Chapter 2
Oscillation of Delay Logistic Models

On earth there is nothing great but man, in man there is nothing great but mind.

William R. Hamilton (1805–1865).

Every problem in the calculus of variations has a solution, provided the word solution is suitably understood.

David Hilbert (1862–1943).

The qualitative study of mathematical models is important in applied mathematics, physics, meteorology, engineering, and population dynamics. In this chapter, we are concerned with the oscillation of solutions of different types of delay logistic models about their positive steady states. One of the main techniques that we will use in the proofs is the so-called linearized oscillation technique. This technique compares the oscillation of a nonlinear delay differential equation with its associated linear equation with a known oscillatory behavior.

In this chapter we establish oscillation results for a variety of autonomous and nonautonomous delay models. It is possible to extend the theory in this chapter to other models, for example, models with impulses and models with distributed delays. Results for other models (which are based on the ideas in this chapter) can be found in the reference list. Chapter 2 presents the current approach in the literature on oscillation of delay equations.

2.1 Models of Hutchinson Type

In this section, we are concerned with the oscillation of an equation of Hutchinson type about the positive equilibrium point. First, we consider the equation

$$N^{'}(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \tag{2.1}$$

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where $N(t)$ is the population at time t, r is the growth rate of the species, and $K > 0$ is called the carrying capacity of the habitat (note that here there is no immigration or emigration). The solution $N(t)$ of (2.1) is said to be oscillatory about the positive steady state K if $N(t_n) - K = 0$, for $n = 0, 1, 2, ..$ and $\lim_{n\to\infty} t_n = \infty$. The solution $N(t)$ of (2.1) is said to be nonoscillatory about K if there exits $t_0 \geq 0$ such that $|N(t) - K| > 0$ for $t \geq t_0$. A solution $N(t)$ is said to be oscillatory (here we mean oscillatory about zero) if there exists a sequence $\{t_n\}$ such that $N(t_n) = 0$, for $n = 0, 1, 2, ...$ and $\lim_{n\to\infty} t_n = \infty$. A solution $N(t)$ is said to be nonoscillatory if there exits $t_0 \geq 0$ such that $|N(t)| > 0$ for $t \geq t_0$.

Together with (2.1), we consider solutions of (2.1) which correspond to the initial condition

$$\begin{cases} N(t) = \phi(t) \text{ for } -\tau \leq t \leq 0, \\ \phi \in C([-\tau, 0], [0, \infty)), \text{ and } \phi(0) > 0. \end{cases} \tag{2.2}$$

Clearly the initial value problem (2.1), (2.2) has a unique positive solution for all $t \geq 0$. This follows by the method of steps. We begin with the usual result in any book on oscillation and we quote here the linearized oscillation theorem taken from [30].

Theorem 2.1.1. *Consider the nonlinear delay differential equation*

$$x'(t) + \sum_{i=1}^{n} p_i f_i(x(t - \tau_i)) = 0, \tag{2.3}$$

where for $i = 1, \ldots, n$,

$$p_i \in (0, \infty),\ \tau_i \in [0, \infty),\ f_i \in C[\mathbf{R}, \mathbf{R}], \tag{2.4}$$

$$u f_i(u) > 0 \ \textit{for}\ u \neq 0 \ \textit{and} \ \lim_{u\to 0} \frac{f_i(u)}{u} = 1, \tag{2.5}$$

and there exits a positive constant δ *such that*

$$\left.\begin{array}{lll} \textit{either} & f_i(u) \leq u & \textit{for } 0 \leq u \leq \delta \quad \textit{and } i = 1, 2, \ldots, n, \\ \textit{or} & f_i(u) \geq u & \textit{for } -\delta \leq u \leq 0 \quad \textit{and } i = 1, 2, \ldots, n. \end{array}\right\} \tag{2.6}$$

Then every solution of (2.3) oscillates if and only if every solution of the linearized equation

$$y'(t) + \sum_{i=1}^{n} p_i y(t - \tau_i) = 0 \tag{2.7}$$

oscillates.

Corollary 2.1.1 ([30]). *Assume that (2.4)–(2.6) hold. Then each one of the following two conditions is sufficient for the oscillation of all solutions of (2.3):*

(*a*) $\sum_{i=1}^{n} p_i \tau_i > \frac{1}{e}$;

(*b*) $\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}} \left(\sum_{i=1}^{n} \tau_i\right) > \frac{1}{e}$;

and when n = 1 the condition $p\tau > 1/e$ is necessary and sufficient for oscillation.

Now, we establish necessary and sufficient condition for the oscillation of all positive solutions of the delay logistic model (2.1) about the positive steady state K.

Theorem 2.1.2. *Every solution of (2.1) oscillates about K if and only if $r\tau > 1/e$.*

Proof. The change of variables

$$N(t) := Ke^{x(t)} \tag{2.8}$$

reduces Eq. (2.1) to the nonlinear delay equation

$$x'(t) + rf(x(t-\tau)) = 0, \tag{2.9}$$

where

$$f(u) = e^u - 1. \tag{2.10}$$

Clearly $f(u)$ satisfies the conditions (2.4)–(2.6). Corollary 2.1.1 completes the proof. ∎

We now consider a generalization of the delay logistic equation (2.1) with several delays of the form

$$N'(t) = N(t)\left[\alpha - \sum_{i=1}^{n} \beta_i N(t-\tau_i)\right], \tag{2.11}$$

where

$$\alpha,\ \beta_1,\ \beta_2, \ldots, \beta_n \in (0,\infty) \text{ and } 0 \le \tau_1 < \tau_2 < \ \tau_3 \ldots < \tau_n \equiv \tau. \tag{2.12}$$

Again with (2.11), we associate the initial condition (2.2) and then it follows by the method of steps that (2.2), (2.11) has a unique solution $N(t)$ and remains positive for all $t \ge 0$.

Theorem 2.1.3. *Assume that (2.12) holds. Then each one of the following conditions implies that every solution of (2.11) oscillates about $N^* = \alpha / \sum_{i=1}^{n} \beta_i$:*

(i) $\alpha e\left(\sum_{i=1}^{n}\beta_i\tau_i\right) > \left(\sum_{i=1}^{n}\beta_i\right)$;

(ii) $\alpha e\left(\prod_{i=1}^{n}\beta_i\right)^{\frac{1}{n}}\left(\sum_{i=1}^{n}\tau_i\right) > \left(\sum_{i=1}^{n}\beta_i\right)$.

Proof. Set

$$N(t) = N^* e^{x(t)}.$$

Then $x(t)$ satisfies Eq. (2.3), where

$$p_i = \beta_i N^*, \quad \text{for } i = 1, 2, \ldots, n \text{ and } f_i(u) = e^u - 1. \tag{2.13}$$

Clearly $f_i(u)$ for $i = 1, 2, .., n$ satisfy the conditions (2.4)–(2.6). The proof follows from Corollary 2.1.1. ■

2.2 Models with Delayed Feedback

In order to observe the influence of a feedback mechanism on fluctuations of a population density $N(t)$ around an equilibrium K via a constant λ, Olach [53] considered a modified nonlinear delay logistic model of the form

$$N'(t) = rN(t)\left|1 - \frac{N(\tau(t))}{K}\right|^{\lambda} sgn\left[\ln\frac{K}{N(\tau(t))}\right], \quad t \geq 0, \tag{2.14}$$

where r, K, $\lambda \in (0, \infty)$ and the term $1 - N(\tau(t))/K$ denotes a feedback mechanism.

We consider those solutions of (2.14) which correspond to the initial condition

$$\begin{cases} N(t) = \phi(t), \text{ for } \tau(0) \leq t \leq 0, \\ \phi \in C([\tau(0), \ 0], [0, \infty)), \ \phi(0) > 0. \end{cases} \tag{2.15}$$

It follows by the method of steps that (2.14), (2.15) has a unique positive solution $N(t)$ for all $t > 0$.

We discuss in this section the nonoscillation of positive solutions of (2.14) around the positive equilibrium point K. We begin with the following lemma.

Lemma 2.2.1. *Consider the nonlinear retarded differential equation*

$$x'(t) + p(t)f(x(\tau(t))) = 0, \quad t \geq t_0 \geq 0, \tag{2.16}$$

such that for $t \geq t_0$,

$$p \in C([t_0,\infty),\mathbf{R}^+),\ \tau \in C([t_0,\infty),\mathbf{R}^+),\ \tau(t) < t,\ \lim_{t\to\infty} \tau(t) = \infty, \tag{2.17}$$

$$f \in C(\mathbf{R},\mathbf{R}),\ \ u f(u) > 0 \ \text{for}\ u \neq 0, \tag{2.18}$$

and

$$\int_{t_0}^{\infty} p(t) = \infty. \tag{2.19}$$

Then every nonoscillatory solution $x(t)$ *of (2.16) satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that (2.16) has a nonoscillatory solution $x(t)$ which we shall assume to be eventually positive (if $x(t)$ is eventually negative the proof is similar). Since $uf(u) > 0$, we note that $x^{'}(t) < 0$ eventually for $t \geq t_1 \geq t_0$. Thus

$$\lim_{t\to\infty} x(t) = L \geq 0, \text{exists.}$$

We claim $L = 0$. If $L > 0$, we have

$$x(t_1) \geq L + \int_{t_1}^{\infty} p(s) f(x(\tau(s)))ds,$$

which with (2.19) gives a contradiction. Thus $\lim_{t\to\infty} x(t) = 0$. The proof is complete. ■

To prove the main oscillation results for Eq. (*2.14)* we need some oscillation results for the equation

$$x^{'}(t) + p(t)\,|x(\tau(t))|^{\lambda}\, sgn x(\tau(t)) = 0,\ \ t \geq t_0 \geq 0. \tag{2.20}$$

Let $C_{loc}([t_0,\infty),\mathbf{R})$ denote the space of continuous functions $x : [t_0,\infty) \to \mathbf{R}$ endowed with the topology of local uniform convergence.

Theorem 2.2.1. *Suppose that (2.17) holds,* $\lambda > 1$ *and for some* $\alpha \in (0,\lambda)$

$$\lim_{t\to\infty} \sup t\,[\tau(t)]^{-\alpha}\,[p(t)]^{(\lambda-\alpha)/\lambda} < \infty. \tag{2.21}$$

Then (2.20) has a nonoscillatory solution.

Proof. According to (2.21) there is a $c > 0$ such that

$$t\,[\tau(t)]^{-\alpha}\,[p(t)]^{(\lambda-\alpha)/\lambda} < c, \ \ \text{for}\ \ t \geq t_0.$$

Set

$$v(t) = c_0 t^{\alpha/(\alpha-\lambda)}, \text{ for } t \geq t_0, \text{ where } c_0 = \left[\frac{\alpha}{\lambda-\alpha} c^{(\lambda-\alpha)/\lambda}\right]^{1/(\lambda-1)}.$$

Let $\mathbf{S} \subset C_{loc}([t_0,\infty),\mathbf{R})$ be the set of functions which satisfy

$$0 \leq x(t) \leq v(t), \text{ for } \quad t \geq t_0$$

and define the operator

$$F : \mathbf{S} \to C_{loc}([t_0,\infty),\mathbf{R})$$

by

$$F(x)(t) = \begin{cases} \int_t^\infty p(s)[x(\tau(s))]^\lambda ds, & \text{for } \; t \geq t_1, \\ v(t) - v(t_1) + F(x)(t_1) & \text{for } \; t \in [t_0,t_1), \end{cases}$$

where $t_1 > t_0$ is such that $\tau(t) \geq t_0$ for all $t \geq t_1$. Note $F(\mathbf{S}) \subset \mathbf{S}$; to see this note if $x \in \mathbf{S}$ and $t \geq t_1$ then

$$F(x(t)) \leq \int_t^\infty p(s)[v(\tau(s))]^\lambda ds = \int_t^\infty p(s)\, c_0^\lambda\, [\tau(s)]^{\frac{\alpha\lambda}{\alpha-\lambda}}\, ds$$
$$\leq c_0^\lambda\, c^{\frac{\lambda}{\lambda-\alpha}} \int_t^\infty s^{\frac{\lambda}{\alpha-\lambda}}\, ds = v(t).$$

We note that $\mathbf{S}$ is a nonempty closed convex subset of $C_{loc}([t_0,\infty),\mathbf{R})$ and the operator F is continuous. The functions belonging to the set $F(\mathbf{S})$ are equicontinuous on compact subintervals of $[t_0,\infty)$. The Tychonov–Schauder Fixed Point Theorem guarantees that the operator F has an element $y \in \mathbf{S}$ such that $y = F(y)$. The proof is complete. ■

Theorem 2.2.2. *Suppose that (2.17)–(2.19) hold, and*

$$\lim_{u\to 0} \frac{f(u)}{|u|^\lambda\, sgn\, u} = 1, \quad \lambda > 1. \tag{2.22}$$

If (2.20) has a nonoscillatory solution then (2.16) also has a nonoscillatory solution.

Proof. Assume that $v(t)$ is a nonoscillatory solution of (2.20) such that $v(\tau(t)) > 0$ for $t \geq t_0$. According to (2.22) there is a $c_1 > 1$ and $\delta > 0$ such that $f(u) \leq c_1 u^\lambda$ for $u \in [0,\delta]$. From Lemma 2.2.1 we have

$$v(t) = \int_t^\infty p(s)[v(\tau(s))]^\lambda ds, \quad t \geq t_0.$$

Now choose $T_0 > t_0$ such that $v(t) < \delta$ for $t \geq T_0$. Let $\mathbf{S} \subset C_{loc}([t_0, \infty), \mathbf{R})$ be the set of functions satisfying

$$0 \leq x(t) \leq c_2 v(t), \text{ for } \quad t \geq T_0,$$

where $c_1 c_2^{\lambda} < c_2 < 1$, and define the operator

$$F : \mathbf{S} \to C_{loc}([t_0, \infty), \mathbf{R})$$

by

$$F(x)(t) = \begin{cases} \int_t^{\infty} p(s)\, f(x(\tau(s)))\, ds, & \text{for } \; t \geq t_1, \\ c_2[v(t) - v(t_1)] + F(x)(t_1), & \text{for } \; t \in [T_0, t_1), \end{cases}$$

where $t_1 > T_0$ is such that $\tau(t) \geq T_0$ for all $t \geq t_1$. Note $F(\mathbf{S}) \subset \mathbf{S}$; to see this note if $x \in \mathbf{S}$ and $t \geq t_1$ then

$$F(x(t)) \leq \int_t^{\infty} p(s)\, c_1\, [x(\tau(s))]^{\lambda}\, ds \leq c_1 c_2^{\lambda} \int_t^{\infty} p(s)\, [v(\tau(s))]^{\lambda}\, ds \leq c_2\, v(t).$$

The remainder of the proof is similar to that of Theorem 2.2.1. ■

Consider (2.14) about the positive steady state K. The transformation $N(t) = Ke^{x(t)}$ transforms Eq. (2.14) to Eq. (2.16) with

$$f(u) = |1 - e^u|\, sgn\, u.$$

Clearly the function $f(u)$ satisfies the hypothesis (2.18) and (2.22) so the above results apply to (2.14).

2.3 α-Delay Models

Aiello [2] considered the nonautonomous delay logistic model

$$N'(t) = r(t)N(t)\left[1 - \frac{N(\tau(t))}{K}\right]\left|1 - \frac{N(\tau(t))}{K}\right|^{\alpha-1}, \quad t > 0, \tag{2.23}$$

where K, α are positive constants, $\alpha \neq 1$, $r(t)$ and $\tau(t)$ are positive continuous functions defined on $[0, \infty)$ such that

$$\tau(t) \leq t, \text{ and } \lim_{t\to\infty} \tau(t) = \infty. \tag{2.24}$$

Our aim in this section is to study the oscillation and nonoscillation of all positive solutions of (2.23) about the positive steady state K. We consider (2.23) with an initial condition

$$\begin{cases} N(t) = \phi(t), \text{ for } \tau(0) \leq t \leq 0, \\ \phi \in C([\tau(0), 0], [0, \infty)), \ \phi(0) > 0. \end{cases} \tag{2.25}$$

The change of variables

$$y(t) = \frac{N(t)}{K} - 1 \tag{2.26}$$

in (2.23) gives us the nonlinear delay equation $y'(t) = -r(t)y(\tau(t))[1 + y(t)] \, |y(\tau(t))|^{\alpha-1}$. Since $N(t) > 0$ in (2.23) then $y(t) > -1$.

In this section we consider

$$y'(t) = -r(t)y(\tau(t))[1 + y(t)] \, |y(\tau(t))|^{\alpha-1}, \ t \geq t_0. \tag{2.27}$$

Assume that

$$\int_{t_0}^{\infty} r(s)ds < \infty \tag{2.28}$$

or

$$\int_{t_0}^{\infty} r(s)ds = \infty. \tag{2.29}$$

From the change of variables (2.26), we see that the oscillation or nonoscillation of (2.23) about K is equivalent to the oscillation or nonoscillation of (2.27) about zero. In the following, we are concerned with the existence of a nonoscillatory solution of (2.27) and the results in this section are adapted from [2].

First, we consider the case when (2.28) holds. Note for $t \geq t_0$ the function $r(t)$ is positive and

$$\int_{t_0}^{\infty} r(s)ds = R, \text{ where } 0 < R < \infty. \tag{2.30}$$

Theorem 2.3.1. *Assume that (2.24), (2.28), and (2.30) hold. Then (2.27) has a positive, nonoscillatory solution bounded away from zero.*

Proof. Note if y is a positive solution of (2.27) then

$$y'(t) = -r(t)[1 + y(t)](y(\tau(t)))^{\alpha}. \tag{2.31}$$

Let $\boldsymbol{\varphi}$ denote the locally convex space of continuous functions on $[t_0, \infty)$ with the topology of uniform convergence on compact sets of **R.** Define the set $\mathbf{S} \subset \boldsymbol{\varphi}$ as

$$\mathbf{S} := \begin{cases} \quad y \text{ is nonincreasing} \\ \quad y(t) = C_\alpha, & t_0 \le t < T \\ y \in \boldsymbol{\varphi} : C_\alpha \ge y(t) \ge C_\alpha \exp\left(-\int_T^t r(s)ds\right), & t \ge T \\ \frac{y(\tau(t))}{y(t)} \le \exp\left(\int_{t_0}^t r(s)ds\right), & t \ge T; \end{cases}$$

here $C_\alpha > 0$ is defined so that

$$[C_\alpha + 1]C_\alpha^{\alpha-1} \le \exp\left(-\int_T^t r(s)ds\right),$$

and T is sufficiently large so that $\tau(t) \ge t_0$ for all $t \ge T$. Such a constant C_α exists since the function

$$h(u) := (u+1)u^{\alpha-1}$$

is monotone increasing and

$$h(0) = 0 \text{ and } \quad h(1) = 2.$$

Since $0 < e^{-R} < 1$ (here R is as in (2.30)) there is a u_0 such that $h(u_0) = e^{-R}$. Then let C_α be any constant satisfying the inequality $0 < C_\alpha < u_0$, and

$$[C_\alpha + 1]C_\alpha^{\alpha-1} \le e^{-R}$$

necessarily follows. Let $R(t) = \int_{t_0}^t r(s)ds$. Note that, since $r(t) \ge 0$ and $T \ge t_0$, we have

$$\int_T^t r(s)ds \le R(t).$$

We can easily see that $\mathbf{S} \subset \boldsymbol{\varphi}$ is nonempty, since $y(t) = C_\alpha$ is in **S.** In addition, **S** is a closed convex subset of $\boldsymbol{\varphi}$. Let $y \in \mathbf{S}$ and define the map

$$F\ y(t) = \begin{cases} C_\alpha, & \text{for } t_0 \le t < T, \\ C_\alpha \exp\left(-\int_T^t \frac{r(s)(1+y(s))(y(\tau(s)))^\alpha}{y(s)} ds\right), & \text{for } \quad t \ge T. \end{cases}$$

Clearly $F\ y(t)$ is continuous, nonincreasing and satisfies

$$F\ y(t) \begin{cases} = C_\alpha, & \text{for } t_0 \le t < T, \\ \le C_\alpha, & \text{for } \quad t \ge T, \end{cases}$$

and since $y(t) \le C_\alpha$, we have by definition that

$$(1 + y(s))(y(\tau(s)))^{\alpha-1} \le e^{-R} < 1, \text{ and } \frac{y(\tau(t))}{y(t)} \le e^{R(t)} \le e^{R}.$$

Then

$$\begin{aligned}\int_T^t \frac{r(s)(1 + y(s))(y(\tau(s)))^\alpha ds}{y(s)} &\le \int_T^t e^{-R} \frac{r(s)y(\tau(s))ds}{y(s)} \\ &\le \int_T^t e^{-R} e^{R(s)} r(s)ds \le \int_T^t r(s)ds,\end{aligned}$$

so,

$$F\,y(t) \ge C_\alpha \exp\left(-\int_T^t r(s)ds\right), \text{ for } t \ge T.$$

Also for $t \ge T$

$$\begin{aligned}\frac{F\,y(\tau(t))}{F\,y(t)} &= \exp\left(\int_{\tau(t)}^t \frac{r(s)(1 + y(s))(y(\tau(s)))^\alpha}{y(s)} ds\right) \\ &\le \exp\left(\int_{\tau(t)}^t e^{-R} \frac{r(s)y(\tau(s))}{y(s)} ds\right) \\ &\le \exp\left(\int_{\tau(t)}^t e^{-R} e^{R(s)} r(s)ds\right) \\ &\le \exp\left(\int_{\tau(t)}^t r(s)ds\right) \le \exp\left(\int_{t_0}^t r(s)ds\right),\end{aligned}$$

so,

$$\frac{F\,y(\tau(t))}{F\,y(t)} \le e^{R(t)}, \text{ for } t \ge T.$$

Thus, $F(\mathbf{S}) \subset \mathbf{S}$. Note $\mathbf{S}$ is bounded above by C_α and bounded below by $C_\alpha e^{-R}$. We now prove that $\{F\,y : y \in \mathbf{S}\}$ is equicontinuous on compact sets of $[t_0, \infty)$. Let T_1 and T_2 be elements in $\mathbf{R}$ and let $T_i^* = \max\{T, T_i\}$ for $i = 1, 2$. Then

$$\begin{aligned}|F\,y(T_1) - F\,y(T_2)| &= |F\,y(T_1^*) - F\,y(T_2^*)| \\ = C_\alpha &\left|\exp\left(-\int_T^{T_1^*} \frac{r(s)(1 + y(s))(y(\tau(s)))^\alpha}{y(s)} ds\right)\right.\end{aligned}$$

$$- \exp\left(\int_{T}^{T_2^*} \frac{-r(s)(1+y(s))(y(\tau(s)))^{\alpha}}{y(s)} ds\right)\Bigg|$$

$$\leq C_{\alpha}\left|1 - \exp\left(\int_{T_1^*}^{T_2^*} \frac{-r(s)(1+y(s))(y(\tau(s)))^{\alpha}}{y(s)} ds\right)\right|$$

$$\leq C_{\alpha}\left|1 - \exp\left(\int_{T_1^*}^{T_2^*} -r(s)ds\right)\right| \to 0, \quad \text{as } T_1 \to T_2,$$

uniformly so $\{Fy : y \in \mathbf{S}\}$ is equicontinuous on every compact set in $[t_0, \infty)$. Apply the Arzela–Ascoli Theorem to conclude that $\overline{F\mathbf{S}}$ is compact in $\mathbf{S}$. The Tychonov-Schauder Fixed Point Theorem guarantees a fixed point y^* of F. This y^* solves (2.31) from the definition of F. The proof is complete. ■

Now, we consider the case when (2.29) holds. First, we prove that every nonoscillatory solution of (2.27) tends to zero as t tends to infinity.

Theorem 2.3.2. *Assume that the conditions of Theorem 2.3.1 hold, except that condition (2.28) is replaced by (2.29) and (2.30) is removed. Then every nonoscillatory solution of (2.27) will satisfy* $\lim_{t\to\infty} y(t) = 0$.

Proof. First, we consider the case when $y(t) > 0$ for all $t > t_1 > 0$. Let

$$v^*(t) = \sup\{s : \tau(s) = t\},$$

and since $\lim_{t\to\infty} \tau(t) = \infty$ there exists $T = v^*(t_1)$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for all $t \geq T$. From (2.27) we have

$$y^{'}(t) = -r(t)[1 + y(t)](y(\tau(t)))^{\alpha} \leq 0. \tag{2.32}$$

Thus,

$$\lim_{t\to\infty} y(t) = \gamma \geq 0 \text{ exists.}$$

Suppose $\gamma > 0$. For all $t \geq T$, $y(t) \geq \gamma$ and $y(\tau(t)) \geq \gamma$ and so (2.32) implies that

$$y^{'}(t) \leq -r(t)[1+\gamma]\gamma^{\alpha},$$

so integration and (2.29) implies that $y(t)$ is negative, and this is a contradiction. Thus $\gamma = 0$. Next, we consider the case when $y(t)$ is negative. Let $y(t)$ be an eventually negative solution of (2.27), such that

$$-1 < y(t) < 0 \text{ and } \quad y(\tau(t)) < 0,$$

for $t \geq T_0$ sufficiently large. Let $T_1 > T_0$ be such that $\tau(t) \geq T_0$ for all $t \geq T_1$. Now, since $y(\tau(t)) < 0$ for $t \geq T_1$, we have from (2.27) that

$$y^{'}(t) = -r(t)[1 + y(t)]y(\tau(t))\,|y(\tau(t))|^{\alpha-1} > 0, \quad t \geq T_1. \tag{2.33}$$

Then

$$\lim_{t\to\infty} y(t) = -\beta \text{ exists, where } 0 \leq \beta < 1.$$

Suppose that $\beta \neq 0$. Since $y^{'}(t) > 0$ and

$$y(\tau(t)) \leq -\beta, \quad t \geq T_1,$$

we have

$$y^{'}(t) \geq -r(t)[1 + y(t)]\beta^{\alpha}, \quad t \geq T_1. \tag{2.34}$$

Now, since $y(t)$ is nonincreasing and $\lim_{t\to\infty} y(t) = -\beta$ then there exists $T_\varepsilon \geq T_1$ such that

$$[1 + y(t)] \geq 1 - \beta - \varepsilon > 0,$$

so with (2.34) we have

$$y^{'}(t) \geq -r(t)[1 - \beta - \varepsilon]\beta^{\alpha}, \quad t \geq T_\varepsilon,$$

which by integration gives a contradiction. Then $\beta = 0$ and this completes the proof. ■

Now, we give sufficient conditions for the existence of nonoscillatory solutions of (2.27) when (2.29) holds and $\alpha \neq 1$.

Theorem 2.3.3. *Assume that (2.24) and (2.29) hold and $\alpha \neq 1$. Furthermore suppose that*

$$\lim_{t\to\infty} \sup \int_{\tau(t)}^{t} r(s)ds < \hbar, \quad \text{where } 0 < \hbar < \infty.$$

Then (2.27) has a nonoscillatory solution.

Proof. Let $\boldsymbol{\varphi}$ denote the locally convex space of continuous functions on $[t_0, \infty)$ with the topology of uniform convergence on compact sets of **R.** Define the set $\mathbf{S} \subset \boldsymbol{\varphi}$ as

$$\mathbf{S} = \begin{cases} y \text{ is nonincreasing} & \\ y(t) = C_\alpha, & t_0 \le t < t_1 \\ y \in \boldsymbol{\varphi} : C_\alpha \ge y(t) \ge C_\alpha \exp\left(-\int_{t_1}^{t} r(s)ds\right), & t_1 \le t < \infty \\ \frac{y(\tau(t))}{y(t)} \le e^{\hbar}, & t \ge t_1 \end{cases}$$

where $0 < C_\alpha < 1$ is defined so that

$$[C_\alpha + 1]C_\alpha^{\alpha-1} \le 1/e^{\hbar},$$

and t_1 is sufficiently large so that

$$\int_{\tau(t)}^{t} r(s)ds < \hbar, \text{ for } t \ge t_1.$$

The remainder of the proof is similar to that of Theorem 2.3.1 and hence is omitted. ■

From the change of variables $y(t) = N(t)/K - 1$ and Theorems 2.3.1–2.3.3 we have the following results on the delay logistic Eq. (2.23).

Theorem 2.3.4. *Assume that (2.24), (2.28), and (2.30) hold. Then (2.23) has a positive, nonoscillatory solution bounded away from K.*

Theorem 2.3.5. *Assume that (2.24) and (2.29) hold. Then every nonoscillatory solution of (2.23) will satisfy* $\lim_{t\to\infty} N(t) = K$.

Theorem 2.3.6. *Assume that (2.24) and (2.29) hold and* $\alpha \ne 1$. *Furthermore suppose that*

$$\lim_{t\to\infty} \sup \int_{\tau(t)}^{t} r(s)ds < \hbar, \ \ where \ \ 0 < \hbar < \infty.$$

Then (2.23) has a nonoscillatory solution.

The following examples illustrate the theory.

Example 1. Consider the nonlinear delay logistic equation

$$N'(t) = \frac{1}{t^2}N(t)(1 - N(t-\tau)/K)\,|1 - N(t-\tau)/K|^2, \ \ t > t_0,$$

where K is a positive constant. Here $r(t) = 1/t^2$, and for $t_0 > 0$,

$$\int_{t_0}^{\infty} (1/s^2)ds = 1/t_0 < \infty.$$

The conditions of Theorem 2.3.4 are satisfied, so there exists a nonoscillatory solution to this equation which is bounded away from K.

Example 2. Consider the nonlinear delay logistic equation

$$N^{'}(t) = rN(t)(1 - N(t-\tau)/K)\,|1 - N(t-\tau)/K|^{2}\,, \;\; t > t_0,$$

where K is a positive constant. Here $r(t) = r > 0$ satisfies

$$\int_{t_0}^{\infty} r ds = \infty.$$

The conditions of Theorem 2.3.5 are satisfied, so there exists a nonoscillatory solution to this equation for any $\tau > 0$ and by Theorem 2.3.5 it tends to K when t tends to infinity.

It is important to establish necessary conditions for the existence of nonoscillatory solutions to (2.23). Li [38] considered this problem and established these conditions by analyzing the generalized characteristic equation corresponding to (2.27). These conditions are equivalent to the sufficient and necessary conditions for the existence of positive solutions of (2.23).

We begin with the following theorem which gives the characteristic equation of (2.27).

Theorem 2.3.7. *A necessary and sufficient condition for the existence of a nonoscillatory solution of (2.27) is that there exist a constant C_α, a function $\lambda(t)$, and t_1 such that*

$$\lambda(t) = |C_\alpha|^{\alpha-1}\left(1 + C_\alpha \exp\left(-\int_{t_1}^{t} r(s)\lambda(s)ds\right)\right)$$
$$\times \exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right). \qquad (2.35)$$

Theorem 2.3.8. *Assume that $\alpha \in (0,1)$. Then (2.29) is a necessary and sufficient condition for every solution of (2.27) to be oscillatory.*

Proof. (*i*) Necessity. If (2.29) does not hold, we can assume that there exists a constant

$$k =: \frac{1}{(2-\alpha)(1+C_\alpha)C_\alpha^{\alpha-1}},$$

where C_α is a positive number, such that

$$\int_{t_0}^{\infty} r(s)ds \le k.$$

Let $T_0 = \inf_{t \geq t_0} \tau(t)$ and let $C([T_0, \infty), \mathbf{R})$ denote the locally convex space of continuous functions on $[T_0, \infty)$ with the topology of uniform convergence on compact sets of $[T_0, \infty)$. Define the subset Ω of $C([T_0, \infty), \mathbf{R})$ by

$$\Omega = \{x \in C([T_0, \infty), \mathbf{R}) : x(t) \geq 0, \quad |x(t)| \leq e(1 + C_\alpha)C_\alpha^{\alpha-1}, \ t \geq T_0\}.$$

Let $x \in \Omega$ and define a mapping F on Ω by

$$(F\,x)(t) = \begin{cases} |C_\alpha|^{\alpha-1}\left(1 + C_\alpha \exp\left(-\int_{T_0}^{t} r(s)x(s)ds\right)\right) \\ \times \exp\left(\int_{\tau(t)}^{t} r(s)x(s)ds + (1-\alpha)\int_{T_0}^{\tau(t)} r(s)x(s)ds\right), & t \geq t_0, \\ (F\,x)(t_0), & t_0 \geq t \geq T_0. \end{cases}$$

Then as in the proof of Theorem 2.3.1 we have $F\,x(t)$ is continuous and $F(\Omega) \subset \Omega$. Also $\{F\,x : x \in \Omega\}$ is equicontinuous and uniformly bounded. Apply the Arzela–Ascoli Theorem to conclude that $\overline{F\,\Omega}$ is compact in Ω. Now, by using the Tychonov–Schauder Fixed Point Theorem, we see that there exists a $\lambda \in \Omega$ such that for $t \geq t_0$ we have

$$\lambda(t) = |C_\alpha|^{\alpha-1}\left(1 + C_\alpha e^{-\int_{T_0}^{t} r(s)\lambda(s)ds}\right) \times \exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{T_0}^{\tau(t)} r(s)\lambda(s)ds\right). \quad (2.36)$$

By Theorem 2.3.7, (2.27) has a nonoscillatory solution.

(*ii*) Sufficiency. If (2.27) has an eventually positive solution, by Theorem 2.3.7 there exit C_α, t_1, and a continuous function $\lambda(t)$ satisfying

$$\begin{aligned} \lambda(t) &= \left(1 + C_\alpha e^{-\int_{t_1}^{t} r(s)\lambda(s)ds}\right) \\ &\quad \times |C_\alpha|^{\alpha-1} \exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right) \\ &\geq |C_\alpha|^{\alpha-1} \exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right) \\ &\geq |C_\alpha|^{\alpha-1} \exp\left((1-\alpha)\int_{t_1}^{t} r(s)\lambda(s)ds\right). \end{aligned}$$

Set

$$z(t) = \exp(-(1-\alpha)\int_{t_1}^{t} r(s)\lambda(s)ds)$$

and note

$$z'(t) \leq -|C_\alpha|^{\alpha-1}(1-\alpha)r(t)z(t_1).$$

Integrate and we have by (2.29) that

$$\lim_{t\to\infty} z(t) = -\infty,$$

a contradiction. Similarly, we can show that (2.27) has no eventually negative solution $y(t)$ with $1 + y(t) > 0$. The proof is complete. ■

Now, we consider the case when $\alpha > 1$.

Theorem 2.3.9. *Assume that $\alpha > 1$. Then a necessary and sufficient condition for the existence of a nonoscillatory solution of (2.27) is that there exists a positive continuous function $\lambda(t)$ such that for $t \geq T$*

$$\exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{T}^{\tau(t)} r(s)\lambda(s)ds\right) \leq m\lambda(t), \tag{2.37}$$

where m and T are some positive constants.

Proof. (i) Sufficiency. We only consider the case (since the other case is similar) when

$$\int_{t_0}^{\infty} r(s)\lambda(s)ds < \infty.$$

Then there exist ϱ, T and $C_\alpha > 0$ such that

$$\int_{T}^{\infty} r(s)\lambda(s)ds < \varrho, \quad (1+C_\alpha)C_\alpha^{\alpha-1} < \frac{1}{m\varrho}.$$

Let $T_0 = \inf_{t\geq t_0}\tau(t)$. Define a mapping F on $C([T_0,\infty),\mathbf{R}^+)$ as follows

$$(F\,y)(t) := \begin{cases} \int_t^\infty r(s)(1+y(s))y^\alpha(\tau(s))ds, & t \geq T \\ (F\,y)(T) + C_\alpha \exp(-\int_{T_0}^t r(s)\lambda(s)ds) & \\ -C_\alpha \exp(-\int_{T_0}^T r(s)\lambda(s)ds), & T_0 \leq t \leq T. \end{cases}$$

Clearly F is an increasing operator. Set

$$y_0 := C_\alpha \exp(-\int_T^t r(s)\lambda(s)ds), \;\; y_{n+1} = F\,y_n, \quad n = 1, 2, \ldots .$$

Then we have that

$$y_0(t) \geq y_1(t) \geq \ldots \geq y_n(t) \geq \ldots \geq 0, \quad \text{for } t \geq T_0. \tag{2.38}$$

In fact

$$\begin{aligned} y_1(t) = (F\,y_0)(t) &\leq \int_t^\infty r(s)\left(1 + C_\alpha \exp(-\int_T^s r(u)\lambda(u)du\right) \\ &\quad\times \left(C_\alpha^\alpha \exp(-\alpha \int_T^{\tau(s)} r(u)\lambda(u)du\right) ds \\ &\leq C_\alpha^\alpha (1 + C_\alpha) m \int_t^\infty r(s)\lambda(s)ds \exp\left(-\int_T^t r(s)\lambda(s)ds\right) \\ &\leq C_\alpha \exp\left(-\int_T^t r(s)\lambda(s)ds\right) = y_0(t), \quad t \geq T. \end{aligned}$$

Continue to obtain (2.38). Then $\lim_{n\to\infty} y_n(t) = y(t) \geq 0$, $t \geq T_0$, exists. From the Lebesgue's Dominated Convergence Theorem

$$y(t) := \begin{cases} \int_t^\infty r(s)(1 + y(s))y^\alpha(\tau(s))ds, & t \geq T \\ \\ (F\,y)(T) + C_\alpha \exp(-\int_{T_0}^t r(s)\lambda(s)ds) \\ -C_\alpha \exp(-\int_{T_0}^T r(s)\lambda(s)ds), & T_0 \leq t \leq T. \end{cases}$$

It is easy to see that $y(t) > 0$ on $[T_0, T]$ and hence $y(t) > 0$ for all $t \geq T_0$. Therefore, $y(t)$ is a positive solution of (2.27) on $[T, \infty)$.

(ii) Necessity. If (2.27) has an eventually positive solution then from Theorem 2.3.7 there exists a continuous positive function $\lambda(t)$ such that

$$\begin{aligned} \lambda(t) &= \left(1 + C_\alpha \exp\left(-\int_{t_1}^t r(s)\lambda(s)ds\right)\right) \\ &\quad\times C_\alpha^{\alpha-1} \exp\left(\int_{\tau(t)}^t r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right) \\ &\geq C_\alpha^{\alpha-1} \exp\left(\int_{\tau(t)}^t r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right). \end{aligned} \tag{2.39}$$

Let $m = 1/C_\alpha^{\alpha-1}$. Then (2.39) implies (2.37). If (2.27) has an eventually negative solution, then

$$\lambda(t) \geq (1-|C_\alpha|) \left|C_\alpha^{\alpha-1}\right| \exp\left(\int_{\tau(t)}^{t} r(s)\lambda(s)ds + (1-\alpha)\int_{t_1}^{\tau(t)} r(s)\lambda(s)ds\right),$$

where $|C_\alpha| < 1$. Thus (2.37) is also true. The proof is complete. ■

From Theorems 2.3.8 and 2.3.9 one can immediately derive some explicit necessary and sufficient conditions for the oscillation and the existence of nonoscillatory solutions of (2.23) about the positive steady state K.

2.4 α-Models with Several Delays

In this section, we consider the nonlinear delay logistic equation with several delays of the form

$$N'(t) = \sum_{k=1}^{m} r_k(t)N(t)\left[1 - \frac{N(h_k(t))}{K}\right]\left|1 - \frac{N(h_k(t))}{K}\right|^{\alpha_k - 1}, \quad t > 0, \tag{2.40}$$

where $\alpha_k < 1$, $k = 1, \ldots, m$ or $\alpha_k > 1$, $k = 1, \ldots, m$ under the conditions:

(b_1) r_k, $k = 1, 2, \ldots, m$, are Lebesgue measurable functions essentially bounded in each finite interval $[0, b]$, $r_k \geq 0$,
(b_2) $h_k : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k = 1, 2, \ldots, m$.

The case $\alpha_k = 1$, $k = 1, \ldots, m$, will be considered in detail in Sect. 2.6.
We consider positive solutions of (2.40) with an initial condition

$$\begin{cases} N(t) = \phi(t), \text{ for } \tau_* \leq t \leq 0, \\ \phi \in C([\tau_*, 0], [0, \infty)), \quad \phi(0) > 0, \end{cases} \tag{2.41}$$

where

$$\tau_* = \min_{1\leq k\leq m}\left(\inf_{t\geq 0}\{h_k(t)\}\right).$$

Clearly the initial value problem (2.40), (2.41) has a unique positive solution for all $t \geq 0$. This follows from the method of steps. In this section we consider

$$x'(t) = -\left[x(t) + 1\right]\sum_{k=1}^{m} r_k(t)x(h_k(t))\left|x(h_k(t))\right|^{\alpha_k - 1}, \; t \geq 0, \tag{2.42}$$

and it is also possible to consider

$$x^{'}(t) = -[x(t)+1]\sum_{k=1}^{m} r_k(t)x(h_k(t))\,|x(h_k(t))|^{\alpha_k-1}\,,\ t \ge t_0,$$

$$x(t) = \varphi(t), \quad t < t_0, \quad \text{and} \quad x(t_0) = x_0 > -1,$$

where

(b_3) $\varphi : (-\infty, t_0) \to$ is a Borel measurable bounded function.

We also consider the delay differential inequalities

$$x^{'}(t) \le -[x(t)+1]\sum_{k=1}^{m} r_k(t)x(h_k(t))\,|x(h_k(t))|^{\alpha_k-1}\,,\ t \ge 0, \tag{2.43}$$

$$x^{'}(t) \ge -[x(t)+1]\sum_{k=1}^{m} r_k(t)x(h_k(t))\,|x(h_k(t))|^{\alpha_k-1}\,,\ t \ge 0. \tag{2.44}$$

In the following we discuss the nonoscillation of solutions of (2.42) which is equivalent to the nonoscillation of positive solutions of (2.40) about K. The results in this section are adapted from [5].

In the following we assume $\alpha_k < 1$, $k = 1, 2, \ldots, m$, and that $(b_1) - (b_2)$ hold and we consider solutions of (2.42), (2.43), and (2.44) for which $1 + x(t) > 0$.

We prove the following comparison theorem.

Theorem 2.4.1. *The following statements are equivalent:*

(*1*) *Either inequality (2.43) has an eventually positive solution or inequality (2.44) has an eventually negative solutions.*

(*2*) *There exist* $t_0 \ge 0$, $\varphi : (-\infty, t_0) \to \mathbf{R}$, *with either* $\varphi(t) \ge 0$, $C > 0$, *or* $\varphi(t) \le 0$, $-1 < C < 0$, *such that the inequality*

$$u(t) \ge \left(1 + C\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)\sum_{k=1}^{m}(F_k u)(t), \tag{2.45}$$

where

$$(F_k u)(t) = \begin{cases} |C|^{\alpha_k-1}\, r_k(t) \times \exp\{\int_{h_k(t)}^{t} u(s)ds\} & \\ \times \exp\{(1-\alpha_k)\int_{t_0}^{h_k(t)} u(s)ds\}, & \text{if } h_k(t) \ge t_0 \\ \frac{r_k(t)}{|C|}\exp\{\int_{t_0}^{t} u(s)ds\}\,|\varphi(h_k(t))|^{\alpha_k}\,, & \text{if } h_k(t) < t_0 \end{cases}$$

has a nonnegative locally integrable solution on $[t_0, \infty)$.

(*3*) *Equation (2.42) has a nonoscillatory solution.*

Proof. 1) $\Rightarrow$ (2) Let x be a solution of (2.43) and $x(t) > 0$ for $t \geq t_1$. Then there exists $t_0 > t_1$ such that $h_k(t) \geq t_1$ for $t \geq t_0$, $k = 1, \ldots, m$. Denote $\varphi(t) = x(t)$, $t < t_0$, and $C = x(t_0)$. Let

$$u(t) = \frac{-x'(t)}{x(t)}, \quad t \geq t_0.$$

Then $u(t) \geq 0$ and

$$x(t) = \begin{cases} C\,\exp\{-\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ \varphi(t), & t < t_0. \end{cases} \tag{2.46}$$

Then by substituting x in (2.43) we obtain inequality (2.45). Similarly (2.45) can be obtained, if $x(t) < 0$ is a solution of (2.44).

2) $\Rightarrow$ 3). Let u_0 be a nonnegative solution of inequality (2.45) with

$$\varphi(t) \leq 0, \ -1 < C < 0.$$

Denote a sequence

$$u_n(t) = \left(1 + C\,\exp\left\{-\int_{t_0}^{t} u_{n-1}(s)ds\right\}\right)\sum_{k=1}^{m}(F_k u_{n-1})(t). \tag{2.47}$$

Inequality (2.45) implies $u_1(t) \leq u_0(t)$. By induction, we have

$$0 \leq u_n(t) \leq u_{n-1}(t) \leq u_0(t).$$

Then there exits a pointwise limit of the nonincreasing nonnegative limit $u_n(t)$. Let

$$\lim_{n\to\infty} u_n(t) = u(t).$$

Then by the Lebesgue Convergence Theorem

$$\lim_{n\to\infty}(F_k u_n)(t) = (F_k u)(t), \quad k = 1, 2, \ldots, m.$$

Thus (2.47) implies that

$$u(t) = \left(1 + C\,\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)\sum_{k=1}^{m}(F_k u)(t).$$

Hence the function $x(t)$ defined by (2.46) is an eventually negative solution of (2.42). Now let u_0 be a nonnegative solution of inequality (2.45) with $\varphi(t) \geq 0$, $C > 0$. Let $C_1 = -C$, $\varphi_1(t) = -\varphi(t)$. Then u is also a solution of (2.45) with

C_1 (respectively $\varphi_1(t)$) instead of C (respectively, $\varphi(t)$). As in the previous case it follows that there exists an eventually negative solution of (2.42). Implication 3) $\Rightarrow$ 1) is evident. The proof is complete. ∎

Corollary 2.4.1. *Suppose there exist t_0 and $A > 1$ such that the inequality*

$$u(t) \geq A \sum_{k=1}^{m} r_k(t) \exp\left\{ \int_{h_k(t)}^{t} u(s)ds \right\} \times \exp\left\{ (1-\alpha_k) \int_{t_0}^{h_k(t)} u(s)ds \right\} \quad (2.48)$$

has a nonnegative, locally integrable solution, where the sum contains only such terms for which $h_k(t) \geq t_0$. Then (2.42) has a nonoscillatory solution.

In the following we give some necessary and sufficient conditions for the existence of nonoscillatory solutions of (2.42).

Theorem 2.4.2. *There exists a nonoscillatory solution of (2.42) if and only if*

$$\int_0^\infty r_k(t)dt < \infty, \quad k = 1, 2, \ldots, m. \quad (2.49)$$

Proof. First, suppose that (2.49) holds. Then there exist t_0 and $A > 1$ such that

$$A \exp\left\{ 2 \int_{t_0}^{\infty} \sum_{k=1}^{m} r_k(t)dt \right\} < 2.$$

For any nonnegative u

$$A \sum_{k=1}^{m} r_k(t) \exp\left\{ \int_{h_k(t)}^{t} u(s)ds \right\} \times \exp\left\{ (1-\alpha_k) \int_{t_0}^{h_k(t)} u(s)ds \right\}$$
$$\leq A \sum_{k=1}^{m} r_k(t) \exp\left\{ \int_{t_0}^{t} u(s)ds \right\}.$$

Let

$$u(t) = 2 \sum_{k=1}^{m} r_k(t).$$

From the above inequalities we see that u is a solution of inequality (2.48). Corollary 2.4.1 implies that (2.42) has an eventually positive solution.

Suppose now that for some i, $1 \leq i \leq m$, we have $\int_0^\infty r_i(t)dt = \infty$. Let x be a positive or negative solution of (2.42) for $t \geq t_1$. There exists $t_0 > t_1$ such that $h_k(t) \geq t_1$, for $t \geq t_0$ and $k = 1, 2, \ldots m$. Let

$$u(t) = \frac{-x'(t)}{x(t)}, \quad t \geq t_0.$$

Then $u(t) \geq 0$ and $x(t)$ satisfies (2.46) where $C = x(t_0)$. Substituting x in (2.42) we obtain for $t \geq t_0$

$$u(t) = \begin{cases} \sum_{k=1}^{m} |C|^{\alpha_k - 1} r_k(t)(1 + C\exp\{-\int_{t_0}^{t} u(s)ds\}) \\ \quad \times \exp\{-\alpha_k \int_{t_0}^{h_k(t)} u(s)ds\} \exp\{\int_{t_0}^{t} u(s)ds\}. \end{cases}$$

Then

$$u(t) \geq \min\{1, 1 + C\} |C|^{\alpha_k - 1} r_i(t) \exp\{(1 - \alpha_i) \int_{t_0}^{t} u(s)ds\}.$$

Hence

$$r_i(t) \leq \frac{|C|^{1-\alpha_i}}{\min\{1, 1 + C\} |C|} u(t) \exp\{-(1 - \alpha_i) \int_{t_0}^{t} u(s)ds\}$$

and so

$$\begin{aligned} \int_{t_0}^{t} r_i(s)ds &\leq \frac{|C|^{1-\alpha_i}}{\min\{1, 1 + C\} |C|} \int_{t_0}^{t} u(s) \exp\{-(1 - \alpha_i) \int_{t_0}^{s} u(\tau)d\tau\} ds \\ &= \frac{|C|^{1-\alpha_i}}{\min\{1, 1 + C\} |C|} \left(1 - \exp\{-(1 - \alpha_i) \int_{t_0}^{t} u(s)ds\}\right) \\ &\leq \frac{|C|^{1-\alpha_i}}{\min\{1, 1 + C\} |C|}. \end{aligned}$$

Hence

$$\int_{t_0}^{\infty} r_i(s)ds < \infty,$$

which gives a contradiction. The proof is complete. ■

It is also possible to establish results when $\alpha_k = 1$ for $k = 1, 2, \ldots, m$ (see Sect. 2.6 where a more general situation is considered).

Next, we consider the case when $\alpha_k > 1$ for $k = 1, 2, \ldots, m$.

Lemma 2.4.1. *If $h \in L_\infty[a, b]$, then the linear integral operator*

$$(Hx)(t) = \begin{cases} \int_{a}^{h(t)} x(s)ds, & \text{if } h(t) \in [a, b] \\ 0, & \text{if } h(t) \notin [a, b] \end{cases}$$

is a completely continuous operator in $L_\infty[a, b]$.

Proof. Let $\epsilon > 0$ be given. Divide $H([a,b]) \cap [a,b]$ into a finite number of subsets F_i, $i = 1, \ldots, n$, such that for every $s_1, s_2 \in F_i$ we have $|s_1 - s_2| < \epsilon$. Let

$$E_i = h^{-1}(F_i),\ i = 1, \ldots, n,\ \ E_0 = \{t \in [a,b] : \ h(t) \notin [a,b]\},$$

$$S = \{x \in L_\infty[a,b] : \ ||x|| = 1\} \ \text{ and } \ M = H(S).$$

For dilatation E_i, $i = 1, 2, \ldots$, we have

$$\sup_{t,s \in E_i} |(Hx)(t) - (Hx)(s)| = \sup_{t,s \in E_i} |\int_{h(t)}^{h(s)} x(w)dw| \leq \sup_{t,s \in E_i} |h(t) - h(s)| < \epsilon.$$

If $i = 0$ then $\sup_{t,s \in E_0} |(Hx)(t) - (Hx)(s)| = 0$. Now Theorem 1.4.10 implies $M = H(S)$ is a compact set. ■

Theorem 2.4.3. *Suppose for some* $\varepsilon > 0$, *there exists a nonoscillatory solution of the linear delay differential equation*

$$x'(t) = -\varepsilon \sum_{k=1}^{m} r_k(t)x(h_k(t)). \tag{2.50}$$

Then there exists a nonoscillatory solution of (2.42).

Proof. Let $t_0 > 0$, C, and $\varphi : (-\infty, t_0) \to \mathbf{R}$ be such that

$$-1 < C < 0, \ \ \varphi(t) \leq 0, \ \ |\varphi(t)| < |C| < \varepsilon^{1/(\alpha_k - 1)},$$

and hence $C \leq \varphi(t) \leq 0$. Now (2.50) with $x(t) = \varphi(t)$, $t < t_0$, and $x(t_0) = x_0$ with $x_0 = C$ has a negative solution $x_0(t) < 0$. Let

$$w_0 = -\frac{x_0'(t)}{x_0(t)}.$$

Then $w_0(t) > 0$ and

$$x_0(t) = C \exp\{-\int_{t_0}^{t} w_0(s)ds\}, \ \ t \geq t_0.$$

By substituting x_0 in (2.50), we have

$$w_0(t) = \varepsilon \sum_{k=1}^{m} r_k(t) \times \begin{cases} \exp\{\int_{h_k(t)}^{t} w_0(s)ds\}, & \text{if } \ h_k(t) \geq t_0, \\ \exp\{\int_{t_0}^{t} w_0(s)ds\} \frac{\varphi(h_k(t))}{C}, & \text{if } \ h_k(t) < t_0. \end{cases}$$

Consider now two sequences

$$w_n(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} w_{n-1}(s)ds\}\right\}\right) \sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |C|^{\alpha_k - 1} \exp\left\{\int_{h_k(t)}^{t} w_{n-1}(s)ds\right\} \\ \times \exp\left\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} v_{n-1}(s)ds\right\}, & \text{if } h_k(t) \geq t_0, \\ \exp\left\{\int_{t_0}^{t} w_{n-1}(s)ds\right\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & \text{if } h_k(t) < t_0, \end{cases}$$

$$v_n(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} v_{n-1}(s)ds\}\right\}\right) \sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |C|^{\alpha_k - 1} \exp\left\{\int_{h_k(t)}^{t} v_{n-1}(s)ds\right\} \\ \times \exp\left\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} w_{n-1}(s)ds\right\}, & if\ h_k(t) \geq t_0, \\ \exp\left\{\int_{t_0}^{t} v_{n-1}(s)ds\right\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & if\ h_k(t) < t_0, \end{cases}$$

where $v_0 = 0$. We have

$$|\varphi(h_k(t))|^{\alpha_k - 1} < |C|^{\alpha_k - 1} < \varepsilon.$$

Then

$$w_0(t) \geq w_1(t), v_1(t) \geq v_0(t) = 0, \quad \text{and} \quad w_0(t) \geq v_0(t).$$

Hence by induction

$$0 \leq w_n(t) \leq w_{n-1}(t) \leq \ldots \leq w_0(t),\ v_n(t) \geq v_{n-1}(t) \geq \ldots \geq v_0(t) = 0,$$

and $w_n(t) \geq v_n(t)$. There exist pointwise limits of the nonincreasing nonnegative sequence $w_n(t)$ and of the nondecreasing sequence $v_n(t)$. If we denote

$$w(t) = \lim_{n\to\infty} w_n(t) \ \text{ and } \ v(t) = \lim_{n\to\infty} v_n(t),$$

then by the Lebesgue Convergence Theorem, we conclude

$$w(t) = \left(1 + C \exp\left\{-\int_{t_0}^{t} w(s)ds\}\right\}\right) \sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |C|^{\alpha_k - 1} \exp\left\{\int_{h_k(t)}^{t} w(s)ds\right\} \\ \times \exp\left\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} v(s)ds\right\}, & \text{if } h_k(t) \geq t_0, \\ \exp\left\{\int_{t_0}^{t} w(s)ds\right\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & \text{if } h_k(t) < t_0, \end{cases} \tag{2.51}$$

$$v(t) = \left(1 + C\exp\left\{-\int_{t_0}^{t} v(s)ds\right\}\right)\sum_{k=1}^{m} r_k(t)$$

$$\times \begin{cases} |C|^{\alpha_k-1}\exp\left\{\int_{h_k(t)}^{t} v(s)ds\right\} \\ \times\exp\left\{-(\alpha_k-1)\int_{t_0}^{h_k(t)} w(s)ds\right\}, & \text{if } h_k(t) \ge t_0, \\ \exp\left\{\int_{t_0}^{t} v(s)ds\right\}\frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & \text{if } h_k(t) < t_0. \end{cases} \quad (2.52)$$

Fix $b \ge t_0$ and denote the operator $F : L_\infty[t_0, b] \to L_\infty[t_0, b]$ by

$$(Fu)(t) = \left(1 + C\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)\sum_{k=1}^{m} r_k(t)$$

$$\times \begin{cases} |C|^{\alpha_k-1}\exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\} \\ \times\exp\left\{-(\alpha_k-1)\int_{t_0}^{h_k(t)} u(s)ds\right\}, & \text{if } h_k(t) \ge t_0, \\ \exp\left\{\int_{t_0}^{t} u(s)ds\right\}\frac{|\varphi(h_k(t))|^{\alpha_k}}{|C|}, & \text{if } h_k(t) < t_0. \end{cases}$$

Note for every function u from the interval $v \le u \le w$, we have $v \le Fu \le w$. Lemma 2.4.1 implies that the operator F is completely continuous on the space $L_\infty[t_0, b]$ (for every $b \ge t_0$). Then by the Schauder Fixed Point Theorem there exists a nonnegative solution of equation $u = Fu$. Let

$$x(t) = \begin{cases} C\exp\{-\int_{t_0}^{t} u(s)ds\}, & t \ge t_0, \\ \varphi(t), & t < t_0. \end{cases}$$

Then $x(t)$ is a negative solution of (2.42), which completes the proof. ■

2.5 Models with Harvesting

In this section we study the dynamics of a population affected by harvesting, i.e.,

$$\frac{dN}{dt} = r(N(t), t)N(t) - E(N(t), t), \quad (2.53)$$

where $E(N, t)$ is a harvesting strategy for the population.

We consider the delay model

$$N^{'}(t) = r(t)N(t)\left[a - \sum_{k=1}^{m} b_k N(h_k(t))\right] - \sum_{l=1}^{n} c_l(t)N(g_l(t)), \quad t \ge 0, \quad (2.54)$$

with

$$N(t) = \varphi(t),\ t < 0,\ N(0) = N_0, \tag{2.55}$$

under the following conditions:

(a_1) $a > 0, b_k > 0$;
(a_2) $r(t) \geq 0, c_l(t) \geq 0$ are Lebesgue measurable and locally essentially bounded functions;
(a_3) $h_k(t),\ g_l(t)$ are Lebesgue measurable functions, $h_k(t) \leq t,\ g_l(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty,\ \lim_{t\to\infty} g_l(t) = \infty$;
(a_4) $\varphi : (-\infty, 0) \to \mathbf{R}$ is a Borel measurable bounded function, $\varphi(t) \geq 0$, $N_0 > 0$.

In this section we obtain sufficient conditions for positiveness, boundedness, and extinction of solutions of equation (2.54). The results in this section are adapted from [14]. An absolutely continuous function N (: $\mathbf{R} \to \mathbf{R}$) on each interval $[0, b]$ is called a solution of problem (2.54), (2.55), if it satisfies equation (2.54) for almost all $t \in [0, \infty)$ and equality (2.55) for $t \leq 0$.

First, we present some lemmas (the proofs can be found in [12, 13], and [30]) which will be used in the proof of the main results. Consider the linear delay differential equation

$$x^{'}(t) + \sum_{l=1}^{n} c_l(t)x(g_l(t)) = 0,\ \ t \geq 0, \tag{2.56}$$

and a corresponding differential inequality

$$y^{'}(t) + \sum_{l=1}^{n} c_l(t)y(g_l(t)) \leq 0,\ \ t \geq 0. \tag{2.57}$$

Lemma 2.5.1. *Suppose that for the functions $c_l,\ g_l$, hypotheses $(a_2) - (a_3)$ hold. Then*

(1) *If $y(t)$ is a positive solution of (2.57) for $t \geq t_0$, then $y(t) \leq x(t)$, $t \geq t_0$, where $x(t)$ is a solution of (2.56) and $x(t) = y(t)$, $t \leq t_0$.*
(2) *For every nonoscillatory solution $x(t)$ of (2.56), we have $\lim_{t\to\infty} x(t) = 0$.*
(3) *If*

$$\sup_{t\geq 0} \sum_{l=1}^{n} \int_{\min_k g_k(t)} c_l(s)ds \leq \frac{1}{e}, \tag{2.58}$$

then equation (2.56) has a nonoscillatory solution.

If in addition, $0 \leq \varphi(t) \leq N_0$, then the solution of the initial value problem (2.56)–(2.55), where $N(t)$ in (2.55) is replaced by $x(t)$, is positive.

Consider also the linear delay equation

$$x'(t) + \sum_{l=1}^{n} c_l(t)x(g_l(t)) - a(t)x(t) = 0, \quad t \geq 0. \tag{2.59}$$

A solution $X(t,s)$ of the problem

$$x'(t) + \sum_{l=1}^{n} c_l(t)x(g_l(t)) - a(t)x(t) = 0, \quad t \geq s,$$

$$x(t) = 0, \quad t < s, \quad x(s) = 1,$$

is called a fundamental function of (2.59).

Lemma 2.5.2. *Suppose for the functions c_l, g_l, hypotheses $(a_2) - (a_3)$ hold, a is a locally bounded function such that $a(t) \geq 0$,*

$$\sum_{l=1}^{n} c_l(t) \geq a(t), \quad \int_0^\infty \left[\sum_{l=1}^{n} c_l(t) - a(t)\right] = \infty, \tag{2.60}$$

and

$$\lim_{t\to\infty} \sup \left[a(t)(t - G(t)) + \sum_{l=1}^{n} c_l(t)(G(t) - g_l(t)) \right] < 1, \tag{2.61}$$

where $G(t) = \max_l g_l(t)$. Then

(1) *If there exists a nonoscillatory solution of (2.59), then for some t_0 and $t \geq t_0$ we have $X(t,s) > 0$ for $t \geq s \geq t_0$, where $X(t,s)$ is a fundamental function of (2.59).*
(2) *For every nonoscillatory solution $x(t)$ of (2.59) we have $\lim_{t\to\infty} x(t) = 0$.*

Let

$$h(t) = \min_k \{h_k(t)\}, \quad g(t) = \min_l \{g_l(t)\}.$$

In addition to $(a_1) - (a_4)$ consider the following hypothesis:
(a_5). $h(t)$ is a nondecreasing continuous function.

If in (2.54) we neglect harvesting terms, i.e., assume $c_l \equiv 0$, then the positive equilibrium becomes $a / \sum_{k=1}^{m} b_k$.

Theorem 2.5.1. *Suppose $(a_1) - (a_5)$ hold,*

$$\varphi(t) \leq N_0 < \frac{a}{\sum_{k=1}^{m} b_k} \quad \text{for} \quad t < 0, \tag{2.62}$$

and

$$\sup_{t>0} \sum_{l=1}^{n} \int_{g(t)}^{t} c_l(s) \exp\left\{ \varkappa(t) \int_{g_l(t)}^{t} r(\tau) d\tau \right\} ds \le \frac{1}{e}, \tag{2.63}$$

where

$$\varkappa(t) = a\left[\exp\left\{a \sup_{t>0} \int_{h(t)}^{t} r(\xi) d\xi\right\} - 1\right].$$

Then for any solution of (2.54)–(2.55), we have

$$0 < N(t) \le \frac{a}{\sum_{k=1}^{m} b_k} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\}. \tag{2.64}$$

Proof. Suppose (2.64) is not valid. Then either there exists a $\bar{t} > 0$ such that

$$0 < N(t) \le \frac{a}{\sum_{k=1}^{m} b_k} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\},\ 0 \le t < \bar{t},$$

$$N(\bar{t}) = \frac{a}{\sum_{k=1}^{m} b_k} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\},\ N'(\bar{t}) > 0, \tag{2.65}$$

or there exists a $\bar{t} > 0$ such that

$$0 < N(t) \le \frac{a}{\sum_{k=1}^{m} b_k} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s) ds \right\},\ 0 \le t < \bar{t},\ N(\bar{t}) = 0. \tag{2.66}$$

Suppose we have the first possibility for a solution $N(t)$ of (2.54)–(2.55). Denote by

$$t_1 < t_2 < \cdots < t_k < \dots,$$

a sequence of all points t_k, such that

$$N(h(t_k)) = \frac{a}{\sum_{i=1}^{m} b_i},\quad N'(h(t_k)) > 0.$$

Now

$$N(0) = N_0 < \frac{a}{\sum_{k=1}^{m} b_k},\ N(\bar{t}) > \frac{a}{\sum_{k=1}^{m} b_k},$$

and (a_5) imply that the set $\{t_k\}$ is not empty. Suppose t^* is a point where we have a local maximum for $N(t)$. We prove that if

$$N(t^*) > \frac{a}{\sum_{i=1}^{m} b_i}, \text{ then } t^* \in \bigcup_k [h(t_k), t_k].$$

Let t_k be the greatest among all points of the sequence $\{t_k\}$ satisfying $h(t_k) < t^*$.

Suppose first

$$N(t) \leq \frac{a}{\sum_{i=1}^{m} b_i},$$

for some t and $h(t_k) < t \leq t_k$. The definition of t_k and t^* imply $t^* < t$ and hence $t^* \in [h(t_k), t_k]$.

Now suppose

$$N(t) > \frac{a}{\sum_{i=1}^{m} b_i}, \text{ for } h(t_k) < t \leq t_k.$$

Suppose there exists a smallest point t' such that

$$N(t') = \frac{a}{\sum_{i=1}^{m} b_i}.$$

Then (2.54) implies $N'(t) < 0$, $t_k \leq t < t'$. Hence in this interval $N(t)$ has no maximal points. Thus $h(t_k) < t^* < t_k$.

If such a t' does not exist then $N'(t) \leq 0$ for $t > t_k$ and so once again $h(t_k) < t^* < t_k$.

Equation (2.54) implies now that

$$N'(t) \leq ar(t)N(t), \quad h(t_k) \leq t \leq t^*, \quad N(h(t_k)) = \frac{a}{\sum_{i=1}^{m} b_i}.$$

Then

$$\begin{aligned} N(t^*) &\leq \frac{a}{\sum_{i=1}^{m} b_i} \exp\left\{ a \int_{h(t_k)}^{t^*} r(s)ds \right\} \\ &\leq \frac{a}{\sum_{i=1}^{m} b_i} \exp\left\{ a \int_{h(t_k)}^{t_k} r(s)ds \right\} \\ &\leq \frac{a}{\sum_{l=1}^{m} b_i} \exp\left\{ a \sup_{t>0} \int_{h(t)}^{t} r(s)ds \right\}, \end{aligned}$$

which contradicts our assumption (2.65).

Suppose now there exists a $\bar{t} > 0$ such that (2.66) holds. After substituting

$$N(t) = \exp\left\{\int_0^t r(s)\left[a - \sum_{k=1}^m b_k N(h_k(s))\right] ds\right\} x(t), \tag{2.67}$$

in (2.54)–(2.55), we have the system

$$x'(t) = -\sum_{l=1}^n c_l(t) \exp\left\{-\int_{g_l(t)}^t r(s)\left[a - \sum_{k=1}^m b_k N(h_k(s))\right] ds\right\} x(g_l(t)), \tag{2.68}$$

for $t > 0$, and (we assume $r(t) = 0,\ t < 0$)

$$x(t) = \varphi(t),\ \text{for}\ \ t < 0,\ x(0) = N_0. \tag{2.69}$$

Consider now the initial value problem

$$y'(t) = -\sum_{l=1}^n p_l(t) y(g_l(t)), t > 0, \tag{2.70}$$

$$y(t) = \psi(t),\ t < 0,\ y(0) = y_0, \tag{2.71}$$

where

$$p_l(t) = c_l(t) \exp\left\{-\int_{g_l(t)}^t r(s)\left[a - \sum_{k=1}^m b_k N(h_k(s))\right] ds\right\}.$$

It is evident that if $\psi(t) = \varphi(t),\ y_0 = N_0$, then the solutions of (2.68)–(2.69) and (2.70)–(2.71) coincide. Inequalities (2.64) and (2.63) imply that

$$\begin{aligned}
&\sum_{l=1}^n \int_{g(t)}^t p_l(s) ds \\
&= \sum_{l=1}^n \int_{g(t)}^t c_l(s) \exp\left\{\int_{g_l(s)}^s r(\tau)\left[\sum_{k=1}^m b_k N(h_k(\tau)) - a\right] d\tau\right\} ds \\
&\le \sup_{t>0} \sum_{l=1}^n \int_{g(t)}^t c_l(s) \exp\left\{\varkappa(t) \int_{g_l(s)}^s r(\tau) d\tau\right\} ds \le \frac{1}{e},
\end{aligned}$$

where

$$\varkappa(t) = a\left[\exp\left\{a \sup_{t>0} \int_{h(t)}^t r(\xi) d\xi\right\} - 1\right].$$

Note (2.62) which say $\varphi(t) \leq N_0$. Thus Lemma 2.5.1 yields that if $\psi(t) = \varphi(t)$, $y_0 = N_0$, then $y(t) > 0, t > 0$. Hence $x(t) > 0,\ t > 0$. Consequently by (2.67) we have $N(t) > 0,\ t > 0$, which contradicts assumption (2.66). The proof is complete. ∎

Theorem 2.5.2. . *Suppose* $(a1) - (a5)$ *hold, then for every eventually positive solution of (2.54)–(2.55) there exists* $t_0 \geq 0$ *such that (2.64) holds for* $t \geq t_0$.

Proof. Suppose $N(t)$ is an eventually positive solution of (2.54)–(2.55). If

$$N(t) \leq \frac{a}{\sum_{k=1}^{n} b_k},$$

for some $t_0 \geq 0$ and $t \geq t_0$, then the statement of the theorem is true.

Suppose now that

$$N(t) > \frac{a}{\sum_{k=1}^{n} b_k},$$

for some $t_1 \geq 0$ and $t \geq t_1$. Now (2.54) implies that

$$N^{'}(t) \leq -\sum_{l=1}^{n} c_l(t) N(g_l(t)), \quad t \geq t_2,$$

for some $t_2 \geq t_1$. Lemma 2.5.1 implies that $0 < N(t) \leq x(t),\ t \geq t_2$, where $x(t)$ is a solution of the equation

$$x^{'}(t) + \sum_{l=1}^{n} c_l(t) x(g_l(t)) = 0, t \geq t_1, \quad x(t) = N(t),\ t \leq t_2,$$

and $\lim_{t\to\infty} x(t) = 0$. Then $\lim_{t\to\infty} N(t) = 0$. We have a contradiction with our assumption.

Hence there exists a sequence $\{t_n\},\ \lim_{n\to\infty} t_n = \infty$, such that

$$N(h(t_n)) = \frac{a}{\sum_{k=1}^{n} b_k}.$$

The end of the proof is similar to the corresponding part of the proof of Theorem 2.5.1. ∎

Consider now

$$N^{'}(t) = r(t)N(t)\left[a - b_0 N(t) - \sum_{k=1}^{m} b_k N(h_k(t))\right] - \sum_{l=1}^{n} c_l(t) N(g_l(t)). \quad (2.72)$$

Theorem 2.5.3. *Suppose* $b_0 > 0$, *hypotheses* $(a_1) - (a_4)$ *hold,*

$$\varphi(t) \le N_0 < \frac{a}{b_0}, \tag{2.73}$$

and

$$\sup_{t>0} \sum_{l=1}^{n} \int_{g(t)}^{t} c_l(s) \exp\left\{ \left[\frac{a \sum_{k=1}^{m} b_k}{b_0} \right] \int_{g_l(s)}^{s} r(u)du \right\} ds \le \frac{1}{e}. \tag{2.74}$$

Then for any solution of (2.72)–(2.73) *we have*

$$0 < N(t) \le \frac{a}{b_0}. \tag{2.75}$$

Proof. We follow the scheme of the proof in Theorem 2.5.1. Suppose (2.75) is not true. Then either there exists $\bar{t} > 0$ such that

$$0 < N(t) \le \frac{a}{b_0},\ 0 \le t < \bar{t},\ N(\bar{t}) = \frac{a}{b_0}, N'(\bar{t}) > 0, \tag{2.76}$$

or there exists $\bar{t} > 0$ such that

$$0 < N(t) \le \frac{a}{b_0},\ \ 0 \le t < \bar{t},\ \ N(\bar{t}) = 0. \tag{2.77}$$

Suppose the first possibility (2.76) holds. Then for $0 < t < \bar{t}$ we have

$$N'(t) \le r(t)N(t)[a - b_0 N(t)],\ N(0) = N_0.$$

Denote by x a solution of the problem

$$x'(t) = r(t)x(t)[a - b_0 x(t)],\ \ x(0) = N_0. \tag{2.78}$$

Then

$$N(t) \le x(t) < \frac{a}{b_0},\ 0 \le t \le \bar{t},$$

since the solution of (2.78) tends to a/b_0 and is always less than a/b_0. We have a contradiction with assumption (2.76).

Suppose now that for $\bar{t} > t_0$ (2.77) holds. Substituting in (2.72),

$$N(t) = \exp\left\{ \int_0^t r(s) \left[a - b_0 N(s) - \sum_{k=1}^{m} b_k N(h_k(s)) \right] ds \right\} x(t), \tag{2.79}$$

we have the system

$$x^{'}(t) = -\sum_{l=1}^{n} p_l(t)x(g_l(t)),\ t > 0, \tag{2.80}$$

$$x(t) = \varphi(t),\ t < 0,\ x(0) = N_0,$$

where

$$p_l(t) = c_l(t)\exp\left\{-\int_{g_l(t)}^{t} r(s)\left[a - b_0N(s) - \sum_{k=1}^{m} b_kN(h_k(s))\right]ds\right\}.$$

Inequalities (2.75) and (2.74) imply that

$$\begin{aligned}
&\sum_{l=1}^{n}\int_{g(t)}^{t} p_l(s)ds\\
&\le \sum_{l=1}^{n}\int_{g(t)}^{t} c_l(s)\\
&\quad \times \exp\left\{\int_{g_l(s)}^{s} r(\tau)\left[\sum_{k=1}^{m} b_kN(h_k(\tau)) + b_0N(\tau) - a\right]d\tau\right\} ds\\
&\le \sup_{t>0}\sum_{l=1}^{n}\int_{g(t)}^{t} c_l(s)\exp\left\{\left[\frac{a\sum_{k=1}^{m} b_k}{b_0}\right]\int_{g_l(s)}^{s} r(\tau)d\tau\right\} ds \le \frac{1}{e}.
\end{aligned}$$

As in the proof of Theorem 2.5.1, Lemma 2.5.1 implies $N(t) > 0,\ 0 \le t \le \bar{t}$. This contradiction proves the theorem. ∎

Similar reasoning to that in Theorem 2.5.2 yields the next result.

Theorem 2.5.4. *Suppose $b_0 > 0$, $(a_1) - (a_4)$ hold. Then for every eventually positive solution of (2.72)–(2.55) there exists a $t_0 \ge 0$ such that (2.75) holds for $t \ge t_0$.*

Now we obtain sufficient extinction conditions for solutions of the logistic equation with harvesting. To this end consider the following equation which is more general than (2.54):

$$N^{'}(t) = N(t)\left[a(t) - \sum_{k=1}^{m} b_k(t)N(h_k(t))\right] - \sum_{l=1}^{n} c_l(t)N(g_l(t)),\ t \ge 0. \tag{2.81}$$

Theorem 2.5.5. *Suppose $a(t) \geq 0$, $b_k \geq 0$ are locally essentially bounded functions and for c_l, h_k, g_l conditions (a_2), (a_3) hold. Suppose in addition (2.60)–(2.61) hold. Then for any solution of (2.81)–(2.55) either*

$$\lim_{t\to\infty} N(t) = 0$$

or there exists $\bar{t} > 0$ such that $N(\bar{t}) < 0$.

Proof. It is sufficient to prove that for every positive solution $N(t)$ of (2.81)–(2.55) we have $\lim_{t\to\infty} N(t) = 0$.

Suppose $N(t) > 0$ is a solution of (2.81)–(2.55). Equation (2.81) implies

$$N^{'}(t) + \sum_{l=1}^{n} c_l(t)N(g_l(t)) - a(t)N(t) \leq 0.$$

Lemma 2.5.2 guarantees that there exists $t_0 \geq 0$, such that the fundamental function $X(t,s)$ of the equation

$$x^{'}(t) + \sum_{l=1}^{n} c_l(t)x(g_l(t)) - a(t)x(t) = 0 \tag{2.82}$$

is positive for $t \geq s \geq t_0$. Then the variation of constant formula [30] implies

$$N(t) = x(t) + \int_{t_0}^{t} X(t,s)f(s)ds,$$

where $x(t)$ is a solution of (2.82) with the initial condition $x(t) = N(t)$, $t \leq t_0$, and $f(t)$ is a nonpositive function. Hence $0 < N(t) \leq x(t)$. Lemma 2.5.2 implies that

$$\lim_{t\to\infty} x(t) = 0.$$

Thus $\lim_{t\to\infty} N(t) = 0$. The proof is complete. ■

2.6 Models with Nonlinear Delays

We return now to Sect. 2.4 when $\alpha_k = 1$, $k = 1, \ldots, m$. Consider the delay logistic model with several delays

$$N^{'}(t) = N(t)\sum_{k=1}^{m} r_k(t)\left[1 - \frac{N(h_k(t))}{K}\right], \quad h_k(t) \leq t. \tag{2.83}$$

Motivated by (2.83) in this section we consider first the scalar delay differential equation

$$x'(t) = -\sum_{k=1}^{m} r_k(t)x(h_k(t))\,[x(t)+1] \tag{2.84}$$

under the following conditions

(c_1) r_k, $k=1,2,\dots,m$, are Lebesgue measurable functions essentially bounded in each finite interval $[0,b]$, $r_k \ge 0$,

$$\int_{t_0}^{\infty} \sum_{k=1}^{m} r_k(t)dt = \infty, \quad \lim_{t\to\infty} \inf \sum_{k=1}^{m} \int_{\max_k h_k(t)}^{t} r_k(s)ds > 0;$$

(c_2) $h_k : [0,\infty) \to \mathbf{R}$ are Lebesgue measurable functions, $h_k(t) \le t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k=1,2,\dots,m$.

Together with (2.84), we consider for each $t_0 \ge 0$ an initial value problem

$$x'(t) = -\sum_{k=1}^{m} r_k(t)x(h_k(t))\,[x(t)+1], \quad t \ge t_0, \tag{2.85}$$

$$x(t) = \varphi(t), \quad t < t_0, \text{ and } x(t_0) = x_0 > -1, \tag{2.86}$$

where

(c_3) $\varphi : (-\infty, t_0) \to \mathbf{R}$ is a Borel measurable bounded function.

Consider the linear delay differential equation

$$x'(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0 \tag{2.87}$$

and the delay differential inequalities

$$x'(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) \le 0, \quad t \ge 0, \tag{2.88}$$

$$x'(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) \ge 0, \quad t \ge 0. \tag{2.89}$$

The following Lemma follows a standard argument (see the proof of Theorem 2.4.1).

Lemma 2.6.1. *Assume that $(c_1)-(c_3)$ hold. Then the following statements are equivalent:*

(1) There exits a nonoscillatory solution of (2.87).
(2) There exists an eventually positive solution of t inequality (2.88).
(3) There exists an eventually negative solution of (2.89).
(4) There exists $t_0 \geq 0$ such that the inequality

$$u(t) \geq \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds\right), \quad t \geq t_0, \; u(t) = 0, \; t < t_0, \tag{2.90}$$

has a nonnegative locally integrable solution.

If $x(t)$, $y(t)$, $z(t)$, $t \geq 0$, are positive solutions of (2.87), (2.88), (2.89), respectively, $x(t) = y(t) = z(t)$, $t < 0$, then $y(t) \leq x(t) \leq z(t)$ for $t \geq 0$.

Lemma 2.6.2. *Assume that for the equation*

$$x'(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad t \geq 0, \tag{2.91}$$

assumptions $(c_1) - (c_2)$ hold.

(i) If $a_k(t) \leq r_k(t)$, $g_k(t) \geq h_k(t)$, and (2.87) has a nonoscillatory solution, then (2.91) has a nonoscillatory solution.
(ii) If $a_k(t) \geq r_k(t)$, $g_k(t) \leq h_k(t)$, and all solutions of (2.87) are oscillatory, then all solutions of (2.91) are oscillatory.

Theorem 2.6.1. *Assume that $(c_1) - (c_3)$ hold. Suppose that for every sufficiently small $\varepsilon \geq 0$ all solutions of the linear delay differential equation*

$$x'(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0, \quad t \geq t_0, \tag{2.92}$$

are oscillatory. Then all solutions of (2.85) are oscillatory.

Proof. Suppose (2.85) has a nonoscillatory solution. Then by the condition $x(t) + 1 > 0$ either there exists a positive solution $x(t) > 0$ for all $t \geq T \geq t_0$ or there exists a solution $x(t)$ such that

$$-1 < x(t) < 0, \quad \text{for } t \geq T.$$

We can assume $h_k(t) \geq t_0$ for all $t \geq T$, since $\lim_{t\to\infty} h_k(t) = \infty$.

First, we suppose that $x(t) > 0$ for $t \geq T$. From (2.85), we have

$$x'(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) \leq 0, \quad t \geq t_0.$$

Lemma 2.6.1 implies for $\varepsilon = 0$ that (2.92) has a nonoscillatory solution, which gives a contradiction.

Suppose now

$$-1 < x(t) < 0, \text{ for } \ t \geq T.$$

Let us introduce the function u as a solution of

$$x'(t) = -u(t)x(t)\left[x(t)+1\right], \quad x(T) = x_0 < 0. \tag{2.93}$$

Now, since $x(t) + 1 > 0$, we have $x'(t) > 0$ and this implies that $u(t) \geq 0$. From (2.93) we obtain

$$x(t) = -\frac{\exp\left(-\int_T^t u(s)ds + c\right)}{1+\exp\left(-\int_T^t u(s)ds + c\right)},$$

where $c = \ln\left[|x_0|/(1+x_0)\right]$. Substituting in (2.85) we have

$$u(t)\frac{\exp\left(-\int_T^t u(s)ds + c\right)}{1+\exp\left(-\int_T^t u(s)ds + c\right)} = \sum_{k=1}^m r_k(t)\frac{\exp\left(-\int_T^{h_k(t)} u(s)ds + c\right)}{1+\exp\left(-\int_T^{h_k(t)} u(s)ds + c\right)}.$$

Hence

$$u(t) = \sum_{k=1}^m r_k(t)\exp\left(\int_{h_k(t)}^t u(s)ds\right)\frac{1+\exp\left(-\int_T^t u(s)ds + c\right)}{1+\exp\left(-\int_T^{h_k(t)} u(s)ds + c\right)}. \tag{2.94}$$

Equality (2.94) implies that $u(t) \geq \sum_{k=1}^m r_k(t)$ and from (c_1) we have

$$\int_T^\infty u(t)dt = \infty.$$

Consequently there exists $T_1 \geq T$ such that

$$\max_{1\leq k\leq m}\frac{1+\exp\left(-\int_T^t u(s)ds + c\right)}{1+\exp\left(-\int_T^{h_k(t)} u(s)ds + c\right)} \geq (1-\varepsilon), \text{ for } t \geq T_1.$$

Then,

$$u(t) \geq (1-\varepsilon)\sum_{k=1}^m r_k(t)\exp\left(\int_{h_k(t)}^t u(s)ds\right).$$

From Lemma 2.6.1, (2.92) has a nonoscillatory solution, which is a contradiction. The proof is complete. ■

From Lemma 2.6.2 and Theorem 2.6.1 we have the following oscillation comparison theorem.

Theorem 2.6.2. *Suppose $a_k(t) \geq r_k(t)$, $g_k(t) \leq h_k(t)$, and the assumptions of Theorem 2.6.1 hold. Then all the solutions of the equation*

$$x'(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t))[1 + x(t)] = 0, \quad t \geq 0, \tag{2.95}$$

are oscillatory.

Theorem 2.6.3. *Assume that $(c_1) - (c_3)$ hold. Suppose for every sufficiently small $\varepsilon \geq 0$ there exists a nonoscillatory solution of the linear delay differential equation*

$$x'(t) + (1+\varepsilon)\sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0, \quad t \geq t_0. \tag{2.96}$$

Then (2.85) has a nonoscillatory solution.

Proof. From Lemma 2.6.1 for some $T \geq t_0$ and for $t \geq T$ there exists a nonnegative solution u_0 of

$$u(t) \geq (1+\varepsilon)\sum_{k=1}^{m} r_k(t)\exp\left(\int_{h_k(t)}^{t} u(s)ds\right), \quad t \geq T. \tag{2.97}$$

This inequality implies that $u_0(t) \geq \sum\limits_{k=1}^{m} r_k(t)$, and hence by (c_1) we have that

$$\int_T^{\infty} u_0(s)ds = \infty.$$

Let c be some negative number. Then there exists $T_1 \geq T$ such that

$$\max_{1\leq k\leq m} \frac{1-\exp\left(-\int_{T_1}^{t} u_0(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} \sum\limits_{k=1}^{m} r_k(s)ds + c\right)} < (1+\varepsilon), \text{ for } t \geq T_1, \tag{2.98}$$

and by (c_1) for $t \geq T_1$, we have

$$\min_{1\leq k\leq m} \exp\left[\int_{h_k(t)}^{t} \sum_{k=1}^{m} r_k(s)ds\right] \frac{1-\exp\left(-\int_{T_1}^{h_k(t)} \sum\limits_{k=1}^{m} r_k(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{t} u_0(s)ds + c\right)} > 1.$$

From (2.97) and (2.98), we have

$$u_0(t) \geq \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right) \frac{1-\exp\left(-\int_{T_1}^{t} u_0(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}. \tag{2.99}$$

Let us fix $t_1 > T_1$ and consider the nonlinear operator

$$(F_1 u)(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds\right) \times \frac{1-\exp\left(-\int_{T_1}^{t} u(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}$$

in the Banach space $L_\infty[T_1, t_1]$. We have

$$(F_1 u)(t) = \sum_{k=1}^{m} r_k(t) \frac{\exp\left(\int_{T_1}^{t} u(s)ds\right)}{\exp\left(\int_{T_1}^{t} \zeta_k(t,s)u(s)ds\right)} \times \frac{1-\exp\left(-\int_{T_1}^{t} u(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}, \tag{2.100}$$

where $\zeta_k(t,s) = 1$, if $s < h_k(t) < t$, and $\zeta_k(t,s) = 0$, if $h_k(t) < s$. The operator F_1 is continuous. Consider all functions $v \in L_\infty[T_1, t_1]$ such that

$$\sum_{k=1}^{m} r_k(t) \leq v(t) \leq u_0(t).$$

We have $(F_1 v)(t) \geq \sum_{k=1}^{m} r_k(t)$. Inequality (2.98) implies that

$$(F_1 v)(t) \leq \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right) \times \frac{1-\exp\left(-\int_{T_1}^{t} u_0(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}$$

$$\leq (1+\varepsilon) \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right) \leq u_0(t).$$

Hence for each v such that

$$\sum_{k=1}^{m} r_k(t) \le v(t) \le u_0(t)$$

we have

$$\sum_{k=1}^{m} r_k(t) \le (F_1 v)(t) \le u_0(t).$$

Then by Knaster's Fixed Point Theorem (see Sect. 1.4), there exists u_1 such that

$$\sum_{k=1}^{m} r_k(t) \le u_1(t) \le u_0(t) \;\text{ and }\; u_1 = F\, u_1.$$

This means that

$$u_1(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_1(s)ds\right) \frac{1 - \exp\left(-\int_{T_1}^{t} u_1(s)ds + c\right)}{1 - \exp\left(-\int_{T_1}^{h_k(t)} \sum_{k=1}^{m} r_k(s)ds + c\right)}. \tag{2.101}$$

Consider the operator

$$(F_2 u)(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds\right) \frac{1 - \exp\left(-\int_{T_1}^{t} u(s)ds + c\right)}{1 - \exp\left(-\int_{T_1}^{h_k(t)} u_1(s)ds + c\right)}.$$

If

$$\sum_{k=1}^{m} r_k(t) \le v(t) \le u_1(t),$$

then (2.101) and (2.98) imply

$$\begin{aligned}(F_2 v)(t) &\le \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_1(s)ds\right) \frac{1 - \exp\left(-\int_{T_1}^{t} u_1(s)ds + c\right)}{1 - \exp\left(-\int_{T_1}^{h_k(t)} u_1(s)ds + c\right)} \\ &\le u_1(t),\end{aligned}$$

and

$$
\begin{aligned}
&(F_2 v)(t) \\
&\geq \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} \sum_{k=1}^{m} r_k(s)ds\right) \frac{1-\exp\left(-\int_{T_1}^{t} \sum_{k=1}^{m} r_k(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} u_0(s)ds + c\right)} \\
&\geq \sum_{k=1}^{m} r_k(t).
\end{aligned}
$$

Hence

$$
\sum_{k=1}^{m} r_k(t) \leq (F_2 v)(t) \leq u_1(t)
$$

and as in the previous case we obtain that there exists a solution u_2 of the equation $u = F_2 u$ such that

$$
\sum_{k=1}^{m} r_k(t) \leq u_2(t) \leq u_1(t).
$$

By induction we prove that there exists a solution u_n of the equation $u = F_n u$ which satisfies

$$
\sum_{k=1}^{m} r_k(t) \leq u_n(t) \leq u_{n-1}(t),
$$

where

$$
(F_n u)(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds\right) \frac{1-\exp\left(-\int_{T_1}^{t} u(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} u_{n-1}(s)ds + c\right)}.
$$

A monotone bounded sequence $\{u_n\}$ has a limit $u = \lim_{n\to\infty} u_n(t)$ and this limit is a solution of the equation

$$
u(t) = \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u(s)ds\right) \frac{1-\exp\left(-\int_{T_1}^{t} u(s)ds + c\right)}{1-\exp\left(-\int_{T_1}^{h_k(t)} u(s)ds + c\right)}.
$$

From this, we have that

$$x(t) = -\frac{\exp\left(-\int_T^t u(s)ds + c\right)}{1 + \exp\left(-\int_T^t u(s)ds + c\right)}$$

(where $c = \ln[|x(T_1)|/(1 + x(T_1))]$) is a positive solution of (2.85) for $T_1 \leq t \leq t_1$. Since t_1 is an arbitrary number, we have a positive solution for all $t \geq T_1$. The proof is complete. ■

For the remainder of this section we consider

$$x'(t) + \sum_{k=1}^{m} r_k(t) f_k[x(h_k(t))] = 0 \tag{2.102}$$

under the following assumptions:

$(a1)$ $r_k(t) \geq 0$, $k = 1, \ldots, m$, are Lebesgue measurable locally essentially bounded functions;
$(a2)$ $h_k : [0, \infty) \to \mathbf{R}$, for $k = 1, \ldots, m$, are Lebesgue measurable functions $h_k(t) \leq t$, $\lim t_{t\to\infty} h_k(t) = \infty$;
$(a3)$ $f_k : \mathbf{R} \to \mathbf{R}$, $k = 1, \ldots, m$, are continuous functions, $x f_k(x) > 0$ for $x \neq 0$.

Together with (2.102), we consider for each $t_0 \geq 0$ an initial value problem

$$x'(t) + \sum_{k=1}^{m} r_k(t) f_k[x(h_k(t))] = 0, \ t \geq t_0, \tag{2.103}$$

$$x(t) = \phi(t), \ t < t_0, \ x(t_0) = x_0. \tag{2.104}$$

We also assume that the following hypothesis holds:

$(a4)$ $\phi : (-\infty, t_0) \to \mathbf{R}$ is a Borel measurable bounded function.

We will also use the following lemma (whose proof is standard) which can be found in [33].

Lemma 2.6.3. *Suppose there exists an index k such that*

$$\int_0^{\infty} r_k(t)dt = \infty \tag{2.105}$$

and $x(t)$ is a nonoscillatory solution of (2.103). Then $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.6.4. *Assume that $(a_1) - (a_4)$ and (2.105) hold. Furthermore assume that*

$$\lim_{u\to\infty} \frac{f_k(u)}{u} = 1, \ k = 1, 2, \ldots, m. \tag{2.106}$$

If for some $\varepsilon > 0$ all solutions of the linear equation

$$x'(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)\, x(h_k(t)) = 0,\ \ t \ge t_0, \tag{2.107}$$

are oscillatory, then all solutions of (2.103) are also oscillatory.

Proof. Assume (2.103) has a nonoscillatory solution $x(t)$. Then, by Lemma 2.6.3 we have that $\lim_{t\to\infty} x(t) = 0$.

Assume that there exists $t_1 \ge t_0$ sufficiently large such that $x(t) > 0$ for $t \ge t_1$ and $h_k(t) \ge t_1$ for $t \ge t_2$. From condition (2.106) there exists $t_3 \ge t_2$ such that

$$f_k(x(h_k(t))) \ge (1-\varepsilon)x(h_k(t)),\ \ t \ge t_3.$$

Hence

$$x^{'}(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)x(h_k(t)) \le 0, \quad t \ge t_3.$$

Now Lemma 2.6.1 implies that (2.107) has a nonoscillatory solution. This is a contradiction.

Suppose now, $x(t) < 0$ for $t \ge t_1$ for some t_1 sufficiently large such that $h_k(t) \ge t_1$ for $t \ge t_2$. Let

$$y(t) := -x(t), \quad g_k(y) = -f_k(-y)$$

and the functions g_k satisfy all the assumptions for f_k, and $y(t)$ is an eventually positive solution of the equation

$$y^{'}(t) + \sum_{k=1}^{m} r_k(t)g_k(y(h_k(t))) = 0.$$

As was shown above, we have

$$y^{'}(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)y(h_k(t)) \le 0,$$

for $t_2 \ge t_1$. Now Lemma 2.6.1 implies that (2.107) has a nonoscillatory solution. This contradiction proves the theorem. ■

Theorem 2.6.5. *Assume that $(a_1)-(a_4)$ hold. Suppose for all $k = 1,\dots,m$, either*

$$f_k(x) \le x \ \text{ for } \ x > 0 \ \text{ or } \ f_k(x) \ge x \ \text{ for } x < 0, \tag{2.108}$$

and there exists a nonoscillatory solution of the linear delay differential equation (2.87).

Then there exists a nonoscillatory solution of (2.103).

Proof. Suppose $f_k(x) \le x$ for $x > 0$, $k = 1, \dots, m$. By Lemma 2.6.1 there exist $t_0 > 0$ and $u_0(t) \ge 0$, $t \ge t_0$, $u_0(t) = 0$, $t < t_0$, such that

$$u_0(t) \ge \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right), \quad t \ge t_0.$$

Let us fix $b > t_0$ and consider the nonlinear operator $F : L_\infty[t_0, b] \to L_\infty[t_0, b]$ given by

$$(Fu)(t) = \sum_{k=1}^{m} r_k(t) f_k\left(\exp\left(-\int_{t_0}^{h_k(t)} u(s)ds\right)\right) \exp\left(\int_{t_0}^{t} u(s)ds\right).$$

For any function u from the interval $0 \le u \le u_0$ we have

$$0 \le (Fu)(t) \le \sum_{k=1}^{m} r_k(t) \exp\left(-\int_{t_0}^{h_k(t)} u(s)ds\right) \exp\left(\int_{t_0}^{t} u(s)ds\right)$$
$$\le \sum_{k=1}^{m} r_k(t) \exp\left(\int_{h_k(t)}^{t} u_0(s)ds\right) \le u_0(t).$$

Hence $0 \le Fu \le u_0$. Lemma 2.4.1 implies that the operator F is completely continuous in $L_\infty[t_0, b]$. Then by the Schauder Fixed Point Theorem, there exists a nonnegative solution of the equation $u = Fu$. Let

$$x(t) = \begin{cases} \exp\left(-\int_{t_0}^{t} u(s)ds\right), & t \ge t_0, \\ 0, & t < t_0. \end{cases}$$

Then $x(t)$ is an eventually positive solution of (2.87).

If $f_k(x) \ge x$, $x \le 0$, $k = 1, \dots, m$, then (2.87) has an eventually negative solution, which completes the proof of the theorem. ■

Consider (2.83). Let $N(t) = Ke^{x(t)}$. Then x is a solution of (2.102) with

$$f_k(x) = f(x) = e^x - 1.$$

Note $f_k(u) \ge u$ for $u \le 0$ and $u f_k(u) > 0$ for $u \ne 0$.

2.7 Hyperlogistic Models

In this section, we are concerned with the oscillation of the delay hyperlogistic models. First, we consider an autonomous delay hyperlogistic model of the form

$$N'(t) = rN(t)\prod_{j=1}^{m}\left[1 - \frac{N(t-\tau_j)}{K}\right]^{\alpha_j}, \quad t \geq 0, \tag{2.109}$$

where $r, K, \tau_j \in (0,\infty)$, and $\alpha_j = p_j/q_j$ are rational numbers with q_j odd, p_j and q_j are co-prime, $1 \leq j \leq m$, and

$$\prod_{j=1}^{m}(-1)^{\alpha_j} = -1.$$

By making a change of variables

$$x(t) = \frac{N(t)}{K} - 1,$$

Eq. (2.109) becomes

$$x'(t) + r\,[1 + x(t)]\prod_{j=1}^{m} x^{\alpha_j}\left(t - \tau_j\right) = 0. \tag{2.110}$$

We are interested in those solutions $x(t)$ of (2.110) satisfying $x(t) \geq -1$ which correspond to solutions $N(t)$ of (2.109) satisfying $N(t) \geq 0$. Thus we consider the initial condition

$$\begin{cases} x(t) = \phi(t) \geq -1, & t \in [t_0 - \tau, t_0], \\ \phi \in C\left([t_0 - \tau, t_0], [-1,\infty)\right) \text{ and } & \phi(t_0) > -1, \end{cases} \tag{2.111}$$

where $\tau = \max\{\tau_1, \ldots, \tau_m\}$. Now (2.110), (2.111) has a unique solution $x(t; t_0, \phi)$ on $[t_0 - \tau, \infty)$ and $x(t) > -1$ for $t \geq t_0$. We will show that all solutions of (2.110) and (2.111) are oscillatory when $\sum_{j=1}^{m}\alpha_j < 1$, but at least one nonoscillatory solution exists when $\sum_{j=1}^{m}\alpha_j > 1$. For the case where $\sum_{j=1}^{m}\alpha_j = 1$, we will establish an equivalence, as far as oscillation is concerned, between (2.110) and its so-called quasilinearized equation

$$y'(t) + r\prod_{j=1}^{m} y^{\alpha_j}(t - \tau_j) = 0. \tag{2.112}$$

The results in this section are adapted from [84].

The case $\sum_{j=1}^{m} \alpha_j < 1.$

Theorem 2.7.1. *If* $\alpha = \sum_{j=1}^{m} \alpha_j < 1$, *then every solution of (2.110)–(2.111) oscillates.*

Proof. Assume that (2.110)–(2.111) has a nonoscillatory solution $x(t)$. We first suppose that $x(t)$ is eventually positive. Then, by (2.110), we eventually have

$$x'(t) = -r(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) < 0,$$

which implies that $x(t)$ is eventually decreasing. Thus

$$x(t - \tau_j) \geq x(t), \quad \text{eventually}, \quad \text{for } j = 1, \ldots, m,$$

and hence (note $\alpha = \sum_{j=1}^{m} \alpha_j$)

$$x'(t) + r(1 + x(t))x^{\alpha}(t) \leq x'(t) + r(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) = 0.$$

Thus

$$\frac{d}{dt} x^{1-\alpha}(t) \leq -(1 - \alpha)r\,[1 + x(t)] \leq -(1 - \alpha)r,$$

which implies that

$$x^{1-\alpha}(t) \to -\infty, \quad \text{as } t \to \infty.$$

This is impossible since $x(t) > 0$ eventually and $1 - \alpha > 0$.

We next suppose that $x(t)$ is eventually negative. Noting that $x(t) > -1$ for $t \geq 0$, we have eventually

$$\begin{aligned} x'(t) &= -r(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) \\ &= r(1 + x(t)) \prod_{j=1}^{m} [-x(t - \tau_j)]^{\alpha_j} > 0, \end{aligned}$$

which implies that $x(t)$ is eventually increasing, so there exists $T_1 > 0$ such that $x(t-\tau_j) \le x(t) < 0$ for $j = 1, \dots, m$ and

$$1 + x(t) > 1 + x(T_1) > 0, \quad \text{for all } t > T_1.$$

Therefore

$$\begin{aligned} & x'(t) + r(1 + x(t))x^{\alpha}(t) \\ \ge\ & x'(t) + r(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0, \quad t > T_1, \end{aligned}$$

and hence

$$\begin{aligned} \frac{d}{dt} x^{1-\alpha}(t) &\le -r(1-\alpha)(1 + x(t)) \\ &< -r(1-\alpha)(1 + x(T_1)) < 0, \ \ t > T_1. \end{aligned}$$

Integrating the above inequality from T_1 to $t > 0$ and letting $t \to \infty$, we get $x^{1-\alpha}(t) \to -\infty$, as $t \to \infty$. This is a contradiction to the fact that $x(t) > -1$ for $t \ge 0$ and completes the proof. ■

The case $\sum_{j=1}^{m} \alpha_j > 1$.

We now recall the following well-known result.

Lemma 2.7.1. *Every solution of (2.112) with* $\sum_{j=1}^{m} \alpha_j = 1$ *oscillates if and only if*

$$r \sum_{j=1}^{m} \alpha_j \tau_j > \frac{1}{e}.$$

Moreover, the above inequality holds if and only if

$$y'(t) + r \prod_{j=1}^{m} y^{\alpha_j}(t-\tau_j) \le 0, \ \textit{has no eventually positive solution,}$$

$$y'(t) + r \prod_{j=1}^{m} y^{\alpha_j}(t-\tau_j) \ge 0, \ \textit{has no eventually negative solution.}$$

Theorem 2.7.2. *If* $\alpha = \sum_{j=1}^{m} \alpha_j > 1$, *then (2.110) has a nonoscillatory solution.*

Proof. Choose rational numbers $\beta_j = \frac{r_j}{s_j} \in [0, \infty)$ with s_j odd, $1 \le j \le m$, such that

$$\beta_j \le \alpha_j, \text{ for } j = 1, \ldots, m, \ \sum_{j=1}^{m} \beta_j = 1, \ \prod_{j=1}^{m} (-1)^{\beta_j} = -1.$$

Let $\epsilon > 0$ satisfy

$$r\epsilon \sum_{j=1}^{m} \beta_j \tau_j \le \frac{1}{e}.$$

Then, by Lemma 2.7.1, the equation

$$x^{'}(t) + r\epsilon \prod_{j=1}^{m} x^{\beta_j}(t - \tau_j) = 0 \tag{2.113}$$

has a positive solution $x(t)$ defined on $[t_0, \infty)$ for some $t_0 \ge 0$. It is clear that $x(t) \to 0$ as $t \to \infty$. Since $\beta_j \le \alpha_j$ and

$$\sum_{j=1}^{m} \beta_j < \sum_{j=1}^{m} \alpha_j,$$

we have

$$\lim_{t\to\infty} (1 + x(t)) \frac{\prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j)}{\prod_{j=1}^{m} x^{\beta_j}(t - \tau_j)} = 0.$$

Thus, there exists $t_1 > t_0$ such that

$$(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) < \epsilon \prod_{j=1}^{m} x^{\beta_j}(t - \tau_j), \ \text{ for } t \ge t_1,$$

and hence for $t \ge t_1$, we see that

$$x^{'}(t) + r\,(1 + x(t)) \prod_{j=1}^{m} x^{\alpha_j}(t - \tau_j) < x^{'}(t) + \epsilon\, r \prod_{j=1}^{m} x^{\beta_j}(t - \tau_j) = 0. \tag{2.114}$$

Set $y(t) = \ln(1 + x(t))$. Then, from (2.114), we have

$$y'(t) + r \prod_{j=1}^{m} \left[e^{y(t-\tau_j)} - 1\right]^{\alpha_j} < 0, \text{ for } t \geq t_1,$$

which yields

$$y(t) > r \int_{t}^{\infty} \prod_{j=1}^{m} \left[e^{y(s-\tau_j)} - 1\right]^{\alpha_j} ds, \quad \text{for } t \geq t_1. \tag{2.115}$$

Define X to be the set of piecewise continuous functions $z : [t_1 - \tau, \infty) \to [0, 1]$ and endow X with the usual pointwise ordering $\leq$, that is,

$$z_1 \leq z_2 \Leftrightarrow z_1(t) \leq z_2(t), \text{ for } t \geq t_1 - \tau.$$

Then $(X; \leq)$ becomes an ordered set. It is obvious that for any nonempty subset M of X, $\inf(M)$ and $\sup(M)$ exist. Thus $(X; \leq)$ is a complete lattice. Define a mapping Ψ on X as follows:

$$(\Psi z)(t) = \begin{cases} \frac{r}{y(t)} \int_{t}^{\infty} \prod_{j=1}^{m} \left[e^{y(s-\tau_j)z(s-\tau_j)} - 1\right]^{\alpha_j} ds, & t \geq t_1, \\ \frac{t}{t_1}(\Psi z)(t) + \left(1 - \frac{t}{t_1}\right), & t_1 - \tau \leq t \leq t_1. \end{cases}$$

For each $z \in X$, we see that

$$0 \leq (\Psi z)(t) \leq \frac{r}{y(t)} \int_{t}^{\infty} \prod_{j=1}^{m} \left[e^{y(s-\tau_j)} - 1\right] ds < 1, \text{ for } t \geq t_1,$$

and

$$0 \leq (\Psi z)(t) \leq 1, \quad \text{for} \quad t \in [t_1 - \tau, t_1].$$

This shows that $\Psi X \subseteq X$. Moreover, it can be easily verified that Ψ is a monotone increasing mapping. Therefore, by the Knaster–Tarski Fixed Point Theorem (see Sect. 1.4), we have that there exists a $z \in X$ such that $\Psi z = z$, that is,

$$z(t) = \begin{cases} \frac{r}{y(t)} \int_{t}^{\infty} \prod_{j=1}^{m} \left[e^{y(s-\tau_j)z(s-\tau_j)} - 1\right]^{\alpha_j} ds, & t \geq t_1, \\ \frac{t}{t_j}(\Psi z)(t_1) + (1 - \frac{t}{t_1}), & t_1 - \tau \leq t \leq t_1. \end{cases} \tag{2.116}$$

By (2.116), $z(t)$ is continuous on $[t_1-\tau,\infty)$. Moreover, since $z(t) > 0$ for $t \in [t_1-\tau,t_1)$, we must have $z(t) > 0$, for all $t \geq t_1$. Set $w(t) = y(t)z(t)$. Then $w(t)$ is positive, continuous on $[t_1-\tau,\infty)$, and satisfies

$$w(t) = r\int_t^\infty \prod_{j=1}^m \left[e^{w(s-\tau_j)}-1\right]^{\alpha_j} ds, \quad \text{for } t \geq t_1. \tag{2.117}$$

Differentiating (2.117) yields

$$\frac{d}{dt}w(t) + r\int_t^\infty \prod_{j=1}^m \left[e^{w(s-\tau_j)}-1\right]^{\alpha_j} = 0, \quad \text{for } t \geq t_1,$$

which shows that $e^{w(t)}-1$ is a positive solution of (2.110) on $[t_1,\infty)$. This completes the proof. ■

The case $\sum_{j=1}^m \alpha_j = 1.$

The following theorem establishes an equivalence between the oscillation of (2.110)–(2.111) and the oscillation of (2.112).

Theorem 2.7.3. *When* $\sum_{j=1}^m \alpha_j = 1$, *every solution of (2.110)–(2.111) oscillates if and only if every solution of (2.112) oscillates.*

Proof. ⇒: Assume that (2.112) has a nonoscillatory solution $y(t)$. Since $-y(t)$ is also a solution of (2.112), we may assume that $y(t)$ is eventually positive. We, will prove that (2.110)–(2.111) has a nonoscillatory solution for some t_0. To this end, we only need to prove that the equation

$$z'(t) + r\prod_{j=1}^m (1-e^{-z(t-\tau_j)})^{\alpha_j} = 0 \tag{2.118}$$

has an eventually positive solution. Let t_0 be such that $y(t-\tau) > 0$ for $t \geq t_0$. Using the inequality $1-e^{-x} \leq x$ for $x \geq 0$, we have for $t \geq t_0$ that

$$y'(t) + r\prod_{j=1}^m (1-e^{-y(t-\tau_j)})^{\alpha_j} \leq y'(t) + r\prod_{j=1}^m y^{\alpha_j}(t-\tau_j) = 0. \tag{2.119}$$

It can be easily shown that $y(t) \to 0$, as $t \to \infty$. Integrating the above inequality from t to ∞, we obtain

$$y(t) \geq r\int_t^\infty \prod_{j=1}^m (1-e^{-y(t-\tau_j)})^{\alpha_j}, \quad \text{for } t \geq t_0.$$

Now an argument similar to the proof of Theorem 2.7.2 shows that (2.119) would have an eventually positive solution $z(t)$ on $[t_0, \infty)$ satisfying $z(t) > 0$ for all $t \geq t_0$.

$\Leftarrow$: Assume, for the sake of contradiction, that (2.110)–(2.111) has a nonoscillatory solution $x(t)$ for every t_0. Then $1 + x(t) > 0$, for $t \geq t_0$. We now distinguish two cases:

Case (i): $x(t)$ is eventually positive.

Then there exists $T \geq t_0$ such that $x(t) > 0$, for $t \geq T$. From (2.110) it follows that

$$x^{'}(t) + r\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \leq x^{'}(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0. \quad (2.120)$$

This, together with Lemma 2.7.1, implies that (2.112) has a nonoscillatory solution, contrary to the assumption that every solution of (2.112) oscillates.

Case (ii): $x(t)$ is eventually negative.

Since $1 + x(t) > 0$ for $t \geq t_0$ and $x(t) < 0$ for $t \geq T$ for some $T \geq t_0$, we have

$$x^{'}(t) = r(1+x(t))\prod_{j=1}^{m}[-x(t-\tau_j)]^{\alpha_j} > 0, \quad \text{for } t \geq T,$$

from which we can easily see that $x(t) \to 0$ as $t \to \infty$. On the other hand, in view of Lemma 2.7.1, we can choose $\epsilon \in (0, 1)$ such that

$$r(1-\epsilon)\sum_{j=1}^{m}\alpha_j\tau_j > \frac{1}{e}. \quad (2.121)$$

Now, let $T_1 > T$ be sufficiently large such that $1 > 1 + x(t) > 1 - \epsilon$, for $t \geq T$. Then by (2.110), we have for $t \geq T + \tau$ that

$$\begin{aligned} &x^{'}(t) + r(1-\epsilon)\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \\ &\geq x^{'}(t) + r(1+x(t))\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) = 0, \end{aligned} \quad (2.122)$$

which is also a contradiction since, by Lemma 2.7.1, (2.122) implies that the inequality

$$x^{'}(t) + r(1-\epsilon)\prod_{j=1}^{m} x^{\alpha_j}(t-\tau_j) \geq 0$$

cannot have an eventually negative solution. This completes the proof. ∎

The following corollary is an immediate result from Theorem 2.7.3 and Lemma 2.7.1.

Corollary 2.7.1. *If* $\sum_{j=1}^{m} \alpha_j = 1$, *then every solution of (2.110)–(2.111) oscillates (or every positive solution of (2.111) oscillates about the steady state K) if and only if*

$$r \sum_{j=1}^{m} \alpha_j \tau > \frac{1}{e}.$$

Next, in the following we consider the nonautonomous hyperlogistic delay model

$$N'(t) = r(t)N(t) \prod_{j=1}^{m} \left[1 - \frac{N(t-\tau_j)}{K}\right]^{\beta_j}, \text{ for } t \geq 0, \tag{2.123}$$

where $0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_m$, $\beta_1, \ldots, \beta_m$ are rational numbers with denominators that are positive odd integers, and

$$r \in C([t_0, \infty), [0, \infty)), \; K > 0.$$

We will establish some sufficient conditions for the oscillation of all positive solutions of (2.123) about K. The results are adapted from [71]. To prove the main results we study the oscillation of the equation

$$x'(t) + p(t) \prod_{j=1}^{m} \left|x(t-\tau_j)\right|^{\alpha_j} sign[x(t-\tau_j)] = 0, \quad t \geq t_0, \tag{2.124}$$

where

$$p \in C([t_0, \infty), [0, \infty)), \; 0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_m, \; \alpha_j > 0, \; j = 1, 2, \ldots, m,$$

and then apply the obtained results on the hyperlogistic model (2.123).

We will consider the equation

$$x'(t) + p(t) f(x(t-\tau_1), \ldots, x(t-\tau_m)) = 0, \text{ for } t \geq t_0, \tag{2.125}$$

where the function f satisfies the following condition (H):

(H). $f \in C(\mathbf{R}^m, \mathbf{R})$, $f(x_1, \ldots, x_m)$ is nondecreasing on each x_i, $i = 1, \ldots, m$, and

$$x_i > 0, \text{ for } i = 1, \ldots, m \Rightarrow f(x_1, \ldots, x_m) > 0,$$

$$x_i < 0, \text{ for } i = 1, \ldots, m \Rightarrow f(x_1, \ldots, x_m) < 0,$$

and

$$\lim_{(x_1,\ldots,x_m)\to(0,\ldots,0)} \frac{|f(x_1,\ldots,x_m)|}{\prod_{j=1}^{m} |x_j|^{\alpha_j}} = M > 0.$$

We will apply the results on the equation

$$x'(t) + \sum_{j=1}^{m} p_j(t)\, x^{\beta_j}(t-\tau_j) = 0, \quad \text{for } t \geq t_0, \tag{2.126}$$

where $\beta_1, \ldots, \beta_m$ are rational numbers with denominators that are positive odd integers and

$$p_j \in C([t_0,\infty),[0,\infty)), \quad \text{for} \quad j = 1, 2, \ldots, m.$$

In the following, we consider the case when

$$\sum_{j=1}^{m} \alpha_j > 1 \tag{2.127}$$

and study the oscillatory behavior of (2.124) in terms of $p(t)$ and the delays $\tau_1, \ldots, \tau_m$.

The following lemma whose proof is standard (see [21]) will be needed to prove the main results.

Lemma 2.7.2. *Assume that* (H) *holds, and for large* t*,*

$$p(s) \neq 0, \quad \text{for } s \in [t, t+\tau], \tag{2.128}$$

where $\tau = \max\{\tau_1, \tau_2, \ldots, \tau_m\}$*. Then (2.125) has an eventually positive solution if and only if the corresponding inequality,*

$$x'(t) + p(t) f(x(t-\tau_1), \ldots, x(t-\tau_m)) \leq 0, \quad t \geq t_0, \tag{2.129}$$

has an eventually positive solution.

Associated with (2.125), we consider the equation

$$x'(t) + q(t) f(x(t-\tau_1), \ldots, x(t-\tau_m)) = 0, \quad \text{for } t \geq t_0, \tag{2.130}$$

where $q \in C([t_0,\infty),[0,\infty))$. Applying Lemma 2.7.2, we have the following lemma.

Lemma 2.7.3. *Assume that* (H) *and (2.128) hold, and that for large t*

$$p(t) \le q(t). \tag{2.131}$$

If every solution of (2.125) oscillates, then every solution of (2.130) oscillates.

Theorem 2.7.4. *Assume that (2.127) holds. Then the following conclusions hold:*

(i) *If there exists* $\lambda > 0$ *such that*

$$\sum_{j=1}^{m} \alpha_j e^{-\lambda \tau_j} < 1, \tag{2.132}$$

and

$$\liminf_{t\to\infty} \left[p(t) \exp\left(-e^{\lambda\tau}\right)\right] > 0, \tag{2.133}$$

then every solution of (2.124) oscillates.

(ii) *If (2.128) holds and there exists* $\mu > 0$ *such that*

$$\sum_{j=1}^{m} \alpha_j e^{-\mu \tau_j} > 1, \tag{2.134}$$

and

$$\limsup_{t\to\infty} \left[p(t) \exp\left(-e^{\mu\tau}\right)\right] < \infty, \tag{2.135}$$

then (2.124) has an eventually positive solution.

Proof. (i) From (2.132) and (2.133), we may choose $\lambda_2 < \lambda_1 < \lambda$ and $T > t_0$ such that

$$\sum_{j=1}^{m} \alpha_j e^{-\lambda \tau_j} < \sum_{j=1}^{m} \alpha_j e^{-\lambda_1 \tau_j} < \sum_{j=1}^{m} \alpha_j e^{-\lambda_2 \tau_j} < 1, \tag{2.136}$$

and

$$p(t) \ge \lambda_1 e^{\lambda_1 t} \exp\left[\frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right) e^{\lambda_1 t}\right], \quad t \ge T. \tag{2.137}$$

Set

$$q(t) = \lambda_1 e^{\lambda_1 t} \exp\left[\frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right) e^{\lambda_1 t}\right]. \tag{2.138}$$

By Lemma 2.7.3, it suffices to prove that every solution of the equation

$$x'(t) + q(t) \prod_{j=1}^{m} \left| x(t-\tau_j) \right|^{\alpha_j} sign\,[x(t-\tau_1)] = 0, \quad t \geq t_0, \tag{2.139}$$

oscillates. Assume the contrary, and let $x(t)$ be an eventually positive solution of (2.139). Then there exists a $T_1 > T$ such that

$$1 > x(t-\tau_m) > 0 \text{ and } x'(t) \leq 0, \text{ for } t \geq T_1.$$

Let $y(t) = -\ln x(t)$ for $t \geq T_1 - \tau_m$. Then $y(t) > 0$ for $t \geq T_1 - \tau_m$, and from (2.139) we have

$$y'(t) = q(t) \exp\left[y(t) - \sum_{j=1}^{m} \alpha_j y(t-\tau_j) \right], \text{ for } t \geq T_1. \tag{2.140}$$

Set $l = \sum_{j=1}^{m} \alpha_j e^{-\lambda_2 \tau_j}$. Then $0 < l < 1$. We consider the following three possible cases.

Case (1): Consider the case when $y(t) \leq \sum_{j=1}^{m} \alpha_j e^{(\lambda_1 - \lambda_2)\tau_j} y(t-\tau_j)$ eventually holds.

Choose $T_2 > T_1$ such that

$$y(t) \leq \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t-\tau_j), \text{ for } t \geq T_2.$$

Consequently, we have for $t \geq T_2$ that

$$\frac{y(t)}{e^{\lambda_1 t}} \leq \sum_{j=1}^{m} \frac{\alpha_j e^{\lambda_1 t - \lambda_2 \tau_j}}{e^{\lambda_1 t}} \frac{y(t-\tau_j)}{e^{\lambda_1 (t-\tau_j)}} = \sum_{j=1}^{m} \alpha_j e^{-\lambda_2 \tau_j} \frac{y(t-\tau_j)}{e^{\lambda_1 (t-\tau_j)}}.$$

Set $z(t) = y(t) e^{-\lambda_1 t}$. Then

$$z(t) \leq \sum_{j=1}^{m} \alpha_j e^{-\lambda_2 \tau_j} z(t-\tau_j), \quad \text{for } t \geq T_2. \tag{2.141}$$

This implies that

$$\lim_{t \to \infty} z(t) = 0. \tag{2.142}$$

From (2.142), it follows that there exists a $T_3 > T_2$ such that

$$y(t) < \frac{1}{2}e^{\lambda_1 t}, \quad t \geq T_3, \tag{2.143}$$

which, together with (2.140), implies for $t \geq T_3$ that

$$\begin{aligned} y'(t) &\geq q(t)\exp\left[\left(1 - \sum_{j=1}^{m}\alpha_j\right)y(t)\right] \\ &\geq q(t)\exp\left[\frac{1}{2}\left(1 - \sum_{j=1}^{m}\alpha_j\right)e^{\lambda_1 t}\right] = \lambda_1 e^{\lambda_1 t}. \end{aligned}$$

It follows that

$$y(t) \geq y(T_3) + e^{\lambda_1 t} - e^{\lambda_1 T_3}, \quad t \geq T_3,$$

which contradicts (2.143).

Case (2): Consider the case when $y(t) - \sum_{j=1}^{m}\alpha_j e^{(\lambda_1-\lambda_2)\tau_j}y(t-\tau_j)$ is oscillatory.

In this case, there exists an increasing infinite sequence $\{t_n\}$ of real numbers with $T_3 < t_1 < t_2 < \ldots$ such that

$$y(t_n) = \sum_{j=1}^{m}\alpha_j e^{(\lambda_1-\lambda_2)\tau_j}y(t_n-\tau_j), \quad n = 1, 2, \ldots, \tag{2.144}$$

and

$$y(t) > \sum_{j=1}^{m}\alpha_j e^{(\lambda_1-\lambda_2)\tau_j}y(t-\tau_j), \quad t \in (t_{2n-1}, t_{2n}), \ n = 1, 2, \ldots \tag{2.145}$$

Set

$$u(t) = y(t) - \sum_{j=1}^{m}\alpha_j e^{(\lambda_1-\lambda_2)\tau_j}y(t-\tau_j).$$

Then $u(t)$ is oscillatory and there exists an increasing infinite sequence $\{\xi_n\}$ of real numbers such that

$$u(\xi_n) = \max\{u(t) \ : t_{2n-1} \leq t \leq t_{2n}\},$$

and $u'(\xi_n) = 0, n = 1, 2, \ldots$. Note

$$u'(\xi_n) = y'(\xi_n) - \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y'(\xi_n - \tau_j),$$

and for $t \geq T_1$

$$y'(t) = q(t) \exp\left[u(t) + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(t - \tau_j)\right]. \tag{2.146}$$

It follows that

$$\begin{aligned}
&q(\xi_n) \exp\left[u(\xi_n) + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_j)\right] \\
&= \sum_{i=1}^{m} \alpha_i e^{(\lambda_1-\lambda_2)\tau_i} q(\xi_n - \tau_i) \\
&\quad \times \exp\left[u(\xi_n - \tau_i) + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_i - \tau_j)\right] \\
&< \lambda_1 e^{\lambda_1 \xi_n} \exp\left[\frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right) e^{\lambda_1(\xi_n - \tau_1)}\right] \\
&\quad \times \exp\left[\max_{1 \leq i \leq m}\{u(\xi_n - \tau_i)\} + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_1 - \tau_j)\right].
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&u(\xi_n) + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_j) \\
&< \max_{1 \leq i \leq m}\{u(\xi_n - \tau_i)\} + \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) y(\xi_n - \tau_1 - \tau_j) \\
&- \frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right)(1 - e^{-\lambda_1 \tau_1}) e^{\lambda_1 \xi_n}, \qquad n = 1, 2, 3, \ldots .
\end{aligned} \tag{2.147}$$

If

$$\limsup_{t\to\infty} u(t) = \limsup_{n\to\infty} u(\xi_n) = \infty,$$

then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that

$$u\left(\xi_{n_k}\right) = \max\{u(t) : T_2 \le t \le \xi_{n_k}\}, \;\; k = 1, 2, \dots .$$

Hence, from (2.147), we have

$$0 < \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) \left[y(\xi_{n_k} - \tau_j) - y(\xi_{n_k} - \tau_1 - \tau_j)\right]$$
$$< -\frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right)(1 - e^{-\lambda_1\tau_1})e^{\lambda_1\xi_{n_k}} < 0, \; k = 1, 2, \dots .$$

This is a contradiction. If

$$\limsup_{t\to\infty} u(t) = \limsup_{n\to\infty} u(\xi_n) < \infty,$$

then from (2.147),

$$0 < \limsup_{n\to\infty} \Bigg(u(\xi_n)$$
$$+ \sum_{j=1}^{m} \alpha_j (e^{(\lambda_1-\lambda_2)\tau_j} - 1) \left[y(\xi_{n_k} - \tau_j) - y(\xi_{n_k} - \tau_1 - \tau_j)\right]\Bigg)$$
$$\le \limsup_{n\to\infty} \Bigg\{ \max_{1\le i\le m} \{u(\xi_n - \tau_i)\}$$
$$- \frac{1}{2}\left(\sum_{j=1}^{m} \alpha_j - 1\right)(1 - e^{-\lambda_1\tau_1})e^{\lambda_1\xi_n} \Bigg\} = -\infty.$$

This is also a contradiction.

Case (3): Consider the case when $y(t) \ge \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t - \tau_j)$ eventually holds.

Let $T_4 > T_3$ be such that

$$y(t) \ge \sum_{j=1}^{m} \alpha_j e^{(\lambda_1-\lambda_2)\tau_j} y(t - \tau_j), \; t \ge T_4.$$

It follows from (2.140) that

$$y'(t) = q(t)\exp\left[y(t) - \sum_{j=1}^{m}\alpha_j y(t-\tau_j)\right]$$
$$\geq q(t)\exp\left[\left(1 - e^{(\lambda_2-\lambda_1)\tau_1}\right) r(t)\right], \text{ for } t \geq T_4.$$

Set $c = 1 - e^{(\lambda_2-\lambda_1)\tau_1}$. Then $0 < c < 1$, and the above inequality reduces to

$$y'(t)e^{-cy(t)} \geq q(t), \text{ for } t \geq T_4.$$

Integrating the above inequality from T_4 to ∞, we obtain

$$\int_{T_4}^{\infty} q(t)dt \leq \int_{T_4}^{\infty} y'(t)e^{-cy(t)}dt \leq \frac{1}{c}e^{-cy(T_4)} < \infty,$$

which contradicts the definition of $q(t)$.

Cases 1, 2, and 3 complete the proof of (i).

(ii) By (2.134) and (2.135), we may choose $\mu_1 > \mu$ and $T > t_0$ such that

$$\sum_{j=1}^{m}\alpha_j e^{-\mu\tau_j} > \sum_{j=1}^{m}\alpha_j e^{-\mu_1\tau_j} > 1, \tag{2.148}$$

and

$$p(t) \leq \mu_1 e^{\mu_1 t}\exp\left[\left(\sum_{j=1}^{m}\alpha_j e^{-\mu_1\tau_j} - 1\right)e^{\mu_1 t}\right], \ t \geq T. \tag{2.149}$$

Set $\varphi(t) = e^{\mu_1 t}$ and $x(t) = e^{-\varphi(t)}$. Then for $t \geq T$,

$$x'(t) + p(t)\prod_{j=1}^{m}\left|x(t-\tau_j)\right|^{\alpha_j} sign\left[x(t-\tau_1)\right]$$

$$= -\varphi(t)e^{-\varphi(t)} + p(t)\prod_{j=1}^{m}e^{-\alpha_j\varphi(t-\tau_j)}$$

$$= \prod_{j=1}^{m}e^{-\alpha_j\varphi(t-\tau_j)}\left\{p(t) - \mu_1 e^{\mu_1 t}\exp\left[\left(\sum_{j=1}^{m}\alpha_j e^{-\mu_1\tau_j} - 1\right)e^{\mu_1 t}\right]\right\} \leq 0.$$

This shows that the inequality

$$x'(t) + p(t)\prod_{j=1}^{m}\left|x(t-\tau_j)\right|^{\alpha_j} sign\,[x(t-\tau_1)] \le 0,\ \ t \ge t_0,$$

has an eventually positive solution. In view of Lemma 2.7.2, the corresponding equation (2.124) also has an eventually positive solution. The proof is complete. ■

Applying Theorem 2.7.4 on the special form

$$x^{'}(t) + p(t)\left|x(t-\tau_j)\right|^{\alpha} sign\,[x(t-\tau)] = 0, \qquad t \ge t_0, \tag{2.150}$$

where

$$p \in C([t_0,\infty),[0,\infty)),\ \tau > 0,\ \alpha > 0,$$

we have immediately the following result.

Corollary 2.7.2. *Assume that $\alpha > 1$. Then the following conclusions hold:*

(i) If there exists $\lambda > \tau^{-1}\ln\alpha$ such that (2.133) holds, then every solution of (2.150) oscillates.
(ii) If $p(t) \not\equiv 0$ on any interval of length τ, and there exists $\mu < \tau^{-1}\ln\alpha$ such that (2.135) holds, then (2.150) has an eventually positive solution.

Note that if $\sum_{j=1}^{m}\alpha_j > 1$, then it follows that there exists a unique $\lambda_0 > 0$ such that

$$\sum_{j=1}^{m}\alpha_j e^{-\lambda_0\tau_j} = 1.$$

Therefore, applying Theorem 2.7.4 to the following equation which is a special form of (2.124)

$$x'(t) + C\exp(e^{\lambda t})\prod_{j=1}^{m}\left|x(t-\tau_j)\right|^{\alpha_j} sign\,[x(t-\tau_1)] = 0,\ \ t \ge t_0, \tag{2.151}$$

where $C > 0$, we have that every solution of (2.151) oscillates if $\lambda > \lambda_0$ and (2.151) has an eventually positive solution in $\lambda < \lambda_0$.

In the following, we apply Theorem 2.7.4 to (2.125), (2.126), and (2.123).

Theorem 2.7.5. *Assume that* (H) *holds and* $\sum_{j=1}^{m}\alpha_j > 1$. *Then the following conclusions hold:*

(i) If there exists $\lambda > 0$ *such that (2.132) and (2.133) hold, then every solution of (2.125) oscillates.*

(ii) If (2.128) and

$$\int_{t_0}^{\infty} p(t)dt = \infty \tag{2.152}$$

hold and there exists $\mu > 0$ *such that (2.134) and (2.135) hold, then (2.125) has an eventually positive solution.*

Proof. (i) Assume the contrary, and let $x(t)$ be an eventually positive solution of (2.125). Then from (2.125) and (2.133), we easily see that $\lim_{t\to\infty} x(t) = 0$. Then from (2.125) and (H) there exists a $T_1 > t_0$ such that

$$1 > x(t-\tau_m) > 0, \text{ and } x^{'}(t) \le 0, \text{ for } \quad t \ge T_1,$$

and

$$f(x(t-\tau_1),\ldots,x(t-\tau_m)) \ge \frac{1}{2}M\prod_{j=1}^{m}\left[x(t-\tau_j)\right]^{\alpha_j}, \quad t \ge T_1. \tag{2.153}$$

Substituting (2.153) into (2.125), we have

$$x'(t) + \frac{1}{2}Mp(t)\prod_{j=1}^{m}\left[x(t-\tau_j)\right]^{\alpha_j} \le 0, \quad \text{for } t \ge T_1. \tag{2.154}$$

This shows that the inequality (2.154) has an eventually positive solution. In view of Lemma 2.7.2, the corresponding equation,

$$x'(t) + \frac{1}{2}Mp(t)\prod_{j=1}^{m}|x(t-\tau_j)|^{\alpha_j} sign\left[x(t-\tau_1)\right] = 0, \quad t \ge t_0, \tag{2.155}$$

also has an eventually positive solution. But, by Theorem 2.7.4, (2.132) and (2.133) imply that every solution of (2.155) oscillates, and this contradiction completes the proof of (i).

(ii) In view of Theorem 2.7.4, (2.128), (2.134), (2.135), and (2.152) imply that the equation

$$x'(t) + 2Mp(t)\prod_{j=1}^{m}|x(t-\tau_j)|^{\alpha_j} sign\,[x(t-\tau_1)] = 0, \quad t \geq t_0, \tag{2.156}$$

has an eventually positive solution $x(t)$ with $\lim_{t\to\infty} x(t) = 0$. From this, (H), and (2.156), there exists a $T_2 > t_0$ such that

$$x(t-\tau_m) > 0, \text{ and } x^{'}(t) \leq 0 \text{ for } \quad t \geq T_2,$$

and

$$f(x(t-\tau_1),\ldots,x(t-\tau_m)) \leq 2M\prod_{j=1}^{m}|x(t-\tau_j)|^{\alpha_j}, \qquad t \geq T_2. \tag{2.157}$$

Substituting (2.157) into (2.156), we have

$$x^{'}(t) + p(t)f(x(t-\tau_1),\ldots,x(t-\tau_m)) \leq 0, \quad t \geq T_2. \tag{2.158}$$

This shows that inequality (2.158) has an eventually positive solution. In view of Lemma 2.7.2, the corresponding equation (2.125) also has an eventually positive solution. The proof is complete. ∎

Theorem 2.7.6. *Assume that* $\sum_{j=1}^{m}\beta_j > m$, *and that there exists* $\lambda > 0$ *such that*

$$\sum_{j=1}^{m}\beta_j e^{-\lambda\tau_j} < m, \tag{2.159}$$

and

$$\liminf_{t\to\infty}\left\{\left[\prod_{j=1}^{m}p_j(t)\right]\exp\left(-me^{\lambda t}\right)\right\} > 0. \tag{2.160}$$

Then every solution of (2.126) oscillates.

Proof. Assume the contrary, and let $x(t)$ be an eventually positive solution of (2.126). It follows from (2.126) that there exists a $T > t_0$ such that

$$x(t-\tau_m) > 0, \text{ and } x^{'}(t) \leq 0, \quad \text{for} \quad t \geq T.$$

From (2.126), we have

$$x^{'}(t) + m\left[\prod_{j=1}^{m} p_j(t)\right]^{\frac{1}{m}} \prod_{j=1}^{m} \left[x(t-\tau_j)\right]^{\frac{\beta_j}{m}} \leq 0, \quad t \geq T. \tag{2.161}$$

This shows that inequality (2.161) has an eventually positive solution. In view of Lemma 2.7.2, the corresponding equation,

$$x^{'}(t) + m\left[\prod_{j=1}^{m} p_j(t)\right]^{\frac{1}{m}} \prod_{j=1}^{m} |x(t-\tau_j)|^{\frac{\beta_j}{m}} sign\left[x\left(t-\tau_1\right)\right] = 0,\ t > t_0, \tag{2.162}$$

also has an eventually positive solution. But Theorem 2.7.4, (2.159), and (2.160) imply that every solution oscillates. This contradiction completes the proof. ■

Now, we consider equation (2.123). Note that if

$$\prod_{j=1}^{m} (-1)^{\beta_j} = -1,$$

then by making a change of variables,

$$x(t) = \ln\left[\frac{N(t)}{K}\right],$$

one can write (2.123) as

$$x^{'}(t) + r(t)\prod_{j=1}^{m}\left[e^{x(t-\tau_j)} - 1\right]^{\beta_j} = 0, \quad \text{for } t \geq 0. \tag{2.163}$$

Set

$$f(x_1, \ldots, x_m) = \prod_{j=1}^{m} (e^{x_j} - 1)^{\beta_j}.$$

Then f satisfies condition (H) for $\beta_1, \ldots, \beta_m$.

Hence, in view of Theorem 2.7.5, we have immediately the following result.

Theorem 2.7.7. *Assume that*

$$\prod_{j=1}^{m} (-1)^{\beta_j} = -1 \textit{ and } \sum_{j=1}^{m} \beta_j > 1.$$

Then the following conclusions hold:

(*i*) *If there exists* $\lambda > 0$ *such that*

$$\sum_{j=1}^{m} \beta_j e^{-\lambda \tau_j} < 1, \tag{2.164}$$

and

$$\lim_{t \to \infty} \inf \left[r(t) \exp \left(-e^{\lambda t} \right) \right] > 0, \tag{2.165}$$

then every positive solution of (2.123) oscillates about K.

(*ii*) *If* $r(t) \neq 0$ *for any interval of length* τ, *where* $\tau = \max \{\tau_1, \dots, \tau_m\}$,

$$\int_0^{\infty} r(s) ds = \infty, \tag{2.166}$$

and there exists $\mu > 0$ *such that*

$$\sum_{j=1}^{m} \beta_j e^{-\mu \tau_j} > 1, \tag{2.167}$$

and

$$\lim_{t \to \infty} \sup \left[r(t) \exp \left(-e^{\mu t} \right) \right] < \infty, \tag{2.168}$$

then (2.123) has a solution greater than K *eventually.*

2.8 Models with a Varying Capacity

In the delay logistic equations we assumed that the carrying capacity $K > 0$ is a constant. The variation of the environment plays an important role in many biological and ecological dynamical systems. It is realistic to assume that the parameters in the models are positive periodic functions of period ω.

Consider the nonautonomous delay logistic model

$$N'(t) = r(t)N(t) \left[1 - \frac{N(t - m\omega)}{K(t)} \right], \tag{2.169}$$

where m is a positive integer and $\omega > 0$. Assume r and K are positive periodic functions of period ω. We consider solutions of (2.169) corresponding to the initial condition

$$\begin{cases} N(t) = \varphi(t), \ \text{for } m\omega < t < 0, \\ \varphi \in C[[-m\omega, 0], \mathbf{R}^+], \ \varphi(0) > 0. \end{cases} \tag{2.170}$$

It is easy to see that there exist a unique positive periodic solution $N^*(t)$ of (2.169).

Theorem 2.8.1. *If*

$$\int_0^\infty \frac{r(t)N^*(t)}{K(t)} dt = \infty, \tag{2.171}$$

then every nonoscillatory solution $N(t)$ of (2.169) satisfies

$$\lim_{t\to\infty} N(t) = N^*(t). \tag{2.172}$$

Proof. Assume that $N(t) > N^*(t)$ for t sufficiently large (the proof when $N(t) < N^*(t)$ is similar and will be omitted). Set

$$N(t) = N^*(t)e^{z(t)}. \tag{2.173}$$

Then $z(t) > 0$ for t sufficiently large, and for t large

$$z'(t) + \frac{r(t)N^*(t)}{K(t)} \left(e^{z(t-m\omega)} - 1\right) = 0, \tag{2.174}$$

so

$$z'(t) = -\frac{r(t)N^*(t)}{K(t)} \left(e^{z(t-m\omega)} - 1\right) < 0.$$

Thus, $z(t)$ is decreasing, and therefore

$$\lim_{t\to\infty} z(t) = \alpha \in [0, \infty).$$

We claim $\alpha = 0$. If $\alpha > 0$, then there exist $\varepsilon > 0$ and $T_\varepsilon > 0$ such that for $t \geq T_\varepsilon$,

$$0 < \alpha - \varepsilon < z(t) < \alpha + \varepsilon.$$

However, then from (2.174), we find

$$z'(t) + \frac{r(t)N^*(t)}{K(t)}(e^{\alpha-\varepsilon} - 1) \leq 0, \quad t \geq T_\varepsilon,$$

By integrating from T_ε to ∞ and using (2.171) we immediately get a contradiction. Hence $\alpha = 0$. Thus

$$\lim_{t\to\infty}(N(t) - N^*(t)) = \lim_{t\to\infty} N^*(t)(e^{z(t)} - 1) = 0.$$

This completes the proof. ∎

Theorem 2.8.2. *Assume that r and K are positive periodic functions of period ω such that (2.171) holds. Suppose for every sufficiently small $\varepsilon \geq 0$ all solutions of the linear delay differential equation*

$$x'(t) + (1-\varepsilon)\frac{r(t)N^*(t)}{K(t)}x(t - m\omega) = 0, \quad t \geq t_0, \tag{2.175}$$

are oscillatory. Then all solutions of (2.169) are oscillatory about $N^*(t)$.

Proof. Assume that (2.169) has a solution which does not oscillate about $N^*(t)$. Without loss of generality we assume that $N(t) > N^*(t)$, so that $z(t) > 0$; here z is defined in Theorem 2.8.1. (The case $N(t) < N^*(t)$ implies that $z(t) < 0$ and the proof is similar. In fact, we will see below that if $z(t)$ is a negative solution of (2.176) then $U(t) = -z(t)$ is positive solution of (2.176)). It is clear that $N(t)$ oscillates about $N^*(t)$ if and only if $z(t)$ oscillates about zero. Also

$$z'(t) + \frac{r(t)N^*(t)}{K(t)}f(z(t - m\omega)) = 0, \tag{2.176}$$

where

$$f(u) = (e^u - 1).$$

Note that

$$\lim_{u\to 0}\frac{f(u)}{u} = 1.$$

Then by Theorem 2.6.4, since every solution of (2.175) oscillates, then every solution of (2.176) oscillates. Thus every positive solution of (2.169) oscillates about $N^*(t)$. The proof is complete. ∎

Next we discuss the oscillation of (2.169) about the positive periodic function $K(t)$. The result is adapted from [86].

Theorem 2.8.3. *Assume the following:*

(i) K is a nonconstant positive differentiable periodic function of period ω.
(ii) r is positive and continuous for $t \geq 0$ such that

$$\liminf_{t\to\infty} r(t) > 0, \textit{ and } \quad \liminf_{t\to\infty} \int_{t-m\omega}^{t} r(s)ds > \frac{1}{e}. \tag{2.177}$$

Then every positive solution of (2.169) is oscillatory about K.

Proof. If we define $y(t) = \ln[N(t)/K(t)]$, then y is governed by

$$y'(t) = r(t)\left[1 - e^{y(t-m\omega)}\right] - \frac{K'(t)}{K(t)}, \tag{2.178}$$

and the oscillation of N about K is equivalent to that of y about zero and thus it is sufficient to consider the usual oscillation of y. We simplify (2.178) by letting

$$Q(t) = \ln\left(\frac{K(t_0)}{K(t)}\right) \tag{2.179}$$

and note that (2.178) becomes

$$y'(t) + r(t)\left[e^{y(t-m\omega)} - 1\right] = Q'(t). \tag{2.180}$$

Suppose now the conclusion of the theorem is false. Then there exists an eventually positive or eventually negative solution for (2.180).

Let us first assume that (2.180) has an eventually positive solution y. Since Q is a nonconstant periodic function, there exist two sequences $\{t_n'\}$ and $\{t_n''\}$ such that $\lim_{n\to\infty} t_n' = \infty$, $\lim_{n\to\infty} t_n''$, and

$$-\infty < q_1 \le Q(t) \le q_2 < \infty,$$

$$q_1 = Q(t_n') \text{ and } q_2 = Q(t_n''), \quad n = 1, 2, \dots . \tag{2.181}$$

Let

$$u(t) = y(t) - Q(t), \quad \text{for } t \ge T,$$

(where $y(t - m\omega) > 0$ for $t \ge T$). Note that (2.180) becomes

$$u'(t) = r(t)\left[1 - e^{y(t-m\omega)}\right] < 0. \tag{2.182}$$

We claim $u(t) + q_1 > 0$. Suppose for some $t \ge T$, $u(t) + q_1 \le 0$. Since $y(t) > 0$, we have $u(t) + Q(t) = y(t) > 0$ and hence $u(t_n') + q_1 = y(t_n') > 0$ showing that $u(t) + q_1 \le 0$ is not possible. Therefore,

$$u(t) + q_1 > 0, \text{ for large } t \ge T. \tag{2.183}$$

Let $z(t) = u(t) + q_1$ and we see that

$$\begin{aligned} z'(t) &= u'(t) = y'(t) - Q'(t) \\ &= r(t)\left[1 - e^{y(t-m\omega)}\right] \\ &= r(t)\left[1 - e^{u(t-m\omega)+Q(t-m\omega)}\right] \\ &\leq -r(t)\left[u(t-m\omega) + Q(t-m\omega)\right] \leq -r(t)z(t-m\omega). \end{aligned} \tag{2.184}$$

Note that (2.184) has an eventually positive solution and this is impossible due to (2.177) (a standard argument is used here).

Let us now consider the case when $y(t)$ is an eventually negative solution of (2.169). This implies that

$$\frac{N(t)}{K(t)} < 1, \quad \text{for large } t. \tag{2.185}$$

The boundedness of K (due to periodicity) and (2.185) imply that $N(t)$ is bounded. It follows from (2.169) that $N'(t) > 0$ eventually and this implies that

$$\lim_{t\to\infty} N(t) = l > 0. \tag{2.186}$$

Integrating (2.169), we have

$$\ln \frac{l}{N(t_0)} = \int_{t_0}^{\infty} r(t)\left(1 - \frac{N(t-m\omega)}{K(t)}\right) dt < \infty. \tag{2.187}$$

Hence

$$\lim_{t\to\infty} \inf r(t)\left(1 - \frac{N(t-m\omega)}{K(t)}\right) = 0.$$

But $\liminf_{t\to\infty} r(t) > 0$, so

$$\lim_{t\to\infty} \sup \frac{N(t-m\omega)}{K(t)} = 1,$$

i.e., there exists a sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} \frac{N(t_k - m\omega)}{K(t_k)} = 1.$$

Since $N(t) < K$, we see that $\lim_{t\to\infty} N(t) = l = \min_{t\in[0,\omega]} K(t)$. But then

$$\begin{aligned}
&\int_{t_0}^{\infty} r(t)\left(1 - \frac{N(t-m\omega)}{K(t)}\right) dt \\
&\geq \frac{\inf r(t)}{\max_{t\in[0,\omega]} K(t)} \int_{t_0}^{\infty} (K(t) - N(t-m\omega))\, dt \\
&\geq \frac{\inf r(t)}{\max_{t\in[0,\omega]} K(t)} \int_{t_0}^{\infty} \left(K(t) - \min_{t\in[0,\omega]} K(t)\right) dt = \infty,
\end{aligned}$$

which contradicts (2.187). This completes the proof. ■

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