Dual Compatible Splines on Nontensor Product Meshes

L. Beirão da Veiga, A. Buffa, G. Sangalli and R. Vázquez

Abstract In this paper we introduce the concept of dual compatible (DC) splines on nontensor product meshes, study the properties of this class, and discuss their possible use within the isogeometric framework. We show that DC splines are linear independent and that they also enjoy good approximation properties.

Keywords Isogeometric analysis · Spline theory · T-splines · Numerical methods for partial differential equations

1 Introduction

Tensor product multivariate spline spaces are easy to construct and their mathematical properties directly extend from the univariate case. However, the tensor product construction restricts the possibility of local refinement which is a severe limitation for their use within the isogeometric framework, i.e., as discretization spaces for the numerical solution of partial differential equations. This is particularly true in problems that exhibit solutions with layers or singularities. In this paper, we discuss an

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extension of splines spaces that go beyond the tensor product structure, and therefore allow local mesh refinement.

Three approaches have emerged in the isogeometric community: T-splines, Locally refinable (LR) splines, and hierarchical splines. T-splines have been proposed in [1] for applications to CAGD and have been adopted for isogeometric methods since [2]. Nowadays, they are likely the most popular approach among engineers: for example, they have been used for shell problems [3], fluid–structure interaction problems [4], and contact mechanics simulation [5]. The algorithm for local refinement has evolved since its introduction (in [6]) and while the first approach was not efficient in isogeometric methods (see for example [7]) the more recent developments (e.g., [8]) overcome the initial limitations. The mathematical literature on T-splines is very recent and mainly restricted to the two-dimensional case. It is based on the notion of Analysis-Suitable (AS) T-splines: these are a subset of T-splines, introduced in [9] and extended to arbitrary degree in [10], for which fundamental properties hold. LR-splines [11] and Hierarchical splines [12] have been proposed more recently in the isogeometric literature and represent a valid alternative to T-splines. However, for reasons of space and because of our expertise, we restrict the presentation to T-splines.

This paper is organized as follows. First, we set up our main notation of Sect. 2. Then, we introduce the notion of Dual-Compatible (DC) set of B-splines. This is a set of multivariate B-splines without a global tensor product structure but endowed with a weaker structure that still guarantees some key properties. The main one is that their linear combination spans a space (named DC space) that can be associated with a dual space by a construction of a dual basis. The existence of a "good" dual space implies other mathematical properties that are needed in isogeometric methods: for example, (local) linear independence and partition of unity of the DC set of B-spline functions, and optimal approximation properties of the DC space. The framework we propose here is an extension of the one introduced in [10], and covers arbitrary space dimension.

2 Preliminaries

Given two positive integers p and n, we say that $\Xi := \{\xi_1, \dots, \xi_{n+p+1}\}$ is a p-open knot vector if

$$\xi_1 = \cdots = \xi_{p+1} < \xi_{p+2} \leq \cdots \leq \xi_n < \xi_{n+1} = \cdots = \xi_{n+p+1},$$

where repeated knots are allowed. Without loss of generality, we assume in the following that $\xi_1 = 0$ and $\xi_{n+p+1} = 1$.

We introduce also the vector $Z = \{\zeta_1, \dots, \zeta_N\}$ of knots without repetitions, also called breakpoints, and denote by m_j, the multiplicity of the breakpoint ζ_j, such that

$$\Xi = \{\underbrace{\zeta_1, \dots, \zeta_1}_{m_1 \text{ times}}, \underbrace{\zeta_2, \dots, \zeta_2}_{m_2 \text{ times}}, \dots, \underbrace{\zeta_N, \dots, \zeta_N}_{m_N \text{ times}}\}, \tag{1}$$

with $\sum_{i=1}^{N} m_i = n + p + 1$. We assume $m_j \leq p+1$, for all internal knots. Note that the points in Z form a partition of the unit interval $I = (0, 1)$, i.e., a mesh, and the local mesh size of the element $I_i = (\zeta_i, \zeta_{i+1})$ is called $h_i = \zeta_{i+1} - \zeta_i$, for $i = 1, \dots, N-1$.

From the knot vector Ξ, B-spline functions of degree p are defined following the well-known Cox-DeBoor recursive formula; we start with piecewise constants ($p = 0$):

$$\widehat{B}_{i,0}(\zeta) = \begin{cases} 1 & \text{if } \xi_i \leq \zeta < \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and for $p \geq 1$ the *B-spline* functions are defined by the recursion

$$\widehat{B}_{i,p}(\zeta) = \frac{\zeta - \xi_i}{\xi_{i+p} - \xi_i} \widehat{B}_{i,p-1}(\zeta) + \frac{\xi_{i+p+1} - \zeta}{\xi_{i+p+1} - \xi_{i+1}} \widehat{B}_{i+1,p-1}(\zeta), \tag{3}$$

where it is here formally assumed that 0/0 = 0.

This gives a set of n B-splines that, among many other properties, are non-negative and form a partition of unity. They also form a basis of the space of *splines*, that is, piecewise polynomials of degree p with $k_j := p - m_j$ continuous derivatives at the points ζ_j, for $j = 1, \dots, N$. Therefore, $-1 \leq k_j \leq p - 1$, and the maximum multiplicity allowed, $m_j = p + 1$, gives $k_j = -1$ which stands for a discontinuity at ζ_j.

We denote the *univariate spline space* spanned by the B-splines by

$$S_p(\Xi) = \text{span}\{\widehat{B}_{i,p}, \quad i = 1, \dots, n\}. \tag{4}$$

Note that the definition of each B-spline $\widehat{B}_{i,p}$ depends only on $p+2$ knots, which are collected in the *local knot vector*

$$\Xi_{i,p} := \{\xi_i, \dots, \xi_{i+p+1}\}.$$

When needed, we will stress this fact by adopting the notation

$$\widehat{B}_{i,p}(\zeta) = \widehat{B}[\Xi_{i,p}](\zeta). \tag{5}$$

Similarly, the support of each basis function is exactly $\text{supp}(\widehat{B}_{i,p}) = [\xi_i, \xi_{i+p+1}]$. Moreover, given an interval $I_j = (\zeta_j, \zeta_{j+1})$ of the partition, which can also be written as (ξ_i, ξ_{i+1}) for a certain (unique) i, we associate the *support extension* $\tilde{I}_j$ defined as

$$\tilde{I}_j := (\xi_{i-p}, \xi_{i+p+1}), \tag{6}$$

that is the interior of the union of the supports of basis functions whose support intersects I_j.

We concentrate now on the construction of interpolation and projection operators onto the space of splines $S_p(\Xi)$. There are several ways to define projections for splines, and here we only describe the one that will be used in this paper.

We will often make use of the following local quasi-uniformity condition on the knot vector, which is a classical assumption in the mathematical isogeometric literature.

Assumption 1 The partition defined by the knots $\zeta_1, \zeta_2, \ldots, \zeta_N$ is locally quasi-uniform, that is, there exists a constant $\theta \geq 1$ such that the mesh sizes $h_i = \zeta_{i+1} - \zeta_i$ satisfy the relation $\theta^{-1} \leq h_i / h_{i+1} \leq \theta$, for $i = 1, \ldots, N-2$.

Since splines are not in general interpolatory, a common way to define projections is by giving a dual basis, i.e.,

$$\Pi_{p,\Xi} : C^\infty([0,1]) \to S_p(\Xi), \qquad \Pi_{p,\Xi}(f) = \sum_{j=1}^{n} \lambda_{j,p}(f) \widehat{B}_{j,p}, \tag{7}$$

where $\lambda_{j,p}$ are a set of dual functionals verifying

$$\lambda_{j,p}(\widehat{B}_{k,p}) = \delta_{jk}, \tag{8}$$

δ_{jk} being the standard Kronecker symbol. It is trivial to prove that, thanks to this property, the quasi-interpolant $\Pi_{p,\Xi}$ preserves splines, that is,

$$\Pi_{p,\Xi}(f) = f, \quad \forall f \in S_p(\Xi). \tag{9}$$

Here, we adopt the dual basis defined in [13, Sect. 4.6]

$$\lambda_{j,p}(f) = \int_{\xi_j}^{\xi_{j+p+1}} f(s) D^{p+1} \psi_j(s)\, \mathrm{d}s, \tag{10}$$

where $\psi_j(\zeta) = G_j(\zeta)\phi_j(\zeta)$, with

$$\phi_j(\zeta) = \frac{(\zeta - \xi_{j+1}) \cdots (\zeta - \xi_{j+p})}{p!},$$

and

$$G_j(\zeta) = g\left(\frac{2\zeta - \xi_j - \xi_{j+p+1}}{\xi_{j+p+1} - \xi_j}\right),$$

where g is the transition function defined in [13, Theorem 4.37]. In the same reference, it is proved that the functionals $\lambda_{j,p}(\cdot)$ are dual to B-splines in the sense of (8) and stable (see [13, Theorem 4.41]), that is

$$|\lambda_{j,p}(f)| \le C(\xi_{j+p+1} - \xi_j)^{-1/2} \|f\|_{L^2(\xi_j, \xi_{j+p+1})}, \tag{11}$$

where the constant C grows exponentially with respect to the polynomial degree p with the upperbound

$$C \le (2p+3)9^p, \tag{12}$$

slightly improved in the literature after the results reported in [13]. Note that these dual functionals are locally defined and only depend on the corresponding local knot vector, that is, adopting a notation as in (5), we can write, when needed:

$$\lambda_{i,p}(f) = \lambda[\Xi_{i,p}](f). \tag{13}$$

The reasons for this choice of the dual basis are mainly historical (in the first paper on the numerical analysis of isogeometric methods [14] the authors used this projection), but also because it verifies the following important stability property:

Proposition 1 *For any non-empty knot span $I_i = (\zeta_i, \zeta_{i+1})$ it holds that*

$$\|\Pi_{p,\Xi}(f)\|_{L^2(I_i)} \le C\|f\|_{L^2(\tilde{I}_i)}, \tag{14}$$

where the constant C depends only on the degree p, and $\tilde{I}_i$ is the support extension defined in (6). Moreover, if Assumption 1 holds, we also have

$$|\Pi_{p,\Xi}(f)|_{H^1(I_i)} \le C|f|_{H^1(\tilde{I}_i)}, \tag{15}$$

with the constant C depending only on p and θ, and where H^1 denotes the Sobolev space of order one, endowed with the standard norm and seminorm.

Proof We first show (14). There exists a unique index j such that $I_i = (\zeta_i, \zeta_{i+1}) = (\xi_j, \xi_{j+1})$, and using the definition of B-splines at the beginning of Sect. 2, and in particular their support, it immediately follows that

$$\left\{\ell \in \{1, 2, \dots, n\} \; : \; \mathrm{supp}(\widehat{B}_{\ell,p}) \cap I_i \neq \emptyset\right\} = \{j-p, j-p+1, \dots, j\}. \tag{16}$$

Let h_i denotes the length of I_i and $\widetilde{h}_i$ indicates the length of $\tilde{I}_i$. First by definition (7), then recalling that the B-spline basis is positive and a partition of unity, we get

$$\begin{aligned}\|\Pi_{p,\Xi}(f)\|_{L^2(I_i)} &= \Big\| \sum_{\ell=j-p}^{j} \lambda_{\ell,p}(f)\widehat{B}_{\ell,p} \Big\|_{L^2(I_i)} \le \max_{j-p\le\ell\le j} |\lambda_{\ell,p}(f)| \, \Big\| \sum_{\ell=j-p}^{j} \widehat{B}_{\ell,p} \Big\|_{L^2(I_i)} \\ &= h_i^{1/2} \max_{j-p\le\ell\le j} |\lambda_{\ell,p}(f)|.\end{aligned}$$

We now apply bound (11) and obtain

$$\|\Pi_{p,\Xi}(f)\|_{L^2(I_i)} \le C h_i^{1/2} \max_{j-p\le\ell\le j} (\xi_{\ell+p+1}-\xi_\ell)^{-1/2} \|f\|_{L^2(\xi_\ell,\xi_{\ell+p+1})}$$
$$\le C h_i^{1/2} \max_{j-p\le\ell\le j} (\xi_{\ell+p+1}-\xi_\ell)^{-1/2} \|f\|_{L^2(\tilde I_i)},$$

that yields (14) since clearly $h_i \le (\xi_{\ell+p+1}-\xi_\ell)$, for all ℓ in $\{j-p,\dots,j\}$.

We now show (15). For any real constant c, since the operator $\Pi_{p,\Xi}$ preserves constant functions and using a standard inverse estimate for polynomials on I_i, we get

$$|\Pi_{p,\Xi}(f)|_{H^1(I_i)} = |\Pi_{p,\Xi}(f)-c|_{H^1(I_i)} = |\Pi_{p,\Xi}(f-c)|_{H^1(I_i)}$$
$$\le C h_i^{-1}\|\Pi_{p,\Xi}(f-c)\|_{L^2(I_i)}.$$

We now apply (14) and a standard approximation estimate for constant functions, yielding

$$|\Pi_{p,\Xi}(f)|_{H^1(I_i)} \le C h_i^{-1}\|f-c\|_{L^2(\tilde I_i)} \le C h_i^{-1}\widetilde{h}_i |f|_{H^1(\tilde I_i)}.$$

Using Assumption 1, it is immediate to check that $\widetilde{h}_i \le C h_i$ with $C = C(p,\theta)$ so that (15) follows.

The operator $\Pi_{p,\Xi}$ can be modified in order to match boundary conditions. We can define, for all $f \in C^\infty([0,1])$:

$$\tilde\Pi_{p,\Xi}(f) = \sum_{j=1}^{n} \tilde\lambda_{j,p}(f)\widehat{B}_{j,p} \quad \text{with} \tag{17}$$
$$\tilde\lambda_{1,p}(f) = f(0),\ \tilde\lambda_{n,p}(f) = f(1),\ \tilde\lambda_{j,p}(f) = \lambda_{j,p}(f), \quad j = 2,\dots,n-1.$$

3 Dual Compatible B-Splines

Consider a set of multivariate B-splines

$$\left\{\widehat{B}_{\mathbf{A},\mathbf{p}}, \quad \mathbf{A} \in \mathcal{A}\right\}, \tag{18}$$

where $\mathcal{A}$ is a set of indices. This is a generalization of the tensor product set of multivariate splines where the functions in (18) have the structure

$$\widehat{B}_{\mathbf{A},\mathbf{p}}(\boldsymbol{\zeta}) = \widehat{B}[\Xi_{\mathbf{A},1,p_1}](\zeta_1)\cdots\widehat{B}[\Xi_{\mathbf{A},d,p_d}](\zeta_d) \tag{19}$$

and have in general uncorrelated local knot vectors, that is, two different local knot vectors $\Xi_{\mathbf{A}',\ell,p_\ell}$ and $\Xi_{\mathbf{A}'',\ell,p_\ell}$ in the ℓ-direction are not in general sub-vectors of a global knot vector. This is equivalent to the definition of *point-based splines* in [1]. We assume that there is a one-to-one correspondence between $\mathbf{A} \in \mathcal{A}$ and $\widehat{B}_{\mathbf{A},\mathbf{p}}$.

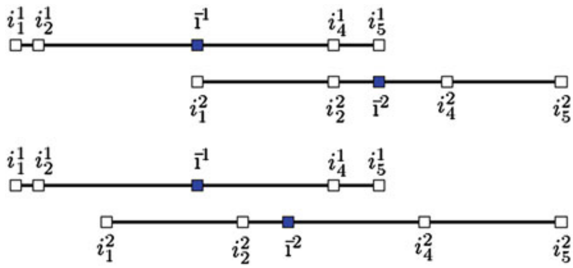

Fig. 1 Overlapping (*left*) and nonoverlapping (*right*) local knot vectors in one dimension

We say that the two p-degree local knot vectors $\Xi' = \{\xi'_1, \dots, \xi'_{p+2}\}$ and $\Xi'' = \{\xi''_1, \dots, \xi''_{p+2}\}$ *overlap* if they are subvectors of the same knot vector (that depends on Ξ' and Ξ''), that is there is a knot vector $\Xi = \{\xi_1, \dots, \xi_k\}$ and k' and k'' such that

$$\begin{aligned} \forall i &= 1, \dots, p+2, \quad \xi'_i = \xi_{i+k'} \\ \forall i &= 1, \dots, p+2, \quad \xi''_i = \xi_{i+k''}, \end{aligned} \tag{20}$$

see Fig. 1.

We now define for multivariate B-splines, the notions of *overlap* and *partial overlap* are as follows.

Definition 1 Two B-splines $\widehat{B}_{\mathbf{A}',\mathbf{p}}$ $\widehat{B}_{\mathbf{A}'',\mathbf{p}}$ in (18) overlap if the local knot vectors in each direction overlap. Two B-splines $\widehat{B}_{\mathbf{A}',\mathbf{p}}$ $\widehat{B}_{\mathbf{A}'',\mathbf{p}}$ in (18) partially overlap if, when $\mathbf{A}' \neq \mathbf{A}''$, there exists a direction ℓ such that the local knot vectors $\Xi_{\mathbf{A}',\ell,p_\ell}$ and $\Xi_{\mathbf{A}'',\ell,p_\ell}$ are different and overlap.

From the previous Definition, overlap implies partial overlap. Examples of B-splines overlapping, only partially overlapping, and not partially overlapping are depicted in Fig. 2.

Definition 2 The set (18) is a DC set of B-splines if each pair of B-splines in it partially overlaps. Its span

$$S_{\mathbf{p}}(\mathcal{A}) = \operatorname{span}\left\{\widehat{B}_{\mathbf{A},\mathbf{p}}, \quad \mathbf{A} \in \mathcal{A}\right\}, \tag{21}$$

is denoted as DC spline space.

Note that the partially overlapping condition in Definition 2 needs to be checked only for those B-spline pairs that have nondisjoint support. Indeed, by Definition 1, any two B-splines with disjoint supports are clearly partially overlapping.

A tensor product space is clearly a DC spline space, since every pair of its multivariate B-splines always overlaps by construction. The next proposition shows how the notion of partial overlap is related with the construction of dual basis.

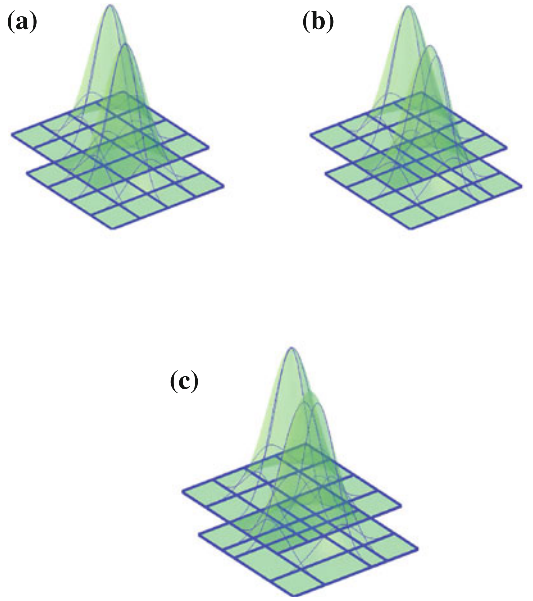

Fig. 2 Example of overlapping, partially overlapping, and not partially overlapping B-splines; *knot lines* are drawn in *blue* **a** Overlapping B-splines, **b** partially overlapping B-splines, **c** not partially overlapping B-splines

Proposition 2 *Assume that* (18) *is a DC set where each* $\widehat{B}_{\mathbf{A},\mathbf{p}}$ *is defined as in* (19)*, i.e., on the local knot vectors* $\Xi_{\mathbf{A},1,p_1}, \ldots, \Xi_{\mathbf{A},d,p_d}$*. Consider an associated set of functionals*

$$\left\{\lambda_{\mathbf{A},\mathbf{p}}, \quad \mathbf{A} \in \mathcal{A}\right\}, \tag{22}$$

where each $\lambda_{\mathbf{A},\mathbf{p}}$ *is*

$$\lambda_{\mathbf{A},\mathbf{p}} = \lambda[\Xi_{\mathbf{A},1,p_1}] \otimes \cdots \otimes \lambda[\Xi_{\mathbf{A},d,p_d}], \tag{23}$$

and $\lambda[\Xi_{\mathbf{A},\ell,p_\ell}]$ *denotes a univariate functional defined in* (10)*. Then* (22) *is a dual basis for* (18)*.*

Remark 1 The set of dual functionals (10) can be replaced by other choices, see, e.g., [15].

Proof Consider any $\widehat{B}_{\mathbf{A}',\mathbf{p}}$ and $\lambda_{\mathbf{A}'',\mathbf{p}}$, with $\mathbf{A}', \mathbf{A}'' \in \mathcal{A}$. We then need to show that

$$\lambda_{\mathbf{A}'',\mathbf{p}}(\widehat{B}_{\mathbf{A}',\mathbf{p}}) = \begin{cases} 1 & \text{if } \mathbf{A}'' = \mathbf{A}', \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Clearly, if $\mathbf{A}' = \mathbf{A}''$, then we have $\lambda_{\mathbf{A}'',\mathbf{p}}(\widehat{B}_{\mathbf{A}',\mathbf{p}}) = 1$ from the definition of dual basis. If $\mathbf{A}' \neq \mathbf{A}''$, thanks to the partial overlap assumption, there is a direction $\bar{\ell}$ such that the local knot vectors $\Xi_{\mathbf{A}',\ell,p_\ell}$ and $\Xi_{\mathbf{A}'',\ell,p_\ell}$ differ and overlap, and then

$$\lambda[\Xi_{\mathbf{A}'',\ell,p_\ell}](\widehat{B}[\Xi_{\mathbf{A}',\ell,p_\ell}]) = 0,$$

and from (23),

$$\lambda_{\mathbf{A}'',\mathbf{p}}(\widehat{B}_{\mathbf{A}',\mathbf{p}}) = \prod_{\ell=1}^{d} \lambda[\Xi_{\mathbf{A}'',\ell,p_\ell}](\widehat{B}[\Xi_{\mathbf{A}',\ell,p_\ell}]) = 0.$$

The existence of dual functionals implies important properties for a DC set (18) and the related space $S_{\mathbf{p}}(\mathcal{A})$ in (21). We list such properties in the following propositions and remarks.

The first result is the linear independence of set (18), therefore forming a *basis*; they are also a partition of unity.

Proposition 3 *The B-splines in a DC set* (18) *are linearly independent. Moreover, if the constant function belongs to* $S_{\mathbf{p}}(\mathcal{A})$*, they form a partition of unity.*

Proof Assume

$$\sum_{\mathbf{A}\in\mathcal{A}} C_{\mathbf{A}} \widehat{B}_{\mathbf{A},\mathbf{p}} = 0$$

for some coefficients $C_{\mathbf{A}}$. Then for any $\mathbf{A}' \in \mathcal{A}$, applying $\lambda_{\mathbf{A}',\mathbf{p}}$ to the sum, using linearity and (24), we get

$$C_{\mathbf{A}'} = \lambda_{\mathbf{A}',\mathbf{p}}\left(\sum_{\mathbf{A}\in\mathcal{A}} C_{\mathbf{A}} \widehat{B}_{\mathbf{A}',\mathbf{p}}\right) = 0.$$

Similarly, let

$$\sum_{\mathbf{A}\in\mathcal{A}} C_{\mathbf{A}} \widehat{B}_{\mathbf{A},\mathbf{p}} = 1$$

for some coefficients $C_{\mathbf{A}}$. For any $\mathbf{A}' \in \mathcal{A}$, applying $\lambda_{\mathbf{A}',\mathbf{p}}$ as above, we get

$$C_{\mathbf{A}'} = \lambda_{\mathbf{A}',\mathbf{p}}\left(\sum_{\mathbf{A}\in\mathcal{A}} C_{\mathbf{A}} \widehat{B}_{\mathbf{A},\mathbf{p}}\right) = 1.$$

To a B-spline set (18), we can associate a parametric domain

$$\widehat{\Omega} = \bigcup_{\mathbf{A}\in\mathcal{A}} \operatorname{supp}(\widehat{B}_{\mathbf{A},\mathbf{p}})$$

Moreover, we give the following extension of the notion of Bézier mesh.

Definition 3 A parametric Bézier mesh in the parametric domain, denoted by $\widehat{\mathcal{M}}$, is the collection of the the maximal open sets $Q \subset \widehat{\Omega}$ such that for all $\mathbf{A} \in \mathcal{A}$, $\widehat{B}_{\mathbf{A},\mathbf{p}}$ is a polynomial in Q; these Q are denoted (Bézier) elements.

Proposition 4 *In a DC set* (18) *there are at most* $(p_1+1)\cdots(p_d+1)$ *B-splines that are non-null in each element* $Q \in \widehat{\mathcal{M}}$.

Proof Given any point $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d) \in \widehat{\Omega}$, denote by $\mathcal{A}(\boldsymbol{\zeta})$ the subset of $\mathbf{A} \in \mathcal{A}$ such that $\widehat{B}_{\mathbf{A},\mathbf{p}}(\boldsymbol{\zeta}) > 0$. It can be easily checked that $\mathcal{A}(\boldsymbol{\zeta})$ only depends on Q, for all $\boldsymbol{\zeta} \in Q$. Recalling (19) and introducing the notation $\Xi_{\mathbf{A},\ell,p_\ell} = \{\xi_{\ell,1}, \ldots, \xi_{\ell,p_\ell+2}\}$, to each $\mathbf{A} \in \mathcal{A}(\boldsymbol{\zeta})$ we can associate a multi-index $(i_{\mathbf{A},1}, \ldots, i_{\mathbf{A},d})$ such that

$$\forall \ell = 1, \ldots, d, \quad 1 \le i_{\mathbf{A},\ell} \le p_\ell + 1 \text{ and } \xi_{\ell,i_{\mathbf{A},\ell}} \le \zeta_\ell < \xi_{\ell,i_{\mathbf{A},\ell}+1}. \tag{25}$$

From the DC assumption, any two $\widehat{B}_{\mathbf{A}',\mathbf{p}}$ and $\widehat{B}_{\mathbf{A}'',\mathbf{p}}$ with $\mathbf{A}' \neq \mathbf{A}''$ partially overlap, that is, there are different and overlapping $\Xi_{\mathbf{A}',\ell,p_\ell}$ and $\Xi_{\mathbf{A}'',\ell,p_\ell}$; then the indices in (25) fulfill

$$\forall \mathbf{A}', \mathbf{A}'' \in \mathcal{A}(\boldsymbol{\zeta}), \quad \mathbf{A}' \neq \mathbf{A}'' \Rightarrow \exists \ell \text{ such that } i_{\mathbf{A}',\ell} \neq i_{\mathbf{A}'',\ell}. \tag{26}$$

The conclusion follows from (26), since by (25) there are at most $(p_1+1)\cdots(p_d+1)$ distinct multi-indices $(i_{\mathbf{A},1}, \ldots, i_{\mathbf{A},d})$.

Assume that each $\lambda_{\mathbf{A},\mathbf{p}}$ is defined on $L^2(\widehat{\Omega})$. An important consequence of Proposition 2 is that we can build a projection operator $\Pi_{\mathbf{p}} : L^2(\widehat{\Omega}) \to S_{\mathbf{p}}(\mathcal{A})$ by

$$\Pi_{\mathbf{p}}(f)(\boldsymbol{\zeta}) = \sum_{\mathbf{A}\in\mathcal{A}} \lambda_{\mathbf{A},\mathbf{p}}(f)\widehat{B}_{\mathbf{A},\mathbf{p}}(\boldsymbol{\zeta}) \qquad \forall f \in L^2(\widehat{\Omega}),\ \forall \boldsymbol{\zeta} \in \widehat{\Omega}. \tag{27}$$

This allows us to prove the approximation properties of $S_{\mathbf{p}}(\mathcal{A})$. The following result will make use of the notion of support extension $\widetilde{Q}$ associated to an element $Q \subset \widehat{\Omega}$ (or a generic open subset $Q \subset \widehat{\Omega}$) and to the B-spline set (18):

$$\widetilde{Q} = \bigcup_{\substack{\mathbf{A}\in\mathcal{A} \\ \mathrm{supp}(\widehat{B}_{\mathbf{A},\mathbf{p}})\cap Q\neq\emptyset}} \mathrm{supp}(\widehat{B}_{\mathbf{A},\mathbf{p}}).$$

Furthermore, we will denote by $\bar{Q}$, the smallest d-dimensional rectangle in $\widehat{\Omega}$ containing $\widetilde{Q}$. Then the following result holds.

Proposition 5 *Let* (18) *be a DC set of B-splines, then the projection operator* $\Pi_{\mathbf{p}}$ *in* (27) *is (locally) h-uniformly* L^2*-continuous, that is, there exists a constant* C *only dependent on* $\mathbf{p}$ *such that*

$$||\Pi_{\mathbf{p}}(f)||_{L^2(Q)} \le C||f||_{L^2(\bar{Q})} \quad \forall Q \subset \widehat{\Omega},\ \forall f \in L^2(\widehat{\Omega}).$$

Proof Let Q be an element in the parametric domain. Since Proposition 4 and since each $B_{\mathbf{A},\mathbf{p}} \leq 1$ we have that, for any $\boldsymbol{\zeta} \in Q$,

$$\sum_{\mathbf{A}\in\mathcal{A}} \left|\widehat{B}_{\mathbf{A},\mathbf{p}}(\boldsymbol{\zeta})\right| \leq C.$$

Therefore, given any point $\boldsymbol{\zeta} \in Q$, denote by $\mathcal{A}(\boldsymbol{\zeta})$ the subset of $\mathbf{A} \in \mathcal{A}$ such that $\widehat{B}_{\mathbf{A},\mathbf{p}}(\boldsymbol{\zeta}) > 0$, and denote by $Q_{\mathbf{A}}$ the common support of $\widehat{B}_{\mathbf{A},\mathbf{p}}$ and $\lambda_{\mathbf{A},\mathbf{p}}$, by $|Q_{\mathbf{A}}|$ its d-dimensional measure, using (11) it follows that

$$\begin{aligned} \left|\Pi_{\mathbf{p}}(f)(\boldsymbol{\zeta})\right|^2 = \Big| \sum_{\mathbf{A}\in\mathcal{A}(\boldsymbol{\zeta})} \lambda_{\mathbf{A},\mathbf{p}}(f)\widehat{B}_{\mathbf{A},\mathbf{p}}(\boldsymbol{\zeta})\Big|^2 &\leq C \max_{\mathbf{A}\in\mathcal{A}(\boldsymbol{\zeta})} \left|\lambda_{\mathbf{A},\mathbf{p}}(f)\right|^2 \\ &\leq C \max_{\mathbf{A}\in\mathcal{A}(\boldsymbol{\zeta})} |Q_{\mathbf{A}}|^{-1}||f||^2_{L^2(Q_{\mathbf{A}})} \\ &\leq C|Q|^{-1}||f||^2_{L^2(\widetilde{Q})}, \end{aligned} \tag{28}$$

where we have used in the last step that $\forall \mathbf{A} \in \mathcal{A}(\boldsymbol{\zeta})$, $Q \subset Q_{\mathbf{A}}$ (and therefore $|Q| \leq |Q_{\mathbf{A}}|$) and that $Q_{\mathbf{A}} \subset \widetilde{Q}$. Since the bound above holds for any $\boldsymbol{\zeta} \in Q$, integrating over Q and applying (28) yields

$$||\Pi_{\mathbf{p}}(f)||^2_{L^2(Q)} \leq C||f||^2_{L^2(\widetilde{Q})}.$$

The continuity of $\Pi_{\mathbf{p}}$ implies the following approximation result in the L^2-norm:

Proposition 6 *Assume that the space of global polynomials of degree* $p = \min_{1\leq\ell\leq d}\{p_\ell\}$ *is included into the space* $S_{\mathbf{p}}(\mathcal{A})$ *and that* $\widehat{\Omega} = [0,1]^d$*. Then there exists a constant C only dependent on* $\mathbf{p}$ *such that for* $0 \leq s \leq p+1$

$$||f - \Pi_{\mathbf{p}}(f)||_{L^2(Q)} \leq C(h_{\bar{Q}})^s|f|_{H^s(\bar{Q})} \quad \forall Q \subset \widehat{\Omega},\ \forall f \in H^s(\widehat{\Omega}),$$

where $h_{\bar{Q}}$ *represents the diameter of* $\bar{Q}$*.*

Proof Let π be any p-degree polynomial. Since $\pi \in S_{\mathbf{p}}(\mathcal{A})$ and $\Pi_{\mathbf{p}}$ is a projection operator, using Proposition 5 it follows that

$$\begin{aligned} ||f - \Pi_{\mathbf{p}}(f)||_{L^2(Q)} &= ||f - \pi + \Pi_{\mathbf{p}}(\pi - f)||_{L^2(Q)} \\ &\leq ||f - \pi||_{L^2(Q)} + ||\Pi_{\mathbf{p}}(\pi - f)||_{L^2(Q)} \\ &\leq (1+C)||f - \pi||_{L^2(\widetilde{Q})} \leq (1+C)||f - \pi||_{L^2(\bar{Q})}. \end{aligned}$$

The result finally follows by a standard polynomial approximation result.

We conclude this section with a final observation: the notion and construction of Greville sites are easily extended to DC sets of B-splines, and the following representation formula holds:

Proposition 7 *Assume that the linear polynomials belong to the space $S_{\mathbf{p}}(\mathcal{A})$. Then we have that*

$$\zeta_\ell = \sum_{\mathbf{A}\in\mathcal{A}} \gamma[\Xi_{\mathbf{A},\ell,p_\ell}]\widehat{B}_{\mathbf{A},\mathbf{p}}(\boldsymbol{\zeta}), \quad \forall\boldsymbol{\zeta} \in \widehat{\Omega},\ 1 \le \ell \le d, \tag{29}$$

where $\gamma[\Xi_{\mathbf{A},\ell,p_\ell}]$ denotes the average of the p_ℓ internal knots of $\Xi_{\mathbf{A},\ell,p_\ell}$.

Proof The identity (29) easily follows from the expansion of $\Pi_{\mathbf{p}}(\zeta_\ell)$ and the definition of dual functionals which is the same as in the tensor product case, yielding $\lambda_{\mathbf{A},\mathbf{p}}(\zeta_\ell) = \gamma[\Xi_{\mathbf{A},\ell,p_\ell}]$.

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