Chapter 1
Topological Vector Spaces

1.1 Introduction

The main objective of this chapter is to present an outline of the basic tools of analysis necessary to develop the subsequent chapters. We assume the reader has a background in linear algebra and elementary real analysis at an undergraduate level. The main references for this chapter are the excellent books on functional analysis: Rudin [58], Bachman and Narici [6], and Reed and Simon [52]. All proofs are developed in details.

1.2 Vector Spaces

We denote by $\mathbb{F}$ a scalar field. In practice this is either $\mathbb{R}$ or $\mathbb{C}$, the set of real or complex numbers.

Definition 1.2.1 (Vector Spaces). A vector space over $\mathbb{F}$ is a set which we will denote by U whose elements are called vectors, for which are defined two operations, namely, addition denoted by $(+) : U \times U \to U$ and scalar multiplication denoted by $(\cdot) : \mathbb{F} \times U \to U$, so that the following relations are valid:

1. $u + v = v + u, \forall u, v \in U$,
2. $u + (v + w) = (u + v) + w, \forall u, v, w \in U$,
3. there exists a vector denoted by θ such that $u + \theta = u$, $\forall u \in U$,
4. for each u $\in U$, there exists a unique vector denoted by $-u$ such that $u + (-u) = \theta$,
5. $\alpha \cdot (\beta \cdot u) = (\alpha \cdot \beta) \cdot u, \forall \alpha, \beta \in \mathbb{F}$, $u \in U$,
6. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v, \forall \alpha \in \mathbb{F}$, $u, v \in U$,
7. $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u, \forall \alpha, \beta \in \mathbb{F}$, $u \in U$,
8. $1 \cdot u = u, \forall u \in U$.

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Remark 1.2.2. From now on we may drop the dot $(\cdot)$ in scalar multiplications and denote $\alpha \cdot u$ simply as αu.

Definition 1.2.3 (Vector Subspace). Let U be a vector space. A set $V \subset U$ is said to be a vector subspace of U if V is also a vector space with the same operations as those of U. If $V \neq U$, we say that V is a proper subspace of U.

Definition 1.2.4 (Finite-Dimensional Space). A vector space is said to be of finite dimension if there exists fixed $u_1, u_2, \ldots, u_n \in U$ such that for each $u \in U$ there are corresponding $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ for which

$$u = \sum_{i=1}^{n} \alpha_i u_i. \tag{1.1}$$

Definition 1.2.5 (Topological Spaces). A set U is said to be a topological space if it is possible to define a collection σ of subsets of U called a topology in U, for which the following properties are valid:

1. $U \in \sigma$,
2. $\emptyset \in \sigma$,
3. if $A \in \sigma$ and $B \in \sigma$, then $A \cap B \in \sigma$,
4. arbitrary unions of elements in σ also belong to σ.

Any $A \in \sigma$ is said to be an open set.

Remark 1.2.6. When necessary, to clarify the notation, we shall denote the vector space U endowed with the topology σ by (U, σ).

Definition 1.2.7 (Closed Sets). Let U be a topological space. A set $A \subset U$ is said to be closed if $U \setminus A$ is open. We also denote $U \setminus A = A^c = \{u \in U \mid u \notin A\}$.

Remark 1.2.8. For any sets $A, B \subset U$ we denote

$$A \setminus B = \{u \in A \mid u \notin B\}.$$

Also, when the meaning is clear we may denote $A \setminus B$ by $A - B$.

Proposition 1.2.9. *For closed sets we have the following properties:*

1. *U and $\emptyset$ are closed,*
2. *if A and B are closed sets, then $A \cup B$ is closed,*
3. *arbitrary intersections of closed sets are closed.*

Proof.

1. Since $\emptyset$ is open and $U = \emptyset^c$, by Definition 1.2.7, U is closed. Similarly, since U is open and $\emptyset = U \setminus U = U^c$, $\emptyset$ is closed.
2. A, B closed implies that A^c and B^c are open, and by Definition 1.2.5, $A^c \cup B^c$ is open, so that $A \cap B = (A^c \cup B^c)^c$ is closed.

3. Consider $A = \cap_{\lambda \in L} A_\lambda$, where L is a collection of indices and A_λ is closed, $\forall \lambda \in L$. We may write $A = (\cup_{\lambda \in L} A_\lambda^c)^c$ and since A_λ^c is open $\forall \lambda \in L$ we have, by Definition 1.2.5, that A is closed.

Definition 1.2.10 (Closure). Given $A \subset U$ we define the closure of A, denoted by $\bar{A}$, as the intersection of all closed sets that contain A.

Remark 1.2.11. From Proposition 1.2.9 item 3 we have that $\bar{A}$ is the smallest closed set that contains A, in the sense that if C is closed and $A \subset C$, then $\bar{A} \subset C$.

Definition 1.2.12 (Interior). Given $A \subset U$ we define its interior, denoted by A°, as the union of all open sets contained in A.

Remark 1.2.13. It is not difficult to prove that if A is open, then $A = A^\circ$.

Definition 1.2.14 (Neighborhood). Given $u_0 \in U$ we say that $\mathscr{V}$ is a neighborhood of u_0 if such a set is open and contains u_0. We denote such neighborhoods by $\mathscr{V}_{u_0}$.

Proposition 1.2.15. *If $A \subset U$ is a set such that for each $u \in A$ there exists a neighborhood $\mathscr{V}_u \ni u$ such that $\mathscr{V}_u \subset A$, then A is open.*

Proof. This follows from the fact that $A = \cup_{u \in A} \mathscr{V}_u$ and any arbitrary union of open sets is open.

Definition 1.2.16 (Function). Let U and V be two topological spaces. We say that $f : U \to V$ is a function if f is a collection of pairs $(u, v) \in U \times V$ such that for each $u \in U$ there exists only one $v \in V$ such that $(u, v) \in f$.

Definition 1.2.17 (Continuity at a Point). A function $f : U \to V$ is continuous at $u \in U$ if for each neighborhood $\mathscr{V}_{f(u)} \subset V$ of $f(u)$, there exists a neighborhood $\mathscr{V}_u \subset U$ of u such that $f(\mathscr{V}_u) \subset \mathscr{V}_{f(u)}$.

Definition 1.2.18 (Continuous Function). A function $f : U \to V$ is continuous if it is continuous at each $u \in U$.

Proposition 1.2.19. *A function $f : U \to V$ is continuous if and only if $f^{-1}(\mathscr{V})$ is open for each open $\mathscr{V} \subset V$, where*

$$f^{-1}(\mathscr{V}) = \{u \in U \mid f(u) \in \mathscr{V}\}. \tag{1.2}$$

Proof. Suppose $f^{-1}(\mathscr{V})$ is open whenever $\mathscr{V} \subset V$ is open. Pick $u \in U$ and any open $\mathscr{V}$ such that $f(u) \in \mathscr{V}$. Since $u \in f^{-1}(\mathscr{V})$ and $f(f^{-1}(\mathscr{V})) \subset \mathscr{V}$, we have that f is continuous at $u \in U$. Since $u \in U$ is arbitrary we have that f is continuous. Conversely, suppose f is continuous and pick $\mathscr{V} \subset V$ open. If $f^{-1}(\mathscr{V}) = \emptyset$, we are done, since $\emptyset$ is open. Thus, suppose $u \in f^{-1}(\mathscr{V})$, since f is continuous, there exists $\mathscr{V}_u$ a neighborhood of u such that $f(\mathscr{V}_u) \subset \mathscr{V}$. This means $\mathscr{V}_u \subset f^{-1}(\mathscr{V})$ and therefore, from Proposition 1.2.15, $f^{-1}(\mathscr{V})$ is open.

Definition 1.2.20. We say that (U,σ) is a Hausdorff topological space if, given u_1, $u_2 \in U$, $u_1 \neq u_2$, there exists $\mathscr{V}_1$, $\mathscr{V}_2 \in \sigma$ such that

$$u_1 \in \mathscr{V}_1\ ,\ u_2 \in \mathscr{V}_2 \text{ and } \mathscr{V}_1 \cap \mathscr{V}_2 = \emptyset. \tag{1.3}$$

Definition 1.2.21 (Base). A collection $\sigma' \subset \sigma$ is said to be a base for σ if every element of σ may be represented as a union of elements of σ'.

Definition 1.2.22 (Local Base). A collection $\hat{\sigma}$ of neighborhoods of a point $u \in U$ is said to be a local base at u if each neighborhood of u contains a member of $\hat{\sigma}$.

Definition 1.2.23 (Topological Vector Space). A vector space endowed with a topology, denoted by (U,σ), is said to be a topological vector space if and only if

1. every single point of U is a closed set,
2. the vector space operations (addition and scalar multiplication) are continuous with respect to σ.

More specifically, addition is continuous if given $u,v \in U$ and $\mathscr{V} \in \sigma$ such that $u+v \in \mathscr{V}$, then there exists $\mathscr{V}_u \ni u$ and $\mathscr{V}_v \ni v$ such that $\mathscr{V}_u + \mathscr{V}_v \subset \mathscr{V}$. On the other hand, scalar multiplication is continuous if given $\alpha \in \mathbb{F}$, $u \in U$ and $\mathscr{V} \ni \alpha \cdot u$, there exists $\delta > 0$ and $\mathscr{V}_u \ni u$ such that $\forall \beta \in \mathbb{F}$ satisfying $|\beta - \alpha| < \delta$ we have $\beta \mathscr{V}_u \subset \mathscr{V}$.

Given (U,σ), let us associate with each $u_0 \in U$ and $\alpha_0 \in \mathbb{F}$ ($\alpha_0 \neq 0$) the functions $T_{u_0} : U \to U$ and $M_{\alpha_0} : U \to U$ defined by

$$T_{u_0}(u) = u_0 + u \tag{1.4}$$

and

$$M_{\alpha_0}(u) = \alpha_0 \cdot u. \tag{1.5}$$

The continuity of such functions is a straightforward consequence of the continuity of vector space operations (addition and scalar multiplication). It is clear that the respective inverse maps, namely T_{-u_0} and M_{1/α_0}, are also continuous. So if $\mathscr{V}$ is open, then $u_0 + \mathscr{V}$, that is, $(T_{-u_0})^{-1}(\mathscr{V}) = T_{u_0}(\mathscr{V}) = u_0 + \mathscr{V}$ is open. By analogy $\alpha_0 \mathscr{V}$ is open. Thus σ is completely determined by a local base, so that the term local base will be understood henceforth as a local base at θ. So to summarize, a local base of a topological vector space is a collection Ω of neighborhoods of θ, such that each neighborhood of θ contains a member of Ω.

Now we present some simple results.

Proposition 1.2.24. *If $A \subset U$ is open, then $\forall u \in A$, there exists a neighborhood $\mathscr{V}$ of θ such that $u + \mathscr{V} \subset A$.*

Proof. Just take $\mathscr{V} = A - u$.

Proposition 1.2.25. *Given a topological vector space (U,σ), any element of σ may be expressed as a union of translates of members of Ω, so that the local base Ω generates the topology σ.*

Proof. Let $A \subset U$ open and $u \in A$. $\mathscr{V} = A - u$ is a neighborhood of θ and by definition of local base, there exists a set $\mathscr{V}_{\Omega u} \subset \mathscr{V}$ such that $\mathscr{V}_{\Omega_u} \in \Omega$. Thus, we may write

$$A = \cup_{u \in A}(u + \mathscr{V}_{\Omega_u}). \tag{1.6}$$

1.3 Some Properties of Topological Vector Spaces

In this section we study some fundamental properties of topological vector spaces. We start with the following proposition.

Proposition 1.3.1. *Any topological vector space U is a Hausdorff space.*

Proof. Pick $u_0, u_1 \in U$ such that $u_0 \neq u_1$. Thus $\mathscr{V} = U \setminus \{u_1 - u_0\}$ is an open neighborhood of zero. As $\theta + \theta = \theta$, by the continuity of addition, there exist $\mathscr{V}_1$ and $\mathscr{V}_2$ neighborhoods of θ such that

$$\mathscr{V}_1 + \mathscr{V}_2 \subset \mathscr{V} \tag{1.7}$$

define $\mathscr{U} = \mathscr{V}_1 \cap \mathscr{V}_2 \cap (-\mathscr{V}_1) \cap (-\mathscr{V}_2)$, thus $\mathscr{U} = -\mathscr{U}$ (symmetric) and $\mathscr{U} + \mathscr{U} \subset \mathscr{V}$ and hence

$$u_0 + \mathscr{U} + \mathscr{U} \subset u_0 + \mathscr{V} \subset U \setminus \{u_1\} \tag{1.8}$$

so that

$$u_0 + v_1 + v_2 \neq u_1, \quad \forall v_1, v_2 \in \mathscr{U}, \tag{1.9}$$

or

$$u_0 + v_1 \neq u_1 - v_2, \quad \forall v_1, v_2 \in \mathscr{U}, \tag{1.10}$$

and since $\mathscr{U} = -\mathscr{U}$

$$(u_0 + \mathscr{U}) \cap (u_1 + \mathscr{U}) = \emptyset. \tag{1.11}$$

Definition 1.3.2 (Bounded Sets). A set $A \subset U$ is said to be bounded if to each neighborhood of zero $\mathscr{V}$ there corresponds a number $s > 0$ such that $A \subset t\mathscr{V}$ for each $t > s$.

Definition 1.3.3 (Convex Sets). A set $A \subset U$ such that

$$\text{if } u, v \in A \text{ then } \lambda u + (1 - \lambda)v \in A, \quad \forall \lambda \in [0, 1], \tag{1.12}$$

is said to be convex.

Definition 1.3.4 (Locally Convex Spaces). A topological vector space U is said to be locally convex if there is a local base Ω whose elements are convex.

Definition 1.3.5 (Balanced Sets). A set $A \subset U$ is said to be balanced if $\alpha A \subset A$, $\forall \alpha \in \mathbb{F}$ such that $|\alpha| \leq 1$.

Theorem 1.3.6. *In a topological vector space U we have:*

1. *every neighborhood of zero contains a balanced neighborhood of zero,*
2. *every convex neighborhood of zero contains a balanced convex neighborhood of zero.*

Proof.

1. Suppose $\mathscr{U}$ is a neighborhood of zero. From the continuity of scalar multiplication, there exist $\mathscr{V}$ (neighborhood of zero) and $\delta > 0$, such that $\alpha\mathscr{V} \subset \mathscr{U}$ whenever $|\alpha| < \delta$. Define $\mathscr{W} = \cup_{|\alpha|<\delta}\alpha\mathscr{V}$; thus $\mathscr{W} \subset \mathscr{U}$ is a balanced neighborhood of zero.
2. Suppose $\mathscr{U}$ is a convex neighborhood of zero in U. Define

$$A = \{\cap \alpha\mathscr{U} \mid \alpha \in \mathbb{C},\ |\alpha| = 1\}. \tag{1.13}$$

As $0 \cdot \theta = \theta$ (where $\theta \in U$ denotes the zero vector) from the continuity of scalar multiplication there exists $\delta > 0$ and there is a neighborhood of zero $\mathscr{V}$ such that if $|\beta| < \delta$, then $\beta\mathscr{V} \subset \mathscr{U}$. Define $\mathscr{W}$ as the union of all such $\beta\mathscr{V}$. Thus $\mathscr{W}$ is balanced and $\alpha^{-1}\mathscr{W} = \mathscr{W}$ as $|\alpha| = 1$, so that $\mathscr{W} = \alpha\mathscr{W} \subset \alpha\mathscr{U}$, and hence $\mathscr{W} \subset A$, which implies that the interior A° is a neighborhood of zero. Also $A^\circ \subset \mathscr{U}$. Since A is an intersection of convex sets, it is convex and so is A°. Now we will show that A° is balanced and complete the proof. For this, it suffices to prove that A is balanced. Choose r and β such that $0 \leq r \leq 1$ and $|\beta| = 1$. Then

$$r\beta A = \cap_{|\alpha|=1} r\beta\alpha\mathscr{U} = \cap_{|\alpha|=1} r\alpha\mathscr{U}. \tag{1.14}$$

Since $\alpha\mathscr{U}$ is a convex set that contains zero, we obtain $r\alpha\mathscr{U} \subset \alpha\mathscr{U}$, so that $r\beta A \subset A$, which completes the proof.

Proposition 1.3.7. *Let U be a topological vector space and $\mathscr{V}$ a neighborhood of zero in U. Given $u \in U$, there exists $r \in \mathbb{R}^+$ such that $\beta u \in \mathscr{V}$, $\forall \beta$ such that $|\beta| < r$.*

Proof. Observe that $u + \mathscr{V}$ is a neighborhood of $1 \cdot u$, and then by the continuity of scalar multiplication, there exists $\mathscr{W}$ neighborhood of u and $r > 0$ such that

$$\beta\mathscr{W} \subset u + \mathscr{V}, \forall \beta \text{ such that } |\beta - 1| < r, \tag{1.15}$$

so that

$$\beta u \in u + \mathscr{V}, \tag{1.16}$$

or

$$(\beta - 1)u \in \mathscr{V}, \text{ where } |\beta - 1| < r, \tag{1.17}$$

and thus

$$\hat{\beta}u \in \mathscr{V}, \forall \hat{\beta} \text{ such that } |\hat{\beta}| < r, \tag{1.18}$$

which completes the proof.

Corollary 1.3.8. *Let $\mathscr{V}$ be a neighborhood of zero in U; if $\{r_n\}$ is a sequence such that $r_n > 0$, $\forall n \in \mathbb{N}$, and $\lim_{n\to\infty} r_n = \infty$, then $U \subset \cup_{n=1}^{\infty} r_n \mathscr{V}$.*

Proof. Let $u \in U$, then $\alpha u \in \mathscr{V}$ for any α sufficiently small, from the last proposition $u \in \frac{1}{\alpha}\mathscr{V}$. As $r_n \to \infty$ we have that $r_n > \frac{1}{\alpha}$ for n sufficiently big, so that $u \in r_n\mathscr{V}$, which completes the proof.

Proposition 1.3.9. *Suppose $\{\delta_n\}$ is a sequence such that $\delta_n \to 0$, $\delta_n < \delta_{n-1}$, $\forall n \in \mathbb{N}$ and $\mathscr{V}$ a bounded neighborhood of zero in U, then $\{\delta_n \mathscr{V}\}$ is a local base for U.*

Proof. Let $\mathscr{U}$ be a neighborhood of zero; as $\mathscr{V}$ is bounded, there exists $t_0 \in \mathbb{R}^+$ such that $\mathscr{V} \subset t\mathscr{U}$ for any $t > t_0$. As $\lim_{n\to\infty} \delta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $\delta_n < \frac{1}{t_0}$, so that $\delta_n \mathscr{V} \subset \mathscr{U}, \forall n$ such that $n \geq n_0$.

Definition 1.3.10 (Convergence in Topological Vector Spaces). Let U be a topological vector space. We say $\{u_n\}$ converges to $u_0 \in U$, if for each neighborhood $\mathscr{V}$ of u_0, then there exists $N \in \mathbb{N}$ such that

$$u_n \in \mathscr{V}, \forall n \geq N.$$

1.4 Compactness in Topological Vector Spaces

We start this section with the definition of open covering.

Definition 1.4.1 (Open Covering). Given $B \subset U$ we say that $\{\mathscr{O}_\alpha, \;\; \alpha \in A\}$ is a covering of B if $B \subset \cup_{\alpha \in A} \mathscr{O}_\alpha$. If $\mathscr{O}_\alpha$ is open $\forall \alpha \in A$, then $\{\mathscr{O}_\alpha\}$ is said to be an open covering of B.

Definition 1.4.2 (Compact Sets). A set $B \subset U$ is said to be compact if each open covering of B has a finite subcovering. More explicitly, if $B \subset \cup_{\alpha \in A} \mathscr{O}_\alpha$, where $\mathscr{O}_\alpha$ is open $\forall \alpha \in A$, then there exist $\alpha_1, \ldots, \alpha_n \in A$ such that $B \subset \mathscr{O}_{\alpha_1} \cup \ldots \cup \mathscr{O}_{\alpha_n}$, for some n, a finite positive integer.

Proposition 1.4.3. *A compact subset of a Hausdorff space is closed.*

Proof. Let U be a Hausdorff space and consider $A \subset U$, A compact. Given $x \in A$ and $y \in A^c$, there exist open sets $\mathscr{O}_x$ and $\mathscr{O}_y^x$ such that $x \in \mathscr{O}_x$, $y \in \mathscr{O}_y^x$, and $\mathscr{O}_x \cap \mathscr{O}_y^x = \emptyset$. It is clear that $A \subset \cup_{x \in A} \mathscr{O}_x$, and since A is compact, we may find $\{x_1, x_2, \ldots, x_n\}$ such that $A \subset \cup_{i=1}^n \mathscr{O}_{x_i}$. For the selected $y \in A^c$ we have $y \in \cap_{i=1}^n \mathscr{O}_y^{x_i}$ and $(\cap_{i=1}^n \mathscr{O}_y^{x_i}) \cap (\cup_{i=1}^n \mathscr{O}_{x_i}) = \emptyset$. Since $\cap_{i=1}^n \mathscr{O}_y^{x_i}$ is open and y is an arbitrary point of A^c we have that A^c is open, so that A is closed, which completes the proof.

The next result is very useful.

Theorem 1.4.4. *Let $\{K_\alpha,\ \alpha \in L\}$ be a collection of compact subsets of a Hausdorff topological vector space U, such that the intersection of every finite subcollection (of $\{K_\alpha,\ \alpha \in L\}$) is nonempty.*

Under such hypotheses

$$\cap_{\alpha \in L} K_\alpha \neq \emptyset.$$

Proof. Fix $\alpha_0 \in L$. Suppose, to obtain contradiction, that

$$\cap_{\alpha \in L} K_\alpha = \emptyset.$$

That is,

$$K_{\alpha_0} \cap [\cap_{\alpha \in L}^{\alpha \neq \alpha_0} K_\alpha] = \emptyset.$$

Thus,

$$\cap_{\alpha \in L}^{\alpha \neq \alpha_0} K_\alpha \subset K_{\alpha_0}^c,$$

so that

$$K_{\alpha_0} \subset [\cap_{\alpha \in L}^{\alpha \neq \alpha_0} K_\alpha]^c,$$

$$K_{\alpha_0} \subset [\cup_{\alpha \in L}^{\alpha \neq \alpha_0} K_\alpha^c].$$

However, K_{α_0} is compact and K_α^c is open, $\forall \alpha \in L$.

Hence, there exist $\alpha_1, \ldots, \alpha_n \in L$ such that

$$K_{\alpha_0} \subset \cup_{i=1}^n K_{\alpha_i}^c.$$

From this we may infer that

$$K_{\alpha_0} \cap [\cap_{i=1}^n K_{\alpha_i}] = \emptyset,$$

which contradicts the hypotheses.

The proof is complete.

Proposition 1.4.5. *A closed subset of a compact space U is compact.*

Proof. Consider $\{\mathcal{O}_\alpha, \alpha \in L\}$ an open cover of A. Thus $\{A^c,\ \mathcal{O}_\alpha,\ \alpha \in L\}$ is a cover of U. As U is compact, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $A^c \cup (\cup_{i=1}^n \mathcal{O}_{\alpha_i}) \supset U$, so that $\{\mathcal{O}_{\alpha_i},\ i \in \{1, \ldots, n\}\}$ covers A, so that A is compact. The proof is complete.

Definition 1.4.6 (Countably Compact Sets). A set A is said to be countably compact if every infinite subset of A has a limit point in A.

Proposition 1.4.7. *Every compact subset of a topological space U is countably compact.*

Proof. Let B an infinite subset of A compact and suppose B has no limit point. Choose $\{x_1, x_2, \ldots.\} \subset B$ and define $F = \{x_1, x_2, x_3, \ldots\}$. It is clear that F has no limit point. Thus, for each $n \in \mathbb{N}$, there exist $\mathcal{O}_n$ open such that $\mathcal{O}_n \cap F = \{x_n\}$. Also, for each $x \in A - F$, there exist $\mathcal{O}_x$ such that $x \in \mathcal{O}_x$ and $\mathcal{O}_x \cap F = \emptyset$. Thus $\{\mathcal{O}_x,\ x \in A - F;\ \mathcal{O}_1, \mathcal{O}_2, \ldots\}$ is an open cover of A without a finite subcover, which contradicts the fact that A is compact.

1.5 Normed and Metric Spaces

The idea here is to prepare a route for the study of Banach spaces defined below. We start with the definition of norm.

Definition 1.5.1 (Norm). A vector space U is said to be a normed space, if it is possible to define a function $\|\cdot\|_U : U \to \mathbb{R}^+ = [0, +\infty)$, called a norm, which satisfies the following properties:

1. $\|u\|_U > 0$, if $u \neq \theta$ and $\|u\|_U = 0 \Leftrightarrow u = \theta$,
2. $\|u+v\|_U \leq \|u\|_U + \|v\|_U, \forall\ u, v \in U$,
3. $\|\alpha u\|_U = |\alpha|\|u\|_U, \forall u \in U, \alpha \in \mathbb{F}$.

Now we present the definition of metric.

Definition 1.5.2 (Metric Space). A vector space U is said to be a metric space if it is possible to define a function $d : U \times U \to \mathbb{R}^+$, called a metric on U, such that

1. $0 \leq d(u,v),\ \ \forall u, v \in U$,
2. $d(u,v) = 0 \Leftrightarrow u = v$,
3. $d(u,v) = d(v,u),\ \ \forall u, v \in U$,
4. $d(u,w) \leq d(u,v) + d(v,w), \forall u, v, w \in U$.

A metric can be defined through a norm, that is,

$$d(u,v) = \|u - v\|_U. \tag{1.19}$$

In this case we say that the metric is induced by the norm.

The set $B_r(u) = \{v \in U \mid d(u,v) < r\}$ is called the open ball with center at u and radius r. A metric $d : U \times U \to \mathbb{R}^+$ is said to be invariant if

$$d(u+w, v+w) = d(u,v), \forall u, v, w \in U. \tag{1.20}$$

The following are some basic definitions concerning metric and normed spaces:

Definition 1.5.3 (Convergent Sequences). Given a metric space U, we say that $\{u_n\} \subset U$ converges to $u_0 \in U$ as $n \to \infty$, if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that if $n \geq n_0$, then $d(u_n, u_0) < \varepsilon$. In this case we write $u_n \to u_0$ as $n \to +\infty$.

Definition 1.5.4 (Cauchy Sequence). $\{u_n\} \subset U$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(u_n, u_m) < \varepsilon, \forall m, n \geq n_0$

Definition 1.5.5 (Completeness). A metric space U is said to be complete if each Cauchy sequence related to $d : U \times U \to \mathbb{R}^+$ converges to an element of U.

Definition 1.5.6 (Limit Point). Let (U, d) be a metric space and let $E \subset U$. We say that $v \in U$ is a limit point of E if for each $r > 0$ there exists $w \in B_r(v) \cap E$ such that $w \neq v$.

Definition 1.5.7 (Interior Point, Topology for (U,d)). Let (U,d) be a metric space and let $E \subset U$. We say that $u \in E$ is interior point if there exists $r > 0$ such that $B_r(u) \subset E$. We may define a topology for a metric space (U,d) by declaring as open all set $E \subset U$ such that all its points are interior. Such a topology is said to be induced by the metric d.

Definition 1.5.8. Let (U,d) be a metric space. The set σ of all open sets, defined through the last definition, is indeed a topology for (U,d).

Proof.

1. Obviously $\emptyset$ and U are open sets.
2. Assume A and B are open sets and define $C = A \cap B$. Let $u \in C = A \cap B$; thus, from $u \in A$, there exists $r_1 > 0$ such that $B_{r_1}(u) \subset A$. Similarly from $u \in B$ there exists $r_2 > 0$ such that $B_{r_2}(u) \subset B$.
   Define $r = \min\{r_1, r_2\}$. Thus, $B_r(u) \subset A \cap B = C$, so that u is an interior point of C. Since $u \in C$ is arbitrary, we may conclude that C is open.
3. Suppose $\{A_\alpha,\ \alpha \in L\}$ is a collection of open sets. Define $E = \cup_{\alpha \in L} A_\alpha$, and we shall show that E is open.
   Choose $u \in E = \cup_{\alpha \in L} A_\alpha$. Thus there exists $\alpha_0 \in L$ such that $u \in A_{\alpha_0}$. Since A_{α_0} is open there exists $r > 0$ such that $B_r(u) \subset A_{\alpha_0} \subset \cup_{\alpha \in L} A_\alpha = E$. Hence u is an interior point of E, since $u \in E$ is arbitrary, we may conclude that $E = \cup_{\alpha \in L} A_\alpha$ is open.

The proof is complete.

Definition 1.5.9. Let (U,d) be a metric space and let $E \subset U$. We define E' as the set of all the limit points of E.

Theorem 1.5.10. *Let (U,d) be a metric space and let $E \subset U$. Then E is closed if and only if $E' \subset E$.*

Proof. Suppose $E' \subset E$. Let $u \in E^c$; thus $u \notin E$ and $u \notin E'$. Therefore there exists $r > 0$ such that $B_r(u) \cap E = \emptyset$, so that $B_r(u) \subset E^c$. Therefore u is an interior point of E^c. Since $u \in E^c$ is arbitrary, we may infer that E^c is open, so that $E = (E^c)^c$ is closed.

Conversely, suppose that E is closed, that is, E^c is open.

If $E' = \emptyset$, we are done.

Thus assume $E' \neq \emptyset$ and choose $u \in E'$. Thus, for each $r > 0$, there exists $v \in B_r(u) \cap E$ such that $v \neq u$. Thus $B_r(u) \not\subset E^c, \forall r > 0$ so that u is not a interior point of E^c. Since E^c is open, we have that $u \notin E^c$ so that $u \in E$. We have thus obtained, $u \in E, \forall u \in E'$, so that $E' \subset E$.

The proof is complete.

Remark 1.5.11. From this last result, we may conclude that in a metric space, $E \subset U$ is closed if and only if $E' \subset E$.

Definition 1.5.12 (Banach Spaces). A normed vector space U is said to be a Banach space if each Cauchy sequence related to the metric induced by the norm converges to an element of U.

Remark 1.5.13. We say that a topology σ is compatible with a metric d if any $A \subset \sigma$ is represented by unions and/or finite intersections of open balls. In this case we say that $d : U \times U \to \mathbb{R}^+$ induces the topology σ.

Definition 1.5.14 (Metrizable Spaces). A topological vector space (U, σ) is said to be metrizable if σ is compatible with some metric d.

Definition 1.5.15 (Normable Spaces). A topological vector space (U, σ) is said to be normable if the induced metric (by this norm) is compatible with σ.

1.6 Compactness in Metric Spaces

Definition 1.6.1 (Diameter of a Set). Let (U,d) be a metric space and $A \subset U$. We define the diameter of A, denoted by $\text{diam}(A)$ by

$$\text{diam}(A) = \sup\{d(u,v) \mid u,v \in A\}.$$

Definition 1.6.2. Let (U,d) be a metric space. We say that $\{F_k\} \subset U$ is a nested sequence of sets if

$$F_1 \supset F_2 \supset F_3 \supset \dots.$$

Theorem 1.6.3. *If (U,d) is a complete metric space, then every nested sequence of nonempty closed sets $\{F_k\}$ such that*

$$\lim_{k \to +\infty} diam(F_k) = 0$$

has nonempty intersection, that is,

$$\cap_{k=1}^{\infty} F_k \neq \emptyset.$$

Proof. Suppose $\{F_k\}$ is a nested sequence and $\lim_{k \to \infty} \text{diam}(F_k) = 0$. For each $n \in \mathbb{N}$, select $u_n \in F_n$. Suppose given $\varepsilon > 0$. Since

$$\lim_{n \to \infty} \text{diam}(F_n) = 0,$$

there exists $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\text{diam}(F_n) < \varepsilon.$$

Thus if $m, n > N$ we have $u_m, u_n \in F_N$ so that

$$d(u_n, u_m) < \varepsilon.$$

Hence $\{u_n\}$ is a Cauchy sequence. Being U complete, there exists $u \in U$ such that

$$u_n \to u \text{ as } n \to \infty.$$

Choose $m \in \mathbb{N}$. We have that $u_n \in F_m, \forall n > m$, so that

$$u \in \bar{F}_m = F_m.$$

Since $m \in \mathbb{N}$ is arbitrary we obtain

$$u \in \cap_{m=1}^{\infty} F_m.$$

The proof is complete.

Theorem 1.6.4. *Let (U,d) be a metric space. If $A \subset U$ is compact, then it is closed and bounded.*

Proof. We have already proved that A is closed. Suppose, to obtain contradiction, that A is not bounded. Thus for each $K \in \mathbb{N}$ there exists $u, v \in A$ such that

$$d(u,v) > K.$$

Observe that

$$A \subset \cup_{u \in A} B_1(u).$$

Since A is compact there exists $u_1, u_2, \ldots, u_n \in A$ such that

$$A = \subset \cup_{k=1}^{n} B_1(u_k).$$

Define

$$R = \max\{d(u_i, u_j) \mid i, j \in \{1, \ldots, n\}\}.$$

Choose $u, v \in A$ such that

$$d(u,v) > R + 2. \tag{1.21}$$

Observe that there exist $i, j \in \{1, \ldots, n\}$ such that

$$u \in B_1(u_i), \; v \in B_1(u_j).$$

Thus

$$\begin{aligned} d(u,v) &\leq d(u,u_i) + d(u_i,u_j) + d(u_j,v) \\ &\leq 2 + R, \end{aligned} \tag{1.22}$$

which contradicts (1.21). This completes the proof.

Definition 1.6.5 (Relative Compactness). In a metric space (U,d), a set $A \subset U$ is said to be relatively compact if $\overline{A}$ is compact.

Definition 1.6.6 (ε-Nets). Let (U,d) be a metric space. A set $N \subset U$ is sat to be a ε-net with respect to a set $A \subset U$ if for each $u \in A$ there exists $v \in N$ such that

$$d(u,v) < \varepsilon.$$

Definition 1.6.7. Let (U,d) be a metric space. A set $A \subset U$ is said to be totally bounded if for each $\varepsilon > 0$, there exists a finite ε-net with respect to A.

Proposition 1.6.8. *Let (U,d) be a metric space. If $A \subset U$ is totally bounded, then it is bounded.*

Proof. Choose $u, v \in A$. Let $\{u_1, \ldots, u_n\}$ be the 1-net with respect to A. Define

$$R = \max\{d(u_i, u_j) \mid i, j \in \{1, \ldots, n\}\}.$$

Observe that there exist $i, j \in \{1, \ldots, n\}$ such that

$$d(u, u_i) < 1, \; d(v, u_j) < 1.$$

Thus

$$\begin{aligned} d(u,v) &\leq d(u,u_i) + d(u_i,u_j) + d(u_j,v) \\ &\leq R + 2. \end{aligned} \tag{1.23}$$

Since $u, v \in A$ are arbitrary, A is bounded.

Theorem 1.6.9. *Let (U,d) be a metric space. If from each sequence $\{u_n\} \subset A$ we can select a convergent subsequence $\{u_{n_k}\}$, then A is totally bounded.*

Proof. Suppose, to obtain contradiction, that A is not totally bounded. Thus there exists $\varepsilon_0 > 0$ such that there exists no ε_0-net with respect to A. Choose $u_1 \in A$; hence $\{u_1\}$ is not a ε_0-net, that is, there exists $u_2 \in A$ such that

$$d(u_1, u_2) > \varepsilon_0.$$

Again $\{u_1, u_2\}$ is not a ε_0-net for A, so that there exists $u_3 \in A$ such that

$$d(u_1, u_3) > \varepsilon_0 \text{ and } d(u_2, u_3) > \varepsilon_0.$$

Proceeding in this fashion we can obtain a sequence $\{u_n\}$ such that

$$d(u_n, u_m) > \varepsilon_0, \text{ if } m \neq n. \tag{1.24}$$

Clearly we cannot extract a convergent subsequence of $\{u_n\}$; otherwise such a subsequence would be Cauchy contradicting (1.24). The proof is complete.

Definition 1.6.10 (Sequentially Compact Sets). Let (U,d) be a metric space. A set $A \subset U$ is said to be sequentially compact if for each sequence $\{u_n\} \subset A$, there exist a subsequence $\{u_{n_k}\}$ and $u \in A$ such that

$$u_{n_k} \to u, \text{ as } k \to \infty.$$

Theorem 1.6.11. *A subset A of a metric space (U,d) is compact if and only if it is sequentially compact.*

Proof. Suppose A is compact. By Proposition 1.4.7 A is countably compact. Let $\{u_n\} \subset A$ be a sequence. We have two situations to consider:

1. $\{u_n\}$ has infinitely many equal terms, that is, in this case we have

$$u_{n_1} = u_{n_2} = \ldots = u_{n_k} = \ldots = u \in A.$$

   Thus the result follows trivially.
2. $\{u_n\}$ has infinitely many distinct terms. In such a case, being A countably compact, $\{u_n\}$ has a limit point in A, so that there exist a subsequence $\{u_{n_k}\}$ and $u \in A$ such that

$$u_{n_k} \to u, \text{ as } k \to \infty.$$

In both cases we may find a subsequence converging to some $u \in A$.

Thus A is sequentially compact.

Conversely suppose A is sequentially compact, and suppose $\{G_\alpha,\ \alpha \in L\}$ is an open cover of A. For each $u \in A$ define

$$\delta(u) = \sup\{r \mid B_r(u) \subset G_\alpha, \text{ for some } \alpha \in L\}.$$

First we prove that $\delta(u) > 0, \forall u \in A$. Choose $u \in A$. Since $A \subset \cup_{\alpha \in L} G_\alpha$, there exists $\alpha_0 \in L$ such that $u \in G_{\alpha_0}$. Being G_{α_0} open, there exists $r_0 > 0$ such that $B_{r_0}(u) \subset G_{\alpha_0}$.

Thus,

$$\delta(u) \geq r_0 > 0.$$

Now define δ_0 by

$$\delta_0 = \inf\{\delta(u) \mid u \in A\}.$$

Therefore, there exists a sequence $\{u_n\} \subset A$ such that

$$\delta(u_n) \to \delta_0 \text{ as } n \to \infty.$$

Since A is sequentially compact, we may obtain a subsequence $\{u_{n_k}\}$ and $u_0 \in A$ such that

$$\delta(u_{n_k}) \to \delta_0 \text{ and } u_{n_k} \to u_0,$$

as $k \to \infty$. Therefore, we may find $K_0 \in \mathbb{N}$ such that if $k > K_0$, then

$$d(u_{n_k}, u_0) < \frac{\delta(u_0)}{4}. \tag{1.25}$$

We claim that

$$\delta(u_{n_k}) \geq \frac{\delta(u_0)}{4}, \text{ if } k > K_0.$$

To prove the claim, suppose

$$z \in B_{\frac{\delta(u_0)}{4}}(u_{n_k}), \forall k > K_0,$$

(observe that in particular from (1.25)

$$u_0 \in B_{\frac{\delta(u_0)}{4}}(u_{n_k}), \forall k > K_0).$$

Since

$$\frac{\delta(u_0)}{2} < \delta(u_0),$$

there exists some $\alpha_1 \in L$ such that

$$B_{\frac{\delta(u_0)}{2}}(u_0) \subset G_{\alpha_1}.$$

However, since

$$d(u_{n_k}, u_0) < \frac{\delta(u_0)}{4}, \text{ if } k > K_0,$$

we obtain

$$B_{\frac{\delta(u_0)}{2}}(u_0) \supset B_{\frac{\delta(u_0)}{4}}(u_{n_k}), \text{ if } k > K_0,$$

so that

$$\delta(u_{n_k}) \geq \frac{\delta(u_0)}{4}, \forall k > K_0.$$

Therefore

$$\lim_{k\to\infty} \delta(u_{n_k}) = \delta_0 \geq \frac{\delta(u_0)}{4}.$$

Choose $\varepsilon > 0$ such that

$$\delta_0 > \varepsilon > 0.$$

From the last theorem since A is sequentially compact, it is totally bounded. For the $\varepsilon > 0$ chosen above, consider an ε-net contained in A (the fact that the ε-net may be chosen contained in A is also a consequence of the last theorem) and denote it by N that is,

$$N = \{v_1, \ldots, v_n\} \in A.$$

Since $\delta_0 > \varepsilon$, there exists

$$\alpha_1, \ldots, \alpha_n \in L$$

such that

$$B_\varepsilon(v_i) \subset G_{\alpha_i}, \forall i \in \{1, \ldots, n\},$$

considering that

$$\delta(v_i) \geq \delta_0 > \varepsilon > 0, \forall i \in \{1, \ldots, n\}.$$

For $u \in A$, since N is an ε-net we have

$$u \in \cup_{i=1}^n B_\varepsilon(v_i) \subset \cup_{i=1}^n G_{\alpha_i}.$$

Since $u \in U$ is arbitrary we obtain

$$A \subset \cup_{i=1}^n G_{\alpha_i}.$$

Thus
$$\{G_{\alpha_1},\dots,G_{\alpha_n}\}$$
is a finite subcover for A of
$$\{G_\alpha,\ \alpha\in L\}.$$
Hence, A is compact.

The proof is complete.

Theorem 1.6.12. *Let (U,d) be a metric space. Thus $A\subset U$ is relatively compact if and only if for each sequence in A, we may select a convergent subsequence.*

Proof. Suppose A is relatively compact. Thus $\overline{A}$ is compact so that from the last theorem, $\overline{A}$ is sequentially compact.

Thus from each sequence in $\overline{A}$ we may select a subsequence which converges to some element of $\overline{A}$. In particular, for each sequence in $A\subset\overline{A}$, we may select a subsequence that converges to some element of $\overline{A}$.

Conversely, suppose that for each sequence in A, we may select a convergent subsequence. It suffices to prove that $\overline{A}$ is sequentially compact. Let $\{v_n\}$ be a sequence in $\overline{A}$. Since A is dense in $\overline{A}$, there exists a sequence $\{u_n\}\subset A$ such that
$$d(u_n,v_n)<\frac{1}{n}.$$
From the hypothesis we may obtain a subsequence $\{u_{n_k}\}$ and $u_0\in\overline{A}$ such that
$$u_{n_k}\to u_0,\ \text{as } k\to\infty.$$
Thus,
$$v_{n_k}\to u_0\in\overline{A},\ \text{as } k\to\infty.$$
Therefore $\overline{A}$ is sequentially compact so that it is compact.

Theorem 1.6.13. *Let (U,d) be a metric space.*

1. *If $A\subset U$ is relatively compact, then it is totally bounded.*
2. *If (U,d) is a complete metric space and $A\subset U$ is totally bounded, then A is relatively compact.*

Proof.

1. Suppose $A\subset U$ is relatively compact. From the last theorem, from each sequence in A, we can extract a convergent subsequence. From Theorem 1.6.9, A is totally bounded.
2. Let (U,d) be a metric space and let A be a totally bounded subset of U.
   Let $\{u_n\}$ be a sequence in A. Since A is totally bounded for each $k\in\mathbb{N}$ we find a ε_k-net where $\varepsilon_k=1/k$, denoted by N_k where
$$N_k=\{v_1^{(k)},v_2^{(k)},\dots,v_{n_k}^{(k)}\}.$$

In particular for $k = 1$ $\{u_n\}$ is contained in the 1-net N_1. Thus at least one ball of radius 1 of N_1 contains infinitely many points of $\{u_n\}$. Let us select a subsequence $\{u_{n_k}^{(1)}\}_{k\in\mathbb{N}}$ of this infinite set (which is contained in a ball of radius 1). Similarly, we may select a subsequence here just partially relabeled $\{u_{n_l}^{(2)}\}_{l\in\mathbb{N}}$ of $\{u_{n_k}^{(1)}\}$ which is contained in one of the balls of the $\frac{1}{2}$-net. Proceeding in this fashion for each $k \in \mathbb{N}$ we may find a subsequence denoted by $\{u_{n_m}^{(k)}\}_{m\in\mathbb{N}}$ of the original sequence contained in a ball of radius $1/k$.

Now consider the diagonal sequence denoted by $\{u_{n_k}^{(k)}\}_{k\in\mathbb{N}} = \{z_k\}$. Thus

$$d(z_n, z_m) < \frac{2}{k}, \text{ if } m, n > k,$$

that is, $\{z_k\}$ is a Cauchy sequence, and since (U, d) is complete, there exists $u \in U$ such that

$$z_k \to u \text{ as } k \to \infty.$$

From Theorem 1.6.12, A is relatively compact.

The proof is complete.

1.7 The Arzela–Ascoli Theorem

In this section we present a classical result in analysis, namely the Arzela–Ascoli theorem.

Definition 1.7.1 (Equicontinuity). Let $\mathscr{F}$ be a collection of complex functions defined on a metric space (U, d). We say that $\mathscr{F}$ is equicontinuous if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in U$ and $d(u, v) < \delta$, then

$$|f(u) - f(v)| < \varepsilon, \forall f \in \mathscr{F}.$$

Furthermore, we say that $\mathscr{F}$ is point-wise bounded if for each $u \in U$ there exists $M(u) \in \mathbb{R}$ such that

$$|f(u)| < M(u), \forall f \in \mathscr{F}.$$

Theorem 1.7.2 (Arzela–Ascoli). *Suppose $\mathscr{F}$ is a point-wise bounded equicontinuous collection of complex functions defined on a metric space (U, d). Also suppose that U has a countable dense subset E. Thus, each sequence $\{f_n\} \subset \mathscr{F}$ has a subsequence that converges uniformly on every compact subset of U.*

Proof. Let $\{u_n\}$ be a countable dense set in (U, d). By hypothesis, $\{f_n(u_1)\}$ is a bounded sequence; therefore, it has a convergent subsequence, which is denoted by $\{f_{n_k}(u_1)\}$. Let us denote

$$f_{n_k}(u_1) = \tilde{f}_{1,k}(u_1), \forall k \in \mathbb{N}.$$

Thus there exists $g_1 \in \mathbb{C}$ such that

$$\tilde{f}_{1,k}(u_1) \to g_1, \text{ as } k \to \infty.$$

Observe that $\{f_{n_k}(u_2)\}$ is also bounded and also it has a convergent subsequence, which similarly as above we will denote by $\{\tilde{f}_{2,k}(u_2)\}$. Again there exists $g_2 \in \mathbb{C}$ such that

$$\tilde{f}_{2,k}(u_1) \to g_1, \text{ as } k \to \infty.$$

$$\tilde{f}_{2,k}(u_2) \to g_2, \text{ as } k \to \infty.$$

Proceeding in this fashion for each $m \in \mathbb{N}$ we may obtain $\{\tilde{f}_{m,k}\}$ such that

$$\tilde{f}_{m,k}(u_j) \to g_j, \text{ as } k \to \infty, \forall j \in \{1, \ldots, m\},$$

where the set $\{g_1, g_2, \ldots, g_m\}$ is obtained as above. Consider the diagonal sequence

$$\{\tilde{f}_{k,k}\},$$

and observe that the sequence

$$\{\tilde{f}_{k,k}(u_m)\}_{k>m}$$

is such that

$$\tilde{f}_{k,k}(u_m) \to g_m \in \mathbb{C}, \text{ as } k \to \infty, \forall m \in \mathbb{N}.$$

Therefore we may conclude that from $\{f_n\}$ we may extract a subsequence also denoted by

$$\{f_{n_k}\} = \{\tilde{f}_{k,k}\}$$

which is convergent in

$$E = \{u_n\}_{n \in \mathbb{N}}.$$

Now suppose $K \subset U$, being K compact. Suppose given $\varepsilon > 0$. From the equicontinuity hypothesis there exists $\delta > 0$ such that if $u, v \in U$ and $d(u,v) < \delta$ we have

$$|f_{n_k}(u) - f_{n_k}(v)| < \frac{\varepsilon}{3}, \forall k \in \mathbb{N}.$$

Observe that

$$K \subset \cup_{u \in K} B_{\frac{\delta}{2}}(u),$$

and being K compact we may find $\{\tilde{u}_1, \ldots, \tilde{u}_M\}$ such that

$$K \subset \cup_{j=1}^{M} B_{\frac{\delta}{2}}(\tilde{u}_j).$$

Since E is dense in U, there exists

$$v_j \in B_{\frac{\delta}{2}}(\tilde{u}_j) \cap E, \forall j \in \{1, \ldots, M\}.$$

Fixing $j \in \{1,\ldots,M\}$, from $v_j \in E$ we obtain that

$$\lim_{k\to\infty} f_{n_k}(v_j)$$

exists as $k \to \infty$. Hence there exists $K_{0_j} \in \mathbb{N}$ such that if $k,l > K_{0_j}$, then

$$|f_{n_k}(v_j) - f_{n_l}(v_j)| < \frac{\varepsilon}{3}.$$

Pick $u \in K$; thus

$$u \in B_{\frac{\delta}{2}}(\tilde{u}_{\hat{j}})$$

for some $\hat{j} \in \{1,\ldots,M\}$, so that

$$d(u, v_{\hat{j}}) < \delta.$$

Therefore if

$$k,l > \max\{K_{0_1},\ldots,K_{0_M}\},$$

then

$$\begin{aligned} |f_{n_k}(u) - f_{n_l}(u)| &\leq |f_{n_k}(u) - f_{n_k}(v_{\hat{j}})| + |f_{n_k}(v_{\hat{j}}) - f_{n_l}(v_{\hat{j}})| \\ &\quad + |f_{n_l}(v_{\hat{j}}) - f_{n_l}(u)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{1.26}$$

Since $u \in K$ is arbitrary, we conclude that $\{f_{n_k}\}$ is uniformly Cauchy on K.
The proof is complete.

1.8 Linear Mappings

Given U, V topological vector spaces, a function (mapping) $f : U \to V$, $A \subset U$, and $B \subset V$, we define

$$f(A) = \{f(u) \mid u \in A\}, \tag{1.27}$$

and the inverse image of B, denoted $f^{-1}(B)$ as

$$f^{-1}(B) = \{u \in U \mid f(u) \in B\}. \tag{1.28}$$

Definition 1.8.1 (Linear Functions). A function $f : U \to V$ is said to be linear if

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v), \forall u,v \in U,\ \alpha,\beta \in \mathbb{F}. \tag{1.29}$$

Definition 1.8.2 (Null Space and Range). Given $f : U \to V$, we define the null space and the range of f, denoted by $N(f)$ and $R(f)$, respectively, as

$$N(f) = \{u \in U \mid f(u) = \theta\} \tag{1.30}$$

and

$$R(f) = \{v \in V \mid \exists u \in U \text{ such that } f(u) = v\}. \tag{1.31}$$

Note that if f is linear, then $N(f)$ and $R(f)$ are subspaces of U and V, respectively.

Proposition 1.8.3. *Let U,V be topological vector spaces. If $f : U \to V$ is linear and continuous at θ, then it is continuous everywhere.*

Proof. Since f is linear, we have $f(\theta) = \theta$. Since f is continuous at θ, given $\mathscr{V} \subset V$ a neighborhood of zero, there exists $\mathscr{U} \subset U$ neighborhood of zero, such that

$$f(\mathscr{U}) \subset \mathscr{V}. \tag{1.32}$$

Thus

$$v - u \in \mathscr{U} \Rightarrow f(v-u) = f(v) - f(u) \in \mathscr{V}, \tag{1.33}$$

or

$$v \in u + \mathscr{U} \Rightarrow f(v) \in f(u) + \mathscr{V}, \tag{1.34}$$

which means that f is continuous at u. Since u is arbitrary, f is continuous everywhere.

1.9 Linearity and Continuity

Definition 1.9.1 (Bounded Functions). A function $f : U \to V$ is said to be bounded if it maps bounded sets into bounded sets.

Proposition 1.9.2. *A set E is bounded if and only if the following condition is satisfied: whenever $\{u_n\} \subset E$ and $\{\alpha_n\} \subset \mathbb{F}$ are such that $\alpha_n \to 0$ as $n \to \infty$ we have $\alpha_n u_n \to \theta$ as $n \to \infty$.*

Proof. Suppose E is bounded. Let $\mathscr{U}$ be a balanced neighborhood of θ in U and then $E \subset t\mathscr{U}$ for some t. For $\{u_n\} \subset E$, as $\alpha_n \to 0$, there exists N such that if $n > N$, then $t < \frac{1}{|\alpha_n|}$. Since $t^{-1}E \subset \mathscr{U}$ and $\mathscr{U}$ is balanced, we have that $\alpha_n u_n \in \mathscr{U}$, $\forall n > N$, and thus $\alpha_n u_n \to \theta$. Conversely, if E is not bounded, there is a neighborhood $\mathscr{V}$ of θ and $\{r_n\}$ such that $r_n \to \infty$ and E is not contained in $r_n\mathscr{V}$, that is, we can choose u_n such that $r_n^{-1}u_n$ is not in $\mathscr{V}$, $\forall n \in \mathbb{N}$, so that $\{r_n^{-1}u_n\}$ does not converge to θ.

Proposition 1.9.3. *Let $f : U \to V$ be a linear function. Consider the following statements:*

1. f is continuous,
2. f is bounded,
3. if $u_n \to \theta$, then $\{f(u_n)\}$ is bounded,
4. if $u_n \to \theta$, then $f(u_n) \to \theta$.

Then,

- 1 *implies* 2,
- 2 *implies* 3,
- *if U is metrizable, then* 3 *implies* 4, *which implies* 1.

Proof.

1. 1 implies 2: Suppose f is continuous, for $\mathscr{W} \subset V$ neighborhood of zero, there exists a neighborhood of zero in U, denoted by $\mathscr{V}$, such that

$$f(\mathscr{V}) \subset \mathscr{W}. \tag{1.35}$$

   If E is bounded, there exists $t_0 \in \mathbb{R}^+$ such that $E \subset t\mathscr{V}$, $\forall t \geq t_0$, so that

$$f(E) \subset f(t\mathscr{V}) = tf(\mathscr{V}) \subset t\mathscr{W}, \;\; \forall t \geq t_0, \tag{1.36}$$

   and thus f is bounded.
2. 2 implies 3: Suppose $u_n \to \theta$ and let $\mathscr{W}$ be a neighborhood of zero. Then, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $u_n \in \mathscr{V} \subset \mathscr{W}$ where $\mathscr{V}$ is a balanced neighborhood of zero. On the other hand, for $n < N$, there exists K_n such that $u_n \in K_n\mathscr{V}$. Define $K = \max\{1, K_1, \ldots, K_n\}$. Then, $u_n \in K\mathscr{V}, \forall n \in \mathbb{N}$ and hence $\{u_n\}$ is bounded. Finally from 2, we have that $\{f(u_n)\}$ is bounded.
3. 3 implies 4: Suppose U is metrizable and let $u_n \to \theta$. Given $K \in \mathbb{N}$, there exists $n_K \in \mathbb{N}$ such that if $n > n_K$, then $d(u_n, \theta) < \frac{1}{K^2}$. Define $\gamma_n = 1$ if $n < n_1$ and $\gamma_n = K$, if $n_K \leq n < n_{K+1}$ so that

$$d(\gamma_n u_n, \theta) = d(Ku_n, \theta) \leq Kd(u_n, \theta) < K^{-1}. \tag{1.37}$$

   Thus since 2 implies 3 we have that $\{f(\gamma_n u_n)\}$ is bounded so that, by Proposition 1.9.2, $f(u_n) = \gamma_n^{-1} f(\gamma_n u_n) \to \theta$ as $n \to \infty$.
4. 4 implies 1: suppose 1 fails. Thus there exists a neighborhood of zero $\mathscr{W} \subset V$ such that $f^{-1}(\mathscr{W})$ contains no neighborhood of zero in U. Particularly, we can select $\{u_n\}$ such that $u_n \in B_{1/n}(\theta)$ and $f(u_n)$ not in $\mathscr{W}$ so that $\{f(u_n)\}$ does not converge to zero. Thus 4 fails.

1.10 Continuity of Operators on Banach Spaces

Let U, V be Banach spaces. We call a function $A : U \to V$ an operator.

Proposition 1.10.1. *Let U, V be Banach spaces. A linear operator $A : U \to V$ is continuous if and only if there exists $K \in \mathbb{R}^+$ such that*

$$\|A(u)\|_V < K\|u\|_U, \forall u \in U.$$

Proof. Suppose A is linear and continuous. From Proposition 1.9.3,

$$\text{if } \{u_n\} \subset U \text{ is such that } u_n \to \theta \text{ then } A(u_n) \to \theta. \tag{1.38}$$

We claim that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u\|_U < \delta$, then $\|A(u)\|_V < \varepsilon$.

Suppose, to obtain contradiction, that the claim is false.

Thus there exists $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there exists $u_n \in U$ such that $\|u_n\|_U \leq \frac{1}{n}$ and $\|A(u_n)\|_V \geq \varepsilon_0$.

Therefore $u_n \to \theta$ and $A(u_n)$ does not converge to θ, which contradicts (1.38).

Thus the claim holds.

In particular, for $\varepsilon = 1$, there exists $\delta > 0$ such that if $\|u\|_U < \delta$, then $\|A(u)\|_V < 1$. Thus given an arbitrary not relabeled $u \in U$, $u \neq \theta$, for

$$w = \frac{\delta u}{2\|u\|_U}$$

we have

$$\|A(w)\|_V = \frac{\delta \|A(u)\|_V}{2\|u\|_U} < 1,$$

that is

$$\|A(u)\|_V < \frac{2\|u\|_U}{\delta}, \forall u \in U.$$

Defining

$$K = \frac{2}{\delta}$$

the first part of the proof is complete. Reciprocally, suppose there exists $K > 0$ such that

$$\|A(u)\|_V < K\|u\|_U, \forall u \in U.$$

Hence $u_n \to \theta$ implies $\|A(u_n)\|_V \to \theta$, so that from Proposition 1.9.3, A is continuous.

The proof is complete.

1.11 Some Classical Results on Banach Spaces

In this section we present some important results in Banach spaces. We start with the following theorem.

Theorem 1.11.1. *Let U and V be Banach spaces and let $A : U \to V$ be a linear operator. Then A is bounded if and only if the set $C \subset U$ has at least one interior point, where*

$$C = A^{-1}[\{v \in V \mid \|v\|_V \leq 1\}].$$

Proof. Suppose there exists $u_0 \in U$ in the interior of C. Thus, there exists $r > 0$ such that

$$B_r(u_0) = \{u \in U \mid \|u - u_0\|_U < r\} \subset C.$$

Fix $u \in U$ such that $\|u\|_U < r$. Thus, we have

$$\|A(u)\|_V \leq \|A(u+u_0)\|_V + \|A(u_0)\|_V.$$

Observe also that

$$\|(u+u_0) - u_0\|_U < r,$$

so that $u + u_0 \in B_r(u_0) \subset C$ and thus

$$\|A(u+u_0)\|_V \leq 1$$

and hence

$$\|A(u)\|_V \leq 1 + \|A(u_0)\|_V, \tag{1.39}$$

$\forall u \in U$ such that $\|u\|_U < r$. Fix an arbitrary not relabeled $u \in U$ such that $u \neq \theta$. From (1.39)

$$w = \frac{u}{\|u\|_U} \frac{r}{2}$$

is such that

$$\|A(w)\|_V = \frac{\|A(u)\|_V}{\|u\|_U} \frac{r}{2} \leq 1 + \|A(u_0)\|_V,$$

so that

$$\|A(u)\|_V \leq (1 + \|A(u_0)\|_V)\|u\|_U \frac{2}{r}.$$

Since $u \in U$ is arbitrary, A is bounded.

Reciprocally, suppose A is bounded. Thus

$$\|A(u)\|_V \leq K\|u\|_U, \forall u \in U,$$

for some $K > 0$. In particular

$$D = \left\{u \in U \mid \|u\|_U \leq \frac{1}{K}\right\} \subset C.$$

The proof is complete.

Definition 1.11.2. A set S in a metric space U is said to be nowhere dense if $\overline{S}$ has an empty interior.

Theorem 1.11.3 (Baire Category Theorem). *A complete metric space is never the union of a countable number of nowhere dense sets.*

Proof. Suppose, to obtain contradiction, that U is a complete metric space and

$$U = \cup_{n=1}^{\infty} A_n,$$

where each A_n is nowhere dense. Since A_1 is nowhere dense, there exist $u_1 \in U$ which is not in $\bar{A}_1$; otherwise we would have $U = \bar{A}_1$, which is not possible since U is open. Furthermore, $\bar{A}_1^c$ is open, so that we may obtain $u_1 \in A_1^c$ and $0 < r_1 < 1$ such that

$$B_1 = B_{r_1}(u_1)$$

satisfies

$$B_1 \cap A_1 = \emptyset.$$

Since A_2 is nowhere dense we have B_1 is not contained in $\bar{A}_2$. Therefore we may select $u_2 \in B_1 \setminus \bar{A}_2$ and since $B_1 \setminus \bar{A}_2$ is open, there exists $0 < r_2 < 1/2$ such that

$$\bar{B}_2 = \bar{B}_{r_2}(u_2) \subset B_1 \setminus \bar{A}_2,$$

that is,

$$B_2 \cap A_2 = \emptyset.$$

Proceeding inductively in this fashion, for each $n \in \mathbb{N}$, we may obtain $u_n \in B_{n-1} \setminus \bar{A}_n$ such that we may choose an open ball $B_n = B_{r_n}(u_n)$ such that

$$\bar{B}_n \subset B_{n-1},$$

$$B_n \cap A_n = \emptyset,$$

and

$$0 < r_n < 2^{1-n}.$$

Observe that $\{u_n\}$ is a Cauchy sequence, considering that if $m, n > N$, then $u_n, u_m \in B_N$, so that

$$d(u_n, u_m) < 2(2^{1-N}).$$

Define

$$u = \lim_{n \to \infty} u_n.$$

Since

$$u_n \in B_N, \forall n > N,$$

we get

$$u \in \bar{B}_N \subset B_{N-1}.$$

Therefore u is not in $A_{N-1}, \forall N > 1$, which means u is not in $\cup_{n=1}^{\infty} A_n = U$, a contradiction.

The proof is complete.

Theorem 1.11.4 (The Principle of Uniform Boundedness). *Let U be a Banach space. Let $\mathscr{F}$ be a family of linear bounded operators from U into a normed linear space V. Suppose for each $u \in U$ there exists a $K_u \in \mathbb{R}$ such that*

$$\|T(u)\|_V < K_u, \forall T \in \mathscr{F}.$$

Then, there exists $K \in \mathbb{R}$ *such that*

$$\|T\| < K, \forall T \in \mathscr{F}.$$

Proof. Define

$$B_n = \{u \in U \mid \|T(u)\|_V \leq n, \forall T \in \mathscr{F}\}.$$

By the hypotheses, given $u \in U$, $u \in B_n$ for all n is sufficiently big. Thus,

$$U = \cup_{n=1}^{\infty} B_n.$$

Moreover each B_n is closed. By the Baire category theorem there exists $n_0 \in \mathbb{N}$ such that B_{n_0} has nonempty interior. That is, there exists $u_0 \in U$ and $r > 0$ such that

$$B_r(u_0) \subset B_{n_0}.$$

Thus, fixing an arbitrary $T \in \mathscr{F}$, we have

$$\|T(u)\|_V \leq n_0, \forall u \in B_r(u_0).$$

Thus if $\|u\|_U < r$ then $\|(u+u_0)-u_0\|_U < r$, so that

$$\|T(u+u_0)\|_V \leq n_0,$$

that is,

$$\|T(u)\|_V - \|T(u_0)\|_V \leq n_0.$$

Thus,

$$\|T(u)\|_V \leq 2n_0, \text{ if } \|u\|_U < r. \tag{1.40}$$

For $u \in U$ arbitrary, $u \neq \theta$, define

$$w = \frac{ru}{2\|u\|_U},$$

from (1.40) we obtain

$$\|T(w)\|_V = \frac{r\|T(u)\|_V}{2\|u\|_U} \leq 2n_0,$$

so that

$$\|T(u)\|_V \leq \frac{4n_0\|u\|_U}{r}, \forall u \in U.$$

Hence

$$\|T\| \leq \frac{4n_0}{r}, \forall T \in \mathscr{F}.$$

The proof is complete.

Theorem 1.11.5 (The Open Mapping Theorem). *Let U and V be Banach spaces and let $A : U \to V$ be a bounded onto linear operator. Thus, if $\mathscr{O} \subset U$ is open, then $A(\mathscr{O})$ is open in V.*

Proof. First we will prove that given $r > 0$, there exists $r' > 0$ such that

$$A(B_r(\theta)) \supset B^V_{r'}(\theta). \tag{1.41}$$

Here $B^V_{r'}(\theta)$ denotes a ball in V of radius r' with center in θ. Since A is onto

$$V = \cup_{n=1}^{\infty} A(nB_1(\theta)).$$

By the Baire category theorem, there exists $n_0 \in \mathbb{N}$ such that the closure of $A(n_0 B_1(\theta))$ has nonempty interior, so that $\overline{A(B_1(\theta))}$ has nonempty interior. We will show that there exists $r' > 0$ such that

$$B^V_{r'}(\theta) \subset \overline{A(B_1(\theta))}.$$

Observe that there exists $y_0 \in V$ and $r_1 > 0$ such that

$$B^V_{r_1}(y_0) \subset \overline{A(B_1(\theta))}. \tag{1.42}$$

Define $u_0 \in B_1(\theta)$ which satisfies $A(u_0) = y_0$. We claim that

$$\overline{A(B_{r_2}(\theta))} \supset B^V_{r_1}(\theta),$$

where $r_2 = 1 + \|u_0\|_U$. To prove the claim, pick

$$y \in A(B_1(\theta))$$

thus there exists $u \in U$ such that $\|u\|_U < 1$ and $A(u) = y$. Therefore

$$A(u) = A(u - u_0 + u_0) = A(u - u_0) + A(u_0).$$

But observe that

$$\begin{aligned} \|u - u_0\|_U &\leq \|u\|_U + \|u_0\|_U \\ &< 1 + \|u_0\|_U \\ &= r_2, \end{aligned} \tag{1.43}$$

so that

$$A(u - u_0) \in A(B_{r_2}(\theta)).$$

This means

$$y = A(u) \in A(u_0) + A(B_{r_2}(\theta)),$$

and hence

$$A(B_1(\theta)) \subset A(u_0) + A(B_{r_2}(\theta)).$$

That is, from this and (1.42), we obtain

$$A(u_0)+\overline{A(B_{r_2}(\theta))} \supset \overline{A(B_1(\theta))} \supset B^V_{r_1}(y_0) = A(u_0)+B^V_{r_1}(\theta),$$

and therefore

$$\overline{A(B_{r_2}(\theta))} \supset B^V_{r_1}(\theta).$$

Since

$$A(B_{r_2}(\theta)) = r_2 A(B_1(\theta)),$$

we have, for some not relabeled $r_1 > 0$, that

$$\overline{A(B_1(\theta))} \supset B^V_{r_1}(\theta).$$

Thus it suffices to show that

$$\overline{A(B_1(\theta))} \subset A(B_2(\theta)),$$

to prove (1.41). Let $y \in \overline{A(B_1(\theta))}$; since A is continuous, we may select $u_1 \in B_1(\theta)$ such that

$$y - A(u_1) \in B^V_{r_1/2}(\theta) \subset \overline{A(B_{1/2}(\theta))}.$$

Now select $u_2 \in B_{1/2}(\theta)$ so that

$$y - A(u_1) - A(u_2) \in B^V_{r_1/4}(\theta).$$

By induction, we may obtain

$$u_n \in B_{2^{1-n}}(\theta),$$

such that

$$y - \sum_{j=1}^{n} A(u_j) \in B^V_{r_1/2^n}(\theta).$$

Define

$$u = \sum_{n=1}^{\infty} u_n,$$

we have that $u \in B_2(\theta)$, so that

$$y = \sum_{n=1}^{\infty} A(u_n) = A(u) \in A(B_2(\theta)).$$

Therefore

$$\overline{A(B_1(\theta))} \subset A(B_2(\theta)).$$

The proof of (1.41) is complete.

To finish the proof of this theorem, assume $\mathcal{O} \subset U$ is open. Let $v_0 \in A(\mathcal{O})$. Let $u_0 \in \mathcal{O}$ be such that $A(u_0) = v_0$. Thus there exists $r > 0$ such that

$$B_r(u_0) \subset \mathcal{O}.$$

From (1.41),

$$A(B_r(\theta)) \supset B^V_{r'}(\theta),$$

for some $r' > 0$. Thus

$$A(\mathcal{O}) \supset A(u_0) + A(B_r(\theta)) \supset v_0 + B^V_{r'}(\theta).$$

This means that v_0 is an interior point of $A(\mathcal{O})$. Since $v_0 \in A(\mathcal{O})$ is arbitrary, we may conclude that $A(\mathcal{O})$ is open.

The proof is complete.

Theorem 1.11.6 (The Inverse Mapping Theorem). *A continuous linear bijection of one Banach space onto another has a continuous inverse.*

Proof. Let $A : U \to V$ satisfying the theorem hypotheses. Since A is open, A^{-1} is continuous.

Definition 1.11.7 (Graph of a Mapping). Let $A : U \to V$ be an operator, where U and V are normed linear spaces. The *graph* of A denoted by $\Gamma(A)$ is defined by

$$\Gamma(A) = \{(u, v) \in U \times V \mid v = A(u)\}.$$

Theorem 1.11.8 (The Closed Graph Theorem). *Let U and V be Banach spaces and let $A : U \to V$ be a linear operator. Then A is bounded if and only if its graph is closed.*

Proof. Suppose $\Gamma(A)$ is closed. Since A is linear, $\Gamma(A)$ is a subspace of $U \oplus V$. Also, being $\Gamma(A)$ closed, it is a Banach space with the norm

$$\|(u, A(u)\| = \|u\|_U + \|A(u)\|_V.$$

Consider the continuous mappings

$$\Pi_1(u, A(u)) = u$$

and

$$\Pi_2(u, A(u)) = A(u).$$

Observe that Π_1 is a bijection, so that by the inverse mapping theorem, Π_1^{-1} is continuous. As

$$A = \Pi_2 \circ \Pi_1^{-1},$$

it follows that A is continuous. The converse is trivial.

1.12 Hilbert Spaces

At this point we introduce an important class of spaces, namely the Hilbert spaces.

Definition 1.12.1. Let H be a vector space. We say that H is a real pre-Hilbert space if there exists a function $(\cdot,\cdot)_H : H \times H \to \mathbb{R}$ such that

1. $(u,v)_H = (v,u)_H,\ \forall u,v \in H,$
2. $(u+v,w)_H = (u,w)_H + (v,w)_H,\ \forall u,v,w \in H,$
3. $(\alpha u,v)_H = \alpha(u,v)_H,\ \forall u,v \in H,\ \alpha \in \mathbb{R},$
4. $(u,u)_H \geq 0,\ \forall u \in H,$ and $(u,u)_H = 0,$ if and only if $u = \theta.$

Remark 1.12.2. The function $(\cdot,\cdot)_H : H \times H \to \mathbb{R}$ is called an inner product.

Proposition 1.12.3 (Cauchy–Schwarz Inequality). *Let H be a pre-Hilbert space. Defining*

$$\|u\|_H = \sqrt{(u,u)_H}, \forall u \in H,$$

we have

$$|(u,v)_H| \leq \|u\|_H \|v\|_H, \forall u,v \in H.$$

Equality holds if and only if $u = \alpha v$ for some $\alpha \in \mathbb{R}$ or $v = \theta$.

Proof. If $v = \theta$, the inequality is immediate. Assume $v \neq \theta$. Given $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} 0 &\leq (u-\alpha v, u-\alpha v)_H \\ &= (u,u)_H + \alpha^2(v,v)_H - 2\alpha(u,v)_H \\ &= \|u\|_H^2 + \alpha^2\|v\|_H^2 - 2\alpha(u,v)_H. \end{aligned} \tag{1.44}$$

In particular, for $\alpha = (u,v)_H/\|v\|_H^2$, we obtain

$$0 \leq \|u\|_H^2 - \frac{(u,v)_H^2}{\|v\|_H^2},$$

that is,

$$|(u,v)_H| \leq \|u\|_H \|v\|_H.$$

The remaining conclusions are left to the reader.

Proposition 1.12.4. *On a pre-Hilbert space H, the function*

$$\|\cdot\|_H : H \to \mathbb{R}$$

is a norm, where as above

$$\|u\|_H = \sqrt{(u,u)}.$$

Proof. The only nontrivial property to be verified, concerning the definition of norm, is the triangle inequality.

Observe that given $u, v \in H$, from the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|u+v\|_H^2 &= (u+v, u+v)_H \\
&= (u,u)_H + (v,v)_H + 2(u,v)_H \\
&\leq (u,u)_H + (v,v)_H + 2|(u,v)_H| \\
&\leq \|u\|_H^2 + \|v\|_H^2 + 2\|u\|_H\|v\|_H \\
&= (\|u\|_H + \|v\|_H)^2. \qquad (1.45)
\end{aligned}$$

Therefore

$$\|u+v\|_H \leq \|u\|_H + \|v\|_H, \forall u, v \in H.$$

The proof is complete.

Definition 1.12.5. A pre-Hilbert space H is to be a Hilbert space if it is complete, that is, if any Cauchy sequence in H converges to an element of H.

Definition 1.12.6 (Orthogonal Complement). Let H be a Hilbert space. Considering $M \subset H$ we define its orthogonal complement, denoted by $M^\perp$, by

$$M^\perp = \{u \in H \mid (u,m)_H = 0, \ \forall m \in M\}.$$

Theorem 1.12.7. *Let H be a Hilbert space and M a closed subspace of H and suppose $u \in H$. Under such hypotheses there exists a unique $m_0 \in M$ such that*

$$\|u - m_0\|_H = \min_{m \in M}\{\|u - m\|_H\}.$$

Moreover $n_0 = u - m_0 \in M^\perp$ so that

$$u = m_0 + n_0,$$

where $m_0 \in M$ and $n_0 \in M^\perp$. Finally, such a representation through $M \oplus M^\perp$ is unique.

Proof. Define d by

$$d = \inf_{m \in M}\{\|u - m\|_H\}.$$

Let $\{m_i\} \subset M$ be a sequence such that

$$\|u - m_i\|_H \to d, \text{ as } i \to \infty.$$

Thus, from the parallelogram law, we have

$$\begin{aligned}
\|m_i - m_j\|_H^2 &= \|m_i - u - (m_j - u)\|_H^2 \\
&= 2\|m_i - u\|_H^2 + 2\|m_j - u\|_H^2 \\
&\quad -2\| - 2u + m_i + m_j\|_H^2 \\
&= 2\|m_i - u\|_H^2 + 2\|m_j - u\|_H^2
\end{aligned}$$

$$
\begin{aligned}
&-4\|-u+(m_i+m_j)/2\|_H^2 \\
&\to 2d^2+2d^2-4d^2=0, \text{ as } i,j\to+\infty.
\end{aligned} \tag{1.46}
$$

Thus $\{m_i\}\subset M$ is a Cauchy sequence. Since M is closed, there exists $m_0\in M$ such that

$$m_i\to m_0, \text{ as } i\to+\infty,$$

so that

$$\|u-m_i\|_H\to\|u-m_0\|_H=d.$$

Define

$$n_0=u-m_0.$$

We will prove that $n_0\in M^\perp$.

Pick $m\in M$ and $t\in\mathbb{R}$, and thus we have

$$
\begin{aligned}
d^2 &\le \|u-(m_0-tm)\|_H^2 \\
&= \|n_0+tm\|_H^2 \\
&= \|n_0\|_H^2+2(n_0,m)_Ht+\|m\|_H^2t^2.
\end{aligned} \tag{1.47}
$$

Since

$$\|n_0\|_H^2=\|u-m_0\|_H^2=d^2,$$

we obtain

$$2(n_0,m)_Ht+\|m\|_H^2t^2\ge 0, \forall t\in\mathbb{R}$$

so that

$$(n_0,m)_H=0.$$

Being $m\in M$ arbitrary, we obtain

$$n_0\in M^\perp.$$

It remains to prove the uniqueness. Let $m\in M$, and thus

$$
\begin{aligned}
\|u-m\|_H^2 &= \|u-m_0+m_0-m\|_H^2 \\
&= \|u-m_0\|_H^2+\|m-m_0\|_H^2,
\end{aligned} \tag{1.48}
$$

since

$$(u-m_0,m-m_0)_H=(n_0,m-m_0)_H=0.$$

From (1.48) we obtain

$$\|u-m\|_H^2>\|u-m_0\|_H^2=d^2,$$

if $m\neq m_0$.

Therefore m_0 is unique.

Now suppose

$$u=m_1+n_1,$$

where $m_1 \in M$ and $n_1 \in M^{\perp}$. As above, for $m \in M$

$$\begin{aligned}\|u-m\|_H^2 &= \|u-m_1+m_1-m\|_H^2 \\ &= \|u-m_1\|_H^2+\|m-m_1\|_H^2, \\ &\geq \|u-m_1\|_H \end{aligned} \tag{1.49}$$

and thus since m_0 such that

$$d = \|u-m_0\|_H$$

is unique, we get

$$m_1 = m_0$$

and therefore

$$n_1 = u - m_0 = n_0.$$

The proof is complete.

Theorem 1.12.8 (The Riesz Lemma). *Let H be a Hilbert space and let $f : H \to \mathbb{R}$ be a continuous linear functional. Then there exists a unique $u_0 \in H$ such that*

$$f(u) = (u, u_0)_H, \forall u \in H.$$

Moreover

$$\|f\|_{H^*} = \|u_0\|_H.$$

Proof. Define N by

$$N = \{u \in H \mid f(u) = 0\}.$$

Thus, as f is a continuous and linear, N is a closed subspace of H. If $N = H$, then $f(u) = 0 = (u, \theta)_H, \forall u \in H$ and the proof would be complete. Thus, assume $N \neq H$. By the last theorem there exists $v \neq \theta$ such that $v \in N^{\perp}$.

Define

$$u_0 = \frac{f(v)}{\|v\|_H^2} v.$$

Thus,if $u \in N$ we have

$$f(u) = 0 = (u, u_0)_H = 0.$$

On the other hand, if $u = \alpha v$ for some $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} f(u) &= \alpha f(v) \\ &= \frac{f(v)(\alpha v, v)_H}{\|v\|_H^2} \\ &= \left(\alpha v, \frac{f(v)v}{\|v\|_H^2}\right)_H \\ &= (\alpha v, u_0)_H. \end{aligned} \tag{1.50}$$

Therefore $f(u)$ equals $(u, u_0)_H$ in the space spanned by N and v. Now we show that this last space (then span of N and v) is in fact H. Just observe that given $u \in H$ we may write

$$u = \left(u - \frac{f(u)v}{f(v)}\right) + \frac{f(u)v}{f(v)}. \tag{1.51}$$

Since

$$u - \frac{f(u)v}{f(v)} \in N$$

we have finished the first part of the proof, that is, we have proven that

$$f(u) = (u, u_0)_H, \forall u \in H.$$

To finish the proof, assume $u_1 \in H$ is such that

$$f(u) = (u, u_1)_H, \forall u \in H.$$

Thus,

$$\begin{aligned} \|u_0 - u_1\|_H^2 &= (u_0 - u_1, u_0 - u_1)_H \\ &= (u_0 - u_1, u_0)_H - (u_0 - u_1, u_1)_H \\ &= f(u_0 - u_1) - f(u_0 - u_1) = 0. \end{aligned} \tag{1.52}$$

Hence $u_1 = u_0$.

Let us now prove that

$$\|f\|_{H^*} = \|u_0\|_H.$$

First observe that

$$\begin{aligned} \|f\|_{H^*} &= \sup\{f(u) \mid u \in H, \ \|u\|_H \leq 1\} \\ &= \sup\{|(u, u_0)_H| \mid u \in H, \ \|u\|_H \leq 1\} \\ &\leq \sup\{\|u\|_H \|u_0\|_H \mid u \in H, \ \|u\|_H \leq 1\} \\ &\leq \|u_0\|_H. \end{aligned} \tag{1.53}$$

On the other hand

$$\begin{aligned} \|f\|_{H^*} &= \sup\{f(u) \mid u \in H, \ \|u\|_H \leq 1\} \\ &\geq f\left(\frac{u_0}{\|u_0\|_H}\right) \\ &= \frac{(u_0, u_0)_H}{\|u_0\|_H} \\ &= \|u_0\|_H. \end{aligned} \tag{1.54}$$

From (1.53) and (1.54)

$$\|f\|_{H^*} = \|u_0\|_H.$$

The proof is complete.

Remark 1.12.9. Similarly as above we may define a Hilbert space H over $\mathbb{C}$, that is, a complex one. In this case the complex inner product $(\cdot,\cdot)_H : H \times H \to \mathbb{C}$ is defined through the following properties:

1. $(u,v)_H = \overline{(v,u)_H}$, $\forall u,v \in H$,
2. $(u+v,w)_H = (u,w)_H + (v,w)_H$, $\forall u,v,w \in H$,
3. $(\alpha u,v)_H = \overline{\alpha}(u,v)_H$, $\forall u,v \in H$, $\alpha \in \mathbb{C}$,
4. $(u,u)_H \geq 0$, $\forall u \in H$, and $(u,u) = 0$, if and only if $u = \theta$.

Observe that in this case we have

$$(u,\alpha v)_H = \alpha(u,v)_H,\ \forall u,v \in H,\ \alpha \in \mathbb{C},$$

where for $\alpha = a + bi \in \mathbb{C}$, we have $\overline{\alpha} = a - bi$. Finally, similar results as those proven above are valid for complex Hilbert spaces.

1.13 Orthonormal Basis

In this section we study separable Hilbert spaces and the related orthonormal bases.

Definition 1.13.1. Let H be a Hilbert space. A set $S \subset H$ is said to be orthonormal if

$$\|u\|_H = 1,$$

and

$$(u,v)_H = 0, \forall u,v \in S, \text{ such that } u \neq v.$$

If S is not properly contained in any other orthonormal set, it is said to be an orthonormal basis for H.

Theorem 1.13.2. *Let H be a Hilbert space and let $\{u_n\}_{n=1}^N$ be an orthonormal set. Then, for all $u \in H$, we have*

$$\|u\|_H^2 = \sum_{n=1}^N |(u,u_n)_H|^2 + \left\| u - \sum_{n=1}^N (u,u_n)_H u_n \right\|_H^2.$$

Proof. Observe that

$$u = \sum_{n=1}^N (u,u_n)_H u_n + \left(u - \sum_{n=1}^N (u,u_n)_H u_n \right).$$

Furthermore, we may easily obtain that

$$\sum_{n=1}^N (u,u_n)_H u_n \text{ and } u - \sum_{n=1}^N (u,u_n)_H u_n$$

are orthogonal vectors so that

$$\begin{aligned}\|u\|_H^2 &= (u,u)_H \\ &= \left\|\sum_{n=1}^{N} |(u,u_n)_H u_n\right\|_H^2 + \left\|u - \sum_{n=1}^{N} (u,u_n)_H u_n\right\|_H^2 \\ &= \sum_{n=1}^{N} |(u,u_n)_H|^2 + \left\|u - \sum_{n=1}^{N} (u,u_n)_H u_n\right\|_H^2 .\end{aligned} \tag{1.55}$$

Corollary 1.13.3 (Bessel Inequality). *Let H be a Hilbert space and let $\{u_n\}_{n=1}^N$ be an orthonormal set. Then, for all $u \in H$, we have*

$$\|u\|_H^2 \geq \sum_{n=1}^{N} |(u,u_n)_H|^2.$$

Theorem 1.13.4. *Each Hilbert space has an orthonormal basis.*

Proof. Define by C the collection of all orthonormal sets in H. Define an order in C by stating $S_1 \prec S_2$ if $S_1 \subset S_2$. Then, C is partially ordered and obviously nonempty, since

$$v/\|v\|_H \in C, \forall v \in H, v \neq \theta.$$

Now let $\{S_\alpha\}_{\alpha \in L}$ be a linearly ordered subset of C. Clearly, $\cup_{\alpha \in L} S_\alpha$ is an orthonormal set which is an upper bound for $\{S_\alpha\}_{\alpha \in L}$.

Therefore, every linearly ordered subset has an upper bound, so that by Zorn's lemma C has a maximal element, that is, an orthonormal set not properly contained in any other orthonormal set.

This completes the proof.

Theorem 1.13.5. *Let H be a Hilbert space and let $S = \{u_\alpha\}_{\alpha \in L}$ be an orthonormal basis. Then for each $v \in H$ we have*

$$v = \sum_{\alpha \in L} (u_\alpha, v)_H u_\alpha,$$

and

$$\|v\|_H^2 = \sum_{\alpha \in L} |(u_\alpha, v)_H|^2.$$

Proof. Let $L' \subset L$ be a finite subset of L. From Bessel's inequality we have

$$\sum_{\alpha \in L'} |(u_\alpha, v)_H| \leq \|v\|_H^2.$$

From this, we may infer that the set $A_n = \{\alpha \in L \mid |(u_\alpha, v)_H| > 1/n\}$ is finite, so that

$$A = \{\alpha \in L \mid |(u_\alpha, v)_H| > 0\} = \cup_{n=1}^{\infty} A_n$$

is at most countable.

Thus $(u_\alpha, v)_H \neq 0$ for at most countably many $\alpha's \in L$, which we order by $\{\alpha_n\}_{n\in\mathbb{N}}$. Since the sequence

$$s_N = \sum_{i=1}^{N} |(u_{\alpha_i}, v)_H|^2,$$

is monotone and bounded, it is converging to some real limit as $N \to \infty$. Define

$$v_n = \sum_{i=1}^{n} (u_{\alpha_i}, v)_H u_{\alpha_i},$$

so that for $n > m$ we have

$$\begin{aligned} \|v_n - v_m\|_H^2 &= \left\| \sum_{i=m+1}^{n} (u_{\alpha_i}, v)_H u_{\alpha_i} \right\|_H^2 \\ &= \sum_{i=m+1}^{n} |(u_{\alpha_i}, v)_H|^2 \\ &= |s_n - s_m|. \end{aligned} \tag{1.56}$$

Hence, $\{v_n\}$ is a Cauchy sequence which converges to some $v' \in H$.

Observe that

$$\begin{aligned} (v - v', u_{\alpha_l})_H &= \lim_{N\to\infty} (v - \sum_{i=1}^{N} (u_{\alpha_i}, v)_H u_{\alpha_i}, u_{\alpha_l})_H \\ &= (v, u_{\alpha_l})_H - (v, u_{\alpha_l})_H \\ &= 0. \end{aligned} \tag{1.57}$$

Also, if $\alpha \neq \alpha_l, \forall l \in \mathbb{N}$, then

$$(v - v', u_\alpha)_H = \lim_{N\to\infty} (v - \sum_{i=1}^{\infty} (u_{\alpha_i}, v)_H u_{\alpha_i}, u_\alpha)_H = 0.$$

Hence

$$v - v' \perp u_\alpha, \ \forall \alpha \in L.$$

If

$$v - v' \neq \theta,$$

then we could obtain an orthonormal set

$$\left\{ u_\alpha, \ \alpha \in L, \frac{v - v'}{\|v - v'\|_H} \right\}$$

which would properly contain the complete orthonormal set

$$\{u_\alpha,\ \alpha \in L\},$$

a contradiction.

Therefore, $v - v' = \theta$, that is,

$$v = \lim_{N\to\infty} \sum_{i=1}^{N} (u_{\alpha_i}, v)_H u_{\alpha_i}.$$

1.13.1 The Gram–Schmidt Orthonormalization

Let H be a Hilbert space and $\{u_n\} \subset H$ be a sequence of linearly independent vectors. Consider the procedure

$$w_1 = u_1,\ \ v_1 = \frac{w_1}{\|w_1\|_H},$$

$$w_2 = u_2 - (v_1, u_2)_H v_1,\ \ v_2 = \frac{w_2}{\|w_2\|_H},$$

and inductively,

$$w_n = u_n - \sum_{k=1}^{n-1} (v_k, u_n)_H v_k,\ \ v_n = \frac{w_n}{\|w_n\|_H}, \forall n \in \mathbb{N}, n > 2.$$

Observe that clearly $\{v_n\}$ is an orthonormal set and for each $m \in \mathbb{N}$, $\{v_k\}_{k=1}^m$ and $\{u_k\}_{k=1}^m$ span the same vector subspace of H.

Such a process of obtaining the orthonormal set $\{v_n\}$ is known as the Gram–Schmidt orthonormalization.

We finish this section with the following theorem.

Theorem 1.13.6. *A Hilbert space H is separable if and only if it has a countable orthonormal basis. If $dim(H) = N < \infty$, the H is isomorphic to $\mathbb{C}^N$. If $dim(H) = +\infty$, then H is isomorphic to l^2, where*

$$l^2 = \left\{ \{y_n\} \mid y_n \in \mathbb{C}, \forall n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} |y_n|^2 < +\infty \right\}.$$

Proof. Suppose H is separable and let $\{u_n\}$ be a countable dense set in H. To obtain an orthonormal basis it suffices to apply the Gram–Schmidt orthonormalization procedure to the greatest linearly independent subset of $\{u_n\}$.

Conversely, if $B = \{v_n\}$ is an orthonormal basis for H, the set of all finite linear combinations of elements of B with rational coefficients are dense in H, so that H is separable.

Moreover, if $dim(H) = +\infty$, consider the isomorphism $F : H \to l^2$ given by

$$F(u) = \{(u_n, u)_H\}_{n\in\mathbb{N}}.$$

Finally, if $dim(H) = N < +\infty$, consider the isomorphism $F : H \to \mathbb{C}^N$ given by

$$F(u) = \{(u_n, u)_H\}_{n=1}^N.$$

The proof is complete.

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