and in view of (20) we also have $3W(t) \ge G(t)$ on $[0, t_1]$. We now use the condition (21) and obtain

$$3W(t_1) \ge G(t_1) > 3W_*.$$

Thus $W(t_1) > W_*$, and necessarily we have $t_1 = T$. As was pointed out in the beginning of the proof, since matter stays strictly away from the centre of symmetry, it follows that $T = \infty$, cf. [8]. For the remaining statements in the theorem we refer to [1]. □

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