Chapter 2
Topological Duality

In this chapter we study the topological duality that exists between relational spaces—that is to say, relational structures endowed with the topology of a Boolean space—and arbitrary Boolean algebras with operators. The duality between the spaces and algebras carries with it a corresponding duality between morphisms: every continuous bounded homomorphism between relational spaces corresponds to a homomorphism between the dual Boolean algebras with operators, and conversely. The duality between the morphisms implies other dualities as well. Here are some examples. Every special open subset of a relational space corresponds to an ideal in the dual algebra, and vice versa. Every relational congruence on a relational space corresponds to a subuniverse of the dual algebra, and vice versa. The equivalence classes of compactifications of the disjoint union of a system of relational spaces correspond to the subalgebras of the direct product of the system of dual algebras that include the weak direct product of the system of dual algebras, and vice versa. In particular, the Stone-Čech compactification of a disjoint union of relational spaces corresponds to the direct product of the dual algebras.

2.1 Topological Duality for Boolean Algebras

To motivate the development that follows, we recall some of the essential features of the topological duality for Boolean algebras that was discovered by Stone [36], [37]. Correlated with every Boolean algebra A is a certain topological space U. The points in the space U are the ultrafilters in A. The clopen subsets of U are the sets of the form

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$$F_r = \{X \in U : r \in X\}$$

for elements r in A, and the open subsets are the unions of arbitrary systems of clopen sets. Under this topology, U becomes a Boolean space, that is to say, it becomes a compact Hausdorff space in which the clopen sets (the sets that are simultaneously closed and open) form a base for the topology. The space U is called the *dual space* of the algebra A.

Inversely, correlated with every Boolean space is a certain Boolean algebra A. The elements in A are the clopen subsets of U, and the Boolean operations are the set-theoretic operations of union and complement. This Boolean algebra is called the *dual algebra* of the space U.

If one starts with a Boolean algebra A, forms its dual space U, and then forms the dual algebra of U, then the result is the Boolean algebra B of sets of the form F_r for r in A. The algebras A and B are isomorphic via the function that maps every element r in A to the clopen set F_r. Similarly, if one starts with a Boolean space U, forms its dual algebra A, and then forms the dual space of A, the result is the Boolean space V in which the points are the ultrafilters of elements in A, that is to say, the ultrafilters of clopen subsets of U; these ultrafilters have the form

$$X_r = \{F \in A : r \in F\}$$

for elements r in $\mathfrak{U}$. The spaces U and V are homeomorphic via the function that maps each point r in U to the ultrafilter X_r. This whole state of affairs may be expressed by saying that each Boolean algebra is isomorphic to its second dual and each Boolean space is homeomorphic to its second dual.

The duality between Boolean algebras and Boolean spaces is accompanied by a corresponding duality—apparently due to Sikorski (see [35])—between the homomorphisms on the algebras and the continuous functions on the spaces. Let U and V be Boolean spaces, and A and B the corresponding dual Boolean algebras. If ϑ is a continuous function from U into V, then there is a natural Boolean homomorphism φ from B into A that is defined by

$$\varphi(F) = \vartheta^{-1}(F) = \{u \in U : \vartheta(u) \in F\}$$

for elements F in B, that is to say, for clopen subsets F of V. This mapping is called the (*first*) *dual* of the function ϑ. Inversely, if φ is a homomorphism from B into A, then there is a natural continuous function ϑ from U to V that is defined by

$$\vartheta(u) = r \qquad \text{if and only if} \qquad r \in \bigcap\{F \in B : u \in \varphi(F)\}.$$

for elements r in U. This function is called the (*first*) *dual* of the homomorphism φ. If one starts with a homomorphism φ from B to A, forms the dual continuous mapping ϑ from U to V, and then forms the dual homomorphism of ϑ, the result is the original homomorphism φ. Similarly, if one starts with a continuous function ϑ from U to V, forms the dual homomorphism φ from B to A, and then forms the dual continuous mapping of φ, the result is the original function ϑ. This whole state of affairs may be expressed by saying that every continuous function between Boolean spaces, and every homomorphism between Boolean algebras, is its own second dual. Furthermore, a continuous mapping between Boolean spaces is one-to-one or onto if and only if its dual homomorphism is onto or one-to-one respectively.

Finally, if ϑ is a continuous function from U to V, and δ a continuous function from V to a Boolean space W, and if φ and ψ are the respective dual homomorphisms of ϑ and δ, then the dual of the composition $\delta \circ \vartheta$ is just the composition $\varphi \circ \psi$. The category of Boolean algebras with homomorphisms as morphisms is therefore dually equivalent to the category of Boolean spaces with continuous functions as morphisms.

The epi-mono duality between morphisms implies a duality between ideals and open sets, and a duality between subuniverses and Boolean congruences; there is a corresponding duality between quotient algebras and subspaces on the one hand, and a corresponding duality between subalgebras and quotient spaces on the other hand. The epi-mono duality also implies a duality between certain subdirect products of Boolean algebras and compactifications of unions of Boolean spaces, and in particular between direct products of Boolean algebras and Stone-Čech compactifications of unions of Boolean spaces. (See, for example, [10], Chapters 34–38, 43, where references to the literature may also be found; or see [23], Chapter 3.)

We shall see that completely analogous dualities hold for Boolean algebras with operators and relational spaces. Only a very basic knowledge of topology is needed to understand much of this duality, but the duality between subdirect products of algebras and compactifications of unions of spaces requires a slightly deeper knowledge of topology.

2.2 Relational Spaces

We begin with the task of defining what is meant by a topological relational structure, or, for short, a relational space. When defining a topological algebra such as a topological group, there are usually some requirements on the fundamental operations—such as continuity or openness—that render these operations compatible with the given topology on the universe of the algebra. An operation of rank n in a topological algebra is defined to be continuous if the inverse image, under the operation, of every open set is open in the product topology on the nth Cartesian power of the universe; and the operation is defined to be open if the image, under the operation, of every open set in the nth Cartesian power of the universe is open in the universe. A relational structure does not have fundamental operations, but rather fundamental relations, so one must clarify what it means for a fundamental relation to be continuous or open. The first task is to clarify what is meant by the image and the inverse image of a set under such a relation.

The definition of the image of a set under a relation is quite natural and straightforward. If R is a relation of rank $n+1$ in a relational structure $\mathfrak{U}$, and if H is a subset of the nth Cartesian power U^n of the universe U, then the *image of H under the relation R* is the set

$$R^*(H) = \{t \in U : R(r_0, \dots, r_{n-1}, t) \text{ for some sequence } (r_0, \dots, r_{n-1}) \in H\}.$$

In particular, if $F_0, \dots, F_{n-1}$ are subsets of U, then

$$R^*(F_0 \times \cdots \times F_{n-1}) = \{t \in U : R(r_0, \dots, r_{n-1}, t) \text{ for some } r_i \in F_i \text{ for } i < n\}.$$

In the case of a sequence of n elements $r_0, \dots, r_{n-1}$ from U, we shall often write simply $R^*(r_0, \dots, r_{n-1})$ instead of $R^*(\{(r_0, \dots, r_{n-1})\})$.

To illustrate these ideas more concrete, consider the case of a ternary relation R. If H a subset of $U \times U$, then

$$R^*(H) = \{t \in U : R(r, s, t) \text{ for some } (r, s) \in H\},$$

and if F and G are subsets of U, then

$$R^*(F \times G) = \{t \in U : R(r, s, t) \text{ for some } r \in F \text{ and } s \in G\}.$$

The definition of the inverse image of a set under a relation is a bit more involved, because there are several possibilities. The one that works in the present context is the following. If H is a subset of U, then the *inverse image of* H *under* R is the set

$$R^{-1}(H) = \{(r_0, \dots, r_{n-1}) \in U^n : R(r_0, \dots, r_{n-1}, t) \text{ implies } t \in H\}.$$

For example, if R is a ternary relation, then

$$R^{-1}(H) = \{(r, s) \in U \times U : R(r, s, t) \text{ implies } t \in H\}.$$

Warning: this definition of inverse image requires the image of the singleton set

$$\{(r_0, \dots, r_{n-1})\} = \{r_0\} \times \cdots \times \{r_{n-1}\}$$

under R to be *entirely include* in H, and not just to have a non-empty intersection with H, before the sequence $(r_0, \dots, r_{n-1})$ is place in the inverse image, so that

$$(r_0, \dots, r_{n-1}) \in R^{-1}(H) \quad \text{if and only if} \quad R^*(r_0, \dots, r_{n-1}) \subseteq H.$$

Lemma 2.1. *If R is a relation of rank n on a set U, and if H and K are subsets of U, then*

$$H \subseteq K \qquad \textit{implies} \qquad R^{-1}(H) \subseteq R^{-1}(K).$$

Proof. Assume that H is included in K, and let $(r_0, \dots, r_{n-1})$ be a sequence in $R^{-1}(H)$. The image set $R^*(r_0, \dots, r_{n-1})$ of this sequence is included in H, by the observation preceding the lemma, and therefore the image set of the sequence is also included in K, by the assumption. Consequently, $(r_0, \dots, r_{n-1})$ belongs to $R^{-1}(K)$, by the observation preceding the lemma. □

One can imagine a definition of the inverse image of H under R in which a sequence $(r_0, \dots, r_{n-1})$ is put into $R^{-1}(H)$ just in case there exists an element t in H such that $R(r_0, \dots, r_{n-1}, t)$ holds. In fact, this is the definition that is adopted by Halmos [15] for binary relations. However, this is not the definition that works in the present context.

With the preceding notions in hand, we can now define what it means for a relation to be open, clopen, or continuous. A relation R of rank $n + 1$ is defined to be *open* if the image under R of every open subset of U^n (in the product topology) is an open subset of U, and *clopen* if the image of every clopen subset of U^n is a clopen subset of U. Note that a unary relation is defined to be open or clopen if it is an open subset or a clopen subset respectively of U. A relation R of rank $n+1$ is defined to be *continuous* if the inverse image of every open subset of U is an open subset of U^n (in the product topology).

Definition 2.2. A *topological relational structure*, or a *relational space* for short, is a relational structure $\mathfrak{U}$, together with a topology on the universe U, such that U is a Boolean space under this topology—that is to say, U is a compact Hausdorff space in which the clopen sets form a base for the topology—and the relations in $\mathfrak{U}$ are clopen, and the relations of rank at least two are continuous. □

Notice that a relation of rank 1 in $\mathfrak{U}$ is always continuous, since the inverse of every subset of U under such a relation is, by definition, a subset of the set $U^0 = \{\varnothing\}$ and is therefore automatically open. Consequently, the final part of the preceding definition amounts to saying that all the relations in $\mathfrak{U}$ are clopen and continuous. In speaking about relational spaces, we shall often use topological terminology. For instance, we may call the elements in the universe "points", and we may use phrases such as "the set F is open in $\mathfrak{U}$" or "the set F is an open subset of $\mathfrak{U}$" to express that F is an open subset of the topological space U.

There have been several earlier definitions of an analogue of a relational space. Halmos [15] is concerned with Boolean algebras with a single unary operator. The corresponding topological structures he considers are Boolean spaces with a single binary relation, say R, that is *Boolean* in the sense that the inverse image (in his sense of the word) of every clopen set under R is clopen, and the image of every point under R is closed (see p. 232 in [15]). Goldblatt [13] is concerned not only with arbitrary Boolean algebras with operators, but more generally with bounded distributive lattices with operators. If we restrict our attention to the relational structures in his paper that correspond to Boolean algebras with operators, then a relational space is for him a relational structure with the topology of a Boolean space and with the property that the inverse images of points under the fundamental relations are closed subsets (in the product topology)—that is to say, the set

$$\{(r_0, \ldots, r_{n-1}) : R(r_0, \ldots, r_{n-1}, t)\}$$

is closed for each fundamental relation R of rank $n+1$ and each point t—and the images of clopen subsets (in the product topology) are clopen sets (see his definition of an ordered relational space on pp. 184–185 in [13]). We shall have more to say about the connection between Goldblatt's definition and our own later on. Hansoul [16] is also concerned with bounded distributive lattices with operators. He deals with multi-algebras instead of relational structures, but it seems that in his definition of the appropriate topological multi-algebra, he requires the inverse images of points to be closed, and the images of clopen sets to be clopen (see Definition 1.6 in [16]). Sambin and Vaccaro [32] are concerned with Boolean algebras with a single unary operator that is distributive, not over addition, but rather over multiplication. They consider relational structures that consist of a single binary relation, and they define such a relation R to be continuous if, for every clopen set F, the set of elements r such that $R(r,t)$ implies that t is in F is always clopen.

2.3 Duality for Algebras

The next step is to correlate with each Boolean algebra with operators $\mathfrak{A}$ a relational space $\mathfrak{U}$. The notion of a canonical extension of a Boolean algebra with operators is implicitly involved in this discussion.

Definition 2.3. A *canonical extension* of a Boolean algebra with operators $\mathfrak{A}$ is a Boolean algebra with operators $\mathfrak{B}$ that has the following properties.

(i) $\mathfrak{B}$ is complete (in particular, the operators in $\mathfrak{B}$ are complete) and atomic, and $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.
(ii) For any two distinct atoms r and s in $\mathfrak{B}$, there is an element t in $\mathfrak{A}$ that *separates* the atoms in the sense that r is below t and s is below $-t$.
(iii) If a subset X of $\mathfrak{A}$ has the supremum 1 in $\mathfrak{B}$, then there is a finite subset Y of X such that 1 is the supremum of Y (in $\mathfrak{A}$ and in $\mathfrak{B}$). □

Condition (ii) is called the *atom separation property*, while condition (iii) is called the *compactness property*. The preceding definition goes back to Jónsson-Tarski [21], where it is shown that every Boolean algebra with operators $\mathfrak{A}$ (even one that is not normal) has a canonical extension, and that any two canonical extensions of $\mathfrak{A}$ are isomorphic via a mapping that is the identity function on $\mathfrak{A}$. This uniqueness up to isomorphisms justifies speaking of *the* canonical extension of $\mathfrak{A}$.

The precise definition of (an isomorphic copy of) the canonical extension plays an important role in the subsequent discussion, so we give some of the details of the construction. Let U be the set of all ultrafilters in $\mathfrak{A}$. With each operator f of rank n in $\mathfrak{A}$, correlate a relation R of rank $n+1$ on U by defining

$$R(X_0, \ldots, X_{n-1}, Z) \qquad \text{if and only if} \qquad \{f(r_0, \ldots, r_{n-1}) : r_i \in X_i \text{ for } i < n\} \subseteq Z$$

for all ultrafilters $X_0, \ldots, X_{n-1}, Z$ in U. If one extends f in the usual way to a *complex operation* of rank n on subsets of $\mathfrak{A}$, by

$$f(X_0, \ldots X_{n-1}) = \{f(r_0, \ldots, r_{n-1}) : r_i \in X_i \text{ for } i < n\},$$

then the definition of the relation R on sequences of elements in $\mathfrak{U}$ assumes the form

$$R(X_0, \ldots, X_{n-1}, Z) \qquad \text{if and only if} \qquad f(X_0, \ldots X_{n-1}) \subseteq Z.$$

For an operator of rank 0, that is to say, for a distinguished constant c, the definition of R takes the form

$$R(Z) \qquad \text{if and only if} \qquad c \in Z.$$

In the case of a binary operator $\circ$ in $\mathfrak{A}$, the relation R is the ternary relation on the set of ultrafilters that is defined on all triples of ultrafilters X, Y, Z by

$$R(X, Y, Z) \qquad \text{if and only if} \qquad X \circ Y \subseteq Z,$$

where $X \circ Y$ is the *complex product* of X and Y, that is to say,

$$X \circ Y = \{r \circ s : r \in X \text{ and } s \in Y\}$$

Let $\mathfrak{U}$ be the relational structure whose universe is the set U of ultrafilters in $\mathfrak{A}$, and whose fundamental relations are the relations on U that are correlated with the operators in $\mathfrak{A}$. The complex algebra $\mathfrak{Cm}(U)$ is a complete and atomic Boolean algebra with operators, and the sum of every set of elements in $\mathfrak{Cm}(U)$ is just the union of that set of elements. For each element r in $\mathfrak{A}$, let F_r be the set of ultrafilters in $\mathfrak{U}$ that contain the element r, and notice that F_r is a subset of $\mathfrak{U}$ and therefore an element in the complex algebra $\mathfrak{Cm}(U)$. Using properties of ultrafilters, one can check that the following equations hold in $\mathfrak{Cm}(U)$. (The occurrence of f on the right side of the equation in (iii) denotes an arbitrary operator in $\mathfrak{A}$, and the occurrence on the left denotes the corresponding operator in $\mathfrak{Cm}(U)$.)

Lemma 2.4. *Let r, s, t, and $r_0, \ldots, r_{n-1}$ be elements in $\mathfrak{A}$.*

(i) $F_r = F_s$ *if and only if* $r = s$.
(ii) $F_r \cup F_s = F_t$ *if and only if* $t = r + s$.
(iii) $\sim F_r = F_t$ *if and only if* $t = -r$.
(iv) $f(F_{r_0}, \ldots, F_{r_{n-1}}) = F_t$ *if and only if* $t = f(r_0, \ldots, r_{n-1})$.

Proof. The implication from right to left in (i) is trivial. The argument establishing the reverse implication proceeds by contraposition. Suppose r and s are distinct. One of the products $r \cdot -s$ and $-r \cdot s$ is then non-zero, say it is $r \cdot -s$. The set consisting of this single element trivially satisfies the finite meet property, so there must be an ultrafilter Z in $\mathfrak{A}$ that contains $r \cdot -s$, and therefore also r and $-s$, by the upward closure of Z. It follows that Z contains r but not s, so Z belongs to the set F_r but not to F_s. These two sets are therefore distinct.

The verifications of (ii) and (iii) use the basic properties of ultrafilters and are not difficult. For example, to establish the implication from right to left in (ii), suppose that $t = r + s$. The element t belongs to an arbitrary ultrafilter Z (in U) if and only if at least one of the summands r and s is in Z, so

$$\begin{array}{lll} Z \in F_r \cup F_s & \text{if and only if} & Z \in F_r \text{ or } Z \in F_s, \\ & \text{if and only if} & r \in Z \text{ or } s \in Z, \\ & \text{if and only if} & t \in Z, \\ & \text{if and only if} & Z \in F_t, \end{array}$$

by the definition of the union of two sets, the definitions of the sets F_r, F_s, and F_t, and the above mentioned property of ultrafilters. Thus, the first equation in (ii) holds. To establish the reverse implication, assume that the first equation in (ii) holds. Write $p = r+s$, and observe that

$$F_p = F_r \cup F_s,$$

by the implication just established. Combine this with the assumed equation in (ii) to arrive at $F_p = F_t$. Apply (i) to conclude that $p = t$. This establishes the equivalence in (ii).

The equivalence in (iii) follows by a similar argument, using the property that a complement $-r$ belongs to an ultrafilter Z if and only if r is not in Z.

The proof of the equivalence in (iv) is similar in character to the preceding arguments, but substantially more complicated in details. Consider the case of a binary operator $\circ$. Let r and s be arbitrary elements in $\mathfrak{A}$, and assume first that $t = r \circ s$, with the goal of showing that

$$F_r \circ F_s = F_t. \tag{1}$$

The definition of the operator $\circ$ in $\mathfrak{Cm}(U)$, and the definitions of the sets F_r and F_s imply that

$$F_r \circ F_s = \{Z \in U : X \circ Y \subseteq Z \\ \text{for some } X, Y \in U \text{ with } r \in X \text{ and } s \in Y\}. \tag{2}$$

Consequently, if Z belongs to the left side of (1), then there must be ultrafilters X and Y in U containing r and s respectively such that the complex product $X \circ Y$ is included in Z, by (2). The product $r \circ s$ belongs to $X \circ Y$, by the definition of the complex product of two sets, so $r \circ s$ must belong to Z and therefore Z must belong to the right side of (1) by the assumption on t and the definition of the set F_t. This proves that the left side of (1) is included in the right side.

To establish the reverse inclusion, suppose Z belongs to the right side of (1). To show that Z belongs to the left side of (1), ultrafilters X and Y containing r and s respectively must be constructed with the property that $X \circ Y$ is included in Z, by (2). The Boolean dual of Z is the maximal Boolean ideal

$$-Z = \{-r : r \in Z\}$$

in $\mathfrak{A}$. Write

$$W_0 = \{v \in A : r \circ (v \cdot s) \in -Z\}. \tag{3}$$

We proceed to show that W_0 is a proper Boolean ideal containing the element $-s$.

First of all,

$$r \circ (-s \cdot s) = r \circ 0 = 0,$$

by Boolean algebra and the assumption that the operators in $\mathfrak{A}$ are normal. The element 0 belongs to the ideal $-Z$, so $r \circ (-s \cdot s)$ belongs to $-Z$, and consequently $-s$ is in W_0, by (3). In particular, W_0 is not empty.

Second, if u is in W_0, and if v is an element in $\mathfrak{A}$ that is below u, then

$$r \circ (v \cdot s) \leq r \circ (u \cdot s),$$

by Boolean algebra and the monotony of the operators in $\mathfrak{A}$. Since the element $r \circ (u \cdot s)$ is in $-Z$, by (3) (with u in place of v), and since $-Z$ is an ideal (and therefore downward closed), the element $r \circ (v \cdot s)$ must also be in $-Z$, and therefore v must be in W_0, by (3). Thus, W_0 is downward closed.

Third, if u and v are in W_0, then the elements

$$r \circ (u \cdot s) \qquad \text{and} \qquad r \circ (v \cdot s)$$

must be in $-Z$, by (3). The sum of these two elements is also in $-Z$, because $-Z$ is an ideal and hence closed under addition. Since

$$r \circ [(u + v) \cdot s] = r \circ (u \cdot s) + r \circ (v \cdot s),$$

by Boolean algebra and the distributivity of the operators in $\mathfrak{A}$, it follows that $r \circ [(u + v) \cdot s]$ is in $-Z$, and therefore $u + v$ is in W_0, by (3). Thus, W_0 is closed under addition.

Finally, W_0 does not contain the unit 1. Indeed, the product $r \circ s$ is in Z, by the assumption that Z is in F_t and the assumption about t, so this product cannot be in $-Z$. Since

$$r \circ (1 \cdot s) = r \circ s,$$

by Boolean algebra, it follows that $r \circ (1 \cdot s)$ is not in $-Z$, and consequently 1 is not in W_0, by (3). This completes the proof that W_0 is a proper Boolean ideal containing $-s$.

Use the Maximal Ideal Theorem for Boolean algebras (see Section 1.2) to extend W_0 to a maximal Boolean ideal W in $\mathfrak{A}$. The Boolean dual of W is the ultrafilter Y that is determined by

$$Y = -W = \{-v : v \in W\} = \{w \in A : w \notin W\} = A \sim W. \qquad (4)$$

The element $-s$ is in W_0 and therefore also in W, so s is in Y, by (4). Also, if v is Y, then v is not in W, by (4), and therefore v is not in W_0.

Consequently, the element $r \circ (v \cdot s)$ cannot be in $-Z$, by (3), so this element must be in Z. Since

$$r \circ (v \cdot s) \le r \circ v,$$

by Boolean algebra and the monotony of the operators in $\mathfrak{A}$, and since Z is a filter, it may be concluded that $r \circ v$ is in Z, by the upward closure of filters. Thus,

$$\{r \circ v : v \in Y\} \subseteq Z. \tag{5}$$

Define a subset V_0 of $\mathfrak{A}$ by

$$V_0 = \{u \in A : (u \cdot r) \circ v \in -Z \text{ for some } v \in Y\}. \tag{6}$$

As before, the set V_0 is a proper Boolean ideal in $\mathfrak{A}$ that contains the element $-r$. For example, $-r$ is in V_0, by (6), because 0 is in the ideal $-Z$, and 1 is in Y, and

$$(-r \cdot r) \circ 1 = 0 \circ 1 = 0,$$

by Boolean algebra and the assumption that $\circ$ is a normal operator.

The argument that V_0 is downward closed is similar to the argument that the set W_0 is downward closed, and is left to the reader. To see that V_0 is closed under addition, assume u and v are in V_0. There must be elements w_1 and w_2 in Y such that

$$(u \cdot r) \circ w_1 \qquad \text{and} \qquad (v \cdot r) \circ w_2 \tag{7}$$

are in $-Z$, by (6). Write $w = w_1 \cdot w_2$, and observe that w is in Y, by the closure of filters under multiplication. The elements

$$(u \cdot r) \circ w \qquad \text{and} \qquad (v \cdot r) \circ w \tag{8}$$

are respectively below the elements in (7), by Boolean algebra and the monotony of the operators in $\mathfrak{A}$, so they, too, belong to the ideal $-Z$, by the downward closure of ideals. The sum of the elements in (8) is therefore also in $-Z$, by the closure of ideals under addition. Since

$$[(u + v) \cdot r] \circ w = (u \cdot r) \circ w + (v \cdot r) \circ w,$$

by Boolean algebra and the distributivity of the operator $\circ$, it follows from the preceding observations and (6) that $u + v$ belongs to V_0. Thus, V_0 is an ideal.

To see that the ideal V_0 is proper, observe that

$$(1 \cdot r) \circ v = r \circ v,$$

by Boolean algebra, and that $r \circ v$ belongs to Z for every v in Y, by (5). Consequently, $(1 \cdot r) \circ v$ cannot be in $-Z$ for any v in Y, so 1 cannot be in V_0, by (6).

Extend V_0 to a maximal Boolean ideal V. The Boolean dual of V is the ultrafilter

$$X = -V = \{-u : u \in V\} = \{w \in A : w \notin V\} = A \sim V. \tag{9}$$

The element $-r$ is in V_0, and V_0 is included in V, so $-r$ must be in V, and therefore r must be in X, by (9). If u is any element in X, then u cannot be in V, by (9), and therefore u cannot be in V_0. Consequently, the element $(u \cdot r) \circ v$ cannot belong to $-Z$ for any v in Y, by (6), so this element must be in Z for every such v. Since Z is a filter, and therefore upward closed, and since

$$(u \cdot r) \circ v \leq u \circ v,$$

it may be concluded that $u \circ v$ belongs to Z for every v in Y. This conclusion holds for all u in X, so the complex product $X \circ Y$ is included in Z, as desired.

This completes the proof of (1) and of the implication from right to left in (iv). The proof of the reverse implication in (iv) is nearly identical to the proof of the corresponding implication in (ii) (with $+$ and $F_r \cup F_s$ replaced by $\circ$ and $F_r \circ F_s$ respectively) and is left to the reader. $\square$

The equivalences in Lemma 2.4 imply that the set B of all subsets of $\mathfrak{U}$ of the form F_r, for r in $\mathfrak{A}$, is closed under the operations of $\mathfrak{Cm}(U)$ and is therefore subuniverse of $\mathfrak{Cm}(U)$. Let $\mathfrak{B}$ be the corresponding subalgebra. The equivalences also imply that the function φ mapping each element r in $\mathfrak{A}$ to the set F_r of ultrafilters containing r is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. It is called the *canonical embedding* of $\mathfrak{A}$ into $\mathfrak{Cm}(U)$.

We proceed to show that $\mathfrak{Cm}(U)$ is the canonical extension of $\mathfrak{B}$. The canonical extension of $\mathfrak{A}$ can then be obtained by applying the Exchange Principle from general algebra to the pair $\mathfrak{B}$ and $\mathfrak{Cm}(U)$, exchanging $\mathfrak{B}$ for $\mathfrak{A}$, and $\mathfrak{Cm}(U)$ for an isomorphic copy of $\mathfrak{Cm}(U)$ that is the desired canonical extension of $\mathfrak{A}$. As regards the atom separation property, observe that the atoms in $\mathfrak{Cm}(U)$ are the singletons of ultrafilters. For any two distinct ultrafilters X and Y, there must be an element r that belongs to one of the ultrafilters but not the other, say r is in X but not in Y. The complement $-r$ belongs to Y (because Y is an ultrafilter that does not contain r), so the atom $\{X\}$ is included in F_r, while the atom $\{Y\}$ is included in F_{-r}. Thus, the atom separation property holds for $\mathfrak{Cm}(U)$ with respect to $\mathfrak{B}$.

To verify the compactness property, argue by contraposition. Consider an arbitrary subset G of $\mathfrak{B}$ with the property that no finite subset of G has the unit $F_1 = U$ in $\mathfrak{Cm}(U)$ as its union. It is to be shown that G cannot have U as its union. The set G must have the form

$$G = \{F_r : r \in I\}$$

for some subset I of $\mathfrak{A}$, by the definition of $\mathfrak{B}$. The assumed property of G and the isomorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$ imply that no finite subset of I has 1 as its sum in $\mathfrak{A}$. Consequently, the product of every finite subset of the set

$$-I = \{-r : r \in I\}$$

is different from 0, by Boolean algebra, that is to say, the set $-I$ has the finite meet property. It follows that the set $-I$ can be extended to an ultrafilter X. Since the complement of every element in I belongs to X, none of the elements r in I can belong to X, and therefore X cannot belong to any of the sets F_r in G. It follows that the union of the sets in G does not contain all of the ultrafilters in U, since it does not contain X. Consequently, the union of G is not the unit U.

We turn now to the problem of correlating a relational space with $\mathfrak{A}$. Let $\mathfrak{U}$ be the relational structure defined above, consisting of the ultrafilters in $\mathfrak{A}$, and let $\mathfrak{B}$ be the subalgebra of $\mathfrak{Cm}(U)$ that is the isomorphic image of $\mathfrak{A}$ under the canonical embedding. There is a natural topology that can be defined on the universe of $\mathfrak{U}$ with the help of the algebra $\mathfrak{B}$. The universe of $\mathfrak{B}$—that is to say, the collection of sets of the form F_r for elements r in $\mathfrak{A}$—is declared to be a base of the topology, so that the open sets in the topology are the unions of arbitrary systems of sets in $\mathfrak{B}$. Using the fact that $\mathfrak{B}$ is closed under intersection and contains the sets $\varnothing = F_0$ and $U = F_1$, it is easy to check that this class of open sets really does satisfy the three conditions for being a topology, namely it contains the empty set and the universal set, it is closed under unions of arbitrary systems of open sets, and it is closed under intersections of finite systems of open sets. We shall refer to this topology as the *topology induced on* $\mathfrak{U}$ *by* $\mathfrak{A}$ (because the topology is completely determined by the universe of $\mathfrak{A}$ and the ultrafilters in $\mathfrak{A}$). Since the algebra $\mathfrak{B}$ is closed under the formation of complements, the sets in the base of the topology are both open and closed; they constitute the *clopen sets* of the topology. We shall return to this point in a moment.

Under this topology, the space $\mathfrak{U}$ is Hausdorff, and in fact any two points in $\mathfrak{U}$ are separated by a clopen set in $\mathfrak{B}$. This is just a topological manifestation of the atom separation property for the canonical extension: distinct points in $\mathfrak{U}$ are distinct ultrafilters X and Y in $\mathfrak{A}$, and for any two such ultrafilters, the atom separation property of $\mathfrak{Cm}(U)$ with respect to $\mathfrak{B}$ requires the existence of an element r in $\mathfrak{A}$ such that X belongs to F_r, but Y does not (see above). The clopen set F_r therefore separates the points X and Y.

The space $\mathfrak{U}$ is also compact, and in fact, compactness is a topological manifestation of the compactness property of the canonical extension. In order to see this, suppose a system of open sets covers the space $\mathfrak{U}$. Each open set in the system is a union of clopen sets, by the definition of the topology, so there must be a system $(F_r : r \in I)$ of clopen sets (where I is some subset of $\mathfrak{A}$) such that each of these clopen sets is contained in one of the open sets in the given open cover, and the union of this system of clopen sets is U. The compactness property for $\mathfrak{Cm}(U)$ with respect to $\mathfrak{B}$ implies the existence of a finite subset I_0 of I such that the union of the system $(F_r : r \in I_0)$ is U. Since each of the clopen sets in this finite subsystem is included in one of the open sets in the given open cover, a finite subsystem of the original open cover must also have U as its union.

It is well known that in a compact topological space, if a given Boolean algebra of clopen sets separates points, then the space is in fact a Boolean space, and the given Boolean algebra of clopen sets is in fact the Boolean algebra of all clopen subsets of the space (see Lemma 1 on p. 305 of [10]). It follows that the topology defined on $\mathfrak{U}$ turns this relational structure into a Boolean space, and the clopen subsets of $\mathfrak{U}$ are precisely the sets of the form F_r for r in $\mathfrak{A}$. In order to show that $\mathfrak{U}$ is in fact a relational space, it remains to prove that the fundamental relations in $\mathfrak{U}$ are clopen, and that the fundamental relations of rank at least two are continuous.

Lemma 2.5. *The fundamental relations in the relational structure* $\mathfrak{U}$ *correlated with a Boolean algebra with operators* $\mathfrak{A}$ *are clopen.*

Proof. Focus on the case of a ternary relation R in $\mathfrak{U}$ and the corresponding binary operator $\circ$ in $\mathfrak{Cm}(U)$. Consider first the special case of the product two clopen subsets of $\mathfrak{U}$, say F_r and F_s. It is to be shown that the image of the set $F_r \times F_s$ under the relation R is clopen. This image is defined to be the set

$$R^*(F_r \times F_s) = \{Z \in U : R(X, Y, Z) \text{ for some } X \in F_r \text{ and } Y \in F_s\}. \tag{1}$$

The sets F_r and F_s both belong to the complex algebra $\mathfrak{Cm}(U)$, so it is possible to form their product $F_r \circ F_s$ in $\mathfrak{Cm}(U)$. Use the definition of $\circ$ in $\mathfrak{Cm}(U)$ and (1) to obtain

$$F_r \circ F_s = \{Z \in U : R(X, Y, Z) \text{ for some } X \in F_r \text{ and } Y \in F_s\} = R^*(F_r \times F_s).$$

If $t = r \circ s$ in $\mathfrak{A}$, then $F_t = F_r \circ F_s$ in $\mathfrak{Cm}(U)$, by Lemma 2.4(iv), so the preceding computation shows that

$$R^*(F_r \times F_s) = F_r \circ F_s = F_t. \tag{2}$$

Since F_t is a clopen subset of $\mathfrak{U}$, by the definition of the topology on $\mathfrak{U}$, the image of $F_r \times F_s$ under R is clopen, by (2).

Consider now the case of an arbitrary clopen subset H of the product space $U \times U$. Because $\mathfrak{U}$ is a Boolean space, every clopen subset of $U \times U$ can be written as the union of finitely many products of clopen subsets of $\mathfrak{U}$. There is consequently a finite set I of pairs of elements from $\mathfrak{A}$ such that

$$H = \bigcup\{F_r \times F_s : (r, s) \in I\}. \tag{3}$$

It follows from (3) and the definition of the image of a set under the relation R that

$$\begin{aligned} R^*(H) &= \{Z : R(X, Y, Z) \text{ for some } (X, Y) \in H\} \\ &= \{Z : R(X, Y, Z) \text{ for some } (r, s) \in I \text{ and } (X, Y) \in F_r \times F_s\} \\ &= \bigcup\{R^*(F_r \times F_s) : (r, s) \in I\}. \end{aligned}$$

It was shown in the preceding paragraph that the sets $R^*(F_r \times F_s)$ are clopen in $\mathfrak{U}$. Since a union of finitely many clopen sets is clopen, it follows that $R^*(H)$ is clopen in $\mathfrak{U}$. $\square$

In order to prove that the fundamental relations in $\mathfrak{U}$ of rank at least two are continuous, it is helpful to prove first that the images of singleton points under these relations are closed sets.

Lemma 2.6. *For every fundamental relation R of rank $n+1$ in the relational structure $\mathfrak{U}$ correlated with the given Boolean algebra with operators $\mathfrak{A}$, and for every sequence of points $X_0, \ldots, X_{n-1}$ in $\mathfrak{U}$, the image set*

$$R^*(X_0, \ldots, X_{n-1}) = \{Z \in U : R(X_0, \ldots, X_{n-1}, Z)\}$$

is a closed, and hence compact, subset of $\mathfrak{U}$.

Proof. Focus on the case of a ternary relation R in $\mathfrak{U}$ that is defined in terms of a binary operator $\circ$ in $\mathfrak{A}$. Let X and Y be points in $\mathfrak{U}$, and write

$$H = R^*(X, Y) = \{Z \in U : R(X, Y, Z)\}. \tag{1}$$

It is to be shown that H is a closed, compact subset of $\mathfrak{U}$. If a point W in $\mathfrak{U}$ does not belongs to H—that is to say, if $R(X, Y, W)$ fails to hold—then the complex product $X \circ Y$ cannot be included in W, by the definition of the relation R in $\mathfrak{U}$. Consequently, there must be elements r_W in X and s_W in Y such that the product $r_W \circ s_W$ (formed in $\mathfrak{A}$) does not belong to W, by the definition of the complex product $X \circ Y$. The set $F_{r_W \circ s_W}$ of all ultrafilters that contain the product $r_W \circ s_W$ is a clopen subset of $\mathfrak{U}$, by the definition of the topology on $\mathfrak{U}$, and this clopen set does not contain W because $r_W \circ s_W$ is not in W. The intersection of a system of clopen sets is closed in the topology on U, so the set

$$F = \bigcap\{F_{r_W \circ s_W} : W \in U \sim H\}$$

is a closed subset of $\mathfrak{U}$ that does not contain W for any point W in $U \sim H$. In other words, F is a closed set that is disjoint from $U \sim H$.

Consider now any point Z in H. Since $R(X, Y, Z)$ holds, by (1), the definition of the relation R in $\mathfrak{U}$ implies that the complex product $X \circ Y$ is included in Z. Consequently, the element $r_W \circ s_W$ must belong to Z for every point W in $U \sim H$, because r_W is in X, and s_W in Y. It follows that the ultrafilter Z belongs to the clopen set $F_{r_W \circ s_W}$ for every W in $U \sim H$, and therefore Z belongs to the intersection F of these clopen sets, by the definition of F. Thus, the set H is included in F. Combine this observation with those of the preceding paragraph to conclude that the set H coincides with F and is therefore closed.

Every closed subset of a compact topological space is compact, so the set H must be compact. □

Lemma 2.7. *Every fundamental relation of rank at least two in the relational structure $\mathfrak{U}$ correlated with the given Boolean algebra with operators $\mathfrak{A}$ is continuous.*

Proof. Focus on the case of a ternary relation R in $\mathfrak{U}$ that is defined in terms of a binary operator $\circ$ in $\mathfrak{A}$. The proof requires a preliminary observation: for every element t in $\mathfrak{A}$, a pair (X, Y) of points from $\mathfrak{U}$ belongs to the inverse image $R^{-1}(F_t)$ of the clopen set F_t if and only if there are elements r in X and s in Y such that $r \circ s \leq t$ (in $\mathfrak{A}$). One direction of the argument is straightforward. Suppose such elements r and s exist. If $R(X, Y, Z)$ holds, then the complex product $X \circ Y$ is included in Z, by the definition of the relation R, and therefore the product $r \circ s$ belongs to Z, by the definition of the complex product and the assumption that r and s are in X and Y respectively. Since $r \circ s$ is below t, and Z is an ultrafilter, the element t must also belong to Z, and therefore Z must belong to the clopen set F_t. This is true for every Z such that $R(X, Y, Z)$ holds, so the entire image set

$$R^*(X, Y) = \{Z \in U : R(X, Y, Z)\} \tag{1}$$

is included in F_t. Consequently, the pair (X, Y) belongs to the inverse image $R^{-1}(F_t)$, by the definition of this inverse image.

To establish the reverse implication, suppose that no such elements r and s exist. We proceed to show that in this case the set

$$X \circ Y \cup \{-t\} = \{r \circ s : r \in X \text{ and } s \in Y\} \cup \{-t\} \tag{2}$$

has the finite meet property. Assume, accordingly, that

$$(r_i : i \in I) \qquad \text{and} \qquad (s_i : i \in I)$$

are finite systems of elements in X and Y respectively, write

$$r = \textstyle\prod_i r_i \qquad \text{and} \qquad s = \textstyle\prod_i s_i,$$

and observe that r belongs to X and s to Y, because X and Y are ultrafilters and are therefore closed under finite Boolean products. The element $r \circ s$ is not below t, by assumption, so

$$-t \cdot (r \circ s) \neq 0.$$

Since $r \le r_i$ and $s \le s_i$ for each i, the monotony of the operator $\circ$ implies that $r \circ s \le r_i \circ s_i$ for each i, and therefore $r \circ s \le \prod_i (r_i \circ s_i)$. It follows that

$$-t \cdot \textstyle\prod_i (r_i \circ s_i) \neq 0,$$

as required.

Every subset of $\mathfrak{A}$ with the finite meet property can be extended to an ultrafilter, so the set in (2) can be extended to an ultrafilter Z in $\mathfrak{A}$, by the observations of the preceding paragraph. The complex product $X \circ Y$ is included in Z, by (2), so $R(X, Y, Z)$ holds, by the definition of R. On the other hand, the complement $-t$ is in Z, again by (2), and Z is an ultrafilter, so t does not belong to Z, and therefore Z does not belong to F_t. Apply the definition of the inverse image to conclude that the pair (X, Y) cannot belong to the inverse image $R^{-1}(F_t)$. This completes the proof of the preliminary observation.

In order to prove that R is continuous, it must be shown that for every open subset H of U, the inverse image set

$$R^{-1}(H) = \{(X, Y) : R(X, Y, Z) \text{ implies } Z \in H\}$$

is open in the product topology on $U \times U$. Consider first the case when H is a clopen subset of U, say $H = F_t$. If a pair (X, Y) belongs to the inverse image $R^{-1}(F_t)$, then there must be elements r in X and s in Y such that $r \circ s \le t$, by the preliminary observation proved above. The ultrafilters X and Y belong to the clopen sets F_r and F_s respectively, by the definition of these clopen sets, so the pair (X, Y) belongs to the product clopen set $F_r \times F_s$. Furthermore, this product clopen set is included in the inverse image $R^{-1}(F_t)$. Indeed, it is not difficult to check that

$$F_r \times F_s \subseteq R^{-1}(F_r \circ F_s) \subseteq R^{-1}(F_t). \tag{3}$$

For the second inclusion, observe that the inequality $r \circ s \le t$ implies that the product $F_r \circ F_s$ (formed in $\mathfrak{Cm}(U)$) is included in F_t, by Lemma 2.4, and therefore the inverse image under R of the product $F_r \circ F_s$ is included in the inverse image under R of F_t, Lemma 2.1. As regards the first inclusion, suppose X_0 is in F_r and Y_0 in F_s. The product $\{X_0\} \circ \{Y_0\}$ in $\mathfrak{Cm}(U)$ of the atoms $\{X_0\}$ and $\{Y_0\}$ is included in the product $F_r \circ F_s$, by the monotony of the operator $\circ$. In other words,

$$Z \in \{X_0\} \circ \{Y_0\} \qquad \text{implies} \qquad Z \in F_r \circ F_s,$$

or put somewhat differently,

$$R^*(X_0, Y_0) \subseteq F_r \circ F_s.$$

This last inclusion is precisely the condition required for (X_0, Y_0) to belong to the inverse image $R^{-1}(F_r \circ F_s)$.

It has been shown that every pair (X, Y) in $R^{-1}(F_t)$ belongs to a clopen subset $F_r \times F_s$ of $U \times U$ that in turn is included in $R^{-1}(F_t)$. Consequently, the inverse image set $R^{-1}(F_t)$ is a union of clopen sets and is therefore open in $U \times U$.

Consider now the general case when H is an arbitrary open subset of $\mathfrak{U}$. The space $\mathfrak{U}$ is Boolean, so H must be the union of a system

$$(F_t : t \in I) \tag{4}$$

of clopen sets. For each pair (X, Y) in the inverse image $R^{-1}(H)$, the image set in (1) is included in H, by the definition of the inverse image set. Consequently, this image set is included in the union of the system (4). The image set in (1) is compact, by Lemma 2.6, so there must be a finite subsystem of (4) whose union covers $R^*(X, Y)$. The union of finitely many clopen sets is clopen, so there is a clopen set F_t such that

$$R^*(X, Y) \subseteq F_t \subseteq H. \tag{5}$$

The first inclusion in (5) implies that the pair (X, Y) belongs to the inverse image $R^{-1}(F_t)$, by the definition of this inverse image. It was shown in the previous paragraph that this inverse image is an open subset of $\mathfrak{U}$. The second inclusion in (5) implies that $R^{-1}(F_t)$ is included in $R^{-1}(H)$, by Lemma 2.1. Conclusion: every pair (X, Y) in the inverse image $R^{-1}(H)$ belongs to an open set $R^{-1}(F_t)$ that is included in $R^{-1}(H)$, so $R^{-1}(H)$ is a union of open sets and is therefore open. Consequently, R is continuous. □

The following theorem contains a summary of what has been accomplished in the preceding discussion and lemmas.

Theorem 2.8. *The relational structure correlated with a Boolean algebra with operators* $\mathfrak{A}$ *is a relational space under the topology induced on the relational structure by* $\mathfrak{A}$.

The relational space correlated with $\mathfrak{A}$ is called the (*first*) *dual*, or the *dual* (*relational*) *space*, of $\mathfrak{A}$. As was mentioned earlier, if $\mathfrak{U}$ is the dual of $\mathfrak{A}$, then we may employ topological terminology in speaking about $\mathfrak{U}$, referring for example to the *points* in $\mathfrak{U}$ (the elements in the universe of $\mathfrak{U}$) and the *clopen sets* in $\mathfrak{U}$ (the clopen sets in the topology on the universe of $\mathfrak{U}$).

We now change footing: starting with an arbitrary relational space $\mathfrak{U}$, we construct a Boolean algebra with operators $\mathfrak{A}$ in terms of $\mathfrak{U}$.

Lemma 2.9. *The set of clopen subsets of a relational space* $\mathfrak{U}$ *is a subuniverse of the complex algebra* $\mathfrak{Cm}(U)$*.*

Proof. Let A be the set of clopen subsets of $\mathfrak{U}$. Clearly, A is a subset of the universe of $\mathfrak{Cm}(U)$. The union of two clopen sets is clopen, and the complement of a clopen set is clopen, so A is closed under the Boolean operations of $\mathfrak{Cm}(U)$, namely union and complement. To see that A is closed under the operators of $\mathfrak{Cm}(U)$, focus on the case of a binary operator $\circ$ that is defined in terms of a ternary relation R in $\mathfrak{U}$. For two clopen subsets F and G in A, the product $F \circ G$ in $\mathfrak{Cm}(U)$ is defined to be the set

$$F \circ G = \{t \in U : R(r, s, t) \text{ for some } r \in F \text{ and } s \in G\}.$$

This is precisely the definition of the image set $R^*(F \times G)$, so

$$F \circ G = R^*(F \times G). \tag{1}$$

The definition of a relational space implies that the relation R is clopen; this means—since F and G are assumed to be clopen subsets of U—that the image set $R^*(F \times G)$ is a clopen subset of U. Thus, $F \circ G$ is a clopen set, by (1), so it belongs to A. It follows that A is closed under the operator $\circ$. □

Observe that the assumed continuity of the relations in $\mathfrak{U}$ of rank at least two is not used in the preceding proof.

The complex algebra $\mathfrak{Cm}(U)$ is a Boolean algebra with operators, and this property is preserved under the passage to subalgebras. In particular, the subalgebra of $\mathfrak{Cm}(U)$ that has as its universe the set of the clopen subsets of $\mathfrak{U}$ is a Boolean algebra with operators. It is called the (*first*) *dual*, or the *dual algebra*, of $\mathfrak{U}$. Start with a Boolean algebra with operators $\mathfrak{A}$, and form the dual space $\mathfrak{U}$ consisting of ultrafilters in $\mathfrak{A}$. The dual algebra of $\mathfrak{U}$ consisting of the clopen subsets of $\mathfrak{U}$ is called the *second dual* of $\mathfrak{A}$.

Theorem 2.10. *The second dual of every Boolean algebra with operators* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{A}$*. More explicitly, if* $\mathfrak{U}$ *is the dual space of* $\mathfrak{A}$*, and* $\mathfrak{B}$ *the dual algebra of* $\mathfrak{U}$*, and if*

$$\varphi(r) = F_r = \{X \in U : r \in X\}$$

for each r *in* $\mathfrak{A}$, *then* φ *is an isomorphism from* $\mathfrak{A}$ *to* $\mathfrak{B}$.

Proof. Essentially, this theorem has already been proved. We summarize the main points of the argument. The relational space $\mathfrak{U}$ that is the dual of $\mathfrak{A}$ has as its universe the set of ultrafilters in $\mathfrak{A}$. Each relation R of rank $n+1$ in $\mathfrak{U}$ is correlated with an operator f of rank n in $\mathfrak{A}$ and is defined by the formula

$$R(X_0, \dots, X_{n-1}, Z) \qquad \text{if and only if} \qquad f(X_0, \dots, X_{n-1}) \subseteq Z$$

for every sequence of ultrafilters $X_0, \dots, X_{n-1}, Z$ (where the occurrence of f on the right side of this equivalence denotes the extension of the operator f in $\mathfrak{A}$ to subsets of $\mathfrak{A}$). The topology on $\mathfrak{U}$ has as a base the set of all sets of the form F_r for r in $\mathfrak{A}$, and these sets prove to be precisely the clopen sets of the topology. The fundamental relations in $\mathfrak{U}$ are continuous (Lemma 2.7) and clopen (Lemma 2.5) under this topology.

The Boolean algebra with operators $\mathfrak{B}$ that is the dual of $\mathfrak{U}$ has as its universe the set of clopen subsets of $\mathfrak{U}$, that is to say, the universe consists of the sets of the form F_r for r in $\mathfrak{A}$. The Boolean operations are the set-theoretic ones of union and complement, while each operator f of rank n in $\mathfrak{B}$ is defined on a sequence of clopen sets $F_{r_0}, \dots, F_{r_{n-1}}$ by

$$f(F_{r_0}, \dots, F_{r_{n-1}}) = R^*(F_{r_0} \times \cdots \times F_{r_{n-1}})$$

(see the proof of Lemma 2.9), and the image set on the right side of this equation coincides with the set F_t when $t = f(r_0, \dots, r_{n-1})$ in $\mathfrak{A}$ (see the proof of Lemma 2.5), so

$$f(F_{r_0}, \dots, F_{r_{n-1}}) = F_t \qquad \text{if and only if} \qquad t = f(r_0, \dots, r_{n-1}).$$

Consequently, $\mathfrak{B}$ is precisely the subalgebra of $\mathfrak{Cm}(U)$ that is the image of $\mathfrak{A}$ under the canonical embedding that maps each element r in $\mathfrak{A}$ to the element F_r in $\mathfrak{B}$. In particular, the canonical embedding is an isomorphism from $\mathfrak{A}$ to its second dual $\mathfrak{B}$. $\square$

Theorem 2.10 has a topological analogue. In order to formulate it, we need some terminology. Start with a relational space $\mathfrak{U}$, and form its dual algebra $\mathfrak{A}$ consisting of the clopen subsets of $\mathfrak{U}$. The dual space of $\mathfrak{A}$, consisting of the ultrafilters in $\mathfrak{A}$ and endowed with the topology induced by $\mathfrak{A}$, is called the *second dual of* $\mathfrak{U}$. We shall prove that there is a bijection ϑ from the universe of $\mathfrak{U}$ to the universe of the second dual of $\mathfrak{U}$ that preserves the algebraic and topological structure of $\mathfrak{U}$ in the sense that ϑ is a relational structure isomorphism and a topological homeomorphism. We shall call such a function a *homeo-isomorphism*, and we shall say that two relational spaces are *homeo-isomorphic* if there exists a homeo-isomorphism mapping one of them onto the other.

Theorem 2.11. *The second dual of every relational space* $\mathfrak{U}$ *is homeo-isomorphic to* $\mathfrak{U}$. *More explicitly, if* $\mathfrak{A}$ *is the dual algebra of* $\mathfrak{U}$, *and* $\mathfrak{V}$ *the dual space of* $\mathfrak{A}$, *and if*

$$\vartheta(r) = \{F \in A : r \in F\}$$

for each r *in* $\mathfrak{U}$, *then* ϑ *is a homeo-isomorphism from* $\mathfrak{U}$ *to* $\mathfrak{V}$.

Proof. The universe of the dual algebra $\mathfrak{A}$ of $\mathfrak{U}$ is defined to be the set of all clopen subsets of $\mathfrak{U}$, and the universe of the dual space $\mathfrak{V}$ of $\mathfrak{A}$ is defined to be the set of all ultrafilters in $\mathfrak{A}$. The set

$$X_r = \{F \in A : r \in F\} \tag{1}$$

is easily seen to be such an ultrafilter in $\mathfrak{A}$, so the function ϑ defined by

$$\vartheta(r) = X_r \tag{2}$$

maps the universe of $\mathfrak{U}$ into the universe of $\mathfrak{V}$.

It is not difficult to see that ϑ is one-to-one. If r and s are distinct points in $\mathfrak{U}$, then there is a clopen set F in $\mathfrak{U}$ that contains r but not s, because $\mathfrak{U}$ has the topological structure of a Boolean space and therefore the clopen sets separate points. The set F—which is an element in $\mathfrak{A}$—belongs to the ultrafilter X_r, but not to the ultrafilter X_s, so

$$\vartheta(r) = X_r \neq X_s = \vartheta(s).$$

To check that ϑ is onto, consider a point Y in the second dual $\mathfrak{V}$. This point must be an ultrafilter in $\mathfrak{A}$, by the definition of $\mathfrak{V}$. In other words, Y must be a maximal class of non-empty clopen subsets of $\mathfrak{U}$ that is upward closed and closed under finite intersections. The compactness of the topology on $\mathfrak{U}$ implies that every non-empty system of closed sets with the finite intersection property has a non-empty intersection. In particular, since the ultrafilter Y has the finite intersection property, the intersection of the system of all sets in Y is non-empty. Let r be any point in this intersection. The point r belongs to every (clopen) set in Y, so every set in Y must belong to the ultrafilter X_r, by (1). Thus, Y is included in X_r. Since Y is assumed to be an ultrafilter, it follows that $Y = X_r$, by the maximality of Y, and therefore

$$\vartheta(r) = X_r = Y.$$

Thus, ϑ is onto.

A bijection from a Boolean space to a compact Hausdorff is a homeomorphism if and only if it maps clopen sets to clopen sets (see Corollary 3 on p. 316 of [10]). Consequently, to prove that ϑ is a homeomorphism, it suffices to show that the image under ϑ of each clopen set in $\mathfrak{U}$ is a clopen set in $\mathfrak{V}$. The image under the mapping ϑ of a clopen set F in $\mathfrak{U}$ is the set

$$\{X_r : r \in F\}, \tag{3}$$

by the definition of ϑ in (2). To say that r belongs to F is equivalent to saying that F belongs to X_r, by (1), so (3) may be rewritten as

$$\{X_r : r \in U \text{ and } F \in X_r\}. \tag{4}$$

The elements in $\mathfrak{V}$ are precisely the sets of the form X_r for r in $\mathfrak{U}$, by the conclusion of the previous paragraph, so (4) may in turn be rewritten as

$$\{Y \in V : F \in Y\}. \tag{5}$$

Because $\mathfrak{V}$ is assumed to be the dual space of the algebra $\mathfrak{A}$, each clopen subset of $\mathfrak{V}$ is determined by an element F in $\mathfrak{A}$ in the sense that it can be written as a set of the form (5) (see the remarks preceding Lemma 2.5). Thus, the set in (5) is clopen in the topology of $\mathfrak{V}$. Combine these observations to conclude that the image under ϑ of the clopen set F in $\mathfrak{U}$ is the clopen set (5) in $\mathfrak{V}$, so the mapping ϑ is clopen and therefore a homeomorphism, as claimed.

It remains to prove that the mapping ϑ is a relational structure isomorphism in the sense that it isomorphically preserves the fundamental relations of the relational structures. Focus on the case of a ternary relation R. In view of (2), it must be shown that the equivalence

$$R(r, s, t) \qquad \text{if and only if} \qquad R(X_r, X_s, X_t) \tag{6}$$

holds for all elements r, s, and t in $\mathfrak{U}$. (The first occurrence of R in (6) refers to the relation in $\mathfrak{U}$, while the second occurrence refers to the relation in $\mathfrak{V}$.) The assertion

$$R(X_r, X_s, X_t) \tag{7}$$

means, by the definition of $\mathfrak{V}$ as the dual space of $\mathfrak{A}$, that the complex product

$$X_r \circ X_s = \{F \circ G : F \in X_r \text{ and } G \in X_s\}$$

is included in the set X_t, that is to say,

$$F \in X_r \quad \text{and} \quad G \in X_s \qquad \text{implies} \qquad F \circ G \in X_t$$

for all elements F and G in $\mathfrak{A}$. This implication may be rewritten in the form

$$r \in F \quad \text{and} \qquad s \in G \qquad \text{implies} \qquad t \in F \circ G \tag{8}$$

for all clopen subsets F and G of $\mathfrak{U}$. Since $\mathfrak{A}$ is the dual algebra of $\mathfrak{U}$, the product $F \circ G$ in $\mathfrak{A}$ is defined to be the image of the pair of clopen sets (F, G) under the relation R in $\mathfrak{U}$,

$$\begin{aligned} F \circ G &= R^*(F \times G) \\ &= \{w \in U : R(p, q, w) \text{ for some } p \in F \text{ and } q \in G\}, \end{aligned}$$

so the implication in (8) is equivalent to the implication

$$r \in F \text{ and } s \in G \text{ implies } R(p, q, t) \text{ for some } p \in F \text{ and } q \in G \tag{9}$$

for all clopen subsets F and G of $\mathfrak{U}$. These observations show that the validity of (7) is equivalent to the validity of (9) for all clopen sets F and G in $\mathfrak{U}$.

The implication from left to right in (6) is now easy to establish. If

$$R(r, s, t) \tag{10}$$

holds, then for all clopen sets F and G in $\mathfrak{U}$ containing the points r and s respectively, there are always points p in F and q in G such that $R(p, q, t)$ holds, namely the points $p = r$ and $q = s$, by (10). Consequently, (9) is valid for all clopen sets F and G in $\mathfrak{U}$, and therefore (7) holds.

In order to establish the reverse implication in (6), assume that (10) fails, with the intention of showing that (7) fails. For every point w in $\mathfrak{U}$, the validity of $R(r, s, w)$ implies that w must be different from t, by the assumption that (10) fails. The clopen subsets of $\mathfrak{U}$ separate points, so for such a point w there must be a clopen set H_w in $\mathfrak{U}$ that contains w, but does not contain t. The union of these clopen sets is an open set

$$H = \bigcup\{H_w : w \in U \text{ and } R(r, s, w)\}$$

in $\mathfrak{U}$ that does not contain t. The inverse image of H under the relation R in $\mathfrak{U}$, that is to say, the set

$$R^{-1}(H) = \{(p, q) \in U \times U : R(p, q, w) \text{ implies } w \in H\}, \tag{11}$$

is open in the product topology on $U \times U$, by the assumed continuity of R. Moreover, the pair (r, s) belongs to this inverse image. Indeed, if w is any point in $\mathfrak{U}$ such that $R(r, s, w)$, then w belongs to the clopen set H_w, by the definition of this set, and therefore w belongs to H, by the definition of H; consequently, (r, s) is in $R^{-1}(H)$, by (11). The products of pairs of clopen subsets of $\mathfrak{U}$ form a base for the product topology on $U \times U$. Since $R^{-1}(H)$ is open in this topology and contains the point (r, s), there must be clopen sets F and G in $\mathfrak{U}$ for which

$$r \in F, \qquad s \in G, \qquad \text{and} \qquad F \times G \subseteq R^{-1}(H). \tag{12}$$

The clopen sets F and G satisfy the hypothesis of (9), by the first part of (12), but not the conclusion of (9), by the last part of (12). Indeed, if we had $R(p, q, t)$ for some p in F and q in G, then the pair (p, q) would belong to $F \times G$, and therefore also to $R^{-1}(H)$, by the final part of (12); this would force the point t to be in H, by (11), in contradiction to the fact that t does not belong to H. Conclusion: if (10) fails, then there are clopen sets F and G for which (9) fails, and therefore (7) fails, as claimed. Thus, ϑ isomorphically preserves the ternary relation R.

The proof that the mapping ϑ isomorphically preserves a unary relation R is somewhat different in character than the preceding argument. It is to be shown that the equivalence

$$R(t) \qquad \text{if and only if} \qquad R(X_t) \tag{13}$$

holds for all elements t in $\mathfrak{U}$, by the definition of ϑ in (2). (The first occurrence of R in (13) refers to the relation in $\mathfrak{U}$, while the second occurrence refers to the relation in $\mathfrak{V}$.) The point X_t belongs to the relation R in $\mathfrak{V}$ just in case the corresponding distinguished constant in $\mathfrak{A}$ belongs to X_t, by the definition of the relational space $\mathfrak{V}$ correlated with the algebra $\mathfrak{A}$. The corresponding distinguished constant in $\mathfrak{A}$ is just the relation R from $\mathfrak{U}$, by the definition of the complex algebra $\mathfrak{Cm}(U)$ and the fact that $\mathfrak{A}$ is a subalgebra of the complex algebra; and R belongs to X_t just in case t belongs to R, by (1). Summarizing, we have

$$\begin{aligned} X_t \in R \qquad &\text{if and only if} \qquad R \in X_t, \\ &\text{if and only if} \qquad t \in R \end{aligned}$$

(where the first occurrence of R refers to the unary relation, or set, in $\mathfrak{V}$, and the second and third to the unary relation, or set, in $\mathfrak{U}$). The equivalence of the first formula with the last is just another way of expressing (13). This completes the proof that the function ϑ is a homeo-isomorphism from the relational space $\mathfrak{U}$ to its second dual $\mathfrak{V}$. □

There is another version of the notion of a relational space that proves to be equivalent to the version given in Definition 2.2, namely (a slightly modified form of) the definition that is used in Goldblatt [13], pp. 184–185, restricted to Boolean algebras with operators. (An earlier version of Goldblatt's approach, for Boolean algebras with a single unary operator, is given in Section 10 of Goldblatt [12], and it is remarked that his construction is an adaptation of one given in Section III of Lemmon [24].)

Definition 2.12. A *weak relational space* is a relational structure $\mathfrak{U}$, together with a topology on the universe U, such that U is a Boolean space under the topology, the relations in $\mathfrak{U}$ are clopen, and the relations R in $\mathfrak{U}$ of rank $n+1$ (with $n \geq 1$) are *weakly continuous* in the sense that the sets

$$\{(r_0, \ldots, r_{n-1}) \in U^n : R(r_0, \ldots, r_{n-1}, t)\}$$

are closed (in the product space U^n) for every element t in $\mathfrak{U}$. □

It turns out that, on the basis of the remaining conditions in the two definitions, a relation in $\mathfrak{U}$ of rank at least two is continuous if and only if it is weakly continuous. One direction of this implication is not difficult to prove.

Theorem 2.13. *Every relational space* $\mathfrak{U}$ *is a weak relational space. In particular, every relation in* $\mathfrak{U}$ *of rank at least two is weakly continuous.*

Proof. Focus on the case of a ternary relation R in $\mathfrak{U}$ that is, by assumption, continuous. To show that R is weakly continuous, fix an element t in $\mathfrak{U}$. The singleton set $\{t\}$ is closed in the topology on $\mathfrak{U}$, because $\mathfrak{U}$ is a Hausdorff space and singleton subsets are always closed in a Hausdorff space. Consequently, the complement $\sim\{t\}$ is open. The inverse image under R of this complement, the set

$$\begin{aligned} R^{-1}(\sim\{t\}) &= \{(r,s) \in U \times U : R(r,s,\bar{t}) \text{ implies } \bar{t} \in \sim\{t\}\} \qquad (1) \\ &= \{(r,s) \in U \times U : R(r,s,\bar{t}) \text{ implies } \bar{t} \neq t\}, \end{aligned}$$

is therefore open in the product topology on $U \times U$, because R is continuous.

It must be shown that the set

$$H_t = \{(r,s) \in U \times U : R(r,s,t)\} \qquad (2)$$

is closed in U, or what amounts to the same thing, the complement $\sim H_t$ is open. This is accomplished by establishing the equality

$$\sim H_t = R^{-1}(\sim\{t\}). \qquad (3)$$

If a pair (r,s) belongs to the complement $\sim H_t$, then $R(r,s,t)$ must fail in $\mathfrak{U}$, by (2), and therefore $R(r,s,\bar{t})$ always implies that $\bar{t} \neq t$. Thus, the pair (r,s) belongs to the inverse image $R^{-1}(\sim\{t\})$, by (1), so the left side of (3) is included in the right side. As regards the reverse inclusion, if a pair (r,s) belongs to the inverse image $R^{-1}(\{\sim\{t\})$, then $R(r,s,\bar{t})$ always implies that $\bar{t} \neq t$, by (1), and therefore $R(r,s,t)$ cannot hold. It follows that the pair (r,s) cannot belong to H_t, by (2), so the pair must belong to the complement $\sim H_t$. This establishes (3). Since the set $R^{-1}(\sim\{t\})$ is open, by the remarks above, the set H_t must closed, by (3). This is true for every element t in $\mathfrak{U}$, so the relation R is weakly continuous. □

The proof of the converse assertion that every weak relational space is a relational space appears to be significantly more involved. We shall give an indirect proof that depends on several of the main results of this section. First of all, Lemma 2.9 remains true for every weak relational space $\mathfrak{U}$. Indeed, as has already been noted, the proof of the lemma does not make use of the assumption that the fundamental relations of rank at least two are continuous (or even weakly continuous). The subalgebra of $\mathfrak{Cm}(U)$ that has as its universe the set of clopen subsets of $\mathfrak{U}$ is therefore a Boolean algebra with operators; it is called the (*first*) *dual* of the weak relational space $\mathfrak{U}$. Write $\mathfrak{A}$ for this dual.

The algebra $\mathfrak{A}$ has a dual relational space $\mathfrak{V}$ (and not just a dual *weak* relational space), by Theorem 2.8. In particular, the fundamental relations of rank at least two in $\mathfrak{V}$ are continuous, by Lemma 2.7 applied to $\mathfrak{V}$. Call $\mathfrak{V}$ the *second dual* of the weak relational space $\mathfrak{U}$. The key step in the argument is to show that $\mathfrak{U}$ is homeo-isomorphic to $\mathfrak{V}$, that is to say, Theorem 2.11 continues to hold for weak relational spaces. The proof that the function ϑ defined in the statement of that theorem is a homeomorphism from the topological space U to the topological space V, and the proof that for a unary relation R,

$$R(t) \qquad \text{if and only if} \qquad R(X_t)$$

remain unchanged. Indeed, the assumption that the fundamental relations of rank at least two are continuous or weakly continuous is not used in those parts of the argument.

The proof that ϑ isomorphically preserves every fundamental relation R of rank $n+1 \geq 2$, and is therefore a homeo-isomorphism, is of course different from the earlier proof, since that proof uses the assumption that R is continuous. The proof below uses only the assumption that R is weakly continuous. (The argument is essentially the one given in Goldblatt [13], starting at the middle of p. 191.)

Consider the case of a ternary relation R. The equivalence

$$R(r, s, t) \qquad \text{if and only if} \qquad R(X_r, X_s, X_t) \tag{6}$$

must be established. As in the proof of Theorem 2.11, the right side of (6) holds in $\mathfrak{V}$ just in case the implication

$$r \in F \quad \text{and} \quad s \in G \qquad \text{implies} \qquad t \in F \circ G \tag{8}$$

is valid for all clopen subsets F and G of $\mathfrak{U}$, where

$$\begin{aligned} F \circ G &= R^*(F \times G) \\ &= \{w \in U : R(p, q, w) \text{ for some } p \in F \text{ and } q \in G\}. \end{aligned}$$

Assume first that the left side of (6) is true in $\mathfrak{U}$, and consider arbitrary clopen subsets F and G of $\mathfrak{U}$. If r is in F and s in G, then since $R(r,s,t)$ holds, the element t must belong to the set $F \circ G$, by the definition of this set, and therefore the implication in (8) holds. Thus, the right side of (6) is true in $\mathfrak{V}$.

Assume next that the left side of (6) is false in $\mathfrak{U}$. In this case, the pair (r,s) does not belong to the set

$$H_t = \{(p,q) \in U \times U : R(p,q,t)\},$$

so (r,s) must belong to the complement $\sim H_t$ (in $U \times U$). The set H_t is closed, by the assumed weak continuity of the relation R, so the complement $\sim H_t$ is open. The products of clopen subsets of $\mathfrak{U}$ form a base for the product topology on $U \times U$, so there must be clopen subsets F and G of $\mathfrak{U}$ such that

$$r \in F, \qquad s \in G, \qquad \text{and} \qquad F \times G \subseteq {\sim}H_t.$$

The pair of sets F and G therefore satisfies the hypothesis of (8), but not the conclusion. Indeed, if p and q are any elements in F and G respectively, then the pair (p,q) belongs to the set $\sim H_t$, and therefore $R(p,q,t)$ is false, by the definition of H_t; consequently, t cannot belong to the set $F \circ G$, by the definition of this set. The failure of (8) for the particular clopen sets F and G implies that the right side of (6) is false in $\mathfrak{V}$.

It has been shown that the function ϑ is a homeo-isomorphism from $\mathfrak{U}$ to $\mathfrak{V}$, so Theorem 2.11 continues to hold for weak relational spaces $\mathfrak{U}$ and their second duals $\mathfrak{V}$. As has already been pointed out, the second dual $\mathfrak{V}$ is a relational space, and not just a weak relational space. In particular, the fundamental relations of rank at least two in $\mathfrak{V}$ are continuous. Since $\mathfrak{U}$ is homeo-isomorphic to $\mathfrak{V}$, it follows that the fundamental relations of rank at least two in $\mathfrak{U}$ must also be continuous, so $\mathfrak{U}$ is in fact a relational space. The following theorem has been proved.

Theorem 2.14. *Every weak relational space* $\mathfrak{U}$ *is a relational space. In particular, the fundamental relations of rank at least two in* $\mathfrak{U}$ *are continuous.*

Goldblatt [13] shows for his notion of a relational space that every bounded distributive lattice with operators is isomorphic to its second dual, and every relational space is homeo-isomorphic to its second dual (see Theorems 2.2.3 and 2.2.4 in [13]). It follows from Theorems 2.13 and 2.14 above that for Boolean algebras with operators, the approach we have taken is equivalent to the approach that Goldblatt has taken.

There is one more topological observation concerning relational spaces that is worth making and that will be used later: the fundamental relations are closed subsets of the appropriate product space. For the unary relations, this follows directly from Definition 2.2, but for relations of rank at least two it requires proof.

Theorem 2.15. *If $\mathfrak{U}$ is a relational space, then every fundamental relation in $\mathfrak{U}$ of rank $n \geq 2$ is a closed subset of the product spaces U^n.*

Proof. Focus on the case when R is a ternary relation in $\mathfrak{U}$. In order to prove that R is a closed subset of $U \times U \times U$, it suffices to shown that the complement of R is open. This is accomplished by demonstrating that for every triple of elements (r, s, t) not in R, there are clopen subsets F, G, and H of $\mathfrak{U}$ such that

$$(r, s, t) \in F \times G \times H \qquad \text{and} \qquad F \times G \times H \subseteq {\sim}R. \tag{1}$$

Assume that the triple (r, s, t) is not in R. The set

$$K = \{(u, v) \in U \times U : R(u, v, t)\} \tag{2}$$

is closed in $U \times U$, by Theorem 2.13, so the complement of K is open; and this complement contains the pair (r, s), by assumption. The products of clopen subsets of $\mathfrak{U}$ form a base for the product topology on $U \times U$, so there must be clopen sets F and G in $\mathfrak{U}$ containing the points r and s respectively such that $F \times G$ is included in ${\sim}K$. The set

$$R^*(F \times G) = \{w \in U : R(u, v, w) \text{ for some } u \in F \text{ and } v \in G\} \tag{3}$$

is clopen in $\mathfrak{U}$, by the assumption that $\mathfrak{U}$ is a relational space (see Definition 2.2). The complement

$$\begin{aligned} H &= {\sim}R^*(F \times G) \\ &= \{w \in U : R(u, v, z) \text{ and } u \in F \text{ and } v \in G \text{ implies } z \neq w\} \end{aligned} \tag{4}$$

is therefore also clopen in $\mathfrak{U}$.

Observe that t belongs to H. For the proof, consider points u in F and v in G, and suppose that z is a point in $\mathfrak{U}$ such that $R(u, v, z)$ holds. Since $F \times G$ is included in ${\sim}K$, the pair (u, v) must belong to ${\sim}K$, and therefore the point z must be different from t, by (2). It follows by (4) that t must be in H. This observation and the definitions of the sets F and G imply that the triple (r, s, t) belongs to the product

$$F \times G \times H. \tag{5}$$

To check that (5) is included in $\sim R$, as is required in (1), consider any triple (u, v, w) in (5). The point w belongs to the set H, by the definition of the product (5), so w must belong to the complement of the set $R^*(F \times G)$, by (4). On the other hand, the points u and v are in F and G respectively, so $R(u, v, w)$ must fail to hold in $\mathfrak{U}$, by (3). This shows that the triple (u, v, w) belongs to the complement of R, which completes the proof of (1). The relation $\sim R$ is therefore an open subset of $U \times U \times U$, so R is closed. □

2.4 Duality for Homomorphisms

The duality between Boolean algebras with operators and relational spaces carries with it a duality between structure preserving functions on the algebras and structure preserving functions on the spaces. The structure preserving functions on the algebras are just homomorphisms (in the algebraic sense of the word), while the structure preserving functions on the spaces are continuous bounded homomorphisms (see Definition 1.8). Recall that a mapping ϑ from a topological space V to a topological space U is said to be *continuous* if the inverse image

$$\vartheta^{-1}(F) = \{u \in V : \vartheta(u) \in F\}$$

of each open subset F of U is open in V. When the spaces in question are Boolean, then it suffices to check that the inverse image of each clopen set is clopen (see Lemma 1 on p. 313 of [10]).

The first task is to show that continuous bounded homomorphisms give rise to homomorphisms. The proof is similar in flavor to the proof of Theorem 1.9. That theorem is purely algebraic in formulation and proof, with no reference to topologies on relational structures. The theorem we want refers also to the topologies and says that if the given bounded homomorphism ϑ is in fact a continuous mapping with respect to the topologies, then an appropriate restriction of the mapping defined in Theorem 1.9 is a homomorphism from the dual algebra of $\mathfrak{U}$ (the subalgebra of $\mathfrak{Cm}(U)$ consisting of the clopen subsets of $\mathfrak{U}$) to the dual algebra of $\mathfrak{V}$ (the subalgebra of $\mathfrak{Cm}(V)$ consisting of the clopen subsets of $\mathfrak{V}$).

Theorem 2.16. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be relational spaces, and $\mathfrak{A}$ and $\mathfrak{B}$ their respective dual algebras. If ϑ is a continuous bounded homomorphism from $\mathfrak{V}$ into $\mathfrak{U}$, then the function φ defined on elements F in $\mathfrak{A}$ by*

$$\varphi(F) = \vartheta^{-1}(F) = \{u \in V : \vartheta(u) \in F\}$$

is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Moreover, φ is one-to-one if and only if ϑ is onto, and φ is onto if and only if ϑ is one-to-one.

Proof. The function ψ defined on subsets X of U by

$$\psi(X) = \vartheta^{-1}(X) = \{u \in V : \vartheta(u) \in X\}$$

is a complete homomorphism from $\mathfrak{Cm}(U)$ to $\mathfrak{Cm}(V)$, by Theorem 1.9. It is clear from this definition that the function φ defined in the statement of the theorem is the restriction of ψ to the set of all clopen subsets of $\mathfrak{U}$. Consequently, φ is a homomorphism from the subalgebra $\mathfrak{A}$ of $\mathfrak{Cm}(U)$ consisting of the clopen subsets of $\mathfrak{U}$ (see Lemma 2.9) into $\mathfrak{Cm}(V)$. The bounded homomorphism ϑ is assumed to be continuous, and the inverse image of a clopen set under a continuous function is again a clopen set (see pp. 312–313 of [10]), so the inverse image $\vartheta^{-1}(F)$ of a clopen subset F of $\mathfrak{U}$ is always a clopen subset of $\mathfrak{V}$. Therefore, φ actually maps $\mathfrak{A}$ into the subalgebra $\mathfrak{B}$ of $\mathfrak{Cm}(V)$ that consists of the clopen subsets of $\mathfrak{V}$.

Consider now the following statements: (1) ϑ is onto; (2) $U \sim \vartheta(V)$ is empty; (3) every clopen subset of $U \sim \vartheta(V)$ is empty; (4) if F is a clopen subset of U such that $\vartheta^{-1}(F)$ is empty, then F must be empty; (5) φ is one-to-one. Each of these statements is equivalent to its neighbor. Indeed, (1) is obviously equivalent to (2), and (2) obviously implies (3). To see that (3) implies (2), observe that $\vartheta(V)$ is the continuous image of the compact set V, so $\vartheta(V)$ is compact and therefore closed in U (see Lemma 3 on p. 314 and Lemma 1 on p. 272 of [10]). It follows that the difference $U \sim \vartheta(V)$ is open, and is therefore the union of a system of clopen sets. If each of these clopen sets is empty, as is asserted in (3), then clearly their union must also be empty, and therefore (2) holds. Statement (4) is really just a rephrasing of (3): to say of a subset F of U that $\vartheta^{-1}(F)$ is empty is to say that F does not contain any elements in the range of ϑ, or what amounts to the same thing, that F is a subset of $U \sim \vartheta(V)$. As regards the equivalence of (4) and (5), recall that a

Boolean homomorphism is one-to-one if and only if its kernel contains only the zero element. The kernel of φ is the set of clopen subsets F of U such that $\vartheta^{-1}(F)$ is empty, so this kernel contains only the zero element $\varnothing$ just in case (4) holds. The equivalence of (1) and (5) proves that φ is one-to-one if and only if ϑ is onto.

To prove the dual assertion, consider the following statements: (1) ϑ is one-to-one; (2) the inverse images under ϑ of the clopen subsets of U separate points in V (in the sense that for any two distinct points, the inverse image of some clopen subset of U contains one of the two points but not the other); (3) every clopen subset of V is the inverse image under ϑ of some clopen subset of U; (4) φ is onto. Each of these statements is equivalent to its neighbor. To establish the equivalence of (1) and (2), consider distinct points u and v in V. If (1) holds, then $\vartheta(u)$ and $\vartheta(v)$ are distinct points in U. The clopen subsets of U separate points (because U is a Boolean space), so there is a clopen subset F of U that contains $\vartheta(u)$ but not $\vartheta(v)$. The inverse image $\vartheta^{-1}(F)$ is a clopen subset of V (because ϑ is continuous), and it contains u but not v, so (2) holds. Conversely, if (2) holds, then there is a clopen subset F of U such that $\vartheta^{-1}(F)$ contains u but not v. Consequently, the set F contains $\vartheta(u)$ but not $\vartheta(v)$, so $\vartheta(u)$ and $\vartheta(v)$ must be distinct. Thus, (1) holds. To see that (2) implies (3), observe that the inverse images under ϑ of the clopen subsets of U constitute a Boolean algebra of clopen subsets of V under the operations of union and complement. In more detail, if F and G are clopen subsets of U, then so are $F \cup G$ and $U \sim F$; since

$$\vartheta^{-1}(F) \cup \vartheta^{-1}(G) = \vartheta(F \cup G) \quad \text{and} \quad \vartheta^{-1}(U \sim F) = V \sim \vartheta^{-1}(F),$$

it follows that the set of inverse images of clopen sets is closed under the operations of union and complement, and is therefore a Boolean algebra under these operations. This must be the Boolean algebra of all clopen subsets of V, by (2) and the fact that in a compact space, a Boolean algebra of clopen sets that separates points must be the Boolean algebra of all clopen subsets of the space (see Lemma 1 on p. 305 of [10]). The reverse implication from (3) to (2) follows from the fact that V is a Boolean space, and therefore the clopen subsets of V separate points. Finally, the equivalence of (3) and (4) follows from the definition of φ. The equivalence of (1) and (4) proves that ϑ is one-to-one if and only if φ is onto. □

Goldblatt [13] shows (at the bottom of p. 193) that the function φ in the statement of Theorem 2.16 is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. He also points out that if ϑ is onto or one-to-one, then φ is one-to-one or onto respectively. (An earlier version of Goldblatt's result, for spaces with a single binary relation, is given in Theorem 5.9 of Goldblatt [12].)

It is also possible to formulate a version of Theorem 2.16 that refers not to relational spaces and their dual algebras, but rather to Boolean algebras with operators and their dual spaces.

Corollary 2.17. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with operators, and $\mathfrak{U}$ and $\mathfrak{V}$ their respective dual spaces. If ϑ is a continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{U}$, then the function ψ defined on elements r in $\mathfrak{A}$ by*

$$\psi(r) = s \qquad \textit{if and only if} \qquad \vartheta^{-1}(F_r) = G_s,$$

where

$$F_r = \{X \in U : r \in X\} \qquad \textit{and} \qquad G_s = \{Y \in V : s \in Y\},$$

is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. Moreover, ψ is one-to-one if and only if ϑ is onto, and ψ is onto if and only if ϑ is one-to-one.

Proof. Let $\mathfrak{A}^*$ and $\mathfrak{B}^*$ be the dual algebras of the dual relational spaces $\mathfrak{U}$ and $\mathfrak{V}$ respectively. Thus, $\mathfrak{A}^*$ is the second dual of $\mathfrak{A}$, and $\mathfrak{B}^*$ is the second dual of $\mathfrak{B}$, by the definition of the second duals. The functions φ_1 and φ_2 defined by

$$\varphi_1(r) = F_r \qquad \text{and} \qquad \varphi_2(s) = G_s \tag{1}$$

for r in $\mathfrak{A}$ and s in $\mathfrak{B}$ are isomorphisms from $\mathfrak{A}$ to $\mathfrak{A}^*$ and from $\mathfrak{B}$ to $\mathfrak{B}^*$ respectively, by Theorem 2.10.

Assume that ϑ is a continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{U}$. The function φ defined by

$$\varphi(F) = \vartheta^{-1}(F) = \{Y \in V : \vartheta(Y) \in F\} \tag{2}$$

is a homomorphism from $\mathfrak{A}^*$ to $\mathfrak{B}^*$, by the first part of Theorem 2.16 (with $\mathfrak{A}^*$ and $\mathfrak{B}^*$ in place of $\mathfrak{A}$ and $\mathfrak{B}$, and Y in place of u). Moreover, φ is one-to-one or onto if and only if ϑ is onto or one-to-one respectively, by the second part of Theorem 2.16. The composition

$$\psi = \varphi_2^{-1} \circ \varphi \circ \varphi_1 \tag{3}$$

is therefore a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (see the diagram below). Moreover, since φ_1 and φ_2 are bijections, the homomorphism ψ is one-to-one or onto if and only the homomorphism φ is one-to-one or onto respectively. Consequently, ψ is one-to-one or onto if and only ϑ is onto or one-to-one respectively.

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\psi} & \mathfrak{B} \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\varphi_2} \\ \mathfrak{A}^* & \xrightarrow{\varphi} & \mathfrak{B}^* \end{array}$$

If $\vartheta^{-1}(F_r) = G_s$, then $\varphi(F_r) = G_s$, by (2), and therefore

$$\begin{aligned} \psi(r) = (\varphi_2^{-1} \circ \varphi \circ \varphi_1)(r) &= \varphi_2^{-1}(\varphi(\varphi_1(r))) \\ &= \varphi_2^{-1}(\varphi(F_r)) = \varphi_2^{-1}(G_s) = s, \end{aligned}$$

by (3) and (1). On the other hand, if $\psi(r) = s$, then

$$\begin{aligned} \varphi(F_r) = (\varphi_2 \circ \psi \circ \varphi_1^{-1})(F_r) &= \varphi_2(\psi(\varphi_1^{-1}(F_r))) \\ &= \varphi_2(\psi(r)) = \varphi_2(s) = G_s, \end{aligned}$$

by (3) and (1), and therefore $\vartheta^{-1}(F_r) = G_s$, by (2). Thus, the homomorphism ψ is determined by the equivalence stated in the corollary. $\square$

The function φ in Theorem 2.16 is called the (*first*) *dual*, or the *dual homomorphism*, of the continuous bounded homomorphism ϑ. The equation defining this dual in the statement of the theorem can be reformulated as an equivalence, namely

$$u \in \varphi(F) \qquad \text{if and only if} \qquad u \in \vartheta^{-1}(F)$$

for all elements u in $\mathfrak{V}$ and F in $\mathfrak{A}$. This equivalence, in turn, may be written in the form

$$u \in \varphi(F) \qquad \text{if and only if} \qquad \vartheta(u) \in F$$

for all elements u in $\mathfrak{V}$ and F in $\mathfrak{A}$. Thus, the dual of ϑ is the function φ from $\mathfrak{A}$ to $\mathfrak{B}$ that is determined by the preceding equivalence.

Theorem 2.16 says that every continuous bounded homomorphism between relational spaces determines a dual homomorphism between the dual algebras of the spaces. The converse is also true: every homomorphism between the dual algebras induces a continuous bounded homomorphism between the relational spaces. We begin by proving the corresponding statement about Boolean algebras with operators and their dual spaces.

Theorem 2.18. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with operators, and $\mathfrak{U}$ and $\mathfrak{V}$ their respective dual spaces. If φ is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, then the function ϑ defined on elements Y in $\mathfrak{V}$ by*

$$\vartheta(Y) = \varphi^{-1}(Y) = \{r \in A : \varphi(r) \in Y\}$$

is a continuous bounded homomorphism from $\mathfrak{V}$ into $\mathfrak{U}$. Moreover, ϑ is one-to-one if and only if φ is onto, and ϑ is onto if and only if φ is one-to-one.

Proof. The first task is to show that the function ϑ defined in the statement of the theorem really does map the universe of $\mathfrak{V}$ into the universe of $\mathfrak{U}$. The elements in $\mathfrak{U}$ and in $\mathfrak{V}$ are, by definition, the ultrafilters in $\mathfrak{A}$ and in $\mathfrak{B}$ respectively. The Boolean homomorphism properties of φ imply that if Y is an ultrafilter in $\mathfrak{B}$, then the inverse image of Y under φ, that is to say, the set

$$X = \varphi^{-1}(Y) = \{r \in A : \varphi(r) \in Y\}, \tag{1}$$

is an ultrafilter in $\mathfrak{A}$, and consequently ϑ does map the set V into the set U. In more detail, if r and s belong to X, then the images $\varphi(r)$ and $\varphi(s)$ belong to Y, by (1), and therefore so does the product of these two images. Since

$$\varphi(r \cdot s) = \varphi(r) \cdot \varphi(s),$$

by the homomorphism properties of φ, it follows from (1) that $r \cdot s$ belongs to X. A similar argument shows that if r is in X, and $r \leq s$, then s is in X. If r is not in X, then the image $\varphi(r)$ cannot be in Y, by (1). In this case, $-\varphi(r)$ must be in Y, because Y is an ultrafilter. Since

$$\varphi(-r) = -\varphi(r),$$

by the homomorphism properties of φ, it follows from (1) that $-r$ is in X. Finally, 0 cannot be in X, by (1), because $\varphi(0) = 0$, and 0 is not in Y. Thus, X is an ultrafilter in $\mathfrak{A}$.

The next step is to show that ϑ is a bounded homomorphism. Focus on the case of a ternary relation R that is defined in terms of a binary operator $\circ$. Suppose Y_1, Y_2, and Y_3 are elements in $\mathfrak{V}$ such that

$$R(Y_1, Y_2, Y_3) \tag{2}$$

holds in $\mathfrak{V}$, with the aim of proving that

$$R(\vartheta(Y_1), \vartheta(Y_2), \vartheta(Y_3)) \tag{3}$$

holds in $\mathfrak{U}$. In view of the definition of the relation R in the dual spaces $\mathfrak{V}$ and $\mathfrak{U}$, the hypothesis in (2) is equivalent to the inclusion

$$Y_1 \circ Y_2 \subseteq Y_3, \tag{4}$$

(where the operation on the left side of this inclusion is the complex operation induced on subsets of $\mathfrak{B}$ by the operation $\circ$ in $\mathfrak{B}$), and the proof of (3) amounts to showing that

$$\vartheta(Y_1) \circ \vartheta(Y_2) \subseteq \vartheta(Y_3),$$

or, equivalently, that

$$\varphi^{-1}(Y_1) \circ \varphi^{-1}(Y_2) \subseteq \varphi^{-1}(Y_3) \tag{5}$$

(where the operation on the left sides of these two inclusions is the complex operation induced on subsets of $\mathfrak{A}$ by the operation $\circ$ in $\mathfrak{A}$). To prove (5), consider elements r in $\varphi^{-1}(Y_1)$ and s in $\varphi^{-1}(Y_2)$. The images $\varphi(r)$ and $\varphi(s)$ belong to the sets Y_1 and Y_2 respectively, so the product $\varphi(r) \circ \varphi(s)$ of the two images belongs to the complex product $Y_1 \circ Y_2$, and therefore also to Y_3, by (4). Since

$$\varphi(r \circ s) = \varphi(r) \circ \varphi(s),$$

by the homomorphism properties of φ, it follows that $\varphi(r \circ s)$ is in Y_3, and consequently that $r \circ s$ is in $\varphi^{-1}(Y_3)$, as was to be shown. This completes the proof of (5), and hence also of the implication from (2) to (3).

In order to show that ϑ is also bounded, consider elements X_1 and X_2 in $\mathfrak{U}$, and Y_3 in $\mathfrak{V}$ such that

$$R(X_1, X_2, \vartheta(Y_3)) \tag{6}$$

in $\mathfrak{U}$. Thus,

$$X_1 \circ X_1 \subseteq \vartheta(Y_3) = \varphi^{-1}(Y_3), \tag{7}$$

by the definition of the relation R in $\mathfrak{U}$, and the definition of ϑ. (The operation on the left side in (7) is the complex operation induced on subsets of $\mathfrak{A}$ by the operation $\circ$ in $\mathfrak{A}$.) Elements Y_1 and Y_2 in $\mathfrak{V}$ are to be constructed such that

$$\vartheta(Y_1) = X_1, \qquad \vartheta(Y_2) = X_2, \qquad \text{and} \qquad R(Y_1, Y_2, Y_3). \tag{8}$$

The construction proceeds stepwise, and the first step is to obtain an ultrafilter Y_1 in $\mathfrak{B}$ with the properties

$$X_1 = \varphi^{-1}(Y_1) \qquad \text{and} \qquad Y_1 \circ \varphi(X_2) \subseteq Y_3. \tag{9}$$

It is clear from the right-hand inclusion in (9) that we must exclude from Y_1 all elements u in $\mathfrak{B}$ with the property that $u \circ \varphi(r)$ is not in Y_3 for some r in X_2. To this end, write

$$W_1 = \{u \in B : u \circ \varphi(r) \notin Y_3 \text{ for some } r \in X_2\}, \tag{10}$$

and observe that W_1 is closed under addition. Indeed, let u and v be elements in W_1, and suppose r and s are elements in X_2 such that $u \circ \varphi(r)$ and $v \circ \varphi(s)$ are not in Y_3. Put $t = r \cdot s$, and observe that t also belongs to X_2, since X_2 is an ultrafilter. Moreover,

$$u \circ \varphi(t) \le u \circ \varphi(r) \qquad \text{and} \qquad v \circ \varphi(t) \le v \circ \varphi(s),$$

by the homomorphism properties of φ and the monotony of the operator $\circ$. Neither of the terms on the right sides of these inequalities is in Y_3, by assumption, so neither of the terms on the left sides of the inequalities can be in Y_3, by the upward closure of Y_3. It follows that the sum $u \circ \varphi(t) + v \circ \varphi(t)$ also cannot be in Y_3, because Y_3 is an ultrafilter. Since

$$u \circ \varphi(t) + v \circ \varphi(t) = (u + v) \circ \varphi(t),$$

we arrive at the conclusion that $(u + v) \circ \varphi(t)$ cannot be in Y_3, and therefore $u + v$ is in W_1, as claimed. As a consequence of this observation, the set of complements of elements in W_1, that is to say, the set

$$-W_1 = \{-u : u \in W_1\},$$

must be closed under multiplication.

Write $\varphi(X_1)$ for the image of the set X_1 under φ, so that

$$\varphi(X_1) = \{\varphi(r) : r \in X_1\}.$$

We proceed to show that the set

$$\varphi(X_1) \cup -W_1 \tag{11}$$

has the finite meet property. The sets X_1 and $-W_1$ are closed under multiplication, and φ preserves this operation, so it suffices to show that there can be no elements r in X_1 and u in W_1 such that

$$\varphi(r) \cdot -u = 0.$$

Assume, to the contrary, that such elements r and u exist. It follows that $\varphi(r) \le u$. Since u is in W_1, there must be an element s in X_2 such that $u \circ \varphi(s)$ is not in Y_3, by (10). But then the product $\varphi(r) \circ \varphi(s)$ cannot be in Y_3, because

$$\varphi(r) \circ \varphi(s) \le u \circ \varphi(s),$$

by the monotony of the operator $\circ$, and the presence of $\varphi(r) \circ \varphi(s)$ in Y_3 would imply that of $u \circ \varphi(s)$, by the upward closure of Y_3. Since

$$\varphi(r \circ s) = \varphi(r) \circ \varphi(s),$$

by the homomorphism properties of φ, it may be concluded that $\varphi(r \circ s)$ is not in Y_3, and therefore that $r \circ s$ is not in $\varphi^{-1}(Y_3)$. However, r is in X_1, and s is in X_2, so $r \circ s$ belongs to the complex product $X_1 \circ X_2$ and therefore also to the inverse image $\varphi^{-1}(Y_3)$, by (7). The desired contradiction has arrived. Conclusion: the set in (11) has the finite meet property.

Every set that has the finite meet property can be extended to an ultrafilter, so there must be an ultrafilter Y_1 in $\mathfrak{B}$ that includes the set in (11). In particular, $\varphi(X_1)$ is included in Y_1, so X_1 is included in the inverse image $\varphi^{-1}(Y_1)$. As both of these last two sets are ultrafilters in $\mathfrak{A}$, it follows that $X_1 = \varphi^{-1}(Y_1)$. The set Y_1 is a proper filter, so no element and its complement can simultaneously belong to Y_1. Since $-W_1$ is included in Y_1, it follows that Y_1 must be disjoint from W_1. This means that if u is in Y_1, then $u \circ \varphi(r)$ belongs to Y_3 for every r in X_2, by (10). Thus, Y_1 possesses the requisite properties stated in (9).

The second step of the construction is to obtain an ultrafilter Y_2 in $\mathfrak{B}$ with the properties

$$X_2 = \varphi^{-1}(Y_2) \qquad \text{and} \qquad Y_1 \circ Y_2 \subseteq Y_3. \tag{12}$$

The argument is similar to the preceding one: we want $\varphi(X_2)$ to be included in Y_2, and we want to exclude from Y_2 all elements v in $\mathfrak{B}$ with the property that $u \circ v$ is not in Y_3 for some u in Y_1. To this end, write

$$W_2 = \{v \in B : u \circ v \notin Y_3 \text{ for some } u \in Y_1\}. \tag{13}$$

Just as before, one proves that W_2 is closed under addition, and concludes that the set of complements,

$$-W_2 = \{-v : v \in W_2\},$$

is closed under multiplication.

Write $\varphi(X_2)$ for the image of the set X_2 under the mapping φ. We proceed to show that the set

$$\varphi(X_2) \cup -W_2 \tag{14}$$

has the finite meet property. The sets X_2 and $-W_2$ are closed under multiplication, and φ preserves this operation, so it suffices to show, as before, that there are no elements s in X_2 and v in W_2 such that

$$\varphi(s) \cdot -v = 0.$$

Assume, to the contrary, that such elements s and v exist. It follows that $\varphi(s) \leq v$. Since v is in W_2, there must be an element u in Y_1 such that $u \circ v$ is not in Y_3, by (13). Consequently, the product $u \circ \varphi(s)$ cannot be in Y_3. In more detail,

$$u \circ \varphi(s) \leq u \circ v,$$

by the monotony of the operator $\circ$, so the presence of $u \circ \varphi(s)$ in Y_3 would imply that of $u \circ v$, by the upward closure of Y_3, in contradiction to the assumption that $u \circ v$ is not in Y_3. The failure of $u \circ \varphi(s)$ to be in Y_3 contradicts the right-hand inclusion in (9). Conclusion: the set in (14) has the finite meet property.

It follows from the preceding conclusion that there is an ultrafilter Y_2 in $\mathfrak{B}$ that includes the set in (14). As $\varphi(X_2)$ is included in Y_2, the set X_2 is included in the inverse image $\varphi^{-1}(Y_2)$. These last two sets are ultrafilters, so $X_2 = \varphi^{-1}(Y_2)$. Also, $-W_2$ is included in Y_2, so W_2 is disjoint from Y_2. This means that if v is in Y_2, then $u \circ v$ belongs to Y_3 for every u in Y_1, by (13). Thus, Y_2 possesses the properties required in (12).

The definition of ϑ, and the left-hand equations in (9) and (12), imply the first two equations in (8). The definition of R in $\mathfrak{V}$ and the right-hand inclusion in (12) imply the last part of (8). Thus, ϑ is a bounded homomorphism, as claimed.

Turn now to the task of showing that the mapping ϑ is continuous. The clopen sets in $\mathfrak{U}$ and in $\mathfrak{V}$ are respectively the sets of the form

$$F_r = \{X \in U : r \in X\} \qquad \text{and} \qquad G_u = \{Y \in V : u \in Y\}$$

for elements r in $\mathfrak{A}$ and u in $\mathfrak{B}$, because $\mathfrak{U}$ and $\mathfrak{V}$ are assumed to be the dual spaces of the algebras $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Every open set in $\mathfrak{U}$ is a union of clopen subsets, and the operation of forming inverse images of sets under a function preserves arbitrary unions. Consequently, it suffices to show that the inverse image under ϑ of every clopen set in $\mathfrak{U}$ is a clopen set in $\mathfrak{V}$. In fact, if F_r is a clopen subset of $\mathfrak{U}$, then

$$\begin{aligned}\vartheta^{-1}(F_r) &= \{Y \in V : \vartheta(Y) \in F_r\} = \{Y \in V : r \in \vartheta(Y)\}\\ &= \{Y \in V : r \in \varphi^{-1}(Y)\} \ = \{Y \in V : \varphi(r) \in Y\} = G_{\varphi(r)},\end{aligned} \tag{15}$$

by the definition of the inverse image under ϑ of a set, the definition of F_r, the definition of the function ϑ, the definition of the inverse image under φ of a set, and the definition of $G_{\varphi(r)}$. The set $G_{\varphi(r)}$ is clopen, so ϑ is continuous.

The argument that ϑ is one-to-one if and only if φ is onto is similar to the corresponding argument given in the proof of Theorem 2.16. Statements (1)–(3) from that argument, and the proof of their equivalence, remain unchanged. The argument continues by establishing the equivalence of the following statements: (3) every clopen subset of V is the inverse image under ϑ of some clopen subset of U; (4) for every element u in $\mathfrak{B}$ there is an element r in $\mathfrak{A}$ such that $G_u = \vartheta^{-1}(F_r)$; (5) for every element u in $\mathfrak{B}$ there is an element r in $\mathfrak{A}$ such that $G_u = G_{\varphi(r)}$; (6) φ is onto. The equivalence of (3) and (4) is obvious from the description of the clopen subsets of U and V given above; the equivalence of (4) and (5) follows from (15); and the equivalence of (5) and (6) is a consequence of the fact that the correspondence mapping each element u in $\mathfrak{B}$ to the clopen set G_u is a bijection from $\mathfrak{B}$ to the set of clopen subsets of $\mathfrak{V}$. Conclusion: ϑ is one-to-one if and only if φ is onto.

The argument that ϑ is onto if and only if φ is one-to-one is also similar to the corresponding argument given in the proof of Theorem 2.16.

Statements (1)–(4) from that argument, and the proof of their equivalence, remain unchanged. The argument continues by establishing the equivalence of the following statements, where we use (15) to write $G_{\varphi(r)}$ for the inverse image $\vartheta^{-1}(F_r)$: (4) if F is a clopen subset of U such that $\vartheta^{-1}(F)$ is empty, then F is empty; (5) if $G_{\varphi(r)}$ is empty, then F_r is empty; (6) if $\varphi(r) = 0$, then $r = 0$; (7) φ is one-to-one. Statement (5) is just a rephrasing of assertion (4) using the introduced notation. The equivalence of (5) and (6) follows from the fact that the only clopen subsets of U and V that are empty are respectively the sets F_r and G_u with $r = 0$ and $u = 0$, by the monomorphism properties of the canonical embeddings. The equivalence of (6) and (7) is just the assertion that a Boolean homomorphism is one-to-one if and only if its kernel contains only the zero element. Conclusion: ϑ is onto if and only if φ is one-to-one. □

The assertion in Theorem 2.18 that, under the hypotheses of the theorem, the mapping ϑ is a continuous bounded homomorphism, and the proof of this assertion, are due to Goldblatt [13] (see Theorem 2.3.2 and its proof in [13]). He also points out that if φ is onto or one-to-one, then ϑ is one-to-one or onto respectively. (An earlier version of Goldblatt's result, for Boolean algebras with a single unary operator, is given in Theorem 10.9 of Goldblatt [12].)

Theorem 2.18 also has a version that refers not to Boolean algebras with operators and their dual spaces, but rather to relational spaces and their dual algebras.

Corollary 2.19. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be relational spaces, and $\mathfrak{A}$ and $\mathfrak{B}$ their respective dual algebras. If φ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then the function δ defined on elements s in $\mathfrak{V}$ by*

$$\delta(s) = r \qquad \textit{if and only if} \qquad \varphi^{-1}(Y_s) = X_r,$$

where

$$X_r = \{F \in A : r \in F\} \qquad \textit{and} \qquad Y_s = \{G \in B : s \in G\},$$

is a continuous bounded homomorphism from $\mathfrak{V}$ into $\mathfrak{U}$. Moreover, δ is one-to-one if and only if φ is onto, and δ is onto if and only if φ is one-to-one.

Proof. Let $\mathfrak{U}^*$ and $\mathfrak{V}^*$ be the dual spaces of the dual algebras $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Thus, $\mathfrak{U}^*$ is the second dual of $\mathfrak{U}$, and $\mathfrak{V}^*$ the second dual of $\mathfrak{V}$, by the definition of the second duals. The functions ϑ_1 and ϑ_2 defined by

$$\vartheta_1(r) = X_r \qquad \text{and} \qquad \vartheta_2(s) = Y_s \tag{1}$$

for r in $\mathfrak{U}$ and s in $\mathfrak{V}$ are homeo-isomorphisms from $\mathfrak{U}$ to $\mathfrak{U}^*$ and from $\mathfrak{V}$ to $\mathfrak{V}^*$ respectively, by Theorem 2.11.

Assume that φ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. The function ϑ defined on elements Y in $\mathfrak{V}^*$ by

$$\vartheta(Y) = \varphi^{-1}(Y) = \{F \in A : \varphi(F) \in Y\} \tag{2}$$

is a continuous bounded homomorphism from $\mathfrak{V}^*$ to $\mathfrak{U}^*$, by the first part of Theorem 2.18 (with $\mathfrak{U}^*$ and $\mathfrak{V}^*$ in place of $\mathfrak{U}$ and $\mathfrak{V}$, and F in place of r). Moreover, ϑ is one-to-one or onto if and only if φ is onto or one-to-one respectively, by the second part of Theorem 2.18. The composition

$$\delta = \vartheta_1^{-1} \circ \vartheta \circ \vartheta_2 \tag{3}$$

is therefore a continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{U}$ (see the diagram below). Moreover, since ϑ_1 and ϑ_2 are bijections, the mapping δ is one-to-one or onto if and only the mapping ϑ is one-to-one or onto respectively. Consequently, δ is one-to-one or onto if and only φ is onto or one-to-one respectively.

$$\begin{tikzcd}
\mathfrak{V} \arrow[r, "\delta"] \arrow[d, "\vartheta_2"'] & \mathfrak{U} \arrow[d, "\vartheta_1"] \\
\mathfrak{V}^* \arrow[r, "\vartheta"] & \mathfrak{U}^*
\end{tikzcd}$$

If $\varphi^{-1}(Y_s) = X_r$, then $\vartheta(Y_s) = X_r$, by (2), and therefore

$$\begin{aligned}
\delta(s) = (\vartheta_1^{-1} \circ \vartheta \circ \vartheta_2)(s) &= \vartheta_1^{-1}(\vartheta(\vartheta_2(s))) \\
&= \vartheta_1^{-1}(\vartheta(Y_s)) = \vartheta_1^{-1}(X_r) = r,
\end{aligned}$$

by (3) and (1). On the other hand, if $\delta(s) = r$, then

$$\begin{aligned}
\vartheta(Y_s) = (\vartheta_1 \circ \delta \circ \vartheta_2^{-1})(Y_s) &= \vartheta_1(\delta(\vartheta_2^{-1}(Y_s))) \\
&= \vartheta_1(\delta(s)) = \vartheta_1(r) = X_r,
\end{aligned}$$

by (3) and (1), and therefore $\varphi^{-1}(Y_s) = X_r$, by (2). Thus, the mapping δ is determined by the equivalence stated in the corollary. $\square$

The function δ in Corollary 2.19 is called the (*first*) *dual*, or the *dual continuous bounded homomorphism*, of the homomorphism φ. The equivalence defining this dual in the statement of the corollary can be formulated in a different way. The right side of this equivalence, namely the equation

$$\varphi^{-1}(Y_s) = X_r,$$

expresses that

$$F \in X_r \qquad \text{if and only if} \qquad \varphi(F) \in Y_s$$

for every element F in $\mathfrak{A}$ (that is to say, for every clopen subset F of $\mathfrak{U}$), or equivalently that

$$r \in F \qquad \text{if and only if} \qquad s \in \varphi(F)$$

for every element F in $\mathfrak{A}$, by the definitions of the ultrafilters X_r and Y_s, and the definition of the inverse image $\varphi^{-1}(Y_s)$. Consequently the function δ defined in the corollary is completely determined by the equivalence

$$\delta(s) \in F \qquad \text{if and only if} \qquad s \in \varphi(F)$$

for all elements s in $\mathfrak{V}$ and F in $\mathfrak{A}$. This equivalence may, in turn, be reformulated as

$$s \in \delta^{-1}(F) \qquad \text{if and only if} \qquad s \in \varphi(F)$$

for all elements s in $\mathfrak{V}$ and F in $\mathfrak{A}$. This last equivalence simply says that the dual δ is determined by the validity of the equation

$$\varphi(F) = \delta^{-1}(F)$$

for every F in $\mathfrak{A}$. This equation is of course the definition of φ in terms of δ, by Theorem 2.16. The point is that the equation is also valid in the context of Corollary 2.19, where we are defining δ in terms of φ. Notice in passing that in view of the second equivalence above, the definition of δ in Corollary 2.19 may be written in the form

$$\delta(s) = r \qquad \text{if and only if} \qquad r \in \bigcap\{F : s \in \varphi(F)\}.$$

We continue with the assumptions of the corollary, namely that $\mathfrak{U}$ and $\mathfrak{V}$ are relational spaces with dual algebras $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

If φ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and δ the dual continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{U}$, then the dual of δ is called the *second dual* of φ. This second dual is a homomorphism φ^* from $\mathfrak{A}$ to $\mathfrak{B}$ that is defined on elements F in $\mathfrak{A}$ by

$$\varphi^*(F) = \delta^{-1}(F),$$

by Theorem 2.16 (with δ and φ^* in place of ϑ and φ respectively). Comparing this definition with the conclusion of the last paragraph, we see that

$$\varphi^*(F) = \delta^{-1}(F) = \varphi(F).$$

In other words, φ is its own second dual. Similarly, if ϑ is a continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{U}$, and φ the dual homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then the dual of φ is called the *second dual* of ϑ. This second dual is a function ϑ^* from $\mathfrak{V}$ to $\mathfrak{U}$ that is determined by the equivalence

$$\vartheta^*(s) \in F \qquad \text{if and only if} \qquad s \in \varphi(F)$$

for all elements s in $\mathfrak{V}$ and F in $\mathfrak{A}$, by the remarks of the preceding paragraph (with ϑ^* in place of δ). On the other hand, since φ is the dual of ϑ, we have

$$s \in \varphi(F) \qquad \text{if and only if} \qquad \vartheta(s) \in F$$

for all s in $\mathfrak{V}$ and F in $\mathfrak{A}$, by the remarks following Corollary 2.17 (with s in place of u). Thus,

$$\vartheta^*(s) \in F \qquad \text{if and only if} \qquad \vartheta(s) \in F$$

for all s in $\mathfrak{V}$ and F in $\mathfrak{A}$, that is to say, for all points s in $\mathfrak{V}$ and all clopen subsets F of $\mathfrak{U}$. Fix the point s for a moment. The space $\mathfrak{U}$ is Boolean, so distinct points are separated by clopen sets. It follows that the image points $\vartheta^*(s)$ and $\vartheta(s)$ must be the same, for otherwise they would be separated by a clopen subset F of $\mathfrak{U}$. Since this is true for every element s in $\mathfrak{V}$, the function ϑ coincides with its own second dual.

Suppose next that $\mathfrak{U}$, $\mathfrak{V}$, and $\mathfrak{W}$ are relational spaces with dual algebras $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ respectively. Let δ be a continuous bounded homomorphism from $\mathfrak{W}$ to $\mathfrak{V}$, and ϑ a continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{U}$. It is easy to check that the composition $\vartheta \circ \delta$ is a continuous bounded homomorphism from $\mathfrak{W}$ to $\mathfrak{U}$. The dual of δ is the homomorphism ϱ from $\mathfrak{B}$ to $\mathfrak{C}$ defined by

$$\varrho(G) = \delta^{-1}(G)$$

for elements G in $\mathfrak{B}$, and the dual of ϑ is the homomorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$ defined by

$$\varphi(F) = \vartheta^{-1}(F)$$

for elements F in $\mathfrak{A}$, by Theorem 2.16 and the definition of the dual of a continuous bounded homomorphism. The composition $\varrho \circ \varphi$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$, and

$$(\varrho \circ \varphi)(F) = \varrho(\varphi(F)) = \delta^{-1}(\vartheta^{-1}(F)) = (\vartheta \circ \delta)^{-1}(F)$$

for all F in $\mathfrak{A}$. Consequently, the composition $\varrho \circ \varphi$ is the dual of the composition $\vartheta \circ \delta$, by the definition of that dual.

The next theorem summarizes the results of this section.

Theorem 2.20. *Let $\mathfrak{U}$, $\mathfrak{V}$, and $\mathfrak{W}$ be relational spaces, and $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ their respective dual algebras. There is a bijective correspondence between the set of continuous bound homomorphisms ϑ from $\mathfrak{V}$ to $\mathfrak{U}$ and the set of homomorphisms φ from $\mathfrak{A}$ to $\mathfrak{B}$ such that the equivalence*

$$u \in \varphi(F) \qquad \textit{if and only if} \qquad \vartheta(u) \in F$$

holds for all sets F in $\mathfrak{A}$ and all elements u in $\mathfrak{V}$. Each of the mappings ϑ and φ is its own second dual. The continuous bounded homomorphism ϑ is one-to-one if and only if its dual homomorphism φ is onto, and ϑ is onto if and only if φ is one-to-one. If δ is a continuous bounded homomorphism from $\mathfrak{W}$ to $\mathfrak{V}$, with dual ϱ, then the dual of the composition $\vartheta \circ \delta$ is the composition $\varrho \circ \varphi$.

Part of the contents of the preceding theorem may be expressed by saying that the correspondence taking each relational space to its dual algebra, and each continuous bounded homomorphism to its dual homomorphism, is a contravariant functor from the category of all relational spaces with continuous bounded homomorphisms as morphisms to the category of all Boolean algebras with operators with homomorphisms as morphisms; and the correspondence taking each Boolean algebra with operators to its dual space, and each homomorphism to its dual continuous bounded homomorphism, is a contravariant functor from the category of all Boolean algebra with operators with homomorphisms as morphisms to the category of all relational spaces with continuous bounded homomorphisms as morphisms; and these two contravariant functors are inverses of one another. Consequently, the two categories are dually equivalent. This last statement is contained in Theorem 2.3.3 of Goldblatt [13]; an earlier version of his theorem, applicable to the categories of modal algebras and descriptive modal frames, is given at the end of Section 10 in Goldblatt [12].

2.5 Duality for Ideals

The duality between homomorphisms and continuous bounded homomorphisms implies a duality between ideals (see Definition 1.15) and open sets: every ideal has a dual open set with a special property, and every open set with this special property has a dual ideal. The special property that the open sets possess is set forth in the next definition.

Definition 2.21. An open subset H of a relational space $\mathfrak{U}$ is called *special* if for every relation R in $\mathfrak{U}$ of rank $n+1$ (with $n \geq 1$), and every sequence of clopen subsets $F_0, \ldots, F_{n-1}$ of $\mathfrak{U}$, if F_i is included in H for some $i < n$, then the clopen image set

$$R^*(F_0 \times \cdots \times F_{n-1}) = \{t \in U : R(r_0, \ldots, r_{n-1}, t) \text{ for some } r_0 \in F_0, \ldots, r_{n-1} \in F_{n-1}\}$$

is included in H. $\square$

For example, if $\mathfrak{U}$ has a single ternary relation R, then an open subset H of $\mathfrak{U}$ is special provided that for every pair of clopen subsets F and G of $\mathfrak{U}$, if one of F and G is included in H, then the clopen image set

$$R^*(F \times G) = \{t \in U : R(r, s, t) \text{ for some } r \in F \text{ and } s \in G\}$$

is included in H.

Suppose $\mathfrak{U}$ is a relational space, and $\mathfrak{A}$ its dual algebra. If M is an ideal in $\mathfrak{A}$, then the union of the clopen sets in M is a special open set in $\mathfrak{U}$. Conversely, if H is an arbitrary special open set in $\mathfrak{U}$, then the set of all clopen subsets of H is an ideal in $\mathfrak{A}$. In order to prove these assertion, it is helpful to make a preliminary observation: if M is a set of elements in $\mathfrak{A}$—that is to say, if M is a set of clopen subsets of $\mathfrak{U}$—and if M satisfies conditions (ii) and (iii) in the definition of an ideal, then an element F in $\mathfrak{A}$ belongs to M just in case F is included in the union of M. One direction of this observation is clear: if F belongs to M, then F is certainly included in the union of all of the sets in M. To prove the reverse direction, suppose F is included in the union of M. The set F, being clopen, is closed and therefore compact in the topology of $\mathfrak{U}$, and the sets in M, being clopen, are open and therefore form an open cover of F (because F is included in their union). Apply compactness to obtain a finite subset M_0 of M such that the union of the sets in M_0 includes F. The set M is closed under finite unions, by condition (ii), so the union of the sets in M_0 is a clopen set G that belongs to M. The set M is also downward closed, by condition (iii), and F is included in G, so F must belong to M.

Lemma 2.22. *Let $\mathfrak{U}$ be a relational space and $\mathfrak{A}$ its dual algebra. If M is an ideal in $\mathfrak{A}$, then the union of the sets in M is a special open subset of $\mathfrak{U}$. Inversely, if H is a special open set in $\mathfrak{U}$, then the set of clopen sets in $\mathfrak{U}$ that are included in H is an ideal in $\mathfrak{A}$.*

Proof. Suppose first that M is an ideal in $\mathfrak{A}$, and let H be the union of the sets in M. Since $\mathfrak{A}$ is the dual algebra of $\mathfrak{U}$, the elements in $\mathfrak{A}$ are the clopen subsets of $\mathfrak{U}$. In particular, M is a set of clopen subsets of $\mathfrak{U}$, so its union H is an open set in $\mathfrak{U}$. To show that H is special, consider the case of a ternary relation R in $\mathfrak{U}$ and the binary operator $\circ$ that is defined in $\mathfrak{A}$ in terms of R. Let F and G be clopen sets in $\mathfrak{U}$, and suppose that F is included in H. The set F must then belong to the ideal M, by the observation preceding the lemma, and therefore the set $F \circ G$ also belongs to M, by condition (iv) in the definition of an ideal. The set $F \circ G$ coincides with the image set $R^*(F \times G)$, by the definition of the dual algebra $\mathfrak{A}$ (see Lemma 2.9 and its proof), so the image set belongs to M and is therefore included in the union H. A similar argument applies if G is included in H. Consequently, H is a special open set, by Definition 2.21.

To prove the second assertion of the lemma, assume that H is an arbitrary special open set in $\mathfrak{U}$, and let M be the set of all clopen subsets of H. It must be shown that M satisfies conditions (i)–(iv) in the definition of an ideal. The empty set is obviously a clopen subset of $\mathfrak{U}$ that is included in H, so the empty set belongs to M. Thus, condition (i) is satisfied. The union of two clopen subsets of $\mathfrak{U}$ that are included in H is again a clopen subset of $\mathfrak{U}$ that is included in H. Consequently, M contains the union of any two of its elements and therefore satisfies condition (ii). The intersection of a clopen subset of $\mathfrak{U}$ that is included in H with an arbitrary clopen subset of $\mathfrak{U}$ is again a clopen subset of $\mathfrak{U}$ that is included in H, so M satisfies condition (iii). To show that M satisfies condition (iv), consider the case of a binary operator $\circ$ in $\mathfrak{A}$ that is defined in terms of a ternary relation R in $\mathfrak{U}$. Let F and G be clopen sets in $\mathfrak{U}$, and suppose F belongs to M. The clopen image set $R^*(F \times G)$ is then included in H, by Definition 2.21 and the assumption that H is a special open set, so this image set must belong to M, by the observation made before the lemma. Since this image set coincides with $F \circ G$, it follows that the latter must belong to M. A similar argument applies if G belongs to M. Consequently, M satisfies conditions (iv) and is therefore an ideal in $\mathfrak{A}$. $\square$

If M is an ideal in the dual algebra $\mathfrak{A}$ of a relational space $\mathfrak{U}$, then the special open set that is the union of the sets in M is called the (*first*) *dual*, or the *dual open set*, of M. Similarly, if H is a special open set in $\mathfrak{U}$, then the ideal of clopen subsets of $\mathfrak{U}$ that are included in H is called the (*first*) *dual*, or the *dual ideal*, of H.

If M is the dual ideal of an arbitrary special open set H in $\mathfrak{U}$, then M is, by definition, the set of clopen subsets of H. The dual open set of M is, by definition, the union of M. This union must coincide with H, because every open set in a Boolean space is the union of its clopen subsets. It follows that the second dual of every special open set in $\mathfrak{U}$ is itself. Similarly, if H is the dual open set of an arbitrary ideal M in $\mathfrak{A}$, then H is, by definition, the union of M. The dual ideal of H is, by definition, the set of all clopen subsets of H. This ideal must coincide with M, because a clopen set is included in H if and only if it belongs to M, by the observation made before the preceding lemma. It follows that the second dual of every ideal in $\mathfrak{A}$ is itself.

If M and N are ideals in $\mathfrak{A}$, and if H and K are their respective dual special open sets, then

$$M \subseteq N \qquad \text{if and only if} \qquad H \subseteq K.$$

The implication from left to right is clear, since H and K are defined to be the unions of M and N respectively. On the other hand, if H is included in K, then every clopen subset of H is also a clopen subset of K. Consequently, M is included in N, because M and N are defined to be the sets of all clopen subsets of H and K respectively.

The principal facts about the duality between ideals and special open sets are summarized in the following duality theorem for ideals.

Theorem 2.23. *The dual of every special open subset of a relational space $\mathfrak{U}$ is an ideal in the dual algebra $\mathfrak{A}$, and the dual of every ideal in $\mathfrak{A}$ is a special open subset of $\mathfrak{U}$. The second dual of every ideal and of every special open set is itself. The function that maps every ideal in $\mathfrak{A}$ to its dual special open set is an isomorphism from the lattice of ideals in $\mathfrak{A}$ to the lattice of special open sets in $\mathfrak{U}$.*

It is illuminating to look at the duality between ideals and special open sets from the perspective of an arbitrary Boolean algebra with operators $\mathfrak{A}$ and its dual space $\mathfrak{U}$, instead of from the perspective of an arbitrary relational space $\mathfrak{U}$ and its dual algebra $\mathfrak{A}$, as in Theorem 2.23. The elements in $\mathfrak{U}$ are the ultrafilters in $\mathfrak{A}$, and every element r in $\mathfrak{A}$ is identified via the canonical isomorphism with an element in the second dual of $\mathfrak{A}$—that is to say, in the dual algebra of $\mathfrak{U}$—namely with the clopen set

$$F_r = \{X \in U : r \in X\}.$$

Every ideal M is $\mathfrak{A}$ is consequently identified with the ideal of clopen sets

$$M_0 = \{F_r : r \in M\}$$

in the second dual. The ideal M determines a special open set in $\mathfrak{U}$, namely the union

$$F_M = \bigcup\{F_r : r \in M\}$$

of the clopen sets belonging to M_0. Furthermore, every special open set H in $\mathfrak{U}$ has the form $H = F_M$ for some ideal M in $\mathfrak{A}$. Indeed, if M_0 is taken to be the set of clopen subsets of H, then M_0 is an ideal in the dual of $\mathfrak{U}$, and the union of the sets in M_0 is just the special open set H, by the preceding theorem. If M is the ideal in $\mathfrak{A}$ that corresponds to M_0 (under the canonical isomorphism), then

$$H = \bigcup M_0 = \bigcup\{F_r : r \in M\} = F_M .$$

Thus, the special open sets in $\mathfrak{U}$ are precisely the sets F_M, where M ranges over the ideals in $\mathfrak{A}$.

The canonical isomorphism from $\mathfrak{A}$ to its second dual obviously induces an isomorphism between the corresponding lattices of ideals. The lattice of ideals of the second dual is isomorphic to the lattice of special open subsets of $\mathfrak{U}$, by Theorem 2.23. Consequently, the correspondence that maps each ideal M in $\mathfrak{A}$ to the special open set F_M in $\mathfrak{U}$ is an isomorphism from the lattice of ideals in $\mathfrak{A}$ to the lattice of special open sets in $\mathfrak{U}$. In this formulation of the duality between ideals and special open sets, the assertion that the second dual of every ideal is itself is not literally true; one must first identify the algebra $\mathfrak{A}$ with its second dual before the assertion becomes true.

Theorem 2.23 has an analogue for filters. (The notion of a filter in a Boolean algebra with operators is defined in a dual manner to that of an ideal.) Let $\mathfrak{U}$ be a relational space, and $\mathfrak{A}$ the dual algebra of $\mathfrak{U}$. A closed subset of $\mathfrak{U}$ is called *special* if it is the complement of a special open set. If N is a filter in $\mathfrak{A}$, then the intersection of the clopen sets in N is a special closed subset of $\mathfrak{U}$; in fact, if M is the ideal that is the dual of N, then the intersection of N is the complement of the special open set that is the union of M. Call this special closed set the (*first*) *dual*, or the *dual closed subset*, of N. Conversely, if H is a special closed subset of $\mathfrak{U}$, then the set of all clopen sets in $\mathfrak{U}$ that include H is a filter in $\mathfrak{A}$. Call this filter the (*first*) *dual*, or the *dual filter*, of H. Each filter in $\mathfrak{A}$ and each special closed set in $\mathfrak{U}$ is its own second dual, and the function that maps each filter in $\mathfrak{A}$ to its dual special closed set is an isomorphism from the lattice of filters in $\mathfrak{A}$ to the lattice of special closed sets in $\mathfrak{U}$. A related result, discovered independently by Celani, is given in Proposition 29 of [2].

2.6 Duality for Quotients

The duality between the special open subsets of a relational space $\mathfrak{U}$ and the ideals in the dual algebra $\mathfrak{A}$ implies a duality between quotients of $\mathfrak{A}$ and certain subspaces of $\mathfrak{U}$. We begin by clarifying the relationship between inner subuniverses (see Definition 1.12 and the remarks following it) and special closed subsets.

Lemma 2.24. *A closed subset V of a relational space $\mathfrak{U}$ is a inner subuniverse of $\mathfrak{U}$ if and only if V is a special closed subset of $\mathfrak{U}$.*

Proof. Focus on the case of a fundamental ternary relation R. Assume first that V is a special closed subset of $\mathfrak{U}$, and observe that the complement $\sim V$ is, by definition, a special open set. To check that V is an inner subuniverse of $\mathfrak{U}$, consider elements r, s, and t in $\mathfrak{U}$ such that $R(r,s,t)$ holds in $\mathfrak{U}$. If at least one of r and s is not in V, then t cannot be in V. Indeed, suppose r is not in V. The clopen sets form a base for the topology of $\mathfrak{U}$, and $\sim V$ is an open set that contains r, so there must be a clopen set F in $\mathfrak{U}$ such that r is in F and F is included in $\sim V$. The set U is also clopen, so the image clopen set

$$R^*(F \times U) = \{w : R(p,q,w) \text{ for some } p \in F \text{ and } q \in U\}$$

is included in $\sim V$, by the assumption that $\sim V$ is a special open set. In particular, since r is in F, and s in U, and since $R(r,s,t)$ holds, it may be concluded that t is in $R^*(F \times U)$ and therefore also in $\sim V$, so t cannot be in V. An analogous argument applies if s is not in V. Conclusion: if $R(r,s,t)$ holds in $\mathfrak{U}$, and if t is in V, then r and s must both belong to V. Consequently, V is an inner subuniverse of $\mathfrak{U}$.

Suppose now that V is an inner subuniverse of $\mathfrak{U}$ that is closed in the topology of $\mathfrak{U}$. The complement $\sim V$ is then an open subset of $\mathfrak{U}$, and it must be shown that this complement is special. To this end, consider clopen sets F and G, and suppose that F is included in $\sim V$. If t belongs to the image set $R^*(F \times G)$, then there must be elements r in F and s in G such that $R(r,s,t)$ is true in $\mathfrak{V}$, by the definition of the image set. Since V is an inner subuniverse of $\mathfrak{U}$, the presence of t in V would force both r and s to belong to V. But r belongs to F, which is included in $\sim V$. Consequently, t cannot be in V, so t belongs to $\sim V$. This is true for every element t in the set $R^*(F \times G)$, so the entire set is included in $\sim V$. A similar argument applies if G is included in $\sim V$, so $\sim V$ is a special open set. $\square$

By a *restriction* of a relational space $\mathfrak{U}$ to a subset V, we understand the relational structure whose universe is V and whose fundamental relations are the restrictions to V of the fundamental relations in $\mathfrak{U}$, together with the topology that V inherits from $\mathfrak{U}$. In general, the restriction of a relational space to a subset is not a relational space. The next lemma gives a sufficient condition for such a restriction to be a relational space.

Lemma 2.25. *If V is a closed subset and an inner subuniverse of a relational space $\mathfrak{U}$, then the restriction of $\mathfrak{U}$ to V is a relational space.*

Proof. The inherited topology turns V into a Boolean space, because V is assumed to be a closed subset of the Boolean space U (see Lemma 2 on p. 306 of [10]). Let $\mathfrak{V}$ be the restriction of $\mathfrak{U}$ to the set V. It must be shown that the relations in $\mathfrak{V}$ are clopen, and those of rank at least two are continuous, in the sense of Definition 2.2 and the remarks preceding it. Focus on the case of a ternary relation R.

Suppose F and G are clopen subsets of $\mathfrak{V}$, with the aim of proving that the image set

$$R^*(F \times G) = \{t \in V : R(r, s, t) \text{ for some } r \in F \text{ and } s \in G\} \tag{1}$$

is clopen. (Since the fundamental relations of $\mathfrak{V}$ are the restrictions to V of the fundamental relations of $\mathfrak{U}$, it does not matter whether the relation R in (1) is viewed as the ternary relation in $\mathfrak{V}$ or as the ternary relation in $\mathfrak{U}$.) The set V is assumed to be closed, so there must be clopen subsets $\bar{F}$ and $\bar{G}$ of $\mathfrak{U}$ such that

$$F = \bar{F} \cap V \qquad \text{and} \qquad G = \bar{G} \cap V \tag{2}$$

(see Lemma 2 on p. 306 of [10]). Because $\bar{F}$ and $\bar{G}$ are clopen in $\mathfrak{U}$, the image set

$$R^*(\bar{F} \times \bar{G}) = \{t \in U : R(r, s, t) \text{ for some } r \in \bar{F} \text{ and } s \in \bar{G}\}. \tag{3}$$

is clopen in $\mathfrak{U}$, by Definition 2.2 and the assumption that $\mathfrak{U}$ is a relational space.

Observe that

$$R^*(F \times G) = R^*(\bar{F} \times \bar{G}) \cap V. \tag{4}$$

Indeed, if t belongs to the left side of (4), then $R(r, s, t)$ holds in $\mathfrak{V}$ for some r in F and s in G, by (1). In particular, t is in V, by (2). Since F is included in $\bar{F}$, and G in $\bar{G}$, by (2), the element r belongs to $\bar{F}$, and s to $\bar{G}$, and therefore t belongs to the right side of (4), by (3) and the assumption that V is a subset of $\mathfrak{U}$. On the other hand, if t belongs to the right side of (4), then t is in V, and $R(r, s, t)$ holds in $\mathfrak{U}$ for some r in $\bar{F}$ and s in $\bar{G}$. Since V is an inner subuniverse of $\mathfrak{U}$, and t is in V, the elements r and s must be in V (see the remarks following Definition 1.12). Consequently, r is in F and s in G, by (2), so t belongs to the left side of (4), by (1).

The equation in (4) shows that $R^*(F \times G)$ is the intersection with V of a clopen subset of $\mathfrak{U}$. It follows that $R^*(F \times G)$ is a clopen subset of $\mathfrak{V}$. Since this is true for every pair of clopen sets F and G in $\mathfrak{V}$, the relation R in $\mathfrak{V}$ is clopen.

The next task is to prove that the relation R in $\mathfrak{V}$ is weakly continuous in the sense of Definition 2.12. To this end, fix an arbitrary element t in $\mathfrak{V}$, with the aim of showing that the set

$$H_t = \{(r,s) : r,s \in V \text{ and } R(r,s,t)\} \tag{5}$$

is closed in the product space $V \times V$. The set

$$\bar{H}_t = \{(r,s) : r,s \in U \text{ and } R(r,s,t)\} \tag{6}$$

is closed in the product space $U \times U$, by Theorem 2.13, because $\mathfrak{U}$ is assumed to be a relational space. It is not difficult to check that

$$H_t = \bar{H}_t \cap (V \times V). \tag{7}$$

Indeed, if a pair (r,s) belongs to the left side of (7), then $R(r,s,t)$ holds in $\mathfrak{V}$, by (5). It follows that $R(r,s,t)$ holds in $\mathfrak{U}$, because the fundamental relations in $\mathfrak{V}$ are the restrictions to V of the fundamental relations in $\mathfrak{U}$; and consequently, the pair (r,s) belongs to the right side of (7), by (6). On the other hand, if a pair (r,s) belongs to the right side of (7), then the elements r and s are in V, as is t (by assumption), and $R(r,s,t)$ holds in $\mathfrak{U}$, by (6). It follows that $R(r,s,t)$ holds in $\mathfrak{V}$, because the fundamental relations in $\mathfrak{V}$ are the restrictions to V of the fundamental relations in $\mathfrak{U}$; and the pair (r,s) therefore belongs to the left side of (7), by (5).

The equation in (7) shows that H_t is the intersection with $V \times V$ of a closed subset of $U \times U$. Since V is assumed to be closed in $\mathfrak{U}$, it follows that H_t is a closed subset of $V \times V$. This is true for every element t in $\mathfrak{V}$, so the relation R in $\mathfrak{V}$ is weakly continuous. It has been shown that the restriction $\mathfrak{V}$ is a Boolean space and that the fundamental relations in $\mathfrak{V}$ are clopen, and those of rank at least two are weakly continuous, under the inherited topology. Apply Theorem 2.14 to conclude that $\mathfrak{V}$ is a relational space. □

In order to state the version of the sub-quotient duality theorem that applies to quotients of Boolean algebras with operators, it is necessary to define the appropriate notion of a subspace for relational spaces.

Definition 2.26. A relational space $\mathfrak{V}$ is an *inner subspace* of a relational space $\mathfrak{U}$ if algebraically $\mathfrak{V}$ is an inner substructure of $\mathfrak{U}$ and if the topology on $\mathfrak{V}$ is the subspace topology inherited from $\mathfrak{U}$. □

Notice that every inner subspace $\mathfrak{V}$ of a relational space $\mathfrak{U}$ must be the restriction of $\mathfrak{U}$ to some subset of $\mathfrak{U}$, namely the subset V that is the universe of $\mathfrak{V}$, by the definition of an inner substructure, the definition of the subspace topology, and the definition of a restriction. The next theorem characterizes those subsets of $\mathfrak{U}$ that lead to inner subspaces.

Theorem 2.27. *For a subset V of a relational space $\mathfrak{U}$, the following conditions are equivalent.*

(i) *V is a special closed subset of $\mathfrak{U}$.*
(ii) *V is a closed subset and an inner subuniverse of $\mathfrak{U}$.*
(iii) *The restriction of $\mathfrak{U}$ to V is an inner subspace of $\mathfrak{U}$.*

Proof. The equivalence of (i) and (ii) follows at once from Lemma 2.24. For the implication from (ii) to (iii), assume that V is a closed subset and an inner subuniverse of $\mathfrak{U}$. The restriction of $\mathfrak{U}$ to V is then a relational space $\mathfrak{V}$, by Lemma 2.25. Algebraically, $\mathfrak{V}$ is an inner substructure of $\mathfrak{U}$, by the assumption that V is an inner subuniverse of $\mathfrak{U}$. Topologically, $\mathfrak{V}$ is a subspace of $\mathfrak{U}$, by the definition of a restriction of a relational space. Consequently, $\mathfrak{V}$ is an inner subspace of $\mathfrak{U}$, by Definition 2.26.

To establish the reverse implication from (iii) to (ii), assume that $\mathfrak{V}$ is an inner subspace of $\mathfrak{U}$. The universe of $\mathfrak{V}$ must then be an inner subuniverse of $\mathfrak{U}$, by Definition 2.26. Topologically, $\mathfrak{V}$ is a subspace of $\mathfrak{U}$, by Definition 2.26, and the universe of $\mathfrak{V}$ is compact because $\mathfrak{V}$ is assumed to be a relational space. Consequently, this universe must be closed in the topology of $\mathfrak{U}$, because a subset of a compact Hausdorff space is compact if and only if it is closed (see pp. 271–272 of [10]). □

Corollary 2.28. *An inner subuniverse V of a relational space $\mathfrak{U}$ is the universe of an inner subspace of $\mathfrak{U}$ if and only if V is closed in the topology of $\mathfrak{U}$.*

The duality between quotient algebras and inner subspaces may now be formulated as follows.

Theorem 2.29. *There is a bijective correspondence between the inner subspaces of a relational space $\mathfrak{U}$ and the quotients of its dual algebra $\mathfrak{A}$. If $\mathfrak{V}$ is an inner subspace of $\mathfrak{U}$, then the dual algebra of $\mathfrak{V}$ is isomorphic to the quotient $\mathfrak{A}/M$, where M is the ideal that is the dual of the special open set $\sim V$. Inversely, if M is an ideal in $\mathfrak{A}$, then the dual space of* *the quotient algebra* $\mathfrak{A}/M$ *is homeo-isomorphic to the inner subspace that is the restriction of* $\mathfrak{U}$ *to* V, *where* V *is the complement of the special open set that is the dual of* M.

Proof. Each ideal M in $\mathfrak{A}$ uniquely determines an inner subspace of $\mathfrak{U}$, by Theorems 2.23 and 2.27, namely the inner subspace whose universe is the special closed set that is the complement of the special open set that is the dual of M. Conversely, each inner subspace $\mathfrak{V}$ of $\mathfrak{U}$ uniquely determines an ideal in $\mathfrak{A}$, namely the ideal that is the dual of the special open set that is the complement of the universe of $\mathfrak{V}$. The correspondence mapping the inner subspace $\mathfrak{V}$ to the quotient $\mathfrak{A}/M$ is therefore a bijection from the set of inner subspaces of $\mathfrak{U}$ to the set of quotients of $\mathfrak{A}$, by Theorem 2.23.

To prove the second assertion of the theorem, suppose $\mathfrak{V}$ is an inner subspace of the relational space $\mathfrak{U}$, and let $\mathfrak{B}$ be the dual algebra of $\mathfrak{V}$. Since $\mathfrak{V}$ is, in particular, an inner substructure of $\mathfrak{U}$, the identity function ϑ on $\mathfrak{V}$ is a bounded monomorphism from $\mathfrak{V}$ into $\mathfrak{U}$, by Corollary 1.14. Also, ϑ is continuous, by the definition of the inherited topology on $\mathfrak{V}$. Indeed, if H is an open set in $\mathfrak{U}$, then

$$\vartheta^{-1}(H) = H \cap V,$$

which is an open set in $\mathfrak{V}$, by the definition of the inherited topology; so the inverse image under ϑ of every open set in $\mathfrak{V}$ is an open set in $\mathfrak{U}$.

The dual of ϑ is the epimorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$ that is defined by

$$\varphi(F) = \vartheta^{-1}(F) = \{u \in V : \vartheta(u) \in F\} = F \cap V$$

for elements F in $\mathfrak{A}$, that is to say, for clopen subsets F of $\mathfrak{U}$, by Theorem 2.16 and the assumption that ϑ is the identity function. The kernel of φ is the set of elements mapped to the empty set by φ. Since

$$\begin{aligned} \varphi(F) = \varnothing \qquad &\text{if and only if} \qquad F \cap V = \varnothing, \\ &\text{if and only if} \qquad F \subseteq {\sim}V, \end{aligned}$$

the kernel of φ is just the ideal of clopen subsets of ${\sim}V$. In other words, the kernel is the ideal M that is the dual of the special open set ${\sim}V$. Because φ is an epimorphism from $\mathfrak{A}$ to $\mathfrak{B}$ with kernel M, the quotient $\mathfrak{A}/M$ is isomorphic to $\mathfrak{B}$ via the function that maps each coset F/M to the intersection $\varphi(F) = F \cap V$, by the First Isomorphism Theorem for Boolean algebras with operators.

To prove the third assertion of the theorem, consider an arbitrary ideal M in $\mathfrak{A}$. The dual of M is the special open set that is the union of the sets in M. If V is the complement of this special open set, and if $\mathfrak{V}$ is the restriction of $\mathfrak{U}$ to V, then $\mathfrak{V}$ is an inner subspace of $\mathfrak{U}$, by Theorem 2.27. Let $\mathfrak{B}$ be the dual algebra of $\mathfrak{V}$. The dual ideal of the special open set $\sim V$ is M, by Theorem 2.23, so $\mathfrak{B}$ is isomorphic to the quotient $\mathfrak{A}/M$, by the observations of the preceding paragraphs. Apply Theorem 2.18 to conclude that the dual space of the quotient $\mathfrak{A}/M$ is homeo-isomorphic to the dual space of $\mathfrak{B}$. Since $\mathfrak{B}$ is the dual algebra of $\mathfrak{V}$, the dual space of $\mathfrak{B}$ is, by definition, the second dual of $\mathfrak{V}$, and $\mathfrak{V}$ is homeo-isomorphic to its second dual, by Theorem 2.11. It follows that the dual space of the quotient $\mathfrak{A}/M$ is homeo-isomorphic to $\mathfrak{V}$, as claimed. $\square$

For some remarks concerning the relationship of Theorem 2.29 to the duality between inner subspaces and homomorphic images that is known from the literature, see the end of Section 2.8.

2.7 Duality for Subuniverses

The other half of the sub-quotient duality involves a duality between subuniverses of Boolean algebras with operators and special congruences on relational spaces. There is a corresponding duality between subalgebras of Boolean algebras with operators and quotients of relational spaces.

Recall that if U is a topological space, and Θ an equivalence relation on U, then the set V of equivalence classes of Θ can be turned into a topological space by declaring a subset F of V to be open just in case the union of the equivalence classes in F is an open subset of U. The set V endowed with this *quotient topology* is called the *quotient space* of U modulo Θ. A quotient of a compact space is necessarily compact, but a quotient of a Hausdorff space or a Boolean space need not be Hausdorff or Boolean. An equivalence relation Θ on U is said to be *Boolean* if for any two elements in $\mathfrak{U}$ that are inequivalent modulo Θ, there is a clopen subset of $\mathfrak{U}$ that is compatible with Θ (that is to say, it is a union of equivalence classes of Θ—see the remarks preceding Lemma 1.21) and that contains one of the two elements but not the other. It is not difficult to check that for any equivalence relation Θ on a Boolean space, the quotient space is Boolean if and only if Θ is a Boolean relation (see, for example, Lemma 1 on p. 362 of [10]). The quotient function that maps each element in a Boolean space U to its equivalence class modulo a Boolean relation Θ is a continuous function from U onto the quotient space V, by the definition of the quotient topology. Conversely, if ϑ is a continuous mapping from a Boolean space U onto a Boolean space W, then the *kernel* of ϑ, that is to say, the relation Θ defined by

$$r \equiv s \mod \Theta \qquad \text{if and only if} \qquad \vartheta(r) = \vartheta(s),$$

is a Boolean relation on U, and the quotient of U modulo Θ is homeomorphic to W via the mapping that take each equivalence class u/Θ to the element $\vartheta(u)$.

To defined the appropriate notion of a congruence on a relational space, we must combine the notions of a Boolean relation and a bounded congruence (see Definition 1.20).

Definition 2.30. A binary relation Θ on a relational space $\mathfrak{U}$ is called a *relational congruence* if Θ is a bounded congruence on $\mathfrak{U}$ and simultaneously a Boolean relation with respect to the topology on $\mathfrak{U}$. $\square$

Every relational congruence on a relational space gives rise to a subuniverse of the dual algebra.

Lemma 2.31. *If Θ is a relational congruence on a relational space $\mathfrak{U}$, then the set of clopen subsets of $\mathfrak{U}$ that are compatible with Θ is a subuniverse of the dual algebra of $\mathfrak{U}$.*

Proof. Let $\mathfrak{A}$ be the dual algebra of $\mathfrak{U}$, and B the set of clopen subsets of $\mathfrak{U}$ that are compatible with Θ. It is to be shown that B is a subuniverse of $\mathfrak{A}$. Clearly, the empty set is clopen and compatible with Θ, so it belongs to B. Also, the union of two clopen sets is clopen, and the complement of a clopen set is clopen, in any topological space; and the union of two sets compatible with Θ is compatible with Θ, and the complement of a set compatible with Θ is compatible with Θ. Consequently, B is closed under the Boolean operations of $\mathfrak{A}$.

To show that B is closed under the operators of $\mathfrak{A}$, consider the case of a binary operator $\circ$ that is defined in terms of a ternary relation R in $\mathfrak{U}$. Let F and G be sets in B. The product of $F \circ G$ in $\mathfrak{A}$, which is the image set defined by

$$F \circ G = R^*(F \times G) = \{w : R(u, v, w) \text{ for some } u \in F \text{ and } v \in G\},$$

is clopen in $\mathfrak{U}$, by the assumption that $\mathfrak{U}$ is a relational space (see Definition 2.2), and it is compatible with Θ because Θ is assumed to be a bounded congruence on $\mathfrak{U}$ (see the argument in the corresponding part of the proof of Lemma 1.21). Consequently, $F \circ G$ belongs to B, so B is in fact a subuniverse of $\mathfrak{A}$. $\square$

There is a type of converse to the preceding lemma that is also true.

Lemma 2.32. *If B is a subuniverse of the dual algebra of a relational space $\mathfrak{U}$, then the relation Θ on $\mathfrak{U}$ that is defined by $r \equiv s \mod \Theta$ if and only if r and s belong to the same sets in B is a relational congruence on $\mathfrak{U}$.*

Proof. We give an indirect proof that makes use of the epi-mono duality. Suppose $\mathfrak{A}$ is the dual algebra of $\mathfrak{U}$, and $\mathfrak{V}$ the dual space of $\mathfrak{A}$. Thus, $\mathfrak{V}$ is the second dual of $\mathfrak{U}$: the elements in $\mathfrak{V}$ are the sets of the form

$$X_r = \{F \in A : r \in F\}, \tag{1}$$

and the function ϱ that maps each element r in $\mathfrak{U}$ to the set X_r is a homeo-isomorphism from $\mathfrak{U}$ to $\mathfrak{V}$, by Theorem 2.11.

Write $\mathfrak{B}$ for the subalgebra of $\mathfrak{A}$ with universe B. The identity function φ on $\mathfrak{B}$ is a monomorphism from $\mathfrak{B}$ into $\mathfrak{A}$, since $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. The dual of φ is the function ϑ on $\mathfrak{V}$ determined by

$$\begin{aligned}\vartheta(X_r) = \varphi^{-1}(X_r) &= \{F \in B : \varphi(F) \in X_r\}\\ &= \{F \in B : F \in X_r\} = \{F \in B : r \in F\},\end{aligned} \tag{2}$$

by the definition of ϑ, the definition of the inverse image of a set under the function φ, the assumption that φ is the identity function on $\mathfrak{B}$, and (1). This dual is a continuous bounded epimorphism from $\mathfrak{V}$ to the dual relational space of $\mathfrak{B}$, by Theorem 2.18. The kernel of ϑ is, by definition, the relation Ψ on $\mathfrak{V}$ that is defined by

$$X_r \equiv X_s \mod \Psi \qquad \text{if and only if} \qquad \vartheta(X_r) = \vartheta(X_s). \tag{3}$$

Since the kernel of a bounded homomorphism is a bounded congruence, by Lemma 1.27, and the kernel of a continuous mapping between Boolean spaces is a Boolean relation (see the remark before Definition 2.30), it follows that Ψ is a relational congruence on $\mathfrak{U}$, by Definition 2.30.

The equations in (2) imply that

$$\vartheta(X_r) = \vartheta(X_s) \qquad \text{if and only if}$$
$$\{F \in B : r \in F\} = \{F \in B : s \in F\}.$$

The right side of this equivalence just says that r and s belong to the same sets in B, which is exactly what it means for the elements r and s to be equivalent modulo Θ. Combine these observations with (3) to conclude that

$$X_r \equiv X_s \mod \Psi \qquad \text{if and only if} \qquad r \equiv s \mod \Theta. \tag{4}$$

The equivalence in (4) implies that the relation Θ is the inverse image, under the homeo-isomorphism ϱ, of the relation Ψ. Homeo-isomorphisms preserve all algebraic and topological properties. In particular, since Ψ is a relational congruence on $\mathfrak{V}$, it follows that Θ must be a relational congruence on $\mathfrak{U}$. $\square$

We continue with the assumption that $\mathfrak{U}$ is a relational space, and $\mathfrak{A}$ its dual algebra. If Θ is a relational congruence on $\mathfrak{U}$, then the subuniverse of $\mathfrak{A}$ consisting of the clopen sets that are compatible with Θ (Lemma 2.31) is called the (*first*) *dual*, or the *dual subuniverse*, of Θ. Similarly, if B is a subuniverse of $\mathfrak{A}$, then the relational congruence on $\mathfrak{U}$ consisting of the pairs of elements that belong to precisely the same sets in B (Lemma 2.32) is called the (*first*) *dual*, or the *dual relational congruence*, of B.

Each relational congruence on $\mathfrak{U}$ is its own second dual, that is to say, if Θ is a relational congruence on $\mathfrak{U}$ with dual subuniverse B, and if Ψ is the dual relational congruence of B, then $\Theta = \Psi$. The proof depends on two facts. First, the sets in B are all compatible with Θ, by the definition of B. In particular, if r and s are elements in $\mathfrak{U}$ that are congruent modulo Θ, then for any set F in B, either r and s are both in F, or neither r nor s is in F, by the compatibility of F with Θ. It follows that r and s belong to the same sets in B, and therefore r and s are congruent modulo Ψ, by the definition of Ψ as the dual of the subuniverse B. Second, Θ is a Boolean relation, so if elements r and s are not congruent modulo Θ, then there must be a clopen subset F of $\mathfrak{U}$ that is compatible with Θ and that contains r but not s. The set F belongs to B, by the definition of B, so the elements r and s do not belong to precisely the same sets in B and consequently r and s are not equivalent modulo Ψ. Conclusion: $\Theta = \Psi$.

Each subuniverse of $\mathfrak{A}$ is also equal to its second dual. For the proof, suppose B is a subuniverse of $\mathfrak{A}$ with dual relational congruence Θ, and let C be the dual subuniverse of Θ. It is to be shown that $B = C$. The proof requires a number of preliminary observations. Write

$$B/\Theta = \{F/\Theta : F \in B\} \qquad \text{and} \qquad C/\Theta = \{F/\Theta : F \in C\},$$

where

$$F/\Theta = \{r/\Theta : r \in F\}.$$

Since Θ is, in particular, a Boolean relation on the topological space U, the quotient space U/Θ is a Boolean space, and the clopen subsets of this quotient space are the quotients of the clopen subsets of U that are compatible with Θ (see Lemma 1 on p. 362 of [10]). By definition, C consists of all clopen subsets of U that are compatible with Θ, so C/Θ is the set of all clopen subsets of U/Θ.

Turn now to the quotient B/Θ. Every set in B is a clopen subset of U, because B is a subuniverse of the algebra $\mathfrak{A}$ of all clopen subsets of U. Furthermore, every set in B is compatible with Θ. Indeed, if r and s are elements in $\mathfrak{U}$ that are equivalent modulo Θ, then r and s belong to exactly the same sets in B, by the definition of Θ as the dual of the subuniverse B. Consequently, if F is a set in B, and if r is in F, then s is in F. The presence of an element r in F therefore implies that the entire equivalence class r/Θ is included in F, so F is compatible with Θ. It follows from these observations that the set B is included in C, and therefore the quotient B/Θ is included in C/Θ.

The next step is to show that B/Θ is closed under the operations of union and complement. To this end, suppose that F and G are sets in B. The union $F \cup G$ and the complement $\sim F$ are also in B, since B is a subuniverse of $\mathfrak{A}$. To show that B/Θ is closed under union, it therefore suffices to check that

$$(F/\Theta) \cup (G/\Theta) = (F \cup G)/\Theta.$$

For an arbitrary element r in U, we have

$$\begin{aligned} r/\Theta \in (F/\Theta) \cup (G/\Theta) \quad &\text{if and only if} \quad r/\Theta \in F/\Theta \text{ or } r/\Theta \in G/\Theta, \\ &\text{if and only if} \quad r \in F \text{ or } r \in G, \\ &\text{if and only if} \quad r \in F \cup G, \\ &\text{if and only if} \quad r/\Theta \in (F \cup G)/\Theta, \end{aligned}$$

by the definition of the union of two sets and by the compatibility of the sets F, G, and $F \cup G$ with the congruence Θ. Consequently, the desired equality does hold. A similar argument shows that

$$\sim(F/\Theta) = (\sim F)/\Theta$$

(where the first complement is formed in B/Θ, and the second in B), so that B/Θ is closed under complement.

The third step is to observe that the sets in B/Θ—which are clopen subsets of the quotient space U/Θ—separate points in the quotient space. Indeed, if r/Θ and s/Θ are distinct points in the quotient space, then the definition of the relation Θ implies that there must be a set F in B that contains the element r but not the element s. The clopen set F/Θ belongs to B/Θ, and it contains the point r/Θ but not the point s/Θ (because F is compatible with Θ), so F/Θ separates these two points.

It has been shown that B/Θ is a Boolean algebra of clopen subsets of U/Θ that separates points in U/Θ. Any Boolean algebra of clopen sets that separates points in a Boolean space is necessarily the Boolean algebra of all clopen sets in the space (see Lemma 1 on p. 305 of [10]). Consequently, B/Θ is the Boolean algebra of all clopen sets in U/Θ, that is to say,

$$B/\Theta = C/\Theta.$$

Each set F that belongs to B or to C is the union of the equivalence classes in the quotient F/Θ, because F is compatible with Θ. The equality at the end of the last paragraph therefore implies that $B = C$. In more detail, if F is in C, then the quotient set F/Θ belongs to C/Θ, and therefore also to B/Θ, by the equality at the end of the last paragraph. Consequently, $F/\Theta = G/\Theta$ for some set G in B. Since

$$F = \bigcup(F/\Theta) = \bigcup(G/\Theta) = G,$$

it follows that F belongs to B. Thus, C is included in B. The reverse inclusion was established above. This completes the proof that B is its own second dual.

Consider now two arbitrary relational congruences on $\mathfrak{U}$, say Θ and Ψ, with dual subuniverses B and C respectively. It is not difficult to check that

$$\Theta \subseteq \Psi \qquad \text{if and only if} \qquad C \subseteq B.$$

For the proof, assume first that C is included in B. The relations Θ and Ψ are their own second duals, so they are the first duals of the subuniverses B and C respectively. If r and s are elements in $\mathfrak{U}$ that are equivalent modulo Θ, then these two elements belong to the same sets in B, by definition of the first dual of B. Since C is assumed to be included in B, it follows that r and s belong to the same sets in C, and therefore r and s are equivalent modulo Ψ, by the definition of first dual of C. Thus Θ is included in Ψ.

To prove the reverse implication, assume that Θ is included in Ψ. The subuniverses B and C are defined to be the sets of clopen subsets of $\mathfrak{U}$ that are compatible with Θ and Ψ respectively. Consequently, if F is an arbitrary element in C, then F must be clopen and compatible with Ψ. It follows from the assumed inclusion that F must also be compatible with Θ, and therefore must belong to B. In more detail, if r is an element in F, then the entire equivalence class r/Ψ is included in F, by the compatibility of F with Ψ. Since Θ is included in Ψ, the equivalence class r/Θ is included in the equivalence class r/Ψ, and therefore r/Θ is also included in F. Thus, F is also compatible with Θ, and therefore F belongs to B, as claimed. This argument shows that C is included in B.

The preceding arguments imply that the function ϱ mapping each relational congruence on $\mathfrak{U}$ to its dual subuniverse of $\mathfrak{A}$ is a dual lattice isomorphism. Indeed, if δ is the function mapping each subuniverse of $\mathfrak{A}$ to its dual relational congruence on $\mathfrak{U}$, then

$$(\delta \circ \varrho)(\Theta) = \Theta \qquad \text{and} \qquad (\varrho \circ \delta)(B) = B$$

for each relational congruence Θ and each subuniverse B, because Θ is the dual of the dual of Θ, and B is the dual of the dual of B. Thus, $\delta \circ \varrho$ is the identity function on the lattice of relational congruences on $\mathfrak{U}$, and $\varrho \circ \delta$ is the identity function on the lattice of subuniverses of $\mathfrak{A}$, so ϱ and δ are bijections and inverses of one another. Since a relational congruence Θ is included in a relational congruence Ψ if and only if the dual of Ψ is included in the dual of Θ, that is to say, if and only if $\varrho(\Psi)$ is included in $\varrho(\Theta)$, the bijection ϱ reverses the lattice partial ordering, and is therefore a dual lattice isomorphism. The results of this section are summarized in the following theorem.

Theorem 2.33. *Suppose $\mathfrak{U}$ is a relational space and $\mathfrak{A}$ its dual algebra. The dual of every relational congruence on $\mathfrak{U}$ is a subuniverse of $\mathfrak{A}$, and the dual of every subuniverse of $\mathfrak{A}$ is a relational congruence on $\mathfrak{U}$.*

The second dual of every relational congruence and of every subuniverse is itself. The function mapping each relational congruence on $\mathfrak{U}$ *to its dual subuniverse of* $\mathfrak{A}$ *is a dual lattice isomorphism from the lattice of relational congruences on* $\mathfrak{U}$ *to the lattice of subuniverses of* $\mathfrak{A}$.

A related result, discovered independently by Celani, is given in Theorem 27 of [2].

2.8 Duality for Subalgebras

The duality between relational congruences and subuniverses implies a corresponding duality between quotient spaces and subalgebras. By the quotient of a relational space, we mean the following.

Definition 2.34. The *quotient* of a relational space $\mathfrak{U}$ modulo a relational congruence Θ is the quotient relational structure of $\mathfrak{U}$ (see Definition 1.26) endowed with the quotient topology. The quotient is denoted by $\mathfrak{U}/\Theta$. The *quotient function*, or *quotient mapping*, from $\mathfrak{U}$ to $\mathfrak{U}/\Theta$ is the function that maps each element u in $\mathfrak{U}$ to its congruence class u/Θ. □

The first observation to make is that every continuous bounded homomorphism gives rise to a relational congruence, namely its kernel.

Lemma 2.35. *If* ϑ *is a continuous bounded homomorphism from a relational space* $\mathfrak{U}$ *to a relational space* $\mathfrak{V}$, *then the kernel of* ϑ *is a relational congruence on* $\mathfrak{U}$.

Proof. The kernel of a continuous mapping between Boolean spaces is a Boolean relation, so the kernel of ϑ—call it Θ—is a Boolean relation. The kernel of a bounded homomorphism is a bounded congruence, by Lemma 1.27, so Θ is a bounded congruence on $\mathfrak{U}$. Therefore, Θ is a relational congruence, by Definition 2.30. □

The converse of the preceding lemma says that every relational congruence gives rise to a continuous bounded epimorphism. This observation is the analogue of Theorem 1.28, and its proof presents no difficulties.

Theorem 2.36. *If Θ is a relational congruence on a relational space $\mathfrak{U}$, then the corresponding quotient mapping is a continuous bounded epimorphism from $\mathfrak{U}$ to the quotient $\mathfrak{U}/\Theta$ that has Θ as its kernel.*

Proof. The quotient mapping ϑ is a bounded epimorphism from $\mathfrak{U}$ to $\mathfrak{U}/\Theta$ with kernel Θ, by Theorem 1.28, and ϑ is continuous because $\mathfrak{U}/\Theta$ is endowed with the quotient topology. □

The proof that quotients of relational spaces are relational spaces is more involved.

Lemma 2.37. *The quotient of a relational space modulo a relational congruence is a relational space.*

Proof. Suppose Θ is a relational congruence on a relational space $\mathfrak{U}$, and ϑ the quotient mapping from $\mathfrak{U}$ onto $\mathfrak{U}/\Theta$. The quotient $\mathfrak{U}/\Theta$ is certainly a relational structure. The congruence Θ is, by definition, a Boolean relation, so the quotient topology on the quotient of the universe of $\mathfrak{U}$ turns that quotient into a Boolean space and turns ϑ into a continuous function. It remains to check that the fundamental relations in the quotient are clopen, and those of rank at least two are continuous, in the quotient topology.

Focus on the case of a ternary relation R. To see that R is clopen in $\mathfrak{U}/\Theta$, consider clopen subsets F and G of $\mathfrak{U}/\Theta$. It is to be shown that the image set

$$R^*(F \times G) = \{t \in U/\Theta : R(r,s,t) \text{ for some } r \in F \text{ and } s \in G\} \tag{1}$$

is clopen in the quotient topology. The inverse images

$$\bar{F} = \vartheta^{-1}(F) \qquad \text{and} \qquad \bar{G} = \vartheta^{-1}(G) \tag{2}$$

are clopen subsets of $\mathfrak{U}$ because the mapping ϑ is continuous, and they are compatible with Θ because ϑ is the quotient mapping, so the image set

$$R^*(\bar{F} \times \bar{G}) = \{w \in U : R(u,v,w) \text{ for some } u \in \bar{F} \text{ and } v \in \bar{G}\} \tag{3}$$

is certainly clopen in $\mathfrak{U}$, by the assumption that $\mathfrak{U}$ is a relational space (see Definition 2.2). Furthermore, the image set in (3) is compatible with Θ. Indeed, consider any element w in the image set, and suppose that $\bar{w}$ belongs to the congruence class w/Θ; it must be shown that $\bar{w}$ is also in the image set. Since w is in (3), there are elements u in $\bar{F}$ and v

in $\bar{G}$ such that $R(u,v,w)$ holds in $\mathfrak{U}$. The congruence Θ is bounded, by definition, and $\bar{w} \equiv w \mod \Theta$, by assumption, so there must be elements $\bar{u}$ and $\bar{v}$ in $\mathfrak{U}$ such that

$$\bar{u} \equiv u \mod \Theta, \qquad \bar{v} \equiv v \mod \Theta, \qquad \text{and} \qquad R(\bar{u},\bar{v},\bar{w}). \tag{4}$$

The sets $\bar{F}$ and $\bar{G}$ are compatible with Θ and contain the elements u and v respectively, so they also contain the elements $\bar{u}$ and $\bar{v}$ respectively, by the definition of compatibility. It follows that $\bar{w}$ belongs to the image set in (3), by the final part of (4). Thus, the image set in (3) is compatible with Θ, as claimed. The quotient of a clopen set in $\mathfrak{U}$ that is compatible with Θ is a clopen set in the quotient $\mathfrak{U}/\Theta$ (see the remarks following Lemma 2.32). Consequently, the quotient set

$$R^*(\bar{F} \times \bar{G})/\Theta = \{w/\Theta : w \in R^*(\bar{F} \times \bar{G})\}$$

is clopen in $\mathfrak{U}/\Theta$.

In view of the preceding observations, to prove that the set in (1) is clopen it suffices to show that

$$R^*(F \times G) = R^*(\bar{F} \times \bar{G})/\Theta. \tag{5}$$

Consider an arbitrary element t in $\mathfrak{U}/\Theta$, and assume first that t belongs to the left side of (5). There are then elements r in F and s in G such that $R(r,s,t)$ holds in $\mathfrak{U}/\Theta$, by (1). The relation R in $\mathfrak{U}/\Theta$ is defined to be the quotient of the corresponding relation in $\mathfrak{U}$, so there must be elements u, v, and w in $\mathfrak{U}$ such that

$$r = u/\Theta, \qquad s = v/\Theta, \qquad t = w/\Theta, \qquad \text{and} \qquad R(u,v,w)$$

in $\mathfrak{U}$. The function ϑ maps each element in $\mathfrak{U}$ to its quotient, so we must have

$$\vartheta(u) = u/\Theta = r, \qquad \vartheta(v) = v/\Theta = s, \qquad \vartheta(w) = w/\Theta = t. \tag{6}$$

Since r and s belong to the sets F and G respectively, it follows from (6) and (2) that u and v belong to the inverse image sets $\bar{F}$ and $\bar{G}$ respectively. Consequently, w belongs to the set in (3), and therefore the quotient w/Θ, which is equal to t, belongs to the quotient on the right side of (5).

Assume now that t belongs to the right side of (5). In this case, t is equal to w/Θ for some element w in $R^*(\bar{F} \times \bar{G})$, so there must elements u in $\bar{F}$ and v in $\bar{G}$ such that $R(u,v,w)$ holds in $\mathfrak{U}$, by (3).

Write $r = u/\Theta$ and $s = v/\Theta$, and observe two things: first, $R(r, s, t)$ must hold in the quotient $\mathfrak{U}/\Theta$, by the definition of the relation R in the quotient; and second, r and s must belong to the sets F and G respectively, by (2), (6), and the fact that u and v are in $\bar{F}$ and $\bar{G}$ respectively. Therefore, t belongs to the left side of (5), by (1). Conclusion: the equation in (5) holds, so the relation R in $\mathfrak{U}/\Theta$ is clopen, as claimed.

The final task is to prove that the relation R in $\mathfrak{U}/\Theta$ is continuous. In view of Theorem 2.14, it suffices to prove that R is weakly continuous in the sense of Definition 2.12. Consider an element t in $\mathfrak{U}/\Theta$, and write

$$H = \{(r, s) \in U/\Theta \times U/\Theta : R(r, s, t)\}. \tag{7}$$

It is to be shown that the set H is closed in the product topology on the space

$$U/\Theta \times U/\Theta. \tag{8}$$

The element t is, by assumption, the quotient of some element in $\mathfrak{U}$, say

$$\vartheta(w) = w/\Theta = t. \tag{9}$$

The set

$$\bar{H} = \{(u, v) \in U \times U : R(u, v, w)\} \tag{10}$$

is certainly closed in $\mathfrak{U}$, because $\mathfrak{U}$ is a relational space and therefore the relation R in $\mathfrak{U}$ is weakly continuous, by Theorem 2.13. The quotient function ϑ is continuous, because U/Θ is endowed with the quotient topology, so the induced function $\bar{\vartheta}$ defined by

$$\bar{\vartheta}(u, v) = (\vartheta(u), \vartheta(v))$$

for u and v in U is a continuous mapping from $U \times U$ to $U/\Theta \times U/\Theta$. A continuous function from a compact space to a Hausdorff space maps closed sets to closed sets (see Corollary 1 on p. 315 of [10]). Since the spaces $U \times U$ and $U/\Theta \times U/\Theta$ are both Boolean (and hence both compact Hausdorff spaces), and since the set $\bar{H}$ is closed, the image set

$$\bar{\vartheta}(\bar{H}) = \{\bar{\vartheta}(u, v) : (u, v) \in \bar{H}\}$$

must also be closed.

In view of the observations of the preceding paragraph, to prove that the set H in (7) is closed in the space (8), it suffices to show that

$$\bar{\vartheta}(\bar{H}) = H. \tag{11}$$

To this end, consider an arbitrary pair (u, v) in $\bar{H}$. Because ϑ is a homomorphism, by Theorem 2.36, and because $R(u, v, w)$ holds in $\mathfrak{U}$, by (10), we have $R(\vartheta(u), \vartheta(v), \vartheta(w))$ in $\mathfrak{U}/\Theta$. Also, $\vartheta(w) = t$, by (9), so it follows from (7) (with r and s replaced by $\vartheta(u)$ and $\vartheta(v)$ respectively) that the pair $(\vartheta(u), \vartheta(v))$ belongs to the set H. Apply the definition of the function $\bar{\vartheta}$ to see that $\bar{\vartheta}(u, v)$ must belong to H. Conclusion: the image set $\bar{\vartheta}(\bar{H})$ is included in H.

To establish the reverse inclusion, assume that (r, s) is an arbitrary pair in H. In this case, $R(r, s, t)$ holds in the quotient $\mathfrak{U}/\Theta$, by (7), and therefore $R(r, s, \vartheta(w))$ holds in the quotient, by (9). The quotient mapping ϑ is a bounded epimorphism, by Theorem 2.36, so there must be elements u and v in $\mathfrak{U}$ such that

$$\vartheta(u) = r, \qquad \vartheta(v) = s, \qquad \text{and} \qquad R(u, v, w).$$

The pair (u, v) belongs to $\bar{H}$, by (10), and

$$\bar{\vartheta}(u, v) = (\vartheta(u), \vartheta(v)) = (r, s),$$

so (r, s) belongs to the image set $\bar{\vartheta}(\bar{H})$. Conclusion: H is included in the set $\bar{\vartheta}(\bar{H})$. This completes the proof of (11), so H is closed in the product space (8), as claimed. Consequently, the relation R in $\mathfrak{U}/\Theta$ is weakly continuous, and therefore continuous. □

The next theorem is the analogue for relational spaces of the First Isomorphism Theorem for algebras. It says that up to homeo-isomorphisms, the only continuous bounded homomorphic images of a relational space are the relational quotients of that space.

Theorem 2.38. *Every continuous bounded homomorphic image of a relational space $\mathfrak{U}$ is homeo-isomorphic to a quotient of $\mathfrak{U}$ modulo a relational congruence on $\mathfrak{U}$. In fact, if ϑ is a continuous bounded epimorphism from $\mathfrak{U}$ to a relational space $\mathfrak{V}$, and if Θ is the kernel of ϑ, then the function mapping u/Θ to $\vartheta(u)$ for each u in $\mathfrak{U}$ is a homeo-isomorphism from $\mathfrak{U}/\Theta$ to $\mathfrak{V}$.*

Proof. The kernel Θ is a relational congruence on $\mathfrak{U}$, by Lemma 2.35. The function that maps each congruence class u/Θ to the element $\vartheta(u)$ is a well-defined homeomorphism from the topological structure of $\mathfrak{U}/\Theta$ to the topological structure of $\mathfrak{V}$, by the First Isomorphism Theorem for topological spaces, and it is an isomorphism from the algebraic structure of $\mathfrak{U}/\Theta$ to the algebraic structure of $\mathfrak{V}$, by First Isomorphism Theorem for relational structures (see Theorem 1.29). Consequently, this function is a homeo-isomorphism from $\mathfrak{U}/\Theta$ to $\mathfrak{V}$. □

We are now ready to establish the duality between quotient relational spaces and subalgebras of relational algebras.

Theorem 2.39. *There is a bijective correspondence between the quotients of a relational space* $\mathfrak{U}$ *and the subalgebras of its dual algebra* $\mathfrak{A}$*. If* Θ *is a relational congruence on* $\mathfrak{U}$*, then the dual algebra of the quotient* $\mathfrak{U}/\Theta$ *is isomorphic to the subalgebra of* $\mathfrak{A}$ *whose universe is the dual subuniverse of* Θ*. Inversely, if* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{A}$*, then the dual space of* $\mathfrak{B}$ *is homeo-isomorphic to the quotient space* $\mathfrak{U}/\Theta$*, where* Θ *is the dual relational congruence of the universe of* $\mathfrak{B}$*.*

Proof. Each relational congruence Θ on $\mathfrak{U}$ uniquely determines a subalgebra of the dual algebra $\mathfrak{A}$, namely the subalgebra whose universe is the dual subuniverse of Θ, that is to say, the subalgebra whose universe is the set of clopen subsets of $\mathfrak{U}$ that are compatible with Θ, by Lemma 2.31. Conversely, each subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ uniquely determines a relational congruence on $\mathfrak{U}$, namely the relational congruence that is the dual of the universe of $\mathfrak{B}$, that is to say, the congruence consisting of the pairs of elements from $\mathfrak{U}$ that belong to exactly the same sets in $\mathfrak{B}$, by Lemma 2.32. Because subuniverses and relational congruences are their own second duals, it follows that the correspondence mapping each quotient space $\mathfrak{U}/\Theta$ to the dual subalgebra $\mathfrak{B}$ is a bijection from the set of quotient spaces of $\mathfrak{U}$ to the set of subalgebras of $\mathfrak{A}$, by Theorem 2.33.

To prove the second assertion of the theorem, suppose Θ is a relational congruence on $\mathfrak{U}$, and let $\mathfrak{C}$ be the dual algebra of the quotient space $\mathfrak{U}/\Theta$. The elements in $\mathfrak{C}$ are the clopen subsets of $\mathfrak{U}/\Theta$, so they are the sets of the form

$$F/\Theta = \{u/\Theta : u \in F\},$$

where F ranges over the clopen subsets of $\mathfrak{U}$ that are compatible with Θ. The quotient function ϑ from $\mathfrak{U}$ to $\mathfrak{U}/\Theta$, which maps each element u to its congruence class u/Θ, is a continuous bounded epimorphism, by Theorem 2.36. The dual of ϑ is the monomorphism φ from $\mathfrak{C}$ into $\mathfrak{A}$ that is defined by

$$\varphi(F/\Theta) = \vartheta^{-1}(F/\Theta)$$

for each element F/Θ in $\mathfrak{C}$, by Theorem 2.16. For clopen sets F that are compatible with Θ, the inverse image under ϑ of the quotient set F/Θ is just F, because F includes the equivalence class of each of its elements. Consequently,

$$\varphi(F/\Theta) = F$$

for each set F/Θ in $\mathfrak{C}$. It follows from these observations that φ maps the universe of $\mathfrak{C}$ bijectively to the subuniverse of $\mathfrak{A}$ consisting of the clopen subsets of $\mathfrak{U}$ that are compatible with Θ, so φ is an isomorphism from $\mathfrak{C}$ to the subalgebra of $\mathfrak{A}$ whose universe is the dual of Θ.

To prove the third assertion of the theorem, consider an arbitrary subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, and let Θ be the relational congruence on $\mathfrak{U}$ that is the dual of the universe of $\mathfrak{B}$. The dual of Θ is, by definition, the second dual of the universe of $\mathfrak{B}$, and the universe of $\mathfrak{B}$ is its own second dual, by Theorem 2.33. Apply the part of the theorem already proved to conclude that the dual algebra of $\mathfrak{U}/\Theta$—call it $\mathfrak{C}$—is isomorphic to $\mathfrak{B}$. The dual relational spaces of these two algebras must therefore be homeo-isomorphic, by Theorem 2.18. The dual relational space of $\mathfrak{C}$ is, by definition, the second dual of the quotient space $\mathfrak{U}/\Theta$, and these two spaces are homeo-isomorphic, by Theorem 2.11. Consequently, $\mathfrak{U}/\Theta$ is homeo-isomorphic to the dual space of $\mathfrak{B}$, as desired. □

As in the case of relational structures and their dual complete and atomic Boolean algebras with operators (see the remarks at the end of Section 1.9), there are weaker, less explicit versions of Theorems 2.29 and 2.39 that are known from the literature and that follow almost immediately from the epi-mono duality between continuous bounded homomorphisms and homomorphisms formulated in Theorem 2.20. To describe these weaker versions, consider two relational spaces $\mathfrak{U}$ and $\mathfrak{V}$ with dual algebras $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

If $\mathfrak{V}$ is homeo-isomorphic to an inner subspace of $\mathfrak{U}$, say via a continuous bounded monomorphism ϑ, then the dual of ϑ is an epimorphism φ from $\mathfrak{A}$ to $\mathfrak{B}$, so that $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. Conversely, if $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, say via an epimorphism φ, then the dual of φ is a continuous bounded monomorphism ϑ from $\mathfrak{V}$ to $\mathfrak{U}$, so that $\mathfrak{V}$ is isomorphic to an inner subspace of $\mathfrak{U}$. Thus, there is a duality between (homeo-isomorphic copies of) inner subspaces of a relational space $\mathfrak{U}$ and homomorphic images of the dual algebra $\mathfrak{A}$.

In a similar vein, if $\mathfrak{V}$ is a continuous bounded homomorphic image of $\mathfrak{U}$, say via a continuous bounded epimorphism ϑ, then the dual of ϑ is a monomorphism φ from $\mathfrak{B}$ to $\mathfrak{A}$, so that $\mathfrak{B}$ is isomorphic to a subalgebra of $\mathfrak{A}$. Conversely, if $\mathfrak{B}$ is isomorphic to a subalgebra of $\mathfrak{A}$, say via a monomorphism φ, then the dual of φ is a continuous bounded epimorphism from $\mathfrak{U}$ to $\mathfrak{V}$, so that $\mathfrak{V}$ is a continuous bounded homomorphic image of $\mathfrak{U}$. Thus, there is a duality between continuous bounded homomorphic images of a relational space $\mathfrak{U}$ and (isomorphic copies of) subalgebras of the dual algebra $\mathfrak{A}$.

For relational spaces with a single binary relation and Boolean algebras with a single unary operator (modal algebras), a version of the dualities just described is implicit in Goldblatt [12] (see, for example, Theorems 5.9 and 10.9). For more general similarity types, a corresponding version of these dualities follows from Theorems 2.3.1 and 2.3.2 in Goldblatt [13]. (See also Theorem 5.28 in Venema [40].)

Theorems 2.29 and 2.39 strengthen these known duality results by making explicit exactly how the dual structures are constructed. First of all, a duality between ideals and special open sets is constructed—the dual of an ideal M is the special open set H that is the union of the sets in M, and the dual of a special open set H is the ideal M of clopen sets that are included in H—and this duality proves to be a lattice isomorphism (Theorem 2.23). Similarly, a duality between relational congruences and subuniverses is constructed—the dual of a relational congruence Θ is the subuniverse B that consists of clopen sets that are compatible with Θ, and the dual of a subuniverse B is the relational congruence Θ consisting of pairs of elements that belong to the same sets in B—and this duality proves to be a dual lattice isomorphism (Theorem 2.33).

Return now to the relational spaces $\mathfrak{U}$ and $\mathfrak{V}$ with their dual algebras $\mathfrak{A}$ and $\mathfrak{B}$. If $\mathfrak{V}$ is homeo-isomorphic to an inner subspace, say $\mathfrak{W}$, of $\mathfrak{U}$, then the dual algebra $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, and in fact it is isomorphic to the quotient of $\mathfrak{A}$ modulo the ideal that is the dual of the special open set that is the complement of the universe of $\mathfrak{W}$. Conversely, if $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, say via an epimorphism φ, then $\mathfrak{B}$ is isomorphic to the quotient of $\mathfrak{A}$ modulo the ideal M that is the kernel of φ, and therefore the dual structure $\mathfrak{V}$ is homeo-isomorphic to the inner subspace of $\mathfrak{U}$ whose universe is the complement of the special open set that is the dual of the ideal M. This is the content of Theorem 2.29.

Similarly, if $\mathfrak{V}$ is a continuous bounded homomorphic image of $\mathfrak{U}$, say via a continuous bounded epimorphism ϑ, then $\mathfrak{V}$ is homeo-isomorphic to the quotient of $\mathfrak{U}$ modulo the relational congruence Θ that is the kernel of ϑ, and therefore the dual algebra $\mathfrak{B}$ is isomorphic to the subalgebra of $\mathfrak{A}$ whose universe is the dual of the congruence Θ. Conversely, if $\mathfrak{B}$ is isomorphic to a subalgebra, say $\mathfrak{C}$, of $\mathfrak{A}$, then the dual space $\mathfrak{V}$ is homeo-isomorphic to the quotient of $\mathfrak{U}$ modulo the relational congruence that is the dual of the universe of the subalgebra $\mathfrak{C}$, so that $\mathfrak{V}$ is a continuous bounded homomorphic image of $\mathfrak{U}$. This is the content of Theorem 2.39.

2.9 Duality for Completeness

What special properties does the dual space of a complete Boolean algebra with operators possess? (In this section, we do not assume that the operators of a complete algebra are necessarily complete.) The answer is that the space must be *complete* in the topological sense that the closure of every open set is open and hence clopen. The proof of the duality between complete algebras and complete spaces is based on the next lemma, which gives a topological characterization of the suprema that happen to be formable in a not necessarily complete Boolean algebra with operators. The lemma has other applications as well.

Lemma 2.40. *If* $(F_i : i \in I)$ *is a system of elements (clopen sets) in the dual algebra* $\mathfrak{A}$ *of a relational space* $\mathfrak{U}$, *and if* $H = \bigcup_i F_i$, *then a necessary and sufficient condition for the given system to have a supremum in* $\mathfrak{A}$ *is that the closure* H^- *of the set* H *in* $\mathfrak{U}$ *be open. If this condition is satisfied, then*

$$\textstyle\sum_i F_i = H^-,$$

that is to say, the algebraic supremum of the given system is the closure of the set-theoretical union.

Proof. The proof is similar to the proof of the analogous result for Boolean algebras (see Lemma 1 on p. 368 of [10]). Assume first that the supremum F of the given system

$$(F_i : i \in I) \tag{1}$$

exists in $\mathfrak{A}$. The set F belongs to $\mathfrak{A}$, by assumption, so it is a clopen subset of $\mathfrak{U}$, by the definition of $\mathfrak{A}$ as the algebra of clopen subsets of $\mathfrak{U}$. Since F is closed and includes each set F_i, it must include the union H of these sets, and therefore it must also include the closure H^- of this union; in more detail,

$$H \subseteq F \qquad \text{implies} \qquad H^- \subseteq F^- = F.$$

Since F is open, the difference $F - H^-$ is also open. If this difference were non-empty, then it would include a non-empty clopen set G, because the clopen sets form a base for the topology on $\mathfrak{U}$. The difference $F - G$ would then be a clopen set—and hence an element in $\mathfrak{A}$—that is properly included in F (because G is non-empty), and that includes each set F_i (because G is included in $F - H^-$ and is therefore disjoint from the union H of the sets F_i). Consequently, F could not be the supremum in $\mathfrak{A}$ of the system of sets in (1), in contradiction to the assumption on F. It follows that $F - H^-$ must be empty, and consequently $H^- = F$. In other word, H^- is the supremum in $\mathfrak{A}$ of the system in (1), and therefore H^- is open.

If, conversely, the closure H^- is open, then H^- is clearly clopen, and of course it includes each of the sets F_i, so it is an upper bound in $\mathfrak{A}$ of the system in (1). If G is any upper bound in $\mathfrak{A}$ of the system in (1), then G is by definition a clopen set that includes each of the sets F_i. The union H of the system in (1) must then be included in G, and therefore the closure of H must be included in G, since G is closed. Thus, H^- is the least upper bound in $\mathfrak{A}$ of the system in (1). In other words, the system of sets in (1) has a supremum, and that supremum is H^-. □

Theorem 2.41. *The dual algebra of a relational space* $\mathfrak{U}$ *is* (*algebraically*) *complete if and only if* $\mathfrak{U}$ *is* (*topologically*) *complete.*

Proof. Let $\mathfrak{A}$ be the dual algebra of the relational space $\mathfrak{U}$, and assume first that $\mathfrak{A}$ is complete. An arbitrary open subset H of $\mathfrak{U}$ is the union of its clopen subsets, because the clopen sets form a base for the topology of $\mathfrak{U}$. The system of these clopen subsets has a supremum in $\mathfrak{A}$, by the assumption that $\mathfrak{A}$ is complete. Apply Lemma 2.40 to conclude that the set H^- must be open. It follows that the space $\mathfrak{U}$ is complete.

Now suppose that the space $\mathfrak{U}$ is complete, and consider an arbitrary system of elements in $\mathfrak{A}$. The elements in this system are clopen subsets of $\mathfrak{U}$, because $\mathfrak{A}$—as the dual algebra of $\mathfrak{U}$—consists of the clopen subsets of $\mathfrak{U}$. Consequently, the union H of this system of elements is open in $\mathfrak{U}$. The closure H^- must also be open in $\mathfrak{U}$, by the assumption that $\mathfrak{U}$ is complete. Apply Lemma 2.40 to conclude that the given system of elements in $\mathfrak{A}$ has a supremum, and that supremum is H^-. It follows that $\mathfrak{A}$ is complete. □

A Boolean algebra with operators $\mathfrak{A}$ may possess a certain degree of completeness without being complete. For instance, $\mathfrak{A}$ is defined to be *countably complete*, or *σ-complete*, if the supremum of every countable set of elements in $\mathfrak{A}$ exists. There is a version of Theorem 2.41 that applies to countably complete algebras. To formulate it, we need the notion of a Baire set. A *Baire set* in a Boolean space is a set that belongs to the countably complete Boolean set algebra generated by the set of clopen subsets of the space. In other words, a Baire set is a set that can be obtained from the class of clopen sets by repeated applications of the operations of forming unions and intersections of countable systems of sets. (The operation of forming complements is not really needed because the complement of a countable union—respectively a countable intersection—of sets is the countable intersection—respectively the countable union—of the complements of the sets, by the infinitary versions of the De Morgan laws, and the complement of a clopen set is again a clopen set.) The main topological result about the structure of open Baire sets is that every open Baire set in a Boolean space is the union of countably many clopen sets (see Corollary 1 on p. 375 of [10]). We shall say that a relational space is *countably complete*, or a *σ-space*, if the closure of every open Baire set is open.

Theorem 2.42. *The dual algebra of a relational space $\mathfrak{U}$ is* (*algebraically*) *countably complete if and only if $\mathfrak{U}$ is* (*topologically*) *countably complete.*

Proof. Let $\mathfrak{A}$ be the dual algebra of the relational space $\mathfrak{U}$, and assume first that $\mathfrak{A}$ is countably complete. If H is an open Baire set in $\mathfrak{U}$, then H must be the union of a countable system of clopen sets, by the remarks preceding the theorem. Since this countable system has a supremum in $\mathfrak{A}$, by the assumed countable completeness of $\mathfrak{A}$, it follows from Lemma 2.40 that H^- is open. Consequently, $\mathfrak{U}$ is a countably complete space, by the definition of such a space.

Assume now that the space $\mathfrak{U}$ is countably complete, and consider any countable system of elements in $\mathfrak{A}$. The elements in this system are clopen subsets of $\mathfrak{U}$ (since they belong to the dual algebra of $\mathfrak{U}$), so their union H is an open set and also a Baire set in $\mathfrak{U}$, by the definition of a Baire set. The closure H^- is therefore open, by the assumed countable completeness of $\mathfrak{U}$. Apply Lemma 2.40 to conclude that the given countable system of elements in $\mathfrak{A}$ has a supremum, and that supremum is H^-. It follows that $\mathfrak{A}$ is a countably complete algebra. $\square$

2.10 Duality for Finite Products

What can one say about the dual space of a direct product of Boolean algebras with operators? This question is easier to answer for products of finite systems than for products of infinite systems: the dual space of the product of finitely many Boolean algebras with operators is homeo-isomorphic to the disjoint union of the dual relational spaces. (A result to this effect is mentioned in passing in Jónsson [19], before Definition 3.2.5, but no details or proofs are given.) Here are the details.

Suppose $(U_i : i \in I)$ is a system of mutually disjoint topological spaces, and U is the union of the sets U_i, for i in I. Every subset F of U can be written in a unique way as a union $F = \bigcup_i F_i$, where F_i is a subset of U_i for each i, namely $F_i = F \cap U_i$; the sets F_i are called the *components* of F. A subset F of U is declared to be open in U if and only if each of its components F_i is open in the corresponding component space U_i. The set of open sets so defined constitutes a topology on U, called the *union topology*, and the resulting topological space is called the *union* of the given system of *component spaces*. Since the complement of a subset F of U is the union of the system of complements of the components F_i (in U_i), it follows that F is closed in U if and only if each component F_i is closed in the component space U_i. Consequently, F is clopen in U if and only if each component is clopen in the corresponding component space.

The union of a disjoint system of topological spaces inherits a number of properties from the component spaces. For instance, the union is a Hausdorff space if and only if each component space is Hausdorff, and the union has a base consisting of clopen sets just in case each component space has a base consisting of clopen sets. If each of the component spaces is compact, then the union is *locally compact* in the sense that every point belongs to the interior of some compact subset. The property of compactness, however, is not inherited by the union unless the given system consists of only finitely many (compact) spaces. If the system is finite in this sense of the word, then the union is a Boolean space if and only if each component space is a Boolean space.

Turn now to the definition of the union of a system of relational spaces.

Definition 2.43. The *union* of a system $(\mathfrak{U}_i : i \in I)$ of mutually disjoint relational spaces is defined to be the structure $\mathfrak{U}$ such that the algebraic part of $\mathfrak{U}$ is the union, in the sense of Definition 1.31, of the relational structures in the given system, and the topology on $\mathfrak{U}$ is the union topology induced by the topologies of the component spaces $\mathfrak{U}_i$. The *disjoint union* of an arbitrary system $(\mathfrak{V}_i : i \in I)$ of relational spaces is defined to be the union of the disjoint system $(\mathfrak{U}_i : i \in I)$ in which $\mathfrak{U}_i$ is the homeo-isomorphic image of the space $\mathfrak{V}_i$ under the function that maps each element r in $\mathfrak{V}_i$ to the element (r, i). $\square$

In connection with the second half of this definition, see the remarks following Definition 1.31.

Lemma 2.44. *If* $\mathfrak{U}$ *is the union of a disjoint system* $(\mathfrak{U}_i : i \in I)$ *of relational spaces, then each component space* $\mathfrak{U}_i$ *is an inner subspace of the union* $\mathfrak{U}$ *in the sense that the topology on* $\mathfrak{U}_i$ *is the one inherited from* $\mathfrak{U}$, *and algebraically* $\mathfrak{U}_i$ *is an inner substructure of* $\mathfrak{U}$. *Moreover, a subset of* $\mathfrak{U}_i$ *is open, closed, clopen, or compact in* $\mathfrak{U}$ *if and only if it has the same property in* $\mathfrak{U}_i$.

Proof. Focus on the case of a ternary relation R in $\mathfrak{U}$. This relation is, by definition, the disjoint union of the corresponding relations, say R_i, in $\mathfrak{U}_i$, for i in I. In particular, the restriction of R to the universe of $\mathfrak{U}_i$ must coincide with the relation R_i. For the same reason, if t is an element in $\mathfrak{U}_i$, and if $R(r, s, t)$ holds in $\mathfrak{U}$, then the triple (r, s, t) must belong to the relation R_i, so that r and s must be in $\mathfrak{U}_i$. Thus, $\mathfrak{U}_i$ is algebraically an inner substructure of $\mathfrak{U}$, by the definition of such a substructure.

If F is any open subset of $\mathfrak{U}$, then for each index i, the component of F in $\mathfrak{U}_i$, which is just the intersection $F \cap U_i$, is open in $\mathfrak{U}_i$, by the definition of the union topology on $\mathfrak{U}$. Conversely, if G is any open subset of $\mathfrak{U}_i$, and if for each index j, the set F_j is defined by

$$F_j = \begin{cases} G & \text{if} \quad i = j, \\ \varnothing & \text{if} \quad i \neq j, \end{cases}$$

then the union $F = \bigcup_j F_j$ is open in $\mathfrak{U}$, since each component of this union is open in the relevant component space. Obviously,

$$G = F = F \cap U_i,$$

so every open set in $\mathfrak{U}_i$ is the intersection with U_i of an open set in $\mathfrak{U}$. Conclusion: the topology on $\mathfrak{U}_i$ is the one inherited from $\mathfrak{U}$.

The preceding argument also shows that every open set G in $\mathfrak{U}_i$ is actually equal to an open set F in $\mathfrak{U}$. Conversely, if F is any open set in $\mathfrak{U}$ that is a subset of $\mathfrak{U}_i$, then F must coincide with its component in $\mathfrak{U}_i$ and therefore must be open in $\mathfrak{U}_i$, by the definition of the union topology. Thus, a subset of $\mathfrak{U}_i$ is open in $\mathfrak{U}_i$ if and only if it is open in $\mathfrak{U}$. Similar arguments apply to closed sets, clopen sets, and compact sets. □

The disjoint union of a system of relational spaces is almost a relational space. The compactness property may fail, but the union is at any rate locally compact.

Lemma 2.45. *Suppose $\mathfrak{U}$ is the union of a disjoint system of relational spaces. The union topology turns the universe of $\mathfrak{U}$ into a locally compact Hausdorff space in which the clopen sets form a base for the topology. Under this topology, the relations in $\mathfrak{U}$ are clopen and continuous.*

Proof. Let $\mathfrak{U}$ be the union of a disjoint system $(\mathfrak{U}_i : i \in I)$ of relational spaces. A subset of $\mathfrak{U}_i$ is open, closed, clopen, or compact in $\mathfrak{U}$ if and only if it is open, closed, clopen, or compact in $\mathfrak{U}_i$, by Lemma 2.44. One consequence of this observation is that the clopen sets in $\mathfrak{U}$ form a base for the topology on $\mathfrak{U}$. In fact, the clopen subsets of the component spaces form a base for the topology on $\mathfrak{U}$. A second consequence is that $\mathfrak{U}$ is a Hausdorff space. Indeed, two points belonging to the same component space $\mathfrak{U}_i$ are separated by a clopen subset of $\mathfrak{U}_i$, while two points belonging to distinct component spaces $\mathfrak{U}_i$ and $\mathfrak{U}_j$ are separated by the clopen sets U_i and U_j; and all these separating sets remain clopen in $\mathfrak{U}$. A third consequence is that $\mathfrak{U}$ is locally compact. In fact, if r is any point in $\mathfrak{U}$, then r belongs to one of the component spaces $\mathfrak{U}_i$, and U_i is an open compact set (both in $\mathfrak{U}_i$ and in $\mathfrak{U}$) that contains r. Thus, $\mathfrak{U}$ has the topology of a locally compact Hausdorff space in which the clopen sets form a base for the topology.

The next task is to show that the relations in $\mathfrak{U}$ are clopen and those of rank at least two are continuous. Focus on the case of a ternary relation R that is the disjoint union of the corresponding ternary relations R_i in $\mathfrak{U}_i$, for i in I. To prove that R is clopen, it must be shown that for any two clopen sets F and G in $\mathfrak{U}$, the image set

$$R^*(F \times G) = \{t \in U : R(r,s,t) \text{ for some } r \in F \text{ and } s \in G\} \qquad (1)$$

is clopen. Write F and G as the unions of their components,

$$F = \bigcup_i F_i \qquad \text{and} \qquad G = \bigcup_i G_i. \tag{2}$$

A set in $\mathfrak{U}$ is clopen if and only if each component of the set is clopen in the relevant component space, so the components F_i and G_i must be clopen sets in $\mathfrak{U}_i$ for each i. Consequently, the set

$$R_i^*(F_i \times G_i) = \{t \in U_i : R_i(r,s,t) \text{ for some } r \in F_i \text{ and } s \in G_i\} \tag{3}$$

is clopen in $\mathfrak{U}_i$ for each i, by the assumption that $\mathfrak{U}_i$ is a relational space. In order to prove that (1) is clopen, it therefore suffices to prove that

$$R^*(F \times G) = \bigcup_i R_i^*(F_i \times G_i), \tag{4}$$

by the definition of the union topology.

The equality in (4) follows rather easily from the fact that

$$R = \bigcup_i R_i \tag{5}$$

is a disjoint union. Indeed, consider an element t in $\mathfrak{U}$. If t belongs to the left side of (4), then there must be elements r in F and s in G such that $R(r,s,t)$ holds (in $\mathfrak{U}$), by (1). In view of (5), we must have $R_i(r,s,t)$ (in $\mathfrak{U}_i$) for some index i, so the elements r and s are in $\mathfrak{U}_i$, and therefore they belong to the sets

$$F \cap U_i = F_i \qquad \text{and} \qquad G \cap U_i = G_i$$

respectively. In view of (3), it follows that t belongs to $R_i^*(F_i \times G_i)$, so t belongs to the right side of (4). Thus, the left side of (4) is included in the right side.

To establish the reverse inclusion, assume that t belongs to the right side of (4). In this case, t is in $R_i^*(F_i \times G_i)$ for some index i. Consequently, there must be elements r in F_i and s in G_i such that $R_i(r,s,t)$ holds. Clearly, r is in F and s in G, by (2), and $R(r,s,t)$ holds in $\mathfrak{U}$, by (5), so t must belong to the left side of (4), by (1). Thus, the right side of (4) is included in the left. This establishes (4) and proves that the relation R is clopen.

To prove that R is continuous, consider an open subset H of $\mathfrak{U}$. It must be shown that the set

$$R^{-1}(H) = \{(r,s) \in U \times U : R(r,s,t) \text{ implies } t \in H\} \tag{6}$$

is open in the product topology on $U \times U$. Write H as the union of its components,

$$H = \bigcup_i H_i. \tag{7}$$

A set is open in $\mathfrak{U}$ if and only if each of its components is open in the relevant component space, so for each index i, the component H_i is open in $\mathfrak{U}_i$. Consequently, the set

$$R_i^{-1}(H_i) = \{(r,s) \in U_i \times U_i : R_i(r,s,t) \text{ implies } t \in H_i\} \tag{8}$$

is open in the product topology on $U_i \times U_i$—and therefore open in the product topology on $U \times U$—by the assumption that $\mathfrak{U}_i$ is a relational space. Also, each of the sets $U_i \times U_j$ is open in $U \times U$, since U_i and U_j are open subsets of $\mathfrak{U}$. Consequently, the union

$$G = (\bigcup_i R_i^{-1}(H_i)) \cup (\bigcup\{U_i \times U_j : i, j \in I \text{ and } i \neq j\}) \tag{9}$$

is open in $U \times U$.

In view of the preceding observations, it suffices to prove that

$$R^{-1}(H) = G. \tag{10}$$

Consider a pair (r,s) in $U \times U$. There are two possibilities: either r and s belong to the same component space or they belong to distinct component spaces. If they belong to distinct component spaces, then the pair (r,s) obviously belongs to G, by (9); and the pair vacuously belongs to $R^{-1}(H)$, by (6), because the relation $R(r,s,t)$ can never hold in $\mathfrak{U}$, by (5). Thus, $R^{-1}(H)$ and G contain the same pairs having coordinates in distinct component spaces.

Suppose now that r and s belong to the same component space, say $\mathfrak{U}_i$, with the goal of showing that the pair (r,s) belongs to the set $R^{-1}(H)$ if and only if it belongs to G. Assume first that (r,s) belongs to $R^{-1}(H)$. In order to prove that (r,s) belongs to $R_i^{-1}(H_i)$, and therefore to G, it must be shown that the hypothesis $R_i(r,s,t)$ implies that t belongs to H_i, by (8). The hypothesis implies that $R(r,s,t)$ must hold, by (5), and therefore t must belong to H, by (6). The hypothesis also implies that t must be in U_i, so t belongs to the intersection $H \cap U_i$, which is just H_i. Assume now that (r,s) belongs to G. Since r and s are assumed to be in the same component space $\mathfrak{U}_i$, the pair (r,s) must belong to $R_i^{-1}(H_i)$, by (9). In order to prove that (r,s) belongs to $R^{-1}(H)$, it must be shown that the hypothesis $R(r,s,t)$ implies t is in H. The hypothesis and the assumption that r and s are in $\mathfrak{U}_i$

imply that $R_i(r, s, t)$ holds, by (5); so t belongs to H_i, by (8), and therefore t belongs to H, by (7). This completes the proof of (10) and shows that R is continuous. □

The lemma implies that the disjoint union of a system of relational spaces possesses all of the properties of a relational space except perhaps compactness. Since the union of finitely many compact spaces is compact, we arrive at the following corollary.

Corollary 2.46. *The disjoint union of finitely many relational spaces is again a relational space.*

The next theorem describes the duality that exists between unions of finitely many disjoint relational spaces and the product of their dual algebras.

Theorem 2.47. *The dual algebra of the union of a finite system of disjoint relational spaces is equal to the internal product of the dual algebras of the system.*

Proof. Let $\mathfrak{U}$ be the union of a finite system

$$(\mathfrak{U}_i : i \in I) \tag{1}$$

of mutually disjoint relational spaces, and let $\mathfrak{A}$ and $\mathfrak{A}_i$ be the dual algebras of $\mathfrak{U}$ and $\mathfrak{U}_i$ respectively. It is to be shown that $\mathfrak{A}$ is the internal product of the system

$$(\mathfrak{A}_i : i \in I) \tag{2}$$

in the sense defined in Section 1.10 (see also the remarks following Definition 2.48 in the next section). The complex algebra $\mathfrak{Cm}(U)$ is equal to the internal product of the system of complex algebras

$$(\mathfrak{Cm}(U_i) : i \in I), \tag{3}$$

by Theorem 1.32. For each index i, the algebra $\mathfrak{A}_i$ is a subalgebra of $\mathfrak{Cm}(U_i)$, by its very construction, so the internal product of the system in (2) is a subalgebra of the internal product of the system in (3). Consequently, the internal product of the system in (2) is a subalgebra of $\mathfrak{Cm}(U)$. The universe of this internal product consists of the sets of the form

$$F = \textstyle\bigcup_i F_i, \tag{4}$$

where for each i, the set F_i belongs to $\mathfrak{A}_i$, that is to say, F_i is a clopen subset of $\mathfrak{U}_i$.

The algebra $\mathfrak{A}$ is also a subalgebra of $\mathfrak{Cm}(U)$, and its universe is the set of clopen subsets of $\mathfrak{U}$. Since $\mathfrak{U}$ is the union of the spaces in (1), the clopen subsets of $\mathfrak{U}$ are just the sets of the form (4), where F_i is a clopen subset of $\mathfrak{U}_i$ for each i. Thus, the universe of $\mathfrak{A}$ coincides with the universe of the internal product of the system in (2). Since $\mathfrak{A}$ and the internal product are both subalgebras of $\mathfrak{Cm}(U)$, and they have the same universe, they must be the same subalgebra. $\square$

The dual version of the preceding theorem says that the dual space of the direct product of a finite system of Boolean algebras with operators is homeo-isomorphic to the disjoint union of the system of dual relational spaces.

2.11 Duality for Subdirect Products

The description of the dual space of a direct product of infinitely many Boolean algebras with operators is more involved. There is in fact a dual correspondence between certain subalgebras of such products and the compactifications of unions of relational spaces. In view of Lemma 2.45, the disjoint union of an infinite system of relational spaces possesses all the properties of a relational space except possibly compactness, which has been replaced by a weaker property, namely local compactness. Let us call such a structure a *locally compact* relational space. Thus, a locally compact relational space is not required to be compact, whereas a relational space is compact by its very definition.

Definition 2.48. A *compactification* of a locally compact relational space $\mathfrak{U}$ is defined to be a relational space $\mathfrak{V}$ such that $\mathfrak{U}$ is a dense inner subspace of $\mathfrak{V}$ in the following sense: algebraically, $\mathfrak{U}$ is an inner substructure of $\mathfrak{V}$; the topology on $\mathfrak{U}$ is the one inherited from $\mathfrak{V}$; and the (topological) closure of the set U in $\mathfrak{V}$ is just the set V. $\square$

Our immediate goal is a description of the relationship between compactifications of disjoint unions of infinite systems relational spaces, and certain subalgebras of direct products of infinite systems of Boolean algebras with operators. We proceed to establish the notation that will be used in this description. Fix a disjoint system $(\mathfrak{U}_i : i \in I)$ of relational spaces for the remainder of this section, and let $\mathfrak{U}$ the union of this system. For each index i, let $\mathfrak{A}_i$ be the dual algebra of the space $\mathfrak{U}_i$, and let $\mathfrak{A}$ be the internal product of the system of dual algebras. The elements in $\mathfrak{A}$ are the subsets of $\mathfrak{U}$ of the form $F = \bigcup_i F_i$, where each set F_i is an element in $\mathfrak{A}_i$, that is to say, F_i is a clopen subset of $\mathfrak{U}_i$. The operations of $\mathfrak{A}$ are performed coordinatewise: if

$$F = \bigcup_i F_i \qquad \text{and} \qquad G = \bigcup_i G_i$$

are elements in $\mathfrak{A}$, then

$$F + G = \bigcup_i (F_i + G_i) \qquad \text{and} \qquad -F = \bigcup_i -F_i,$$

and if f is an operator of rank n, and if

$$F_0 = \bigcup_i F_{0,i}\ , \qquad \dots \qquad , F_{n-1} = \bigcup_i F_{n-1,i}$$

is a sequence of elements in $\mathfrak{A}$, then

$$f(F_0, \dots, F_{n-1}) = \bigcup_i f(F_{0,i}, \dots, F_{n-1,i}),$$

where the operations on the left sides of the equations above are those of $\mathfrak{A}$, while the ones on the right sides, after the union symbols, are the operations of the factor algebras $\mathfrak{A}_i$.

Lemma 2.49. *If $\mathfrak{V}$ is a compactification of $\mathfrak{U}$, then each component space $\mathfrak{U}_i$ is an inner subspace of $\mathfrak{V}$ in the sense of Definition 2.26. The universe of $\mathfrak{U}$ is an open subset of $\mathfrak{V}$. A subset of $\mathfrak{U}_i$ is open, closed, clopen, or compact in $\mathfrak{V}$ if and only if it has this same property in $\mathfrak{U}_i$.*

Proof. For each index i, the relational space $\mathfrak{U}_i$ is algebraically an inner substructure of $\mathfrak{U}$, by Lemma 2.44, and $\mathfrak{U}$ is an inner substructure of $\mathfrak{V}$, by Definition 2.48, so $\mathfrak{U}_i$ is an inner substructure of $\mathfrak{V}$, by the transitivity of the relation of being an inner substructure. Similarly, the universe of $\mathfrak{U}_i$ is topologically a subspace of the universe of $\mathfrak{U}$, by Lemma 2.44, and the universe of $\mathfrak{U}$ is a topologically a subspace of the universe of $\mathfrak{V}$, by Definition 2.48, so the universe of $\mathfrak{U}_i$ is topologically a subspace of the universe of $\mathfrak{V}$, by the transitivity of the relation of being a topological subspace. Combine these observations to conclude that $\mathfrak{U}_i$ is a subspace of $\mathfrak{V}$ in the sense of Definition 2.26.

Every dense, locally compact subspace of a Hausdorff space is open in that space (see Corollary 1 on p. 400 of [10]). Since the universe of $\mathfrak{U}$ is topologically a dense, locally compact subspace of $\mathfrak{V}$, it follows that this universe must be open in the topology of $\mathfrak{V}$.

The universe of $\mathfrak{U}_i$ is open in $\mathfrak{U}$, by Lemma 2.44, and the universe of $\mathfrak{U}$ is open in $\mathfrak{V}$, by the observations of the preceding paragraph, so the universe of $\mathfrak{U}_i$ is also open in $\mathfrak{V}$. Consequently, a subset of $\mathfrak{U}_i$ is open in $\mathfrak{V}$ if and only if it is open in $\mathfrak{U}_i$. From this it also follows that a subset F of $\mathfrak{U}_i$ is compact in $\mathfrak{V}$ if and only if it is compact in $\mathfrak{U}_i$. The reason is that every open cover of F in $\mathfrak{U}_i$ remains an open cover of F in $\mathfrak{V}$, and inversely, the intersection with the universe of $\mathfrak{U}_i$ of any open cover of F in $\mathfrak{V}$ yields an open cover of F in $\mathfrak{U}_i$. In compact Hausdorff spaces, the closed sets coincide with the compact sets. Consequently, a subset of $\mathfrak{U}_i$ is closed in $\mathfrak{V}$ if and only if it is closed in $\mathfrak{U}_i$, by the preceding remark. Combine these observations to conclude that a subset of $\mathfrak{U}_i$ is clopen in $\mathfrak{V}$ if and only if it is clopen in $\mathfrak{U}_i$. □

There is a subalgebra of the internal product $\mathfrak{A}$ that will play a special role in our discussion, namely the one that is generated by the union $\bigcup_i A_i$ of the universes of the factor algebras. The elements of this subalgebra are precisely those sets $F = \bigcup_i F_i$ in $\mathfrak{A}$ such that the system $(F_i : i \in I)$ is *constant almost everywhere* in the following sense: there is a term τ in the language of Boolean algebras with operators that is built up from the distinguished constant symbols (symbols for operations of rank 0, including 0 and 1) and the operation symbols, without using any variables, such that F_i coincides with the value of τ in $\mathfrak{A}_i$ for all but finitely many indices i. We denote this subalgebra by $\mathfrak{D}$ and call it the *weak internal product* of the system $(\mathfrak{A}_i : i \in I)$. We now prove that the dual of every compactification of $\mathfrak{U}$ corresponds to a subalgebra of $\mathfrak{A}$ that includes $\mathfrak{D}$.

Lemma 2.50. *If* $\mathfrak{B}$ *is the dual algebra of a compactification of* $\mathfrak{U}$, *then the set*

$$B_0 = \{F \cap U : F \in B\}$$

is a subuniverse of $\mathfrak{A}$ *that includes the universe of* $\mathfrak{D}$. *Moreover,* $\mathfrak{B}$ *is isomorphic to the corresponding subalgebra* $\mathfrak{B}_0$ *via the function that maps* F *to* $F \cap U$ *for each* F *in* $\mathfrak{B}$.

Proof. Let $\mathfrak{V}$ be a compactification of $\mathfrak{U}$, and $\mathfrak{B}$ the dual algebra of $\mathfrak{V}$. Algebraically, $\mathfrak{U}$ is an inner substructure of $\mathfrak{V}$, by Definition 2.48, so the identity function ϑ on $\mathfrak{U}$ is a bounded monomorphism from $\mathfrak{U}$ to $\mathfrak{V}$, by Corollary 1.14. The algebraic dual of ϑ is, by Theorem 1.9, the complete epimorphism ψ from $\mathfrak{Cm}(V)$ to $\mathfrak{Cm}(U)$ that is defined by

$$\psi(F) = \vartheta^{-1}(F)$$

for elements F in $\mathfrak{Cm}(V)$, that is to say, for subsets F of $\mathfrak{V}$. Since

$$\vartheta^{-1}(F) = \{r \in U : \vartheta(r) \in F\} = \{r \in U : r \in F\} = F \cap U,$$

by the definition of the inverse image under ϑ of a set, and the definition of ϑ as the identity function on $\mathfrak{U}$, it may be concluded that

$$\psi(F) = F \cap U \tag{1}$$

for each F in $\mathfrak{Cm}(V)$. The dual algebra $\mathfrak{B}$ is, by construction, a subalgebra of $\mathfrak{Cm}(V)$. The appropriate restriction of ψ therefore maps $\mathfrak{B}$ onto a subalgebra of $\mathfrak{Cm}(U)$, namely the subalgebra $\mathfrak{B}_0$ with universe B_0, by (1) and the homomorphism properties of ψ. We shall also use the symbol ψ to refer to this restriction.

The dual algebra $\mathfrak{A}_i$ of the relational space $\mathfrak{U}_i$ is a subalgebra of $\mathfrak{Cm}(U_i)$, by construction. The internal product $\mathfrak{A}$ of the system

$$(\mathfrak{A}_i : i \in I) \tag{2}$$

is therefore a subalgebra of the internal product of the system

$$(\mathfrak{Cm}(U_i) : i \in I).$$

Since the internal product of the latter system is just $\mathfrak{Cm}(U)$, by Theorem 1.32, it follows that $\mathfrak{A}$ is a subalgebra of $\mathfrak{Cm}(U)$. We proceed to show that B_0 is a subset of the universe of $\mathfrak{A}$. Since $\mathfrak{B}_0$ and $\mathfrak{A}$ are both subalgebras of $\mathfrak{Cm}(U)$, it then follows that $\mathfrak{B}_0$ is a subalgebra of $\mathfrak{A}$.

An arbitrary element in B_0 has the form $F \cap U$ for some set F in $\mathfrak{B}$, by the definition of B_0. The set F is clopen in $\mathfrak{V}$, by the definition of $\mathfrak{B}$ as the dual of $\mathfrak{V}$, and the universe U_i of $\mathfrak{U}_i$ is clopen in $\mathfrak{V}$, by Lemma 2.49, so the intersection of F with U_i, the set

$$F_i = F \cap U_i,$$

is clopen in $\mathfrak{V}$ and therefore also in $\mathfrak{U}_i$, by Lemma 2.49. Consequently, F_i belongs to $\mathfrak{A}_i$, by the definition of $\mathfrak{A}_i$ as the dual algebra of $\mathfrak{U}_i$. Since

$$F \cap U = F \cap (\textstyle\bigcup_i U_i) = \bigcup_i (F \cap U_i) = \bigcup_i F_i,$$

it follows from the definition of $\mathfrak{A}$ as the internal product of the system in (2) that $F \cap U$ belongs to $\mathfrak{A}$. Thus, B_0 is a subset of $\mathfrak{A}$, so $\mathfrak{B}_0$ is a subalgebra of $\mathfrak{A}$.

The next step is to show that $\mathfrak{D}$ is a subalgebra of $\mathfrak{B}_0$. Since both of these algebras are subalgebras of $\mathfrak{A}$, it suffices to show that every element in a set of generators of $\mathfrak{D}$ belongs to $\mathfrak{B}_0$. The algebra $\mathfrak{D}$ is, by definition, generated by the union of the universes of the algebras $\mathfrak{A}_i$, and the elements in these universes are just the clopen subsets of the component spaces $\mathfrak{U}_i$. An arbitrary clopen subset F of $\mathfrak{U}_i$ remains clopen in $\mathfrak{V}$, by Lemma 2.49, and therefore belongs to $\mathfrak{B}$. Since the intersection of F with U is just F, it follows that F belongs to $\mathfrak{B}_0$. Thus, every element in a set of generators for $\mathfrak{D}$ does belong to $\mathfrak{B}_0$, so $\mathfrak{D}$ is a subalgebra of $\mathfrak{B}_0$.

The function ψ maps $\mathfrak{B}$ homomorphically onto $\mathfrak{B}_0$. In order to show that the two algebras are isomorphic, it suffices to prove that ψ restricted to $\mathfrak{B}$ is one-to-one. Suppose F and G are elements in $\mathfrak{B}$. Observe that

$$(F \cap U)^- = F \qquad \text{and} \qquad (G \cap U)^- = G, \tag{3}$$

where X^- denotes the topological closure in $\mathfrak{V}$ of a subset X of $\mathfrak{V}$. In more detail, $(F \cap U)^- = F \cap U^-$ because F is a clopen set (see Exercise 13(e) on p. 62 of [10]), and therefore

$$(F \cap U)^- = F \cap U^- = F \cap V = F,$$

because U is dense in $\mathfrak{V}$, and F is a subset of $\mathfrak{V}$. If $\psi(F) = \psi(G)$, then

$$F \cap U = G \cap U,$$

by (1), and therefore

$$F = (F \cap U)^- = (G \cap U)^- = G,$$

by (3). Thus, ψ is one-to-one. $\square$

We shall refer to the algebra $\mathfrak{B}_0$ in Lemma 2.50 as the *relativization of* $\mathfrak{B}$ *to* U, and we shall call the isomorphism ψ from $\mathfrak{B}$ to $\mathfrak{B}_0$ the *relativization isomorphism.*

The next task is to determine how the subalgebras of $\mathfrak{A}$ that correspond to various compactifications of $\mathfrak{U}$ are related to one another.

Lemma 2.51. *Suppose* $\mathfrak{V}$ *and* $\mathfrak{W}$ *are compactifications of* $\mathfrak{U}$, *with dual algebras* $\mathfrak{B}$ *and* $\mathfrak{C}$ *respectively. The relativization of* $\mathfrak{C}$ *to* U *is a subalgebra of the relativization of* $\mathfrak{B}$ *to* U *if and only if there is a continuous bounded epimorphism from* $\mathfrak{V}$ *to* $\mathfrak{W}$ *that is the identity function on* U.

Proof. Let $\mathfrak{B}_0$ be the relativization of $\mathfrak{B}$ to U, and ψ the corresponding relativization isomorphism defined by

$$\psi(F) = F \cap U \tag{1}$$

for each set F in $\mathfrak{B}$. Similarly, let $\mathfrak{C}_0$ be the relativization of $\mathfrak{C}$ to U, and ϱ the corresponding relativization isomorphism defined by

$$\varrho(F) = F \cap U \tag{2}$$

for each set F in $\mathfrak{C}$.

Assume first that there is a continuous bounded epimorphism ϑ from $\mathfrak{V}$ to $\mathfrak{W}$ that maps each element in $\mathfrak{U}$ to itself. The dual of ϑ is the monomorphism φ from $\mathfrak{C}$ to $\mathfrak{B}$ that is defined by

$$\varphi(F) = \vartheta^{-1}(F) = \{r \in V : \vartheta(r) \in F\} \tag{3}$$

for every set F in $\mathfrak{C}$, by Theorem 2.16. The composition

$$\delta = \psi \circ \varphi \circ \varrho^{-1} \tag{4}$$

is a monomorphism from $\mathfrak{C}_0$ to $\mathfrak{B}_0$ (see the diagram below).

$$\begin{tikzcd}
\mathfrak{B} \arrow[d, "\psi"'] & \mathfrak{C} \arrow[l, "\varphi"'] \arrow[d, "\varrho"] \\
\mathfrak{B}_0 & \mathfrak{C}_0 \arrow[l, "\delta"]
\end{tikzcd}
\qquad\qquad
\mathfrak{V} \xrightarrow{\ \vartheta\ } \mathfrak{W}$$

We proceed to show that δ is the identity function on $\mathfrak{C}_0$. From this it follows at once that $\mathfrak{C}_0$ is a subalgebra of $\mathfrak{B}_0$. For each element G in $\mathfrak{C}_0$, there is a unique element F in $\mathfrak{C}$ such that

$$G = \varrho(F) = F \cap U, \tag{5}$$

by (2) and the fact that ϱ is an isomorphism from $\mathfrak{C}$ to $\mathfrak{C}_0$. An easy computation yields

$$\begin{aligned}
\delta(G) = \psi(\varphi(\varrho^{-1}(G))) &= \psi(\varphi(F)) \\
&= \psi(\vartheta^{-1}(F)) = \vartheta^{-1}(F) \cap U = F \cap U = G.
\end{aligned}$$

The first equality follows from (4), the second and final equalities from (5), the third equality from (3), and the fourth equality from (1).

For the fifth equality, observe that an element r is in $\vartheta^{-1}(F) \cap U$ just in case r is in U and $\vartheta(r)$ is in F, by (3). Since ϑ is the identity function on U, this last condition is equivalent to saying that r is in $F \cap U$.

To prove the converse direction of the lemma, assume that $\mathfrak{C}_0$ is a subalgebra of $\mathfrak{B}_0$. The identity function on $\mathfrak{C}_0$—call it δ—is then a monomorphism from $\mathfrak{C}_0$ to $\mathfrak{B}_0$. The composition

$$\varphi = \psi^{-1} \circ \delta \circ \varrho \tag{6}$$

is a monomorphism from $\mathfrak{C}$ to $\mathfrak{B}$ (see the diagram above). For each set F belonging to any one of the factor algebras $\mathfrak{A}_i$, we have

$$\varphi(F) = \psi^{-1}(\delta(\varrho(F))) = \psi^{-1}(\delta(F \cap U)) = \psi^{-1}(F \cap U) = F,$$

by (6), (2), the assumption that δ is the identity function on $\mathfrak{C}_0$, and (1). Thus, φ is the identity function on the universe of $\mathfrak{A}_i$ for each index i.

The dual of φ is the continuous bounded epimorphism ϑ from $\mathfrak{V}$ to $\mathfrak{W}$ that is determined by

$$\vartheta(r) \in F \qquad \text{if and only if} \qquad r \in \varphi(F) \tag{7}$$

for all elements r in $\mathfrak{V}$ and sets F in $\mathfrak{C}$, by Theorem 2.20. It remains to show that ϑ is the identity function on $\mathfrak{U}$. An element r in $\mathfrak{U}$ necessarily belongs to one of the spaces $\mathfrak{U}_i$, by the definition of $\mathfrak{U}$ as the union of these spaces, and φ is the identity function on the sets in $\mathfrak{A}_i$, by the observations of the preceding paragraph. Consequently, for such an element r, and for sets F in $\mathfrak{A}_i$, the equivalence in (7) assumes the form

$$\vartheta(r) \in F \qquad \text{if and only if} \qquad r \in F. \tag{8}$$

The set X of all sets in $\mathfrak{A}_i$ that contain the element r is an ultrafilter in $\mathfrak{A}_i$, and r is the only element in $\mathfrak{U}_i$ that belongs to each of the sets in X (since two distinct elements in $\mathfrak{U}_i$ are always separated by a clopen set). On the other hand, the sets in X all contain $\vartheta(r)$, by (8), so we must have $\vartheta(r) = r$. $\square$

Notice that the argument in the final paragraph proves a somewhat stronger assertion: if the dual of a continuous bounded homomorphism ϑ from $\mathfrak{V}$ to $\mathfrak{W}$ is the identity function on elements in $\mathfrak{A}_i$ for each index i, then ϑ is the identity function on elements in $\mathfrak{U}$.

Lemma 2.52. *Suppose* $\mathfrak{V}$ *and* $\mathfrak{W}$ *are compactifications of* $\mathfrak{U}$, *with dual algebras* $\mathfrak{B}$ *and* $\mathfrak{C}$ *respectively. The spaces* $\mathfrak{V}$ *and* $\mathfrak{W}$ *are homeo-isomorphic via a function that is the identity function on* $\mathfrak{U}$ *if and only if the relativizations of* $\mathfrak{B}$ *and* $\mathfrak{C}$ *to* U *are equal*, *or equivalently*, *if and only if* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are isomorphic via a function that is the identity function on* $\mathfrak{A}_i$ *for each* i.

Proof. Write $\mathfrak{B}_0$ and $\mathfrak{C}_0$ for the respective relativizations of $\mathfrak{B}$ and $\mathfrak{C}$ to the set U. If there is a homeo-isomorphism from $\mathfrak{V}$ to $\mathfrak{W}$ that is the identity function on $\mathfrak{U}$, then the relativizations $\mathfrak{B}_0$ and $\mathfrak{C}_0$ are subalgebras of each other, by Lemma 2.51, and are therefore equal.

Suppose now that $\mathfrak{B}_0$ and $\mathfrak{C}_0$ are equal. There are then continuous bounded epimorphisms ϑ from $\mathfrak{V}$ to $\mathfrak{W}$, and δ from $\mathfrak{W}$ to $\mathfrak{V}$, that are the identity function on $\mathfrak{U}$, by Lemma 2.51. The composition $\delta\circ\vartheta$ is a continuous bounded epimorphism from $\mathfrak{V}$ to $\mathfrak{V}$ that is the identity function on $\mathfrak{U}$, and the identity function on $\mathfrak{V}$ is also a continuous bounded epimorphism from $\mathfrak{V}$ to $\mathfrak{V}$ that is the identity function on $\mathfrak{U}$. The universe of $\mathfrak{U}$ is a dense subset of $\mathfrak{V}$, by Definition 2.48. Two continuous functions from a topological space to a Hausdorff space that agree on a dense subset agree on the whole space (see Corollary 2 on p. 315 of [10]), so $\delta\circ\vartheta$ must be the identity function on $\mathfrak{V}$. A symmetric argument shows that $\vartheta\circ\delta$ is the identity function on $\mathfrak{W}$. It follows that ϑ is a bijection and δ its inverse.

Since each of ϑ and its inverse δ is a continuous bounded epimorphism, the function ϑ must be a homeo-isomorphism from $\mathfrak{V}$ to $\mathfrak{W}$. For example, consider the case of a ternary relation R. If $R(r,s,t)$ holds in $\mathfrak{V}$, then $R(\vartheta(r),\vartheta(s),\vartheta(t))$ must hold in $\mathfrak{W}$, by the homomorphism properties of ϑ. Conversely, if $R(\vartheta(r),\vartheta(s),\vartheta(t))$ holds in $\mathfrak{W}$, then $R(\delta(\vartheta(r)),\delta(\vartheta(s)),\delta(\vartheta(t)))$ must hold in $\mathfrak{V}$, by the homomorphism properties of δ. Since $\delta\circ\vartheta$ is the identity function on $\mathfrak{V}$, it may be concluded that $R(r,s,t)$ holds in $\mathfrak{V}$. Consequently, the function ϑ isomorphically preserves the relation R.

To establish the second equivalence of the lemma, assume that $\mathfrak{C}$ is isomorphic to $\mathfrak{B}$ via a function φ that is the identity function on each algebra $\mathfrak{A}_i$. The dual of φ is a homeo-isomorphism ϑ from $\mathfrak{V}$ to $\mathfrak{W}$, by Corollary 2.19, and ϑ is the identity function on $\mathfrak{U}$, by the remark following Lemma 2.51.

Assume now that $\mathfrak{V}$ is homeo-isomorphic to $\mathfrak{W}$ via a function ϑ that is the identity function on $\mathfrak{U}$. The relativizations $\mathfrak{B}_0$ and $\mathfrak{C}_0$ are then equal, by the first part of the lemma. If ψ is the relativization isomorphism from $\mathfrak{B}$ to $\mathfrak{B}_0$, and ϱ the relativization isomorphism from $\mathfrak{C}$ to $\mathfrak{C}_0$, then the composition

$$\varphi = \psi^{-1} \circ \varrho$$

is an isomorphism from $\mathfrak{C}$ to $\mathfrak{B}$, and it is easy to check that φ is the identity function on each of the algebras $\mathfrak{A}_i$. Indeed, if F is a set in $\mathfrak{A}_i$, then

$$\varphi(F) = \psi^{-1}(\varrho(F)) = \psi^{-1}(F \cap U) = F,$$

by the definitions of ψ and ϱ. $\square$

The final lemma says that every algebra between $\mathfrak{D}$ and $\mathfrak{A}$ comes from the dual of some compactification of $\mathfrak{U}$.

Lemma 2.53. *Every subalgebra of $\mathfrak{A}$ that includes $\mathfrak{D}$ is the relativization to U of the dual algebra of some compactification of $\mathfrak{U}$.*

Proof. Consider a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ that includes $\mathfrak{D}$, and let $\mathfrak{V}$ be the dual space of $\mathfrak{C}$. The idea of the proof is to construct a dense inner subspace of $\mathfrak{V}$ (in the sense of Definition 2.48) that is the image of $\mathfrak{U}$ under a homeo-isomorphism ϱ, and that has the property that the relativization of the dual algebra of $\mathfrak{V}$ to this dense inner subspace is the image of $\mathfrak{C}$ under the isomorphism induced by ϱ. An application of the general algebraic Exchange Principle then yields a compactification of $\mathfrak{U}$ with the property that the dual algebra of the compactification, when relativized to U, coincides with the given subalgebra $\mathfrak{C}$.

Because $\mathfrak{V}$ is assumed to be the dual space of $\mathfrak{C}$, the topology on $\mathfrak{V}$ is the one induced by $\mathfrak{C}$. Thus, the elements in $\mathfrak{V}$ are the ultrafilters of elements in $\mathfrak{C}$, the clopen subsets of $\mathfrak{V}$ are the sets of the form

$$G_F = \{Y \in V : F \in Y\}, \tag{1}$$

where F ranges over the elements in $\mathfrak{C}$, and the open subsets of $\mathfrak{V}$ are the unions of the clopen sets. Each element F in $\mathfrak{C}$ comes from $\mathfrak{A}$ and is therefore a subset of $\mathfrak{U}$; in fact, F has the form $F = \bigcup_i F_i$, where for each index i, the component $F_i = F \cap U_i$ belongs to $\mathfrak{A}_i$ and is therefore a clopen subset of $\mathfrak{U}_i$. For every element r in $\mathfrak{U}$, the set

$$Y_r = \{F \in C : r \in F\}$$

is easily seen to be an ultrafilter of elements in $\mathfrak{C}$; in fact, if r belongs to the component space $\mathfrak{U}_i$, then Y_r consists of precisely those elements F

in $\mathfrak{C}$ such that r belongs to the component F_i. Every ultrafilter of this form is a point in the dual space $\mathfrak{V}$ (but there are other points—other ultrafilters—in $\mathfrak{V}$ as well). Notice that distinct points r and s in $\mathfrak{U}$ yield distinct ultrafilters Y_r and Y_s. Indeed, suppose r is in $\mathfrak{U}_i$ and s in $\mathfrak{U}_j$. If $i \neq j$, then the clopen set U_i—which belongs to $\mathfrak{A}_i$ and is therefore in $\mathfrak{D}$, and hence also in $\mathfrak{C}$—belongs to Y_r, while its complement in $\mathfrak{C}$ belongs to Y_s. If $i = j$, then there is a clopen subset of $\mathfrak{U}_i$ that contains r, but not s, because the clopen subsets of $\mathfrak{U}_i$ separate points; and that clopen set (which belongs to $\mathfrak{A}_i$ and is therefore in $\mathfrak{D}$ and in $\mathfrak{C}$) belongs to Y_r, while its complement in $\mathfrak{U}_i$ belongs to Y_s. Notice also that

$$G_{U_i} = \{Y \in V : U_i \in Y\}$$

is a clopen subset of $\mathfrak{V}$ (because U_i belongs to $\mathfrak{D}$ and therefore also to $\mathfrak{C}$).

Define a subset W of $\mathfrak{V}$ by

$$W = \{Y_r : r \in U\}.$$

It is easy to check that W is a dense subset of $\mathfrak{V}$. For the proof, it suffices to show that every non-empty clopen set in $\mathfrak{V}$ has a non-empty intersection with W. An arbitrary non-empty clopen set in $\mathfrak{V}$ has the form G_F for some non-empty set F in $\mathfrak{C}$, by the remarks of the preceding paragraph. It follows from (1) and the definitions of the set W and the ultrafilters Y_r that

$$G_F \cap W = \{Y_r : F \in Y_r\} = \{Y_r : r \in F\}. \tag{2}$$

The set F is not empty, so the intersection in (2) cannot be empty. In fact, it contains the element Y_r for every r in F. Thus, the set W is dense in $\mathfrak{V}$, as claimed. We shall eventually prove that the restriction of $\mathfrak{V}$ to W is an inner subspace of $\mathfrak{V}$ that is homeo-isomorphic to $\mathfrak{U}$.

Write

$$W_i = G_{U_i} \cap W = \{Y_r : r \in U_i\}, \tag{3}$$

and observe that for $i \neq j$, the sets W_i and W_j are disjoint, since the sets U_i and U_j are disjoint. The space $\mathfrak{U}$ is the disjoint union of the component spaces $\mathfrak{U}_i$, so obviously W is the disjoint union of the component sets W_i (in $\mathfrak{V}$), by the definitions of W and W_i, and by the observation made earlier that distinct elements r and s in $\mathfrak{U}$ lead to distinct ultrafilters Y_r and Y_s. The immediate goal is to prove that W_i is a compact subset of $\mathfrak{V}$. Assume for a moment that this has been accomplished. It then follows that W, as the disjoint union of compact subsets of $\mathfrak{V}$, is locally compact. A dense, locally compact subspace of a Hausdorff space is necessarily open (see Corollary 1 on p. 400 of [10]), so W must be open in $\mathfrak{V}$. Thus, W_i is the intersection of two open sets in $\mathfrak{V}$, namely G_{U_i} and W, so W_i must also be open in $\mathfrak{V}$. Of course, W_i is also closed in $\mathfrak{V}$, because it is a compact subset of the Hausdorff space $\mathfrak{V}$, so W_i must in fact be clopen.

The proof that W_i is compact is somewhat involved and uses some of the duality theorems that were proved earlier. Because $\mathfrak{A}$ is the internal product of the system of algebras $(\mathfrak{A}_i : i \in I)$, the projection from $\mathfrak{A}$ to the factor algebra $\mathfrak{A}_i$ is the epimorphism φ_i defined by

$$\varphi_i(F) = F \cap U_i = F_i$$

for every set F in $\mathfrak{A}$, where F_i is the component of F in $\mathfrak{A}_i$. Because $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, the restriction of φ_i to $\mathfrak{C}$ is a homomorphism from $\mathfrak{C}$ into $\mathfrak{A}_i$. Every element in $\mathfrak{A}_i$ belongs to $\mathfrak{D}$ and therefore also to $\mathfrak{C}$. For each set F in $\mathfrak{A}_i$, we have

$$\varphi_i(F) = F \cap U_i = F,$$

since F is a subset of $\mathfrak{U}_i$. Thus, φ_i maps $\mathfrak{C}$ homomorphically onto $\mathfrak{A}_i$. We shall refer to the restriction of φ_i to $\mathfrak{C}$ by using the same symbol φ_i.

The restriction of φ_i to $\mathfrak{C}$ induces a continuous bounded monomorphism ϑ_i from the dual space of $\mathfrak{A}_i$ to the dual space of $\mathfrak{C}$ that is defined by

$$\vartheta_i(X) = \varphi_i^{-1}(X)$$

for each element X in the dual space of $\mathfrak{A}_i$, by Theorem 2.18. The dual space of $\mathfrak{C}$ is $\mathfrak{V}$, by assumption. The dual space of $\mathfrak{A}_i$—call it $\bar{\mathfrak{U}}_i$—is the second dual of the relational space $\mathfrak{U}_i$, and $\mathfrak{U}_i$ is homeo-isomorphic $\bar{\mathfrak{U}}_i$ via the function δ_i that maps each element r in $\mathfrak{U}_i$ to the ultrafilter

$$X_r = \{F \in A_i : r \in F\},$$

by Theorem 2.11. In particular, the elements in $\bar{\mathfrak{U}}_i$ are precisely the ultrafilters of the form X_r for elements r in $\mathfrak{U}_i$, and distinct elements in $\mathfrak{U}_i$ correspond to distinct ultrafilters. The definition of ϑ_i may therefore be written in the form

$$\vartheta_i(X_r) = \varphi_i^{-1}(X_r)$$

for each element r in $\mathfrak{U}_i$. Observe that for each such r,

$$\begin{aligned}\varphi_i^{-1}(X_r) &= \{F \in C : \varphi_i(F) \in X_r\} = \{F \in C : F \cap U_i \in X_r\} \\ &= \{F \in C : r \in F \cap U_i\} = \{F \in C : r \in F\} = Y_r,\end{aligned}$$

by the definition of the inverse image under φ_i of a set, the definition of φ_i, the definition of X_r, the assumption that r is in $\mathfrak{U}_i$, and the definition of Y_r. Combine these observations to conclude that ϑ_i is the continuous bounded monomorphism from $\bar{\mathfrak{U}}_i$ to $\mathfrak{V}$ that is determined by

$$\vartheta_i(X_r) = Y_r \tag{4}$$

for every r in $\mathfrak{U}_i$.

The set W_i is the image (in $\mathfrak{V}$) under the continuous mapping ϑ_i of the compact set $\bar{U}_i$, by (3) and (4). The continuous image of a compact set is compact, so W_i must be a compact subset of $\mathfrak{V}$. The argument presented earlier now implies that W_i is a clopen subset, and W an open subset, of $\mathfrak{V}$. From this, it is not difficult to see that the subspace topology on W coincides with the union topology that W inherits from the components W_i. Indeed, the open sets in W under the subspace topology are just the subsets of W that are open in $\mathfrak{V}$, because W itself is open in $\mathfrak{V}$. For any subset H of W, write $H_i = H \cap W_i$, and observe that

$$H = H \cap W = H \cap \left(\bigcup_i W_i\right) = \bigcup_i (H \cap W_i) = \bigcup_i H_i.$$

If H is open in $\mathfrak{V}$, then H_i is open in $\mathfrak{V}$, because W_i is open in $\mathfrak{V}$; consequently, H is a union of open subsets of the components W_i, so H is open in the union topology, by the definition of that topology. On the other hand, if H is open in the union topology, then each set H_i is open in W_i, and therefore also open in $\mathfrak{V}$, by the definition of the union topology; consequently, H is a union of open sets in $\mathfrak{V}$, so H is open in $\mathfrak{V}$.

The fact that ϑ_i is a continuous bounded monomorphism from $\bar{\mathfrak{U}}_i$ into $\mathfrak{V}$ implies that the image set W_i is an inner subuniverse of $\mathfrak{V}$, by Lemma 1.13, and also that ϑ_i is algebraically an isomorphism from $\bar{\mathfrak{U}}_i$ to the corresponding inner substructure that is the restriction of $\mathfrak{V}$ to W_i. The set W_i is closed in $\mathfrak{V}$, so the restriction of $\mathfrak{V}$ to W_i is actually a relational space $\mathfrak{W}_i$ that is an inner subspace of $\mathfrak{V}$, by Lemma 2.25. It is clear that ϑ_i is continuous with respect to the topology on $\mathfrak{W}_i$. Indeed, W_i is open in $\mathfrak{V}$, so every open subset of $\mathfrak{W}_i$

is open in $\mathfrak{V}$, and therefore the inverse image under ϑ_i of every open subset of $\mathfrak{W}_i$ must be open in $\bar{\mathfrak{U}}_i$ (because ϑ_i is continuous with respect to the topology on $\mathfrak{V}$). A continuous bijection from a compact space to a Hausdorff space is necessarily a homeomorphism (see Lemma 5 on p. 316 of [10]), so ϑ_i is a homeo-isomorphism from $\bar{\mathfrak{U}}_i$ to $\mathfrak{W}_i$.

Write $\mathfrak{W}$ for the restriction of $\mathfrak{V}$ to the set W. It is not difficult to check that $\mathfrak{V}$ is a compactification of $\mathfrak{W}$ in the sense of Definition 2.48. First of all, the topology on $\mathfrak{W}$ is, by definition, the topology inherited from $\mathfrak{V}$. Second, we have already seen that the universe of $\mathfrak{W}$ is a dense subset of $\mathfrak{V}$. Third, $\mathfrak{W}$ is algebraically an inner substructure of $\mathfrak{V}$. To see this, consider the case of a ternary relation R. Let r and s be any elements in $\mathfrak{V}$, and t any element in $\mathfrak{W}$, and suppose that $R(r,s,t)$ holds in $\mathfrak{V}$. The universe of $\mathfrak{W}$ is the union of the universes of the spaces $\mathfrak{W}_i$, so the element t must belong to $\mathfrak{W}_i$ for some i. It was shown in the preceding paragraph that $\mathfrak{W}_i$ is an inner subspace of $\mathfrak{V}$, so the elements r and s must belong to $\mathfrak{W}_i$, and therefore these elements must also belong to $\mathfrak{W}$.

In a similar way, we show that $\mathfrak{W}$ is the union, in the sense of Definition 2.43, of the disjoint system of relational spaces $(\mathfrak{W}_i : i \in I)$. First of all, we have already seen that the subspace topology on $\mathfrak{W}$ coincides with the union topology inherited from the given system of spaces. Second, to check that $\mathfrak{W}$ is algebraically the union of the relational structures $\mathfrak{W}_i$, consider as an example the case of a ternary relation R. If $R(r,s,t)$ holds in $\mathfrak{W}$, then this relationship must also hold in $\mathfrak{V}$, because $\mathfrak{W}$ is a restriction of $\mathfrak{V}$. The element t belongs to $\mathfrak{W}_i$ for some i, and $\mathfrak{W}_i$ is an inner subspace of $\mathfrak{V}$, so r and s must be in $\mathfrak{W}_i$. Consequently, $R(r,s,t)$ holds in $\mathfrak{W}_i$, because $\mathfrak{W}_i$ is a restriction of $\mathfrak{V}$. Conversely, if $R(r,s,t)$ holds in $\mathfrak{W}_i$, then this relationship also holds in $\mathfrak{V}$, because $\mathfrak{W}_i$ is a restriction of $\mathfrak{V}$. The elements r, s, and t clearly belong to $\mathfrak{W}$, which is a restriction of $\mathfrak{V}$, so $R(r,s,t)$ must hold in $\mathfrak{W}$. Conclusion: the relation R in $\mathfrak{W}$ is the (disjoint) union of the corresponding relations in the spaces $\mathfrak{W}_i$, so $\mathfrak{W}$ is the union of the given system, as claimed.

The composition $\varrho_i = \vartheta_i \circ \delta_i$ of the homeo-isomorphism δ_i from $\mathfrak{U}_i$ to $\bar{\mathfrak{U}}_i$ and the homeo-isomorphism ϑ_i from $\bar{\mathfrak{U}}_i$ to $\mathfrak{W}_i$ is a homeo-isomorphism from $\mathfrak{U}_i$ to $\mathfrak{W}_i$. The union of these compositions is the bijection ϱ from $\mathfrak{U}$ to $\mathfrak{W}$ that is defined by

$$\varrho(r) = \varrho_i(r) = \vartheta_i(\delta_i(r)) = \vartheta_i(X_r) = Y_r \tag{5}$$

whenever r is an element in $\mathfrak{U}$ that belongs to $\mathfrak{U}_i$. It is not difficult to check that this disjoint union of homeo-isomorphisms is itself a homeo-isomorphism. To check that ϱ is a homeomorphism, consider any open subset $F = \bigcup_i F_i$ of $\mathfrak{U}$. The component sets F_i must be open in $\mathfrak{U}_i$, by the definition of the union topology, and each mapping ϱ_i is a homeo-isomorphism, so each set $\varrho_i(F_i)$ must be open in $\mathfrak{W}_i$. Consequently, the union of these sets is open in $\mathfrak{W}$ (because the topology on $\mathfrak{W}$ coincides with the union topology). That union is just $\varrho(F)$, since the image of F under ϱ is the set

$$\varrho(F) = \varrho(\bigcup_i F_i) = \bigcup_i \varrho(F_i) = \bigcup_i \varrho_i(F_i),$$

so the image under ϱ of an open set in $\mathfrak{U}$ is an open set in $\mathfrak{W}$. A completely analogous argument shows that the inverse image under ϱ of an open set in $\mathfrak{W}$ is an open set in $\mathfrak{U}$. Thus, ϱ is a homeomorphism, as claimed.

To check that ϱ isomorphically preserves the fundamental relations, consider the case of a ternary relation R. Let r, s, and t be elements in $\mathfrak{U}$. If $R(r, s, t)$ holds in $\mathfrak{U}$, then this relationship must hold in $\mathfrak{U}_i$ for some index i, because $\mathfrak{U}$ is the union of the component spaces $\mathfrak{U}_i$. Since ϱ_i is a homeo-isomorphism from $\mathfrak{U}_i$ to $\mathfrak{W}_i$, the relationship

$$R(\varrho_i(r), \varrho_i(s), \varrho_i(t))$$

must hold in $\mathfrak{W}_i$. Consequently, $R(\varrho(r), \varrho(s), \varrho(t))$ must hold in $\mathfrak{W}_i$ and therefore also in $\mathfrak{W}$, by the definition of ϱ and the fact that $\mathfrak{W}$ is the union of the component spaces $\mathfrak{W}_i$. A completely analogous argument shows that if $R(\varrho(r), \varrho(s), \varrho(t))$ holds in $\mathfrak{W}$, then $R(r, s, t)$ holds in $\mathfrak{U}$. Thus, ϱ isomorphically preserves the relation R. Conclusion: ϱ is an isomorphism, and therefore a homeo-isomorphism, from $\mathfrak{U}$ to $\mathfrak{W}$.

The homeo-isomorphism ϱ from $\mathfrak{U}$ to $\mathfrak{W}$ induces an isomorphism $\bar{\varrho}$ from $\mathfrak{Cm}(U)$ to $\mathfrak{Cm}(W)$ that is defined by

$$\bar{\varrho}(F) = \{\varrho(r) : r \in F\}$$

for every subset F of $\mathfrak{U}$, by Corollary 1.7. In view of (5), it is clear that the definition of $\bar{\varrho}$ may be written in the form

$$\bar{\varrho}(F) = \{Y_r : r \in F\}. \tag{6}$$

The algebra $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, by assumption, and $\mathfrak{A}$ is a subalgebra of $\mathfrak{Cm}(U)$, by Theorem 1.32, so $\mathfrak{C}$ is a subalgebra of $\mathfrak{Cm}(U)$.

It therefore makes sense to restrict the isomorphism $\bar{\varrho}$ to $\mathfrak{C}$, and this restriction must map $\mathfrak{C}$ isomorphically to a subalgebra of $\mathfrak{Cm}(W)$.

Consider now the dual algebra of $\mathfrak{V}$—call it $\mathfrak{B}$. This algebra is the second dual of $\mathfrak{C}$, and $\mathfrak{C}$ is isomorphic to its second dual via the canonical isomorphism ζ that maps each set F in $\mathfrak{C}$ to the set G_F defined in (1), by Theorem 2.10. In particular, the elements in $\mathfrak{B}$ are just the sets G_F for F in $\mathfrak{C}$. The algebra $\mathfrak{B}$ is, in turn, isomorphic to its relativization $\mathfrak{B}_0$ via the relativization isomorphism ψ that maps each element in $\mathfrak{B}$ to its intersection with W, by Lemma 2.50 (with $\mathfrak{W}$ as the locally compact relational space, and $\mathfrak{V}$ as the compactification of $\mathfrak{W}$). In this connection, recall from the proof of Lemma 2.50 that ψ is the restriction of the relativization homomorphism from $\mathfrak{Cm}(V)$ to $\mathfrak{Cm}(W)$ that maps each subset of $\mathfrak{V}$ to its intersection with the set W. Composing ψ with ζ, and using (2), we arrive at

$$(\psi \circ \zeta)(F) = \psi(\zeta(F)) = \psi(G_F) = G_F \cap W = \{Y_r : r \in F\} \tag{7}$$

for each set F in $\mathfrak{C}$. Compare (6) with (7) to conclude that

$$\bar{\varrho} = \psi \circ \zeta$$

(where the left side of this equation actually denotes the restriction of $\bar{\varrho}$ to $\mathfrak{C}$). Thus, the restriction of $\bar{\varrho}$ maps $\mathfrak{C}$ isomorphically to the algebra $\mathfrak{B}_0$ that is the relativization to W of the algebra $\mathfrak{B}$, which in turn is the dual of $\mathfrak{V}$.

Here is a summary of what has been accomplished. First, the locally compact union space $\mathfrak{U}$ has been mapped homeo-isomorphically by ϱ to a space $\mathfrak{W}$ of which $\mathfrak{V}$ is a compactification. Second, the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ has been mapped isomorphically by $\bar{\varrho}$—the mapping induced on $\mathfrak{C}$ by ϱ—to the relativization of the dual algebra $\mathfrak{B}$ (of $\mathfrak{V}$) to the set W. If we now use the homeo-isomorphism ϱ to identify $\mathfrak{U}$ with $\mathfrak{W}$, and we use the induced isomorphism $\bar{\varrho}$ to identify $\mathfrak{C}$ with the relativization of $\mathfrak{B}$ to W, then we arrive at the desired goal: $\mathfrak{V}$ is a compactification of the union space $\mathfrak{U}$, and the relativization to U of the dual algebra of $\mathfrak{V}$ is just $\mathfrak{C}$.

The technical tool for carrying out this identification is a version of the general algebraic Exchange Principle that applies to structures such as $\mathfrak{U}$. The elements in $\mathfrak{V}$ that come from $\mathfrak{W}$ are replaced by the corresponding elements from $\mathfrak{U}$ (under the correspondence that is the inverse of ϱ), the remaining elements in $\mathfrak{V}$ being modified if necessary so that they do not occur in $\mathfrak{U}$. Once this is accomplished, the function ϱ becomes the identity function on $\mathfrak{U}$, and therefore the mapping $\bar{\varrho}$ on $\mathfrak{C}$ induced by ϱ becomes the identity function on $\mathfrak{C}$. Consequently, $\mathfrak{C}$ coincides with $\mathfrak{B}_0$, which is the relativization of $\mathfrak{B}$ to W. $\square$

Consider the function that maps each compactification $\mathfrak{V}$ of the union space $\mathfrak{U}$ to the isomorphic copy of the dual algebra of $\mathfrak{V}$ that is obtained by relativizing the dual algebra to the set U. This function maps the class of compactifications of $\mathfrak{U}$ onto the class of algebras intermediate between $\mathfrak{D}$ and $\mathfrak{A}$, by Lemmas 2.50 and 2.53. In general, this function is not one-to-one, as Lemma 2.52 makes clear; distinct compactifications of $\mathfrak{U}$ may be mapped to the same intermediate algebra. Such compactifications do not differ from one another in any material way, and it is natural to identify them by grouping them together in one class. Motivated by these considerations, we define two compactifications of $\mathfrak{U}$ to be *equivalent* if there is a homeo-isomorphism from one compactification to the other that is the identity function on $\mathfrak{U}$. It is easy to check that the relation defined in this way is an equivalence relation on the class of compactifications of $\mathfrak{U}$. Equivalent compactifications have dual algebras that are isomorphic via a function that is the identity on the universes of the factor algebras $\mathfrak{A}_i$, and the isomorphic copies of these dual algebras obtained by relativization to U are in fact equal, by Lemma 2.52. Thus, one may speak with some justification of the dual algebra of the equivalence class. The correspondence that maps each equivalence class of compactifications of $\mathfrak{U}$ to the relativization of its dual algebra is a well-defined bijection from the class of equivalence classes of compactifications of $\mathfrak{U}$ to the set of subalgebras of $\mathfrak{A}$ that include $\mathfrak{D}$. It turns out that this bijection is actually a lattice isomorphism.

The set of algebras between $\mathfrak{D}$ and $\mathfrak{A}$ is partially ordered by the relation of being a subalgebra, and under this partial ordering the set becomes a complete lattice with zero $\mathfrak{D}$ and unit $\mathfrak{A}$. The partial ordering on the class of equivalence classes of compactifications of $\mathfrak{U}$ is more complicated to describe, but it is equally natural. Define a binary relation $\leq$ on the class of compactifications of $\mathfrak{U}$ by writing $\mathfrak{W} \leq \mathfrak{V}$ just in case there is a continuous bounded epimorphism from $\mathfrak{V}$ to $\mathfrak{W}$ that is the identity function on $\mathfrak{U}$. This relation is preserved by the relation of equivalence in the following sense: if compactifications $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are equivalent, and if compactifications $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are also equivalent, then

$$\mathfrak{W}_1 \leq \mathfrak{V}_1 \qquad \text{if and only if} \qquad \mathfrak{W}_2 \leq \mathfrak{V}_2.$$

For the proof, suppose δ is a homeo-isomorphism from $\mathfrak{V}_1$ to $\mathfrak{V}_2$ that is the identity function on $\mathfrak{U}$, and ϱ a homeo-isomorphism from $\mathfrak{W}_1$ to $\mathfrak{W}_2$ that is the identity function on $\mathfrak{U}$. If ϑ is a continuous bounded epimorphism from $\mathfrak{V}_1$ to $\mathfrak{W}_1$ that is the identity function on $\mathfrak{U}$, then the composition $\varrho\circ\vartheta\circ\delta^{-1}$ is a continuous bounded epimorphism from $\mathfrak{V}_2$ to $\mathfrak{W}_2$ that is the identity function on $\mathfrak{U}$ (see the diagram below). This shows that $\mathfrak{W}_1 \leq \mathfrak{V}_1$ implies $\mathfrak{W}_2 \leq \mathfrak{V}_2$. The reverse implication is established by a symmetric argument.

$$\begin{tikzcd}
\mathfrak{V}_1 \arrow[r, "\delta"] \arrow[d, "\vartheta"'] & \mathfrak{V}_2 \arrow[d] \\
\mathfrak{W}_1 \arrow[r, "\varrho"'] & \mathfrak{W}_2
\end{tikzcd}$$

We shall say of two compactifications $\mathfrak{V}$ and $\mathfrak{W}$ of $\mathfrak{U}$ that the equivalence class of $\mathfrak{W}$ is *less than or equal to* the equivalence class of $\mathfrak{V}$ if $\mathfrak{W} \leq \mathfrak{V}$. The remarks in the preceding paragraph imply that the relation on equivalence classes defined in this manner is well defined in the sense that it does not depend on the particular choice of the representatives of the equivalence classes involved. One can prove without difficulty that the relation is a partial ordering on the class of equivalence classes. For instance, to prove that the relation is antisymmetric, assume that the equivalence class of $\mathfrak{W}$ is less than or equal to the equivalence class of $\mathfrak{V}$, and vice versa. It must be shown that the two equivalence classes are equal. The assumption implies that $\mathfrak{V} \leq \mathfrak{W}$ and $\mathfrak{W} \leq \mathfrak{V}$. Consequently, there is a continuous bounded epimorphism ϑ from $\mathfrak{V}$ to $\mathfrak{W}$ and a continuous bounded epimorphism δ from $\mathfrak{W}$ to $\mathfrak{V}$ such that both mappings are the identity function on $\mathfrak{U}$, by the definition of $\leq$. The composition $\delta\circ\vartheta$ is therefore a continuous bounded epimorphism from $\mathfrak{V}$ to $\mathfrak{V}$ that is the identity function on $\mathfrak{U}$. The identity function on $\mathfrak{V}$ is also a continuous bounded epimorphism from $\mathfrak{V}$ to $\mathfrak{V}$ that is the identity function on $\mathfrak{U}$. The universe of $\mathfrak{U}$ is a dense subset of $\mathfrak{V}$, by the assumption that $\mathfrak{V}$ is a compactification of $\mathfrak{U}$. Two continuous functions from a topological space to a Hausdorff space that agree on a dense subset must agree on the entire space (see Corollary 2 on p. 315 of [10]), so the composition $\delta\circ\vartheta$ must be the identity function on $\mathfrak{V}$. A similar argument shows that $\vartheta\circ\delta$ is the identity function on $\mathfrak{W}$. Consequently, ϑ is a bijection from $\mathfrak{V}$ to $\mathfrak{W}$ and δ is its inverse. A bounded epimorphism that is a bijection is an isomorphism, by the remark following Definition 1.8, so ϑ is a continuous isomorphism with a continuous inverse δ. Conclusion: ϑ is a homeo-isomorphism that is the identity function on $\mathfrak{U}$. This implies that the compactifications $\mathfrak{V}$ and $\mathfrak{W}$ are equivalent, and therefore their equivalence classes are equal, as desired.

We have seen that the class of equivalence classes of compactifications of $\mathfrak{U}$ is partially ordered by the less-than-or-equal-to relation defined in the preceding paragraph, and the class of subalgebras of $\mathfrak{A}$ that include $\mathfrak{D}$ is partially ordered by the relation of being a subalgebra. We have also seen that the function ζ mapping each equivalence class of compactifications to the relativization to U of the dual algebra of the equivalence class is a bijection from the class of equivalence classes of compactifications of $\mathfrak{U}$ to the class of subalgebras of $\mathfrak{A}$ that include $\mathfrak{D}$. We now show that ζ preserves the partial ordering in a strong sense. Consider compactifications $\mathfrak{V}$ and $\mathfrak{W}$ of $\mathfrak{U}$, and let $\mathfrak{B}$ and $\mathfrak{C}$ be their respective dual algebras. Write $\mathfrak{B}_0$ and $\mathfrak{C}_0$ for the relativizations of $\mathfrak{B}$ and $\mathfrak{C}$ to U. The function ζ maps the equivalence class of $\mathfrak{V}$ to the subalgebra $\mathfrak{B}_0$, and the equivalence class of $\mathfrak{W}$ to the subalgebra $\mathfrak{C}_0$. Since the equivalence class of $\mathfrak{W}$ is less than or equal to the equivalence class of $\mathfrak{V}$ just in case $\mathfrak{W} \leq \mathfrak{V}$, it suffices to prove that

$$\mathfrak{W} \leq \mathfrak{V} \qquad \text{if and only if} \qquad \mathfrak{C}_0 \subseteq \mathfrak{B}_0 .$$

If $\mathfrak{W} \leq \mathfrak{V}$, then there must be a continuous bounded epimorphism ϑ from $\mathfrak{V}$ to $\mathfrak{W}$ that is the identity function on $\mathfrak{U}$, by the definition of the relation $\leq$. Apply Lemma 2.51 to conclude that the relativization of $\mathfrak{C}$ to U is a subalgebra of the relativization of $\mathfrak{B}$ to U, that is to say, $\mathfrak{C}_0$ is a subalgebra of $\mathfrak{B}_0$. Conversely, if $\mathfrak{C}_0$ is a subalgebra of $\mathfrak{B}_0$, then there must be a continuous bounded epimorphism from $\mathfrak{V}$ to $\mathfrak{W}$ that is the identity function on U, again by Lemma 2.51, so $\mathfrak{W} \leq \mathfrak{V}$. The following theorem summarizes what has been proved.

Theorem 2.54. *Let $\mathfrak{U}$ be the union of a disjoint system $(\mathfrak{U}_i : i \in I)$ of relational spaces. For each index i, let $\mathfrak{A}_i$ be the dual algebra of $\mathfrak{U}_i$, and let $\mathfrak{A}$ be the internal product, and $\mathfrak{D}$ the weak internal product, of the system $(\mathfrak{A}_i : i \in I)$ of algebras. Equivalent compactifications of $\mathfrak{U}$ have, up to isomorphism, the same dual algebra, and that dual algebra is isomorphic via relativization to a subalgebra of $\mathfrak{A}$ that includes $\mathfrak{D}$. The function that maps each equivalence class of compactifications of $\mathfrak{U}$ to the corresponding subalgebra of $\mathfrak{A}$ that includes $\mathfrak{D}$ is a lattice isomorphism from the lattice of equivalence classes of compactifications of $\mathfrak{U}$ to the lattice of subalgebras of $\mathfrak{A}$ that include $\mathfrak{D}$.*

2.12 Duality for Infinite Direct Products

The internal product $\mathfrak{A}$ of the system of dual algebras $(\mathfrak{A}_i : i \in I)$ of a given disjoint system $(\mathfrak{U}_i : i \in I)$ of relational spaces is the maximum element in the lattice of subalgebras of $\mathfrak{A}$ that include the weak internal product. The equivalence class of the dual space of $\mathfrak{A}$ must therefore be the maximum element in the lattice of equivalence classes of compactifications of the union space $\mathfrak{U}$, by Theorem 2.54. It is natural to look for a topological characterization of this maximum compactification. The maximum compactification of an arbitrary locally compact Hausdorff space U is the Stone-Čech compactification. This space—call it V—is characterized by the property that every continuous mapping from U into a compact Hausdorff space W can be extended (in a unique way) to a continuous mapping from V into W. These considerations motivate the following definition.

Definition 2.55. A *Stone-Čech compactification* of a locally compact relational space $\mathfrak{U}$ is defined to be a compactification $\mathfrak{V}$ of $\mathfrak{U}$ with the property that every continuous bounded homomorphism from $\mathfrak{U}$ into a relational space $\mathfrak{W}$ can be extended to a continuous bounded homomorphism from $\mathfrak{V}$ into $\mathfrak{W}$. □

Observe that if $\mathfrak{V}$ is a Stone-Čech compactification of a locally compact relational space $\mathfrak{U}$, then the continuous bounded homomorphism on $\mathfrak{V}$ that extends a given continuous bounded homomorphism from $\mathfrak{U}$ to a relational space $\mathfrak{W}$ must be unique. The reason is that the universe of $\mathfrak{U}$ is a dense subset of $\mathfrak{V}$, by Definition 2.48, and two continuous functions that agree on a dense subset of a space agree everywhere on the space. Observe also that if such a compactification $\mathfrak{V}$ exists, then the equivalence class of $\mathfrak{V}$ must be the maximum element in the lattice of equivalence classes of compactifications of $\mathfrak{U}$, by the next lemma.

Lemma 2.56. *If $\mathfrak{V}$ is a Stone-Čech compactification of a locally compact relational space $\mathfrak{U}$, then every compactification of $\mathfrak{U}$ is a continuous bounded homomorphic image of $\mathfrak{V}$ via a mapping that is the identity function on $\mathfrak{U}$.*

Proof. Suppose $\mathfrak{W}$ is a compactification of $\mathfrak{U}$. The space $\mathfrak{U}$ is then an inner subspace of $\mathfrak{W}$ in the sense that it is algebraically an inner substructure of $\mathfrak{W}$ and topologically a subspace of $\mathfrak{W}$ (see Definition 2.48). The identity function ϑ on $\mathfrak{U}$ is therefore a continuous bounded homomorphism from $\mathfrak{U}$ to $\mathfrak{W}$. In more detail, ϑ is a bounded monomorphism from $\mathfrak{U}$ to $\mathfrak{W}$, by Corollary 1.14; and if H is an open subset of $\mathfrak{W}$, then the inverse image of H under ϑ is the set

$$\vartheta^{-1}(H) = \{r \in U : \vartheta(r) \in H\} = \{r \in U : r \in H\} = H \cap U,$$

which is an open subset of $\mathfrak{U}$, by the definition of the subspace topology; consequently, ϑ is continuous. Apply Definition 2.55 and the assumption that $\mathfrak{V}$ is a Stone-Čech compactification of $\mathfrak{U}$ to conclude that there is a continuous bounded homomorphism δ from $\mathfrak{V}$ to $\mathfrak{W}$ that extends ϑ. A continuous mapping from a compact space to a Hausdorff space maps closed sets to closed sets (see Corollary 1 on p. 315 of [10]), so the image set

$$\delta(V) = \{\delta(r) : r \in V\}$$

must be closed in $\mathfrak{W}$. This image set includes the set U, because δ is an extension of ϑ, and ϑ is the identity function on $\mathfrak{U}$. Since U is a dense subset of $\mathfrak{W}$, the closure of U in $\mathfrak{W}$ must be W. It follows that

$$W = U^{-} \subseteq \delta(V)^{-} = \delta(V),$$

so δ maps $\mathfrak{V}$ onto $\mathfrak{W}$. Conclusion: $\mathfrak{W}$ is the image of $\mathfrak{V}$ under a continuous bounded homomorphism δ that is the identity function on $\mathfrak{U}$. □

Corollary 2.57. *A Stone-Čech compactification of a locally compact relational space* $\mathfrak{U}$, *if it exists*, *is unique up to homeo-isomorphisms that are the identity function on* $\mathfrak{U}$.

Proof. Suppose $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are Stone-Čech compactifications of a locally compact relational space $\mathfrak{U}$. There is a continuous bounded epimorphism δ_1 from $\mathfrak{V}_1$ to $\mathfrak{V}_2$ that is the identity function on $\mathfrak{U}$, and also a continuous bounded epimorphism δ_2 from $\mathfrak{V}_2$ into $\mathfrak{V}_1$ that is the identity function on $\mathfrak{U}$, by Lemma 2.56. Just as in the argument given before Theorem 2.54 (showing that the relation $\leq$ is antisymmetric), the compositions $\delta_1 \circ \delta_2$ and $\delta_2 \circ \delta_1$ must be the identity functions on the spaces $\mathfrak{V}_2$ and $\mathfrak{V}_1$ respectively, so δ_1 and δ_2 are bijections and inverses of one another; consequently, δ_1 is a homeo-isomorphism from $\mathfrak{V}_1$ to $\mathfrak{V}_2$ that is the identity function on $\mathfrak{U}$. □

The preceding corollary justifies speaking about *the* Stone-Čech compactification of a locally compact relational space $\mathfrak{U}$. The next theorem establishes the existence of the Stone-Čech compactification in the case in which we are interested, and it simultaneously shows that this compactification is essentially the dual space of the direct product.

Theorem 2.58. *Let* $\mathfrak{U}$ *be the union of a disjoint system* $(\mathfrak{U}_i : i \in I)$ *of relational spaces, and for each index* i*, let* $\mathfrak{A}_i$ *be the dual algebra of* $\mathfrak{U}_i$*. The Stone-Čech compactification of* $\mathfrak{U}$ *exists, and its dual algebra is isomorphic to the internal product of the system* $(\mathfrak{A}_i : i \in I)$ *via the relativization function.*

Proof. We begin by introducing some notation and making some preliminary observations. Let $\mathfrak{A}$ be the internal product of the system $(\mathfrak{A}_i : i \in I)$ of dual algebras of the given disjoint system $(\mathfrak{U}_i : i \in I)$ of relational spaces. The definition of the internal product implies that the elements in $\mathfrak{A}$ are the subsets of the union space $\mathfrak{U}$ that can be written in the form $F = \bigcup_i F_i$, where F_i is an element in $\mathfrak{A}_i$ and therefore a clopen subset of $\mathfrak{U}_i$. The clopen subsets of the space $\mathfrak{U}$ are also the unions of the clopen subsets of the component spaces $\mathfrak{U}_i$, by the remarks at the beginning of Section 2.10 concerning union spaces. It follows that the elements in $\mathfrak{A}$ are precisely the clopen subsets of $\mathfrak{U}$.

Observe, as in the proof of Theorem 2.47, that for each index i, the dual algebra $\mathfrak{A}_i$ is, by its very construction, a subalgebra of the complex algebra $\mathfrak{Cm}(U_i)$. The internal product of the system of dual algebras is therefore a subalgebra of the internal product of the system of complex algebras

$$(\mathfrak{Cm}(U_i) : i \in I).$$

The first product is equal to $\mathfrak{A}$, by assumption, and the second product is equal to the complex algebra $\mathfrak{Cm}(U)$, by Theorem 1.32, so $\mathfrak{A}$ is a subalgebra of $\mathfrak{Cm}(U)$. Conclusion: $\mathfrak{A}$ is the subalgebra of $\mathfrak{Cm}(U)$ whose universe is the set clopen subsets of $\mathfrak{U}$, by the previous observations.

Let $\mathfrak{V}$ be the maximum compactification of the union space $\mathfrak{U}$, which exists by Theorem 2.54 and the fact that $\mathfrak{A}$ has a maximum subalgebra, namely itself. The dual algebra of $\mathfrak{V}$—call it $\mathfrak{B}$—is isomorphic to $\mathfrak{A}$ via the relativization function ψ that is defined by

$$\psi(G) = G \cap U$$

for every element G in $\mathfrak{B}$, by Theorem 2.54. The inverse function ψ^{-1} is therefore the isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ that is defined by

$$\psi^{-1}(F) = G \qquad \text{if and only if} \qquad F = G \cap U, \tag{1}$$

for every element F in $\mathfrak{A}$.

Consider now an arbitrary continuous bounded homomorphism ϑ from $\mathfrak{U}$ into a relational space $\mathfrak{W}$. It is to be shown that ϑ can be extended to a continuous bounded homomorphism δ from $\mathfrak{V}$ to $\mathfrak{W}$. As a bounded homomorphism, ϑ has an algebraic dual, by Theorem 1.9, namely the complete homomorphism φ from $\mathfrak{Cm}(W)$ to $\mathfrak{Cm}(U)$ that is defined by

$$\varphi(H) = \vartheta^{-1}(H) = \{s \in U : \vartheta(s) \in H\} \tag{2}$$

for every element H in $\mathfrak{Cm}(W)$, that is to say, for every subset H of $\mathfrak{W}$. The assumed continuity of ϑ implies that the inverse image under ϑ of every clopen subset of $\mathfrak{W}$ is a clopen subset of $\mathfrak{U}$. Thus, the algebraic dual φ defined in (2) maps clopen subsets of $\mathfrak{W}$ to clopen subsets of $\mathfrak{U}$. The (topological) dual algebra of the relational space $\mathfrak{W}$ is the subalgebra $\mathfrak{C}$ of $\mathfrak{Cm}(W)$ whose universe is the set of all clopen subsets of $\mathfrak{W}$, and the internal product $\mathfrak{A}$ is, by the observations made above, the subalgebra of $\mathfrak{Cm}(U)$ whose universe is the set of all clopen subsets of $\mathfrak{U}$. It follows that the restriction of φ to $\mathfrak{C}$—for which we shall also write φ—is a homomorphism from $\mathfrak{C}$ into $\mathfrak{A}$.

The composition of the homomorphism φ from $\mathfrak{C}$ to $\mathfrak{A}$ with the isomorphism ψ^{-1} from $\mathfrak{A}$ to $\mathfrak{B}$ is a homomorphism ϱ from $\mathfrak{C}$ into $\mathfrak{B}$ that is determined by

$$\varrho(H) = \psi^{-1}(\varphi(H))$$

for every element H in $\mathfrak{C}$. In view of (1) and the definition of ϱ, this means that

$$\varrho(H) = G \qquad \text{if and only if} \qquad \varphi(H) = G \cap U. \tag{3}$$

Because $\mathfrak{C}$ is the dual algebra of the relational space $\mathfrak{W}$, and $\mathfrak{B}$ is the dual algebra of the relational space $\mathfrak{V}$, and ϱ is a homomorphism from $\mathfrak{C}$ into $\mathfrak{B}$, the topological duality theorem for homomorphisms in the form of Corollary 2.19 (with $\mathfrak{U}$, $\mathfrak{A}$, and φ replaced by $\mathfrak{W}$, $\mathfrak{C}$, and ϱ respectively) may be applied to the homomorphism ϱ to obtain a dual continuous bounded homomorphism δ from $\mathfrak{V}$ to $\mathfrak{W}$ that is defined by

$$\delta(s) = r \qquad \text{if and only if} \qquad \varrho^{-1}(Y_s) = X_r,$$

where

$$X_r = \{H \in C : r \in H\} \qquad \text{and} \qquad Y_s = \{G \in B : s \in G\}.$$

It remains to show that δ is an extension of ϑ.

The remarks following Corollary 2.19 (with ϱ and H in place of φ and F) imply that the definition of the dual mapping δ may equivalently be written in the form

$$\delta(s) \in H \qquad \text{if and only if} \qquad s \in \varrho(H) \tag{4}$$

for all elements s in $\mathfrak{V}$ and H in $\mathfrak{C}$. Fix an element H in $\mathfrak{C}$, that is to say, fix a clopen subset of $\mathfrak{W}$, and write $\varrho(H) = G$. For any element s in $\mathfrak{U}$, we have

$$\begin{aligned} s \in \varrho(H) \qquad &\text{if and only if} \qquad s \in G, \\ &\text{if and only if} \qquad s \in G \cap U, \\ &\text{if and only if} \qquad s \in \varphi(H), \\ &\text{if and only if} \qquad s \in \vartheta^{-1}(H), \\ &\text{if and only if} \qquad \vartheta(s) \in H, \end{aligned}$$

by the definition of the set G, the assumption that s is in $\mathfrak{U}$, the equivalence in (3) and the assumption that $\varrho(H) = G$, the definition of φ in (2), and the definition of the inverse image under ϑ of the set H. The preceding equivalences show that

$$\vartheta(s) \in H \qquad \text{if and only if} \qquad s \in \varrho(H).$$

Combine this with (4) to see that if s belongs to $\mathfrak{U}$, then

$$\delta(s) \in H \qquad \text{if and only if} \qquad \vartheta(s) \in H \tag{9}$$

for every clopen subset H of $\mathfrak{W}$. The clopen subsets of $\mathfrak{W}$ separate points, so the equivalence in (9) can only hold if $\delta(s) = \vartheta(s)$. Thus, δ agrees with ϑ on elements in $\mathfrak{U}$. Conclusion: δ is a continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{W}$ that extends ϑ. $\square$

The preceding theorem does not say that the topology on the Stone-Čech compactification $\mathfrak{V}$ of the union space $\mathfrak{U}$ is the Stone-Čech compactification of the topology on $\mathfrak{U}$ in the standard sense that this term is used in topology, namely that every continuous function from (the universe of) $\mathfrak{U}$ into an arbitrary compact Hausdorff space W can be extended to a continuous function from (the universe of) $\mathfrak{V}$ into W. The theorem only says that condition set forth in Definition 2.55 is satisfied, namely every continuous bounded homomorphism from $\mathfrak{U}$

into a *relational space* $\mathfrak{W}$ can be extended to a continuous bounded homomorphism from $\mathfrak{V}$ to $\mathfrak{W}$. In particular, the topology on $\mathfrak{W}$ is that of a Boolean space, not that of an arbitrary compact Hausdorff space. Showing that the topology on $\mathfrak{V}$ really is the standard Stone-Čech topology requires a separate argument.

Theorem 2.59. *If $\mathfrak{V}$ is the Stone-Čech compactification of the union $\mathfrak{U}$ of a disjoint system of relational spaces, then the topology on $\mathfrak{V}$ is the standard Stone-Čech compactification of the topology on $\mathfrak{U}$.*

Proof. Suppose $\mathfrak{U}$ is the union of a disjoint system

$$(\mathfrak{U}_i : i \in I) \tag{1}$$

of relational spaces. Let

$$(\mathfrak{A}_i : i \in I) \tag{2}$$

be the system of dual algebras corresponding to (1), and let $\mathfrak{A}$ be the internal product of the system of algebras in (2). The dual algebra of the Stone-Čech compactification $\mathfrak{V}$ (of $\mathfrak{U}$)—call it $\mathfrak{B}$—is isomorphic to $\mathfrak{A}$ via the relativization function ψ that maps each set H in $\mathfrak{B}$ to the intersection $H \cap U$, by Theorem 2.58. Essential use of this fact is needed to establish a preliminary observation, namely that two disjoint closed subsets of the union space $\mathfrak{U}$ are always separated by a clopen subset of $\mathfrak{V}$.

For the proof, consider two closed subsets F_1 and F_2 of $\mathfrak{U}$ that are disjoint. For each index i, the intersections

$$F_1 \cap U_i \qquad \text{and} \qquad F_2 \cap U_i \tag{3}$$

are disjoint closed subsets of $\mathfrak{U}_i$, because $\mathfrak{U}_i$ is topologically a subspace of $\mathfrak{U}$, by Lemma 2.44; and therefore the two sets in (3) are compact, because closed subsets of a compact space are compact. A rather straightforward compactness argument, using the fact that the topology on $\mathfrak{U}_i$ is Boolean, produces a clopen subset G_i of $\mathfrak{U}_i$ that separates the sets in (3) in the sense that

$$F_1 \cap U_i \subseteq G_i \qquad \text{and} \qquad F_2 \cap G_i = \varnothing. \tag{4}$$

In more detail, in a compact Hausdorff space, any two closed sets can be separated by an open set that includes the first closed set and is disjoint from the second (see Exercise 33 on p. 280 of [10]). If the space is Boolean, then the separating open set is the union of a system of clopen sets, so the compactness of the first closed set implies that there is a finite subsystem of the clopen sets whose union includes the first closed set, and that union remains disjoint from the second closed set. The desired conclusion now follows from the observation that a union of finitely many clopen sets is clopen.

For each index i, the clopen set G_i belongs to the dual algebra $\mathfrak{A}_i$, by the definition of the dual algebra, so the system of clopen sets

$$(G_i : i \in I) \tag{5}$$

has a supremum in the internal product $\mathfrak{A}$, by the definition of the internal product. In fact, the supremum in $\mathfrak{A}$ of this system is just the union

$$G = \bigcup_i G_i .$$

Every clopen subset of $\mathfrak{U}_i$ remains a clopen subset of $\mathfrak{V}$, by Lemma 2.49, and obviously the relativization isomorphism ψ maps each such subset of $\mathfrak{U}_i$ to itself. Consequently, the image of each set G_i under ψ is G_i, and therefore

$$\psi^{-1}(G_i) = G_i . \tag{6}$$

Since the inverse isomorphism ψ^{-1} preserves arbitrary suprema, and since the system in (5) has a supremum in $\mathfrak{A}$, the image of this system under ψ^{-1} must have a supremum in $\mathfrak{B}$. The image system is again just (5), by (6); consequently, the system in (5) has a supremum in $\mathfrak{B}$. Apply Lemma 2.40 (with F_i, H, $\mathfrak{A}$, and $\mathfrak{U}$ respectively replaced by G_i, G, $\mathfrak{B}$, and $\mathfrak{V}$) to conclude that the closure G^- is open and hence clopen in the topology of $\mathfrak{V}$, and that G^- is the supremum of (5) in $\mathfrak{B}$, that is to say

$$G^- = \sum_i G_i .$$

An easy computation shows that the clopen set G^- separates the sets F_1 and F_2 in $\mathfrak{V}$. Indeed,

$$F_1 = F_1 \cap U = F_1 \cap \big(\bigcup_i U_i\big) = \bigcup_i (F_1 \cap U_i) \subseteq \bigcup_i G_i \subseteq G^- ,$$

and

$$F_2 \cap G^- = F_2 \cap \big(\sum_i G_i\big) = \sum_i (F_2 \cap G_i) = \sum_i \varnothing = \varnothing ,$$

by (4) and the distributive law for multiplication over arbitrary sums in $\mathfrak{B}$.

Turn now to the main task of the proof, which is to demonstrate that the topology on $\mathfrak{V}$ is the Stone-Čech compactification of the topology on $\mathfrak{U}$. We use italic letters to refer to topological spaces having no algebraic structure. Consider an arbitrary continuous function ϑ from U (the universe of $\mathfrak{U}$) into a compact Hausdorff space W. It is to be shown that ϑ can be extended to a continuous function δ from V (the universe of $\mathfrak{V}$) into W. Every point in V is completely determined by the clopen sets to which it belongs, since clopen sets separate points in a Boolean space. Thus, if s is a point in V, and if N_s is the set of clopen subsets of V that contain s, then N_s is an ultrafilter in $\mathfrak{B}$, and the intersection of the sets in N_s is just the singleton $\{s\}$. It is natural to define $\delta(s)$ to be the intersection of the class of image sets

$$\{\vartheta(F) : F \in N_s\}.$$

Two problems arise with this approach. First, ϑ is defined only on points in U (not on points in V), so it is necessary to replace the set $\vartheta(F)$ with the set

$$\vartheta(F \cap U) = \{\vartheta(t) : t \in F \cap U\}.$$

Second, $\vartheta(F \cap U)$ may not be closed in W, so there is no assurance that the intersections of all of the sets of this form will be non-empty. The solution is to pass to the closure $\vartheta(F \cap U)^-$.

Given any point s in V, put

$$M_s = \{\vartheta(F \cap U)^- : F \in N_s\}. \tag{7}$$

We shall prove that the intersection of the sets in M_s contains exactly one point. In order to show that the intersection is not empty, it suffices to prove that the sets in M_s have the finite intersection property in the sense that the intersection of finitely many of these sets is always non-empty; the desired conclusion then follows by compactness, because the intersection of a system of closed sets with the finite intersection property is always non-empty in a compact space (see p. 271 of [10]). The finite intersection property is a direct consequence of two observations. The first is that the sets in M_s are not empty. For the proof, consider a clopen set F in N_s, and notice that F is not empty because it contains s. The set U is a dense subset of V, because V is a compactification of U, so every non-empty open set has a non-empty intersection with U; in particular, F has a non-empty intersection with U. It follows that the image set $\vartheta(F \cap U)$ cannot be empty, so the closure of this image set cannot be empty. The second observation is that the intersection of any finite system of sets in M_s includes a set from M_s and is therefore not empty. For the proof, consider a finite system

$$(F_j : 0 \leq j < n)$$

of sets in N_s. The intersection F of this system belongs to N_s, because N_s is a Boolean filter and is therefore closed under finite intersections. Since F is included in F_j for $j < n$, the set $\vartheta(F \cap U)^-$ must be included in the set $\vartheta(F_j \cap U)^-$ for each j, so that

$$\vartheta(F \cap U)^- \subseteq \bigcap_j \vartheta(F_j \cap U)^-,$$

as desired.

The argument that the intersection of M_s cannot contain two distinct points proceeds by showing that for any two distinct points in W, there is a set in M_s that does not contain at least one of the two points. Let r_1 and r_2 be distinct points in W. Since W is assumed to be a compact Hausdorff space, there must exist open sets H_1 and H_2 in W containing r_1 and r_2 respectively such that the closures H_1^- and H_2^- are disjoint (see Corollary 2 on p. 273 of [10]). The inverse images

$$\vartheta^{-1}(H_1^-) \qquad \text{and} \qquad \vartheta^{-1}(H_2^-) \tag{8}$$

are then obviously disjoint, and they are closed subsets of U, by the assumed continuity of the mapping ϑ. The preliminary observation at the beginning of the proof implies the existence of a clopen subset G of V that separates the two closed sets in (8) in the sense that

$$\vartheta^{-1}(H_1^-) \subseteq G \qquad \text{and} \qquad \vartheta^{-1}(H_2^-) \cap G = \varnothing. \tag{9}$$

The element s belongs to exactly one of the sets G and $\sim G$. If s is in G, then G is in N_s, by the definition of N_s; so the set $\vartheta(G \cap U)^-$ belongs to M_s, by (7), and this set does not contain r_2. In more detail, if r_2 belonged to the closure of $\vartheta(G \cap U)$, then every open set containing r_2 would have a non-empty intersection with $\vartheta(G \cap U)$. In particular, the open set H_2 would have a non-empty intersection with $\vartheta(G \cap U)$. It follows that $\vartheta^{-1}(H_2)$ would have a non-empty intersection with $G \cap U$ and therefore also with G, in contradiction to the right-hand equation in (9). A similar argument applies if s is assumed to be in $\sim G$: the assumption implies that the set $\vartheta(\sim G \cap U)^-$

is in M_s, and this set does not contain r_1. For if it did contain r_1, then H_1 would have a non-empty intersection with $\vartheta(\sim G \cap U)^-$, and therefore $\vartheta^{-1}(H_1)$ would have a non-empty intersection with $\sim G$, in contradiction to the left-hand inclusion in (9).

Define the function δ on each point s in V by taking $\delta(s)$ to be the unique point that belongs to the intersection of M_s. This function is a well-defined mapping from V to W, by the observations of the preceding two paragraphs. To see that δ is an extension of ϑ, assume that s belongs to U. In this case, s belongs to the intersection $F \cap U$ for every set F in N_s, so $\vartheta(s)$ belongs to $\vartheta(F \cap U)$ and therefore also to $\vartheta(F \cap U)^-$, for every set F in N_s. It follows that $\vartheta(s)$ is the unique element in the intersection of M_s, by (7). Consequently, $\delta(s) = \vartheta(s)$, by the definition of δ.

It remains to prove that δ is continuous. To this end, consider an arbitrary open subset H of W. It must be shown that the inverse image of H under δ is open in V, and for this it suffices to prove that every element s in $\delta^{-1}(H)$, belongs to some clopen set F that is included in $\delta^{-1}(H)$. Fix a point s in $\delta^{-1}(H)$, and observe that $\delta(s)$ belongs to H. The definition of $\delta(s)$ specifies that this value is the unique element in the intersection $\bigcap M_s$, so this intersection must be included in H. A routine compactness argument shows that the intersection of some finite subset of M_s must already be included in H (see Exercise 24 on p. 279 of [10]). The intersection of any finite subset of M_s includes an element from M_s, by the argument used to prove that M_s has the finite intersection property. Consequently, there is a clopen set F in N_s such that

$$\vartheta(F \cap U)^- \subseteq H, \tag{10}$$

by (7). The element s belongs to F, by the definition of N_s and the fact that F is in N_s. To see that F is included in $\delta^{-1}(H)$, consider an arbitrary element t in F. Since F is in N_t, by the definition of N_t, the set $\vartheta(F \cap U)^-$ must belong to M_t, by (7) (with t in place of s). Consequently, the intersection of the sets in M_t is included in $\vartheta(F \cap U)^-$ and therefore also in H, by (10). The value $\delta(t)$ is the unique point in the intersection of M_t, so $\delta(t)$ belongs to H, and therefore t belongs to $\delta^{-1}(H)$. This is true for every element t in F, so F is included in $\delta^{-1}(H)$. $\square$

In view of Theorem 2.59, we know that the maximum compactification $\mathfrak{V}$ of a union space $\mathfrak{U}$ has the topology of the classic Stone-Čech compactification, that is to say, every continuous function ϑ from $\mathfrak{U}$

into a compact Hausdorff space W can be extended to a continuous function from $\mathfrak{V}$ into W. It may happen that a compact Hausdorff space W is the universe of a relational structure $\mathfrak{W}$, and that a continuous function ϑ from $\mathfrak{U}$ to $\mathfrak{W}$ is also a homomorphism (though not necessarily a bounded homomorphism). What topological conditions must the relations in $\mathfrak{W}$ satisfy in order for the continuous extension of ϑ to also be a homomorphism? The answer is that the relations in $\mathfrak{W}$ must be closed subsets of the appropriate product space. Define a *closed space* to be a relational structure $\mathfrak{W}$ with a compact Hausdorff topology such that the relations in $\mathfrak{W}$ are closed in the product topology, that is to say, if R is a relation of rank n in $\mathfrak{W}$, then R is a closed subset of the product space W^n. Observe that every relational space is a closed space, by Theorem 2.15 and the remark preceding that theorem.

Theorem 2.60. *If* $\mathfrak{V}$ *is the Stone-Čech compactification of the union* $\mathfrak{U}$ *of a disjoint system of relational spaces*, *then every continuous homomorphism from* $\mathfrak{U}$ *into a closed space* $\mathfrak{W}$ *can be extended to a continuous homomorphism from* $\mathfrak{V}$ *into* $\mathfrak{W}$.

Proof. Suppose ϑ is a continuous homomorphism from $\mathfrak{U}$ into a closed space $\mathfrak{W}$. The topology on $\mathfrak{W}$ is, by definition, compact and Hausdorff, so there is a uniquely determined continuous function δ from $\mathfrak{V}$ into $\mathfrak{W}$ that extends ϑ, by Theorem 2.59. It must be demonstrated that δ preserves the fundamental relations of $\mathfrak{V}$. Focus on the case of a ternary relation R.

Assume u, v, and w are elements in $\mathfrak{V}$ such that

$$R(u, v, w) \tag{1}$$

holds in $\mathfrak{V}$, and write

$$r = \delta(u), \qquad s = \delta(v), \qquad t = \delta(w). \tag{2}$$

The goal is to prove that $R(r, s, t)$ holds in $\mathfrak{W}$. The strategy is to show that every open subset of the product space

$$W \times W \times W \tag{3}$$

that contains the triple (r, s, t) must have a non-empty intersection with R. Since the relation R is a closed subset of (3), by the assumption that $\mathfrak{W}$ is a closed space, it then follows that (r, s, t) belongs to R, so that $R(r, s, t)$ does hold. The sets of the form

$$H_1 \times H_2 \times H_3, \tag{4}$$

with H_k open in $\mathfrak{W}$ for $k = 1, 2, 3$, form a base for the product topology on (3), so it suffices to prove that every set of form (4) which contains the triple (r, s, t) has a non-empty intersection with R.

Consider open sets H_1, H_2, and H_3 in $\mathfrak{W}$ that contain the points r, s, and t respectively. The inverse images $\delta^{-1}(H_1)$ and $\delta^{-1}(H_2)$ are open subsets of $\mathfrak{V}$ that contain the points u and v respectively, by (2) and the continuity of δ. The clopen sets form a base for the topology of $\mathfrak{V}$, so there must be clopen sets F and G in $\mathfrak{V}$ such that

$$u \in F \qquad \text{and} \qquad v \in G, \tag{5}$$

and

$$F \subseteq \delta^{-1}(H_1) \qquad \text{and} \qquad G \subseteq \delta^{-1}(H_2). \tag{6}$$

The image set

$$R^*(F \times G) = \{z \in V : R(x, y, z) \text{ for some } x \in F \text{ and } y \in G\} \tag{7}$$

is clopen in $\mathfrak{V}$, because $\mathfrak{V}$ is a relational space (see Definition 2.2); and the point w belongs to this set, by (1), (5) and (7). Also, the inverse image $\delta^{-1}(H_3)$ is open and contains w, by (2), the continuity of δ, and the assumption that t is in H_3. Consequently, the intersection

$$R^*(F \times G) \cap \delta^{-1}(H_3) \tag{8}$$

is an open set in $\mathfrak{V}$ that contains w.

The universe of $\mathfrak{U}$ is a dense subset of $\mathfrak{V}$, because $\mathfrak{V}$ is a compactification of $\mathfrak{U}$ (see Definition 2.48), so the open set in (8) must intersect the universe of $\mathfrak{U}$ in some point $\bar{w}$. Since $\bar{w}$ belongs to the set in (7), there must be points $\bar{u}$ in F and $\bar{v}$ in G such that

$$R(\bar{u}, \bar{v}, \bar{w}) \tag{9}$$

holds in $\mathfrak{V}$. Now $\mathfrak{U}$ is an inner subspace of $\mathfrak{V}$ (because $\mathfrak{V}$ is a compactification of $\mathfrak{U}$) and consequently $\mathfrak{U}$ is algebraically an inner substructure of $\mathfrak{V}$, by Definition 2.48. Since $\bar{w}$ belongs to $\mathfrak{U}$, it follows that the points $\bar{u}$ and $\bar{v}$ must also belong to $\mathfrak{U}$, and (9) must hold in $\mathfrak{U}$, by the definition of an inner substructure. The continuous function ϑ is assumed to be a homomorphism from $\mathfrak{U}$ to $\mathfrak{W}$, so (9) implies that

$$R(\vartheta(\bar{u}), \vartheta(\bar{v}), \vartheta(\bar{w}))$$

holds in $\mathfrak{W}$. The function δ agrees with ϑ on $\mathfrak{U}$, so

$$\delta(\bar{u}) = \vartheta(\bar{u}), \qquad \delta(\bar{v}) = \vartheta(\bar{v}), \qquad \delta(\bar{w}) = \vartheta(\bar{w}),$$

and therefore

$$R(\delta(\bar{u}), \delta(\bar{v}), \delta(\bar{w})) \tag{10}$$

holds in $\mathfrak{W}$.

The points $\bar{u}$ and $\bar{v}$ belong to the sets F and G respectively, so they must also belong to the inverse images $\delta^{-1}(H_1)$ and $\delta^{-1}(H_2)$ respectively, by (6). Consequently, $\delta(\bar{u})$ is in H_1 and $\delta(\bar{v})$ in H_2, by the definition of the inverse image of a set. Also, the point $\bar{w}$ belongs to $\delta^{-1}(H_3)$, so $\delta(\bar{w})$ belongs to H_3. It follows that the triple

$$(\delta(\bar{u}), \delta(\bar{v}), \delta(\bar{w}))$$

belongs to the set in (4), and therefore to the intersection of this set with R, by (10). Thus, the set in (4) has a non-empty intersection with R, as was to be shown. Conclusion: the function δ preserves the fundamental relations of the relational structures, so it is a homomorphism; consequently, δ is a continuous homomorphism from $\mathfrak{V}$ to $\mathfrak{W}$ that extends ϑ. $\square$

Ian Hodkinson has kindly pointed out to us the following theorem due to Goldblatt that is apparently related to some of the results in this section: the ultrafilter space of the direct limit of a direct system of modal algebras is isomorphic to the inverse limit of the inverse system of ultrafilter spaces of the modal algebras in the direct system. According to Hodkinson, this theorem is a consequence of Theorems 10.7, 11.2, and 11.6 in [12].

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