Chapter 2
Stabilization of T–S Fuzzy Systems with Constrained Controls

2.1 Introduction

It is known that the qualitative knowledge of a system can be represented by a nonlinear model. This idea has allowed the emergence of a new design approach in the fuzzy control field. The nonlinear system can be represented by a Takagi–Sugeno (T–S) fuzzy model [1, 2]. The control design is then carried out using known or recently developed methods from control theory [3–8].

A main problem, which is always inherent to all dynamical systems, is the presence of actuator saturations. Even for linear systems, this problem has been an active area of research for many years. Two main approaches have been developed in the literature: The first one is the so-called positive invariance approach, which is based on the design of controllers that work inside a region of linear behavior where saturations do not occur (see [9–13] and the references therein). This approach has been extended to systems modeled by T–S systems [4, 14]. The second approach, allows saturations to take effect, while guaranteeing asymptotic stability (see [15, 16] and the references therein). This method has been extended to T–S continuous-time fuzzy systems in [17]. The main challenge in these two approaches is to obtain a large enough domain of initial states that ensures asymptotic stability of the system despite the presence of saturations [18].

In this chapter, the saturations on the control signal are taken into account with the fuzzy model. The concept of positive invariance is used to obtain sufficient conditions of asymptotic stability for the global fuzzy system with constrained control inside a subset of the state space. The main idea of [19] representing the nonlinear system by a set of uncertain linear subsystems is used in this chapter. The problem is then to design a controller which is "robust" with respect to the upper bound extreme subsystems by taking into account the saturations on the control. Both a common Lyapunov function and a piecewise Lyapunov function as used in [19] and [20] are used to analyze and to design the controllers which ensure the asymptotic stability of the nonlinear system despite the presence of saturations on the control. Hence, a set of Linear Matrix Inequalities (LMIs) is proposed to built stabilizing controllers together

A. Benzaouia and A. El Hajjaji, *Advanced Takagi–Sugeno Fuzzy Systems*,
Studies in Systems, Decision and Control 8, DOI: 10.1007/978-3-319-05639-5_2,

with their corresponding region of asymptotic stability and positive invariance. The results of this chapter were published for the first time in [4, 14, 21, 22].

2.2 Problem Presentation

Consider the following nonlinear system with constrained control that can be described by the T–S fuzzy model as detailed in Chap. 1:

$$\dot{x}(t) = A(z)x(t) + B(z)u(t) \tag{2.1}$$

with,

$$A(z) = \sum_{i=1}^{r} h_i(z(t))A_i; \tag{2.2}$$

$$B(z) = \sum_{i=1}^{r} h_i(z(t))B_i; \tag{2.3}$$

with, $h_i(z(t))$ is the normalized membership function satisfying:

$$h_i(z(t)) \geq 0, i = 1, \ldots, r; \quad \sum_{i=1}^{r} h_i(z(t)) = 1 \tag{2.4}$$

$x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control which is constrained as follows:

$$u \in \Omega = \left\{u \in \mathbb{R}^m, -q_2 \leq u \leq q_1;\ q_1,\ q_2 \in \mathbb{R}^m\right\}. \tag{2.5}$$

Using the PDC control defined in Chap. 1

$$\begin{aligned} u(t) &= F(z)x(t) \qquad (2.6)\\ &= \sum_{i=1}^{r} h_i(z(t))F_i x(t) \end{aligned}$$

This control leads to the following system in closed-loop,

$$\dot{x}(t) = [A(z) + B(z)F(z)]\, x(t) \tag{2.7}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t)) \left(A_i + B_i F_j\right) x(t) \tag{2.8}$$

The main objective of this chapter is to design controller $F(z)$ such that the global system is asymptotically stable at the origin despite the presence of constraints on the control. To achieve this objective, two techniques will be used: The first one concerns the use of the so-called positive invariance approach which will enable one to construct regions of linear behavior for the system with saturations on the control. The second consists in rewriting equivalently the initial system (2.1) by using a state space repartition allowing to introduce r like uncertain subsystems as used before by many authors.

2.3 Preliminary Results

In this section, we remind the approach of positive invariance as known in the literature applied to a linear time-invariant system. For more details, one can consult [11]. Consider the following system given by,

$$\dot{x}(t) = Ax(t) \tag{2.9}$$

Let the state be constrained as follows,

$$\mathscr{D} = \{x \in \mathbb{R}^n / -\delta_2 \leq x \leq \delta_1; \quad \delta_1, \delta_2 \in \mathbb{R}^n\}; \tag{2.10}$$

In the following, we remind the approach proposed in [9, 10, 23].

Definition 2.1 A subset $\mathscr{D}$ of $\mathbb{R}^n$ is said to be positively invariant with respect to (w.r.t.) the motion of the system (2.9) if for every initial state $x_o \in \mathscr{D}$, the motion $x(x_o, \ t) \in \mathscr{D}$, for every t.

The necessary and sufficient condition of domain $\mathscr{D}$ to be positively invariant w.r.t system (2.9) is given by [10, 23]:

Theorem 2.1 *The set $\mathscr{D}$ is positively invariant w.r.t system* (2.9) *if and only if:*

$$\widetilde{A}\delta \leq 0 \tag{2.11}$$

where,

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}; \quad A_1 = \begin{cases} a_{ii} \\ a_{ij}^+ & \text{for } i \neq j \end{cases},$$
$$\text{and } A_2 = \begin{cases} 0 \\ a_{ij}^- & \text{for } i \neq j \end{cases}, \quad \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix},$$

Remark 2.1 If the constraints are symmetric, i.e., $\delta_1 = \delta_2$, the condition of positive invariance of the set $\mathscr{D}$ w.r.t the system (2.9) becomes,

$$\hat{A}\delta_1 \leq 0 \tag{2.12}$$

where matrix $\hat{A}$ is given by,

$$\hat{A} = \begin{cases} a_{ii} \\ |a_{ij}| \text{ for } i \neq j \end{cases}$$

Note that $A = A_1 - A_2$; $|A| = A_1 + A_2$.

Consider now the following time-invariant system given by:

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{2.13}$$

The control vector is constrained in domain Ω defined by (2.5). We propose a control law given by,

$$u(t) = Fx(t) \tag{2.14}$$

The system in closed-loop follows readily,

$$\dot{x}(t) = (A + BF)x(t) \tag{2.15}$$

We follow the same approach proposed in [9, 10, 23]. Remind that this approach consists in giving conditions allowing the choice of stabilizing controller (2.14) in such a way that model (2.15) remains valid every time. This is only possible if the state is constrained to evolve in a specified region defined by the set,

$$\mathscr{D} = \{x \in \mathbb{R}^n / -q_2 \leq Fx \leq q_1; \quad q_1, q_2 \in \mathbb{R}^m\}; \tag{2.16}$$

The necessary and sufficient condition of each domain $\mathscr{D}$ to be positively invariant w.r.t system (2.15) is given by [10, 23]:

Theorem 2.2 *Set $\mathscr{D}$ is positively invariant w.r.t system (2.15) if and only if, there exists matrix $H \in \mathbb{R}^{m \times m}$ such that:*

$$F(A + BF) = HF \tag{2.17}$$
$$\widetilde{H}q \leq 0 \tag{2.18}$$

where,

$$\tilde{H} = \begin{bmatrix} H_1 & H_2 \\ H_2 & H_1 \end{bmatrix}; \quad H_1 = \begin{cases} h_{ii} \\ h_{ij}^{+} & \text{for} i \neq j \end{cases};$$

$$H_2 = \begin{cases} 0 \\ h_{ij}^{-} & \text{for} i \neq j \end{cases} \text{ and } q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

An efficient algorithm to built such controllers is given by the resolution of algebraic equations $XA + XBX = HX$ [24] where matrix H is first given according to conditions (2.18). Note that the obtained controller is stabilizing the system in the closed-loop (2.15) while the control is admissible for all $x_0 \in \mathscr{D}$. This technique is so-called the inverse procedure. The resolution of this algebraic equation necessitates that matrices A admit at least $n - m$ stable eigenvalues as required by assumption H2. If not, one has to use the technique of augmentation [24] described below.

Rewrite the system (2.13) under the equivalent form:

$$\dot{x}(t) \;=\; Ax(t) \;+\; B_a w(t), \tag{2.19}$$

with matrix B_a given by:

$$B_a = \begin{bmatrix} B \; \odot \end{bmatrix},$$

where $\odot \in \mathbb{R}^{n\times(n-m)}$ represents the null matrix. This augmentation technique leads to the introduction of $n-m$ fictitious entries together with their fictitious constraints given by: $-\varphi_2 \leq v \leq \varphi_1$. In this case, the control law is also modified and becomes

$$w(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix},$$

$w(t) = Kx(t)$ and $v(t) = Ex(t)$. Note K and g as follows:

$$K = \begin{bmatrix} F \\ E \end{bmatrix}, g_1 = \begin{bmatrix} q_1 \\ \varphi_1 \end{bmatrix}, g_2 = \begin{bmatrix} q_2 \\ \varphi_2 \end{bmatrix}, g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

where g represents the new vector constraint. Note that the system in closed-loop given with augmented control $w(t)$ remains the same as (2.13) while the set of admissible constraints becomes, with this augmentation,

$$\Omega_a \;= \left\{ w \;\in\; \mathbb{R}^n / - g_2 \;\leq\; w \;\leq\; g_1 \right\}$$

It is worth noting that this technique does not modify the system, but introduces new degree of freedom with φ, which are used to satisfy conditions (2.18), but in return, reduces domain $\mathscr{D}$ which is transformed to the following bounded and convex set:

$$\mathscr{G} = \left\{ x \in \mathbb{R}^n / - g_2 \leq Kx \leq g_1; \quad g_1, g_2 \in \mathbb{R}^n \right\};$$

Obviously, conditions (2.17) and (2.18) are to be rewritten with matrices K and B_a, matrices H become of $n \times n$ size:

$$KA \;+\; KB_a K \;=\; HK; H \in \mathbb{R}^{n\times n} \tag{2.20}$$

$$\tilde{H} g \;\leq 0, \tag{2.21}$$

Remind now a result of asymptotic stability of the fuzzy system in closed-loop (2.8).

Theorem 2.3 [2] *Unconstrained system* (2.8) *is asymptotically stable if there exists a common positive definite matrix P such that,*

$$(A_i + BF_i)^{\mathrm{T}} P + P(A_i + BF_i) < 0 \qquad j \in [1,\ r]$$

2.4 Conditions of Stabilizability Using Positive Invariance Approach

In this section, we present an approach which consists in ensuring that the global control is always admissible. The condition of stability of global system (2.8) is then derived. These results appeared for the first time in [4].

Consider the system in closed-loop (2.8) where matrices A_i and B_i are constant of appropriate size and satisfy the following assumptions:

(H1) Each pair (A_i, B_i) is stabilizable.
(H2) Each matrix A_i admits $n - m$ stable eigenvalues.
(H3) $B_i = B, i = 1, \ldots, r$.

Define the following change of coordinates,

$$y_j(t) = F_j x(t) \qquad j \in [1,\ r]$$

The corresponding dynamical system is then,

$$\begin{aligned} \dot{y}_j(t) &= F_j \dot{x}(t) \\ &= \sum_{i=1}^{r} h_i(z(t)) F_j \left(A_i + BF_i\right) x(t) \end{aligned}$$

If there exist matrices $H_{ij} \in \mathbb{R}^{m \times m}$ such that,

$$F_j(A_i + BF_i) = H_{ij} F_j; \qquad i = 1, \ldots, r$$

Then, dynamical system (2.8) is transformed into the following reduced order dynamical system,

$$\dot{y}_j(t) = \sum_{i=1}^{r} h_i(z(t)) H_{ij} y_j(t) \tag{2.22}$$

With the same transformation, set $\mathscr{D}_j$ defined by:

$$\mathscr{D}_j = \{x \in \mathbb{R}^n / -q_2 \le F_j x \le q_1; \quad q_1, q_2 \in \mathbb{R}^m\}; \tag{2.23}$$

is transformed into the following set,

$$\Psi_j = \{y_j \in \mathbb{R}^n / -q_2 \leq y_j \leq q_1; \quad q_1, q_2 \in \mathbb{R}^m\}; \tag{2.24}$$

Now, we are able to apply the result of positive invariance into the set Ψ_j w.r.t each system (2.22).

Theorem 2.4 *Each set Ψ_j is positively invariant w.r.t the corresponding system (2.22) if there exist, r matrices $H_{ij} \in \mathbb{R}^{m\times m}$ such that:*

$$F_j(A_i + BF_i) = H_{ij}F_j; \qquad i = 1, \dots, r; \tag{2.25}$$

$$\widetilde{H}_{ij}q \ \leq 0; \qquad i = 1, \dots, r \tag{2.26}$$

where, matrices $\tilde{H}_{ij}$ and vector q are defined by Theorem 2.3.

Proof According to Theorem 2.3, the necessary and sufficient condition of domain Ψ_j to be positively invariant w.r.t the dynamical system (2.22) is given by,

$$\widetilde{L}q \leq 0 \tag{2.27}$$

where $L = \sum_{i=1}^{r} h_i(z(t))H_{ij}$. Recall that $h_i(z(t)) > 0$. According to the definition of $\tilde{L}$, it is easy to obtain,

$$\widetilde{L}q \leq \sum_{i=1}^{r} h_i(z(t))\tilde{H}_{ij}q$$

Taking into account of conditions (2.26), condition (2.27) holds. Then, the set Ψ_j is positively invariant w.r.t system (2.22). □

Define now the common set for all the sets $\mathscr{D}_j$ by,

$$\mathscr{D} = \bigcap_{j=1}^{r} \mathscr{D}_j \tag{2.28}$$

Corollary 2.1 *If each set Ψ_j is positively invariant w.r.t system (2.22), then the global control (2.6) is admissible for all $x_0 \in \mathscr{D}$.*

Proof Let each set Ψ_j be positively invariant w.r.t system (2.22). This implies that,

$$-q_2 \leq F_jx \leq q_1, \forall j \in [1,\ r], \forall x_0 \in \mathscr{D}, \forall t$$

By taking into account (2.4), it follows that the global control satisfies,

$$-q_2 \leq \sum_{j=1}^{r} h_j(t)F_jx \leq q_1, \quad \forall t, \forall x_0 \in \mathscr{D}$$ □

It is worth noting that the direct idea is to ensure that the following set,

$$\Gamma = \left\{ x \in \mathbb{R}^n / - q_2 \leq \sum_{j=1}^{r} h_j(z(t)) F_j x \leq q_1, \right\} \tag{2.29}$$

is positively invariant w.r.t the system (2.22). However, this property is very difficult to obtain.

The problem now is to stabilize the global system. The stability of the global system with constrained control is then stated by the following result,

Theorem 2.5 *If there exist matrices $H_{ij} \in \mathbb{R}^{m \times m}$ and a common definite positive matrix $P \in \mathbb{R}^{n \times n}$ such that:*

$$F_j(A_i + BF_i) = H_{ij} F_j; \quad i = 1, \ldots, r; \; j = 1, \ldots, r \tag{2.30}$$

$$\widetilde{H}_{ij} q \leq 0; \quad i = 1, \ldots, r; \; j = 1, \ldots, r \tag{2.31}$$

$$(A_i + BF_i)^{\mathrm{T}} P + P(A_i + BF_i) < 0; \quad i = 1, \ldots, r \tag{2.32}$$

then, the system (2.8) is asymptotically stable $\forall x_0 \in \mathscr{D}$.

Proof According to Theorem 2.5 conditions (2.30) and (2.31) ensure that each set Ψ_i defined by (2.24) is positively invariant w.r.t every system (2.22). By virtue of Corollary 2.1, global control (2.6) is also admissible $\forall x_0 \in \mathscr{D}$ allowing to system (2.8) to be valid despite the presence of saturations. Since all matrices F_i are assumed to be computed according to (2.30) and (2.31), then global system in the closed-loop (2.8) is asymptotically stable if condition (2.32) holds $\forall x_0 \in \mathscr{D}$. □

Remark 2.2 In order to compute matrices H_{ij} and F_j, one can follow two steps:

(i) For a given $j \in [1, r]$, give matrix H_{jj} such that condition (2.31) is satisfied. Compute matrix F_j by using the resolution given by [24]. The obtained matrix is unique and of full rank. Note that one can take all matrices H_{jj} identical, i.e, $H_{jj} = H_0$. In this case, all the matrices in closed-loop $A_j + BF_j$ will have the same spectrum.
(ii) The computation of matrices $H_{ij}, i \neq j$ is given by Lemma 2.1

Pose $R_{ij} = F_j(A_i + BF_i)$.

Lemma 2.1 [25] *Matrix $H_{ij}, i \neq j$ solution of equation (2.30) exists if and only if,*

$$\operatorname{rank} \begin{bmatrix} R_{ij} \\ F_j \end{bmatrix} = m, \tag{2.33}$$

where,

$$R_{ij} \in \mathbb{R}^{m \times n}, F_j \in \mathbb{R}^{m \times n}, \operatorname{rank} F = m$$

In this case and without loss of generality, it is always possible to decompose matrix F_j, which is of full rank, as follows: $F_j = \begin{bmatrix} F_j^1 & F_j^2 \end{bmatrix}$,
where $F_j^1 \in \mathbb{R}^{m\times m}$, rank$F_j^1 = m$, $F_j^2 \in \mathbb{R}^{m\times(n-m)}$.
Decompose matrix R_{ij},

$$R_{ij} = \begin{bmatrix} R_{ij}^1 & R_{ij}^2 \end{bmatrix}, R_{ij}^1 \in \mathbb{R}^{m\times m}, R_{ij}^2 \in \mathbb{R}^{m\times(n-m)}.$$

Hence, matrix H_{ij} will be given by,

$$H_{ij} = R_{ij}^1 \left(F_j^1\right)^{-1} \tag{2.34}$$

The following algorithm presents the necessary steps to use the result of Theorem 2.5.

Algorithm 2.1

- *Step1: Give r matrices H_{jj} satisfying conditions* (2.31). *One can resolve the following linear programming for each j:*

$$(LP1)\begin{cases} \min \varepsilon \\ \text{s.t.} \\ \begin{bmatrix} H_1 & H_2 \\ H_2 & H_1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \leq -\varepsilon \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \\ \varepsilon > 0 \\ H_1(i,i) < 0, H_2(i,i) = 0; \\ H_1(i,j) > 0, H_2(i,j) > 0, i \neq j \end{cases} \tag{2.35}$$

  where δ_1 and δ_2 are design positive vectors. Each matrix H is given by $H = H_1 - H_2$.
- *Step 2:Compute gain matrices F_j solution of equations $F_j(A_j + BF_j) = H_{jj}F_j$; $j = 1,\ldots,r$ by using the method given in* [24]. *Note that solution F_j is of full rank.*
- *Step 3: Compute matrices H_{ij}; $i \neq j = 1,\ldots,r$; given by* (2.34).
- *Step 4: If conditions* (2.31) *for $i \neq j = 1,\ldots,r$. are satisfied continue, else return to Step* 1 *to change matrices H_{jj}.*
- *Step 5: Compute matrices $A_i + BF_i$; $i = 1,\ldots,r$.*
- *Step 6: Compute matrix P by resolving the LMI constraints* (2.32).

2.4.1 Example

Consider the following constrained nonlinear system,

$$\ddot{y} + a_1(1 - y^2)\dot{y} + a_2 y(t) = b_1 u(t)$$

with, $a_1 = 2.2165$, $b_1 = a_2 = 12.7388$ and $-10 \le u \le 15$. This system admits the following state representation,

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -a_2 x_1(t) - a_1 x_2(t) + a_1 x_1^2(t) x_2(t) + b_1 u(t) \end{cases}$$

Now we give the exact representation of the nonlinear system by a T–S fuzzy model. For this, assume that $x_1(t) \in [-\gamma, \gamma]$, then one can write,

$$x_1^2(t) = M_1^1(x_1(t)) \cdot 0 + M_1^2(x_1(t)) \cdot \gamma^2$$

with,

$$M_1^1(x_1(t)) = \frac{\gamma^2 - x_1^2(t)}{\gamma^2} = h_1(t)$$

$$M_1^2(x_1(t)) = 1 - M_1^1(x_1(t)) = \frac{x_1^2(t)}{\gamma^2} = h_2(t)$$

The fuzzy model which represents exactly the nonlinear system is given by,

$$\text{If } x_1(t) \text{ is } M_1^1 \text{ Then } \dot{x}(t) = A_1 x(t) + B_1 u(t); \ -10 \le u \le 15$$
$$\text{If } x_1(t) \text{ is } M_1^2 \text{ Then } \dot{x}(t) = A_2 x(t) + B_2 u(t); \ -10 \le u \le 15$$

where matrices A_1, A_2, B_1 and B_2 are given by,

$$A_1 = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1(1-\gamma^2) \end{bmatrix};$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ b_1 \end{bmatrix}.$$

Note that matrix A_1 admits two complex eigenvalues and matrix A_2 admits two unstable eigenvalues for $\gamma = 3$. For this, we apply the technique of augmentation described previously.

Let matrices H_1 and H_2 be chosen according to conditions (2.21) with $\varphi_1 = 15$; $\varphi_2 = 10$ as follows,

$$H_1 = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}; \quad H_2 = \begin{bmatrix} -3 & 1 \\ 0 & -3.5 \end{bmatrix}$$

The resolution of the algebraic equations (2.20) leads to the following solutions,

$$K_1 = \begin{bmatrix} 0.529 & -0.2185 \\ 2.369 & 1.1845 \end{bmatrix}; \quad K_2 = \begin{bmatrix} 0.1758 & -1.9022 \\ 20.5005 & 6.8335 \end{bmatrix}$$

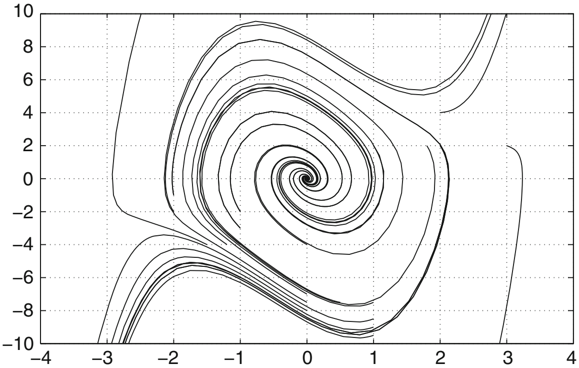


Fig. 2.1 Evolution of the nonlinear system in open-loop

The matrices $H_{ij}; i \neq j$ are given by:

$$H_{12} = \begin{bmatrix} -1.6607 & 1.339 \\ -1.839 & -4.839 \end{bmatrix}; \ H_{21} = \begin{bmatrix} -3 & 0.5825 \\ 0 & -2 \end{bmatrix}$$

Note that these matrices satisfy condition (2.31).

$$\tilde{H}_{12}g = \begin{bmatrix} -4.82 \\ -54.196 \\ -3.214 \\ -20.80 \end{bmatrix}; \ \tilde{H}_{21}g = \begin{bmatrix} -36.26 \\ -30.00 \\ -24.175 \\ -20 \end{bmatrix}$$

The matrices in closed-loop are as follows:

$$G_{11} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}; \ G_{22} = \begin{bmatrix} 0 & 1 \\ -10.5 & -6.5 \end{bmatrix}$$

A feasible solution to LMIs (2.32) is presented

$$P = \begin{bmatrix} 0.904 & 0.0456 \\ 0.0456 & 0.0754 \end{bmatrix}$$

The results of simulation are given by the following figures.

Figure 2.1 plots the evolution of the nonlinear system, while Fig. 2.2 presents the common set of positive invariance together with the set Γ defined by (2.29). Figure 2.3 plots the evolution of the state of the system in closed-loop inside the common set of positive invariance for different initial states. Finally, Fig. 2.4 presents the evolution of the corresponding control.

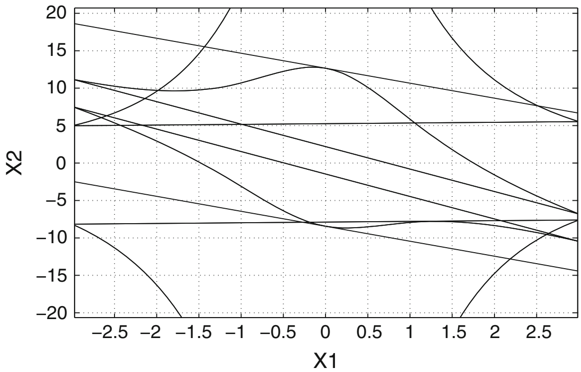


Fig. 2.2 Common set of positive invariance and the set Γ

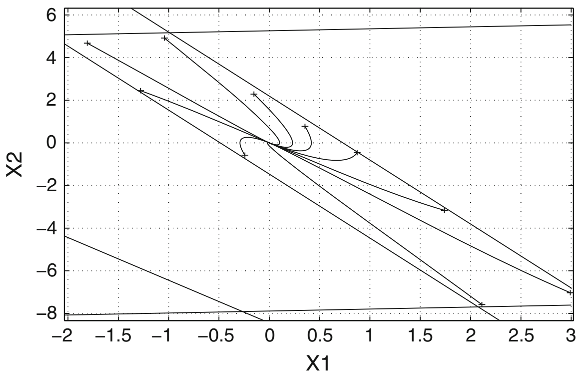


Fig. 2.3 Evolution of the state of the system in closed-loop inside the common set of positive invariance for different initial states

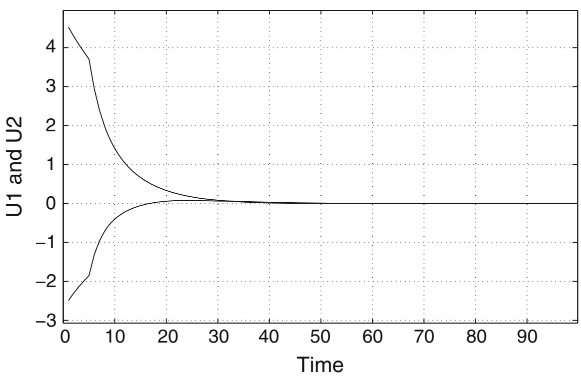


Fig. 2.4 Evolution of the control of the system for $x_0 = [-1.0019 \quad 4.9189]^{\mathrm{T}}$

2.5 Conditions of Stabilizability Using Uncertainty Approach

In this section, we propose sufficient conditions of asymptotic stability for the system with constrained control, by using both a common Lyapunov and a piecewise Lyapunov function. These results are based on the technique of rewriting equivalently fuzzy system (2.1) under the form of r like uncertainty subsystems as proposed in [19]. The results of this section were published in [4].

Consider the nonlinear system with constrained control that can be described by the T–S fuzzy model (2.1). Following the idea of [19], one can divide the input space into fuzzy subspaces and build a linear model, called the local model, in each subspace. Then, the membership function is used to connect smoothly the local models together to form a global fuzzy model of the nonlinear system. Let us define the r subspaces in the state space as follows:

$$S_j = \{x/h_j(x) \geq h_i(x), i = 1, 2, \ldots, r, i \neq j\}, j = 1, 2, \ldots, r \tag{2.36}$$

The characteristic function of S_j is defined by:

$$\eta_j = \begin{cases} 1, & x \in S_j \\ 0, & x \notin S_j \end{cases}; \qquad \sum_{j=1}^{r} \eta_j = 1 \tag{2.37}$$

See Fig. 2.6 of the Example for a repartition of the state space on two subspaces S_1 and S_2 related to the corresponding membership function.

On every S_j subspace, the fuzzy system can be denoted by:

$$\dot{x}(t) = (A_j + \Delta A_j(t))x(t) + (B_j + \Delta B_j(t))u(t) \tag{2.38}$$

with,

$$\begin{aligned} \Delta A_j(t) &= \sum_{i=1, i\neq j}^{r} h_i(z(t))(A_i - A_j); \\ \Delta B_j(t) &= \sum_{i=1, i\neq j}^{r} h_i(z(t))(B_i - B_j) \end{aligned} \tag{2.39}$$

It is assumed that if the jth subsystem is in the jth subspace, it will stay in this subspace for a $t_j > \tau$, $\tau > 0$ time is a fixed constant. The number of traversing time instants among the regions is also assumed to be finite.

Remark 2.3 It is useful to note that $\Delta A_j(t)$ and $\Delta B_j(t)$ are known at any time and the studied system is not an uncertain system. However, in order to obtain simpler stability conditions, this technique assumes that terms $\Delta A_j(t)$ and $\Delta B_j(t)$ are like uncertain terms and are bounded.

Following the idea of [19], we assume that an upper bound of each like uncertainty term is known and is given by,

$$-E_{1j} \leq \Delta A_j(t) \leq E_{1j};\ E_{1j} \geq 0, \quad \forall t \succeq 0;\ j = 1, \ldots, r \tag{2.40}$$
$$-E_{2j} \leq \Delta B_j(t) \leq E_{2j};\ E_{2j} \geq 0, \quad \forall t \succeq 0;\ j = 1, \ldots, r \tag{2.41}$$

This type of inequality bounds can always be transformed to the following quadratic bounds,

$$[\Delta A_j(t)]^{\mathrm{T}}[\Delta A_j(t)] \leq E_{1j}^{\mathrm{T}} E_{1j}, \quad \forall t \succeq 0;\ j = 1, \ldots, r \tag{2.42}$$
$$[\Delta B_j(t)]^{\mathrm{T}}[\Delta B_j(t)] \leq E_{2j}^{\mathrm{T}} E_{2j}, \quad \forall t \succeq 0;\ j = 1, \ldots, r \tag{2.43}$$

Note that the details about the estimation of the upper bounds according to (2.40)–(2.41) are widely developed in [19]. We obtain r distinct linear time-varying subsystems. The stabilization problem of fuzzy system (2.1) without saturation constraints has been studied in [19] by using extreme subsystems obtained with the upper bounds of the like uncertainty terms (2.40)–(2.41). In our case, the upper uncertainty bounds are also used to obtain asymptotic stability conditions, while the like uncertain subsystems are used directly to built necessary and sufficient conditions of positive invariance.

The control is constrained as follows:

$$u \in \Omega = \left\{ u \in \mathbb{R}^m / -q_2 \ \leq\ u \ \leq\ q_1;\ q_1,\ q_2\ \in\ \mathbb{R}^m \right\}. \tag{2.44}$$

The idea of this approach is to choose on every S_j, $j \in 1, \ldots, r$ subspace, fuzzy subsystem (2.38) and consider that the interaction of the corresponding system with all the remainder $r-1$ subsystems is taken into account by uncertainty terms $\Delta A_j(t)$ and $\Delta B_j(t)$. The objective is then to design for such a subsystem a feedback control given by:

$$u(t) = F_j x(t), x(t) \in S_j \tag{2.45}$$

which guarantees the asymptotic stability of the like uncertain subsystem (2.38) despite the presence of the saturations (2.44). The subsystem in closed-loop is given by:

$$\dot{x}(t) = \left[(A_j + B_j F_j) + (\Delta A_j(t) + \Delta B_j(t) F_j) \right] x(t) \tag{2.46}$$

Note that the control in system (2.1) can be considered in this approach as a switching control formed by all the subsystem controls and given by,

$$u(t) = \sum_{j=1}^{r} \eta_j F_j x(t) \tag{2.47}$$

In the constrained case, we follow the approach proposed in [9, 10, 23]. Recall that this approach consists in giving conditions allowing the choice of a stabilizing controller (2.45) in such a way that model (2.46) remains valid every time. This is only possible if the state is constrained to evolve in a specified region defined by

$$\mathscr{D}_j = \{x \in \mathbb{R}^n / -q_2 \le F_j x \le q_1;\quad q_1, q_2 \in \mathbb{R}^m\}; \tag{2.48}$$

Note that these domains are convex and unbounded for $m < n$.

The result of stabilizability of the fuzzy system without constrained control, using the idea of [19] based on the upper extreme subsystems to obtain conditions of asymptotic stability for the fuzzy system (2.1), is reminded below according to the following definition.

Definition 2.2 The system (2.1) is said to be quadratically stabilizable if there exists a control law (2.6), a positive symmetric matrix P and a scalar $\gamma > 0$ such that the following condition is satisfied:

$$\begin{aligned}\dot{V}(x(t)) = x(t)^{\mathrm{T}} \Big\{ & [A(z) + B(z)F(z)]^{\mathrm{T}}\, P \\ & + P\,[A(z) + B(z)F(z)]\Big\}\, x(t) \le -\gamma \|x\|^2\end{aligned} \tag{2.49}$$

$\forall x(t) \in \mathbb{R}^n, \forall t > 0$ where $V(x) = x^{\mathrm{T}} P x$ is a Lyapunov function.

It is worth noting that if the system (2.1) is quadratically stabilizable, then function $V(x)$ is a Lyapunov function for the closed-loop system (2.7). Then, equilibrium point $x = 0$ will be uniformly asymptotically stable.

Lemma 2.2 [19]*: Fuzzy system* (2.1) *is quadratically stabilizable if and only if there exists a set of feedback gains* $(F_1, F_2, \ldots, F_r)$ *such that the following closed-loop subsystems with the accurate upper bounds are quadratically stable:*

$$\dot{x}(t) = (A_j + E_{j1})x(t) + (B_j + E_{j2})F_j x(t),\ x(t) \in S_j,\ j = 1, \ldots, r \tag{2.50}$$

Reminding that the stability result obtained by [19] is based on the use of Lemma 2.2 and a piecewise Lyapunov function candidate, as used by [20], given by,

$$V(x(t)) = x^{\mathrm{T}}(t)\Big(\sum_{j=1}^{r} \eta_j P_j\Big)x(t) \tag{2.51}$$

In our case, we first consider a common Lyapunov function for the application of Lemma 2.2, that is, $P_1 = \cdots = P_r$. In this case, function (2.51) becomes $V(x) = x^{\mathrm{T}} P x$. Define its level set by the following,

$$\varepsilon(P, \rho) = \Big\{x | x^{\mathrm{T}} P x \le \rho, \rho \succ 0\Big\} \tag{2.52}$$

The use of lemma 2.2 and the result of [26] enable us to state the main result of this chapter concerning the asymptotic stability of fuzzy system (2.1) with saturations (2.5).

Theorem 2.6 *If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{nxn}$ and a positive scalar ρ such that:*

$$\begin{aligned} (A_j + B_j F_j)^{\mathrm{T}} P + P(A_j + B_j F_j) + \\ (E_{j1} + E_{j2} F_j)^{\mathrm{T}} P + P(E_{j1} + E_{j2} F_j) < 0; \\ j = 1, \ldots, r; \end{aligned} \tag{2.53}$$

$$\varepsilon(P, \rho) \subset \mathscr{D}_j,\ j = 1, \ldots, r, \tag{2.54}$$

then, fuzzy system (2.1) *with feedback control* (2.47) *is asymptotically stable* $\forall x_0 \in \varepsilon(P, \rho)$.

Proof Conditions (2.53) imply that function $V(x) = x^{\mathrm{T}} P x$ is a common Lyapunov function of all the upper bound extreme subsystems (2.50). Reminding that level set $\varepsilon(P, \rho)$ of the common Lyapunov function is positively invariant w.r.t the upper bound extreme subsystems. According to lemma 2.2 and definition 2.2, this set is also a level set (region of stability) for uncertain subsystems (2.46), that is, set $\varepsilon(P, \rho)$ is also positively invariant w.r.t uncertain subsystems (2.46). Thus, the control is always admissible i.e., $-q_2 \le F_j x(t) \le q_1, \forall t \succeq 0$ by virtue of conditions (2.54). Consequently, each control $u(t) = F_j x(t)$ is admissible $\forall x_0 \in \varepsilon(P, \rho)$ and linear subsystem (2.46) is always valid inside this region of linear behavior. Hence, it is obvious that by applying the switching control (2.47) to the like uncertain fuzzy system (2.38), the control remains admissible by virtue of the following,

$$-q_2 \le F_j x(t) \le q_1, \quad \forall t \succeq 0,$$

implies

$$-q_2 \le \sum_{j=1}^{r} \eta_j F_j x(t) \le q_1, \quad \forall t \succeq 0;\ j = 1, \ldots, r$$

where η_j is defined by (2.37). In order to guarantee that this implication remains satisfied even if the state switches from a subspace S_j to a different subspace S_i, $i \neq j$, it is necessary to take the initial state inside the common domain $\varepsilon(P, \rho)$. The positive invariance property of the set $\varepsilon(P, \rho)$, implies that all the uncertain subsystems (2.46) remain linear despite the presence of the saturations. This fact allows the application of Lemma 2.2 and Definition 2.2 to these like uncertain subsystems to obtain r upper bound extreme subsystems by using the assumptions (2.39). If in addition the feedback controllers F_j satisfy conditions (2.53), then global fuzzy system (2.1) with feedback control (2.47) is asymptotically stable at origin $\forall x_0 \in \varepsilon(P, \rho)$ despite the presence of saturations. □

Note that another condition (2.53) were presented by [19] based on the well-known separation lemma

$$X^{\mathrm{T}} Y + Y^{\mathrm{T}} X \le \varepsilon X^{\mathrm{T}} X + \frac{1}{\varepsilon} Y^{\mathrm{T}} Y$$

for any positive scalar ε and matrices X, Y. In our case, condition (2.53) is easily resolved by the LMI technique.

It is worth noting that to include a symmetric ellipsoid inside a nonsymmetrical polyhedral, it is sufficient to realize this only inside the symmetrical part of the polyhedral. This means in our case, to realize (2.54) only with $\bar{q} = \min(q_1, q_2)$. It is well known that to obtain condition (2.54), one has only to satisfy the following inequalities [27],

$$\rho F_j^i P^{-1} (F_j^i)^{\mathrm{T}} \leq \bar{q}_i^2, \quad j = 1, \ldots, r; \; i = 1, \ldots, m, \tag{2.55}$$

where F_j^i is the ith row of matrix F_j, $\bar{q} = \min(q_1, q_2)$. These inequalities can be transformed by the use of Schur complement to the following LMI,

$$\begin{bmatrix} \beta_i & Y_j^i \\ * & X \end{bmatrix} \geq 0, \quad i = 1, \ldots, m \tag{2.56}$$

where Y_j^i is the ith row of matrix $Y_j = F_j X$, $X = P^{-1}$ and $\beta_i = \bar{q}_i^2 / \rho$.

The result of Theorem 2.6 is now used for control synthesis.

Theorem 2.7 *If there exist a symmetric matrix X, r matrices $Y_1, \ldots, Y_r$ and a positive scalar ρ solutions of the following LMIs:*

$$X(A_j + E_{1j})^{\mathrm{T}} + Y_j^{\mathrm{T}} (B_j + E_{2j})^{\mathrm{T}} + (A_j + E_{1j}) X + (B_j + E_{2j}) Y_j < 0, \tag{2.57}$$

$$\begin{bmatrix} \beta_i & Y_j^i \\ * & X \end{bmatrix} \geq 0, \tag{2.58}$$

$$X > 0,$$

$$j = 1, \ldots, r; \; i = 1, \ldots, m$$

where $\beta_i = \bar{q}_i^2 / \rho$, Y_j^i is the ith row of matrix Y_j;
then, fuzzy system (2.1) with feedback control (2.47) with,

$$F_j = Y_j X^{-1} \tag{2.59}$$

$$P = X^{-1} \tag{2.60}$$

is asymptotically stable at origin $\forall x_0 \in \varepsilon(P, \rho)$.

Proof Follows readily from Theorem 2.6. □

This result is easily applied to design controllers: solving LMIs (2.57)–(2.58) by any common available software (in our case we used the Matlab LMI control toolbox), matrix P, and controllers gains F_i can be computed easily according to equalities (2.59) and (2.60). Nevertheless, a common Lyapunov function for all the r upper bound extreme subsystems does not always exists. We can then attempt to use a piecewise Lyapunov function candidate as used by [19]. The use of this type of function is not easy when the system is, in addition, with constrained control. The following result proposes a sufficient condition of asymptotic stability based on a piecewise function.

Define the following polyhedral set,

$$\Gamma(\delta) = \left\{x \in \mathbb{R}^n / - \delta \le x \le \delta; \delta \succ 0\right\} \tag{2.61}$$

In this approach, we would like to design all controller gains F_j such that all the level sets associated to matrices P_j, $j = 1, \ldots, r$ contain the same $\Gamma(\delta)$ polyhedra. This is possible if we add the following constraint to our problem,

$$\Gamma(\delta) \subset \varepsilon(P_j, \rho_j); \quad j = 1, \ldots, r. \tag{2.62}$$

Remark 2.4 Condition (2.62) can also be given under LMI form. For this, redefine polyhedral set $\Gamma(\delta)$ in the equivalent form,

$$\Gamma(\delta) = \mathrm{cov}\{v_1, v_2, \ldots, v_\kappa\},$$

where $v_l \in \mathbb{R}^n$ states for the vertex of the bounded polyhedron $\Gamma(\delta)$. Note that $\kappa = 2^n$. With this, condition (2.62) is equivalent to,

$$v_l^{\mathrm{T}} P_j v_l \le \rho_j, \quad l = 1, \ldots, \kappa; \;\; j = 1, \ldots, r.$$

By virtue of Schur complement, the latter is equivalent to,

$$\begin{bmatrix} \rho_j & v_l^{\mathrm{T}} \\ * & X_j \end{bmatrix} \ge 0, \tag{2.63}$$

$$j = 1, \ldots, r; \;\; l = 1, \ldots, \kappa. \tag{2.64}$$

with $X_j = P_j^{-1}$.

The following result ensures to realize this objective.

Theorem 2.8 *For given positive scalars $\rho_1, \ldots, \rho_r$ and positive vector δ, if there exist symmetric definite positive matrices $X_1, \ldots, X_r$ and matrices $Y_1, \ldots, Y_r$, solutions of the following LMIs:*

$$\begin{aligned} X_j(A_j + E_{1j})^{\mathrm{T}} + Y_j^{\mathrm{T}}(B_j + E_{2j})^{\mathrm{T}} + \\ (A_j + E_{1j})X_j + (B_j + E_{2j})Y_j < 0, \end{aligned} \tag{2.65}$$

$$\begin{bmatrix} \bar{q}_i^2/\rho_j & Y_j^i \\ * & X_j \end{bmatrix} \ge 0,$$

$$\begin{bmatrix} \rho_j & v_l^{\mathrm{T}} \\ * & X_j \end{bmatrix} \geq 0, \tag{2.66}$$
$$X_j > 0,$$
$$j = 1, \ldots, r; i = 1, \ldots, m; l = 1, \ldots, \kappa.$$

such that, the matrices in closed-loop satisfy,

$$\hat{A}_{cj}\delta + |E_{cj}|\delta \leq 0, \tag{2.67}$$

where, $A_{cj} = A_j + B_j Y_j X_j^{-1}$; $E_{cj} = E_{1j} + E_{2j} Y_j X_j^{-1}$ *and* $v_1, \ldots, v_\kappa$ *the corresponding vertices to vector* δ *;*
then, the fuzzy system (2.1) *with the feedback control* (2.47) *is asymptotically stable at the origin* $\forall x_0 \in \Gamma(\delta)$.

Proof Based on Lemma 2.2 and the use of piecewise Lyapunov function candidate (2.51), the feasibility of LMIs (2.65)–(2.66), give symmetric positive definite matrices $P_j = X_j^{-1}$ and gain controllers $F_j = Y_j X_j^{-1}$ ensuring the asymptotic stability at the origin of fuzzy system (2.1) with feedback control (2.47) which is always admissible by virtue of conditions (2.66), that is, each level set $\varepsilon(P_j, \rho_j) \subset \mathscr{D}_j$. Note also, that all obtained level sets $\varepsilon(P_j, \rho_j) \supset \Gamma(\delta)$. The new problem in this approach with a piecewise function and a switching control, is, even if inside any level set $\varepsilon(P_j, \rho_j)$, the control is admissible, this property may be lost when a switch occurs according to strategy (2.37). This problem can be solved if we can ensure for the system that common set $\Gamma(\delta)$ is positively invariant w.r.t the all r uncertain subsystems (2.38). This fact is realized with condition (2.67), which is a direct application of Theorem 2.1. In this case, the state of the system belongs to inside all the sets $\mathscr{D}_j$, that is, the state feedback control is always admissible, $\forall x_0 \in \Gamma(\delta)$. □

The result of this theorem can be used in two steps: the first step consists in computing the solutions of LMIs (2.65)–(2.66). With these solutions, matrices in closed-loop are computed. The second step consists in testing conditions (2.67) with vector δ as a design parameter.

2.5.1 Examples

In this section, we apply our results to two examples. The first one is the one studied in [19].

Example 2.1 Consider the problem of balancing an inverted pendulum on a cart presented by Fig. 2.5.

The equations of the motion for the pendulum are,

$$\dot{x}_1 = x_2$$

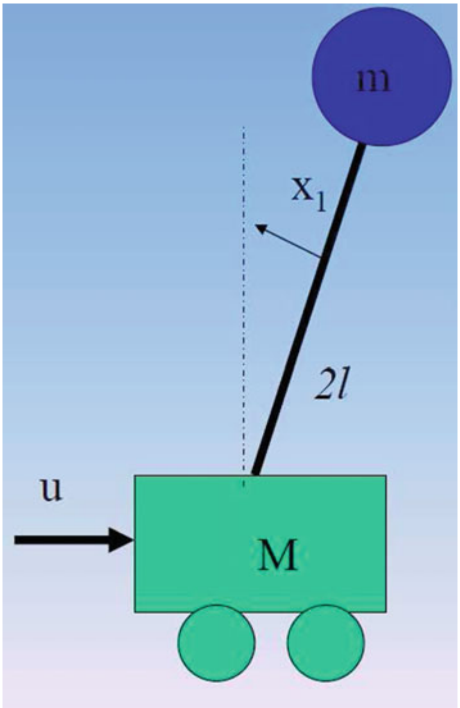


Fig. 2.5 Inverted pendulum system

$$\dot{x_2} = \frac{g\sin(x_1) - am_p l x_2^2 \sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - am_p l\cos^2(x_1)}$$

where x_1 denotes the angle of the pendulum from the vertical, and x_2 is the angular velocity. g is the gravity acceleration, m_p the mass of the pendulum, m_c is the mass of the cart, $2l$ is the length of the pendulum and u is the force applied to the cart. $a = 1/(m_p + m_c)$. The following data are chosen: $m_p = 2\,\text{kg}$; $m_c = 8$ kg and $2l = 1\,\text{m}$. We also add the following saturation on the control,

$$-3000 \le u \le 3500$$

The following fuzzy model is used to design a fuzzy controller.

$$\begin{aligned}
&\text{Rule 1:} \quad \text{IF} \quad x_1(t) \text{ is about } \ 0 \\
&\qquad\qquad \text{THEN} \quad \dot{x}(t) = A_1 x(t) + B_1 u(t) \\
&\text{Rule 2:} \quad \text{IF} \quad x_1(t) \text{ is about } \ \pm\pi/2 \\
&\qquad\qquad \text{THEN} \quad \dot{x}(t) = A_2 x(t) + B_2 u(t)
\end{aligned}$$

with,

$$A_1 = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 1 \\ 9.3648 & 0. \end{bmatrix};$$

$$B_1 = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0 \\ -0.0054 \end{bmatrix}$$

The membership functions and the upper bounds as used by [19] are as follows:

$$\begin{aligned} M_1^1(x_1(t)) &= \left(1 - \frac{1}{1+e^{-7(x_1(t)-\pi/4)}}\right) \cdot \left(\frac{1}{1+e^{-7(x_1(t)+\pi/4)}}\right) \\ &= h_1(t) \\ M_1^2(x_1(t)) &= 1 - M_1^1(x_1(t)) = h_2(t) \end{aligned}$$

$$E_{11} = E_{12} = 0.1|A_1 - A_2|; \, E_{21} = E_{22} = 0.01|B_1 - B_2|.$$

By applying Theorem 2.7, we find the following result.

$$P = \begin{bmatrix} 14.9944 & 4.7034 \\ 4.7034 & 1.4756 \end{bmatrix};$$

The obtained gain controllers are given by,

$$F_1 = \begin{bmatrix} 849.7047 & 253.0999 \end{bmatrix}; \, F_2 = \begin{bmatrix} 3156.8 & 990.5 \end{bmatrix}$$

Common set $\varepsilon(P, \rho = 10)$ of asymptotic stability is given by Figure 2.6 together with sets $\mathscr{D}_j$.

Figure 2.6 presents the evolution of the state of the system in closed-loop (in red color) inside the set of asymptotic stability $\varepsilon(P, \rho)$ (in magneta color) for different initial states, the evolution of the control and the membership function together with the sets S_1 and S_2.

The application of Theorem 2.8 leads to nonfeasible LMIs due to the structure of matrices A_1, A_2, B_1, B_2 which are under Compagnon form. With any feedback controller, the matrices of the system in closed-loop remain under the same form. Hence, condition (2.67) cannot be satisfied. To overcome this problem, one has to apply any non singular transformation to the initial linear subsystems.

Example 2.2 Consider now the following constrained nonlinear system,

$$\begin{aligned} \dot{x}_1(t) &= -2.1x_1 + 1.5x_2(t) + 2.5u_1(t) + 0.5u_2(t) \\ \dot{x}_2(t) &= 3.5x_1(t) - 0.5\left[0.5 + \ln(x_1^2 + 1)\right]x_2(t) + u_1(t) - 1.5u_2(t) \end{aligned}$$

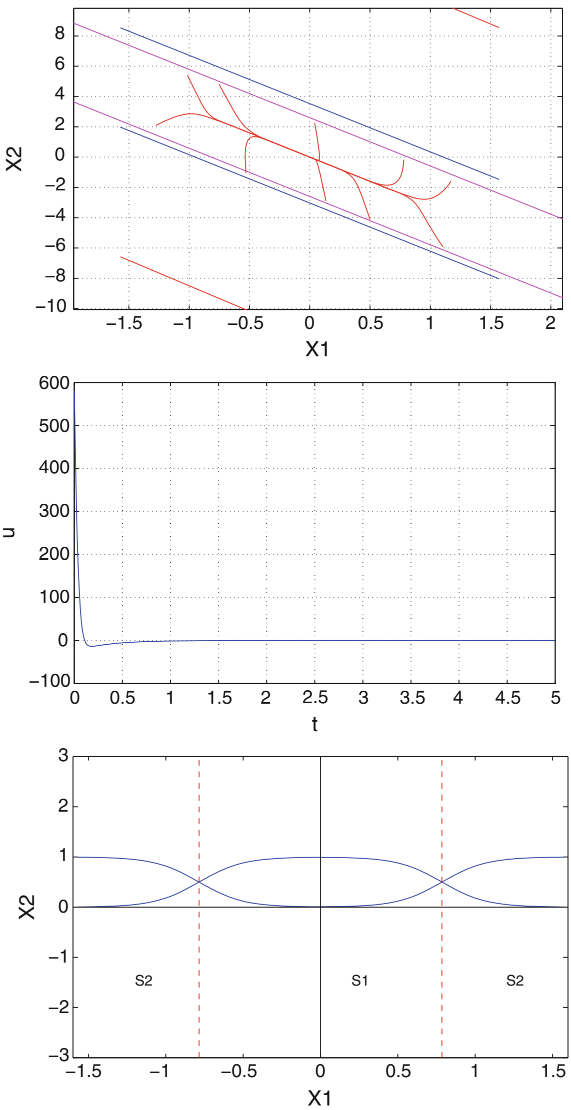


Fig. 2.6 Evolution of the state, control and membership functions

where the control is constrained as follows:

$$-q_2 \leq u \leq q_1; q_1 = \begin{bmatrix} 35 \\ 45 \end{bmatrix}; q_2 = \begin{bmatrix} 40 \\ 45 \end{bmatrix}$$

Now we give the exact approximation of the nonlinear system by a T–S model. For this, assume that $x_1(t) \in [-\gamma, \gamma]$, then one can write,

$$\ln(x_1^2 + 1) = M_1^1(x_1(t)) \cdot 0 + M_1^2(x_1(t)) \cdot \ln(\gamma^2 + 1) \tag{2.68}$$

with,

$$M_1^1(x_1(t)) = \frac{\ln(\gamma^2+1) - \ln(x_1^2+1)}{\ln(\gamma^2+1)} = h_1(t)$$
$$M_1^2(x_1(t)) = 1 - M_1^1(x_1(t)) = \frac{\ln(x_1^2+1)}{\ln(\gamma^2+1)} = h_2(t)$$

The fuzzy model which represents exactly the nonlinear system is given by:

$$\text{If } x_1(t) \text{ is } M_1^1 \text{ Then } \dot{x}(t) = A_1x(t) + B_1u(t); -q_2 \le u \le q_1$$

$$\text{If } x_1(t) \text{ is } M_1^2 \text{ Then } \dot{x}(t) = A_2x(t) + B_2u(t); -q_2 \le u \le q_1$$

where matrices A_1, A_2, B_1 and B_2 are given by,

$$A_1 = \begin{bmatrix} -2.1 & 1.5 \\ 3.5 & -0.25 \end{bmatrix}; \quad A_2 = \begin{bmatrix} -2.1 & 1.5 \\ 3.5 & -0.5(0.5+\ln(\gamma^2+1)) \end{bmatrix};$$
$$B_1 = B_2 = \begin{bmatrix} 2.5 & 0.5 \\ 1 & -1.5 \end{bmatrix}.$$

For this fuzzy system composed of two subsystems, one can take the following upper bounds:

$$E_{11} = 0.5|A_2 - A_1|; E_{21} = 0; E_{12} = 0.5|A_1 - A_2|; E_{22} = 0.$$

Choose $\gamma = 12$.

The asymptotic stability is guaranteed by the existence of one symmetric positive definite matrix by resolving the LMI (2.57)–(2.58):

$$P = \begin{bmatrix} 0.0224 & -0.0016 \\ -0.0016 & 0.0127 \end{bmatrix};$$

The obtained gain controllers are given by,

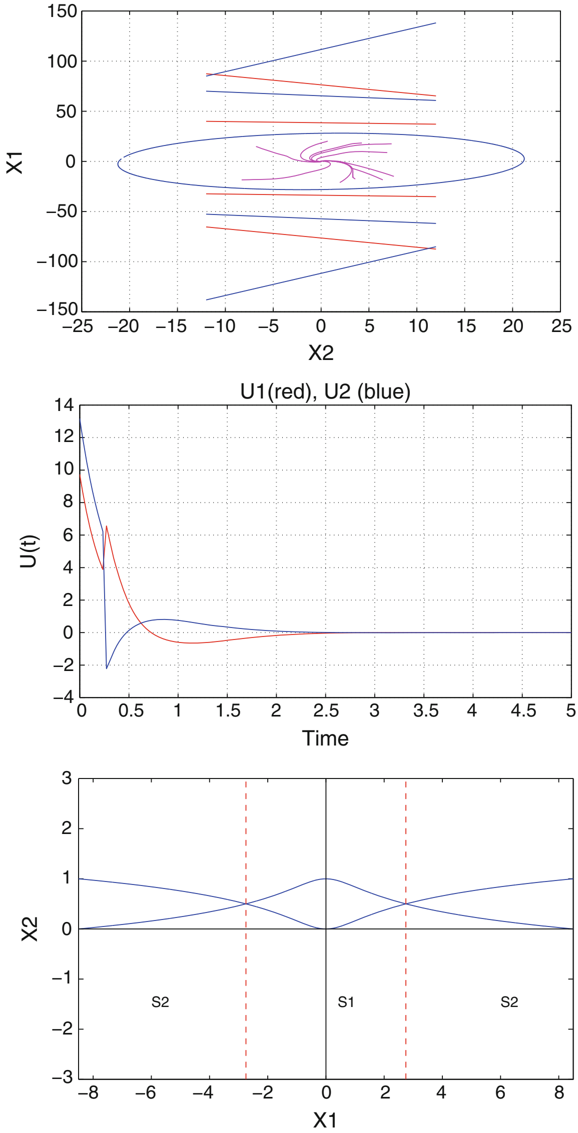
$$F_1 = \begin{bmatrix} -0.1243 & -1.0373 \\ 0.5409 & 0.5891 \end{bmatrix}; \quad F_2 = \begin{bmatrix} -0.2354 & -0.6114 \\ 0.8910 & -0.4031 \end{bmatrix}.$$

Common set $\varepsilon(P, \rho = 10)$ of asymptotic stability is given by Fig. 2.7 together with sets $\mathscr{D}_j$.

Figure 2.7 presents the evolution of the state of the system in closed-loop inside the set (in blue) of asymptotic stability $\varepsilon(P, \rho)$ for different initial states, the evolution of the control for an initial state inside $\varepsilon(P, \rho)$ and the membership function together with the sets S_1 and S_2.

Now, we apply the results of Theorem 2.8. The resolution of LMIs (2.65)–(2.66) leads to the following solutions only for a reduced upper bound and interval of

Fig. 2.7 Evolution of the sate, control, and membership functions



evolution of x_1: :

$$E_{11} = 0.25|A_2 - A_1|;\ E_{21} = 0;\ E_{12} = 0.25|A_1 - A_2|;\ E_{22} = 0;\ \gamma = 8.5.$$

$$P_1 = \begin{bmatrix} 0.0141 & -0.0005 \\ -0.0005 & 0.0066 \end{bmatrix}; \qquad P_2 = \begin{bmatrix} 0.0132 & -0.0053 \\ -0.0053 & 0.0085 \end{bmatrix}.$$

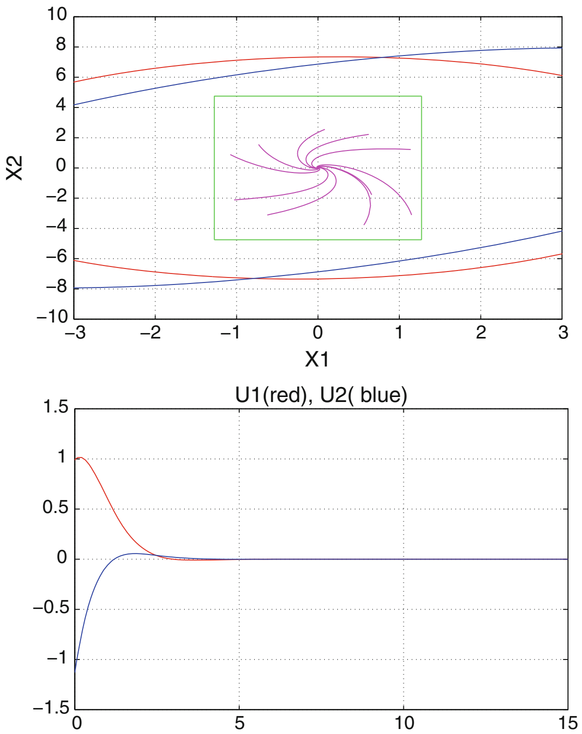


Fig. 2.8 Evolution of the state and control

The obtained controller gains are given by,

$$F_1 = \begin{bmatrix} 0.0184 & -0.7839 \\ 0.9094 & 0.0645 \end{bmatrix}; F_2 = \begin{bmatrix} 0.2772 & -0.5679 \\ 0.8088 & -0.4785 \end{bmatrix}.$$

For given polyhedral set $\Gamma(\delta)$, with $\delta = [1.2735 \;\; 4.7475]^{\mathrm{T}}$, condition (2.67) is also satisfied.

$$\left(\hat{A}_{c1} + |E_{c1}|\right)\delta = \begin{bmatrix} -0.0070 \\ -0.0761 \end{bmatrix}; \left(\hat{A}_{c2} + |E_{c2}|\right)\delta = \begin{bmatrix} -0.5216 \\ -4.8546 \end{bmatrix}.$$

Common set $\Gamma(\delta)$ of positive invariance is given by Fig. 2.8 together with sets $\mathscr{D}_j$ and $\varepsilon(P_j, \rho_j)$; $\rho_1 = 8$, $\rho_2 = 9$.

Figure 2.8 presents the evolution of the state of the system in closed-loop inside the common set of positive invariance $\Gamma(\delta)$ for different initial states, the time evolution of the control for an initial state inside the common set of positive invariance $\Gamma(\delta)$.

The study of these two examples shows that the result of Theorem 2.7 are less conservative than the results of Theorem 2.8. This fact is due to the more constraining condition (2.67) which is needed with the use of a piecewise Lyapunov function candidate. Consequently, a common Lyapunov function, when it exists, is more adequate to the design of fuzzy controllers for a nonlinear systems with constrained control.

2.6 Improved Conditions of Stabilizability

In this section, we follow the approach proposed in [10, 23, 26]. This approach uses the following piecewise smooth quadratic Lyapunov function candidate:

$$V(x(t)) = x^{\mathrm{T}}(t)Px(t) \tag{2.69}$$

where $P = \sum_{j=1}^{r} \eta_j P_j$. Let us define the level set of this function by:

$$\varepsilon(P, \rho) = \left\{x \in \mathbb{R}^n \middle| V(x) \leq \rho;\ \ \rho \succ 0\right\}$$

In the previous section, the same methodology was used with a common Lyapunov function for all the r upper bound extreme subsystems and a piecewise Lyapunov function. Controller gains F_j were designed such that all the level sets associated to matrices P_j, $j = 1, \ldots, r$ contain a same predefined polyhedral Γ to ensure the asymptotic stability inside a common region. Nevertheless, in this section, we show that even if a piecewise Lyapunov function is used, no common region is needed at all to guarantee the asymptotic stability of the fuzzy system despite the presence of constraints on the control.The aim of this approach consists in giving conditions allowing the choice of stabilizing controller (2.45) in such a way that:

- $V(x(t))$ is Lyaponuv function of the fuzzy system.
- There exist a positive scalar ρ such that $\varepsilon(P, \rho) \subseteq \bigcap \mathscr{D}_j$.

Hence, for all $x \in \varepsilon(P, \rho)$ the system trajectory converges to the origin and the control never saturates.

For this, we remind below the result of stabilizability of the unconstrained fuzzy system, using the idea of [19] based on the upper extreme subsystems. The conditions of asymptotic stability for fuzzy system (2.1) are given according to Definition 2.2.

The use of Lemma 2.2 and the result of [26] enable us to state the main result of this paper concerning the asymptotic stability of the fuzzy system (2.1) with saturations (2.5).

Theorem 2.9 *If there exist a set of symmetric positives definite matrix $P_i \in \mathbb{R}^{n\times n}$ and a positive scalar ρ such that:*

$$\begin{aligned}(A_j + B_jF_j)^{\mathrm{T}}P_j + P_j(A_j + B_jF_j) + (E_{j1} + E_{j2}F_j)^{\mathrm{T}}P_j \\ + P_j(E_{j1} + E_{j2}F_j) < 0; \quad j = 1, \ldots, r;\end{aligned} \tag{2.70}$$

$$\varepsilon(P, \rho) \subset \mathscr{D}_j, \quad j = 1, \ldots, r, \tag{2.71}$$

then, the fuzzy system (2.1) *with the feedback control* (2.47) *is asymptotically stable* $\forall x_0 \in \varepsilon(P, \rho)$.

Proof Conditions (2.70) imply that function $V(x) = x^{\mathrm{T}} P x$ is a Lyapunov functions of all the upper bound extreme subsystems (2.50). Reminding that level set $\varepsilon(P, \rho)$ of the Lyapunov function is positively invariant w.r.t the upper bound extreme subsystems. According to Lemma 2.2 and Definition 2.2, this set is also a level set (region of stability) for uncertain subsystems (2.46), that is, set $\varepsilon(P, \rho)$ is also positively invariant w.r.t uncertain subsystems (2.46). Thus, the control is always admissible, i.e., $-q_2 \le F_j x(t) \le q_1, \forall t \succeq 0$ by virtue of conditions (2.54). Consequently, each control $u(t) = F_j x(t)$ is admissible $\forall x_0 \in \varepsilon(P, \rho)$ and linear subsystem (2.46) is always valid inside this region of linear behavior. Hence, it is obvious that by applying switching control (2.47) to uncertain fuzzy system (2.38), the control remains admissible by virtue of the following,

$$-q_2 \le F_j x(t) \le q_1, \quad \forall t \succeq 0 \text{ implies}$$
$$-q_2 \le \sum_{j=1}^{r} \eta_j F_j x(t) \le q_1, \quad \forall t \succeq 0; \;\; j = 1, \ldots, r$$

where η is defined by (2.37). In order to guarantee that this implication remains satisfied even if the state switches from a subspace S_j to another subspace S_i, $i \neq j$, it is necessary to take the initial state inside common domain $\varepsilon(P, \rho)$. The positive invariance property of the set $\varepsilon(P, \rho)$, implies that all the uncertain subsystems (2.46) remain linear despite the presence of the saturations. This fact allows the application of the Lemma 2.2 and Definition 2.2 to these uncertain subsystems to obtain r upper bound extreme subsystems by using assumptions (2.39). If in addition feedback controllers F_j satisfy conditions (2.70), then global fuzzy system (2.1) with the feedback control (2.47) is asymptotically stable at the origin $\forall x_0 \in \varepsilon(P, \rho)$ despite the presence of saturations. □

It is worth noting that to include a symmetric ellipsoid inside a nonsymmetrical polyhedral, it is sufficient to realize this only inside the symmetrical part of the polyhedral. This means in our case, to realize (2.54) only with $\bar{q} = \min(q_1, q_2)$. It is well known that to obtain condition (2.54), one has only to satisfy the following inequalities [27],

$$\rho F_j^i P^{-1} (F_j^i)^{\mathrm{T}} \le \bar{q}_i^2, \quad j = 1, \ldots, r; \;\; i = 1, \ldots, m, \tag{2.72}$$

where F_j^i is the ith row of matrix F_j, $\bar{q} = \min(q_1, q_2)$. These inequalities can be transformed by the use of Schur complement to the following LMI,

$$\begin{bmatrix} \beta_i & Y_j^i \\ * & X \end{bmatrix} \ge 0, \quad i = 1, \ldots, m \tag{2.73}$$

where Y_j^i is the ith row of matrix $Y_j = F_j X$, $X = P^{-1}$ and $\beta_i = \bar{q}_i^2/\rho$.

The result of Theorem 2.9 is now used for the control synthesis.

Theorem 2.10 *For given positive scalars ρ, if there exist symmetric definite positive matrices $X_1, \ldots, X_r$ and matrices $Y_1, \ldots, Y_r$, solutions of the following LMIs:*

$$X_j(A_j + E_{1j})^{\mathrm{T}} + Y_j^{\mathrm{T}}(B_j + E_{2j})^{\mathrm{T}} + (A_j + E_{1j})X_j + (B_j + E_{2j})Y_j < 0, \tag{2.74}$$

$$\begin{bmatrix} \beta_i & Y_j^i \\ * & X_s \end{bmatrix} \geq 0, \tag{2.75}$$

$$X_s > 0,$$

$$j = 1, \ldots, r; i = 1, \ldots, r; s = 1, \ldots, r$$

where $\beta_i = \bar{q}_i^2/\rho$, Y_j^i is the ith row of matrix Y_j;
then, fuzzy system (2.1) with feedback control (2.47) with,

$$F_j = Y_j X^{-1} \tag{2.76}$$

$$P_i = X_i^{-1} \tag{2.77}$$

is asymptotically stable at the origin $\forall x_0 \in \varepsilon(P, \rho)$.

Proof Follows readily from Theorem 2.9. □

This result is easily applied to design controllers: solving LMIs (2.74)–(2.75) by any common available software (in our case we used Matlab LMI control toolbox), matrices P_i and the controllers gains F_i can be computed easily according to equalities (2.76) and (2.77).

2.6.1 Example

Let us consider the same constrained nonlinear system studied in Example 2.2.

Solving the LMI (2.74)–(2.75) for $\gamma = 15$ we find:

$$P_1 = \begin{bmatrix} 0.1044 & 0.0050 \\ 0.0050 & 0.0356 \end{bmatrix}; P_2 = \begin{bmatrix} 0.0796 & -0.0395 \\ -0.0395 & 0.0511 \end{bmatrix}$$

The obtained gain controllers are given by,

$$F_1 = \begin{bmatrix} -0.3501 & -0.7210 \\ 1.0798 & 0.1654 \end{bmatrix}; F_2 = \begin{bmatrix} 0.1946 & -0.4226 \\ 0.7014 & -0.4580 \end{bmatrix}$$

The set of positive invariance $\varepsilon(P, \rho)$ is depicted in Fig. 2.9 together with sets $\mathscr{D}_j$ while Figs. 2.10, 2.11 present the evolution of the states and the control, respectively.

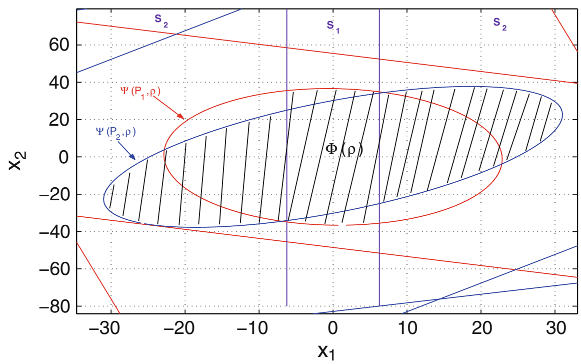


Fig. 2.9 The set $\Phi(\rho)$ representation

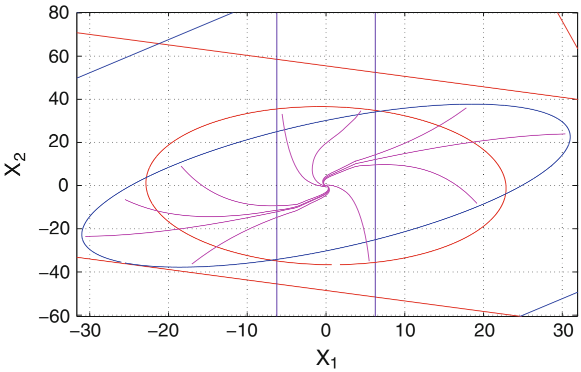


Fig. 2.10 Evolution of the state of the system in closed-loop inside the common set of positive invariance $\Phi(\rho)$ for different initial states

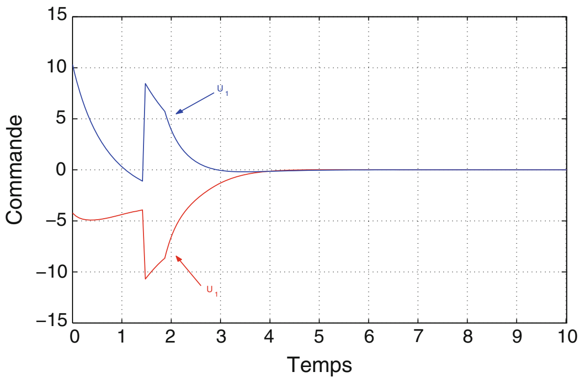


Fig. 2.11 Control time evolution for an initial state inside the common set of positive invariance $\Phi(\rho)$

Figure 2.9 plots the set $\Phi(\rho)$ together with sets $\varepsilon(P_1, \rho_1)$ and $\varepsilon(P_2, \rho_2)$ noted $\Psi(P_1, \rho_1)$ and $\Psi(P_2, \rho_2)$ respectively. Figure 2.10 presents the evolution of the state of the system in closed-loop inside the common set of positive invariance $\Phi(\rho)$ for different initial states. The corresponding control is depicted in Fig 2.11.

2.7 Stabilization of Saturated Discrete-Time T–S Fuzzy Systems

The objective of this section is to extend the results of [17] to discrete-time T–S fuzzy systems subject to actuator saturations. Thus, two directions are explored, based on two different methods, one direct and one indirect, leading to two different sets of LMIs. It is then shown, by application to a real plant model, that the indirect method, which uses the idea in [28] is less restrictive than the direct one, that uses [17]. The results of this section were published in [21]. The case of discrete-time T–S fuzzy systems is considered in this section.

2.7.1 Preliminaries

This section presents some preliminary results on which our work is based. Define the following subset of $\mathbb{R}^n$:

$$\mathscr{L}(F) = \left\{x \in \mathbb{R}^n \middle| \left|F_l x\right| \leq 1,\ l \in [1, m]\right\}, \tag{2.78}$$

with $F \in \mathbb{R}^{m\times n}$ and F_l stands for the l^{th} row of matrix F. $\mathscr{L}(F)$ is a polyhedral set where the saturations do not occur. Further, the set $\varepsilon(P, \rho)$ defined by (2.52), which is an ellipsoid, will be used as a level set of the Lyapunov function $V(x(k)) = x^{\mathrm{T}}(k)Px(k)$.

Lemma 2.3 [16] *Let $F, H \in \mathbb{R}^{m\times n}$ be given matrices, for $x \in \mathbb{R}^n$, if $x \in \mathscr{L}(H)$ then*

$$\mathrm{sat}(Fx) = \mathrm{co}\left\{D_i F x + D_i^- H x \ :\ i \in \left[1,\ 2^m\right]\right\},$$

with $D_i \in \mathscr{V}$ where

$$\mathscr{V} = \left\{G \in \mathbb{R}^{m\times m} / G = \mathrm{diag}\left\{\zeta_1, \ldots, \zeta_l, \ldots \zeta_m\right\}\right\},$$

with $\zeta_l = 1$ or 0, $D_i^- = \mathrm{i} - D_i$ and co *stands for the convex hull function.*

The main idea of [16] based on Lemma 2.3, is to build a third set with matrix H as $\mathscr{L}(H)$. This polyhedral set will be the set where saturations of the control are allowed without destabilizing the system. It is generally shown that set $\mathscr{L}(H)$ is larger than set $\mathscr{L}(F)$ [16].

Lemma 2.4 [17] *Suppose that matrices $G_i \in \mathbb{R}^{m\times n}$ $i = 1, 2, \ldots, r$ and a positive semi-definite matrix $P \in \mathbb{R}^{m\times m}$ are given:*

if

$$\sum_{i=1}^{r} h_i(k) = 1, \;\; 0 \leq h_i(k) \leq 1,$$

then

$$\left(\sum_{i=1}^{r} h_i(k) G_i^{\mathrm{T}}\right) P \left(\sum_{i=1}^{r} h_i(k) G_i\right) \leq \sum_{i=1}^{r} h_i(k) G_i^{\mathrm{T}} P G_i.$$

Lemma 2.5 [29] *Let* $x \in \mathbb{R}^n,\ H \in \mathbb{R}^{m\times n},\ P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ *such that* $\mathrm{rank}(H) = \sigma < n$. *The following statements are equivalent:*
(i) $x^{\mathrm{T}} P x < 0, \forall x \neq 0, Hx = 0$
(ii) $\exists X \in \mathbb{R}^{n\times m}: \;\; P + XH + H^{\mathrm{T}} X^{\mathrm{T}} < 0.$

2.7.2 Problem Statement

This section presents the problem to be solved. Consider the discrete-time T–S fuzzy system described by:

$$x(k+1) = A(z)x(k) + B(z)\mathrm{sat}(u(k)), \tag{2.79}$$

where

$$A(z) = \sum_{i=1}^{r} h_i(k) A_i, \;\; B(z) = \sum_{i=1}^{r} h_i(k) B_i,$$

The saturation function is defined as follows:

$$\mathrm{sat}(u_i(k)) = \begin{cases} 1, \;\; \text{if} \;\; u_i(k) > 1 \\ u(k), \;\; \text{if} \;\; -1 \leq u_i(k) \leq 1 \\ -1 \;\; \text{if} \;\; u_i(k) < -1 \end{cases} \tag{2.80}$$

Based on the Parallel Distribution Control (PDC) structure [2], we consider the following fuzzy control law for the T–S fuzzy system (2.79):

$$u(k) = \sum_{i=1}^{r} h_i(k) F_i x(k). \tag{2.81}$$

The objective of this work is to develop sufficient conditions of asymptotic stability of the T–S fuzzy system in closed-loop in presence of saturated control. These conditions will enable one to obtain a large set of initial values where the saturations of the control are allowed.

2.7.3 Conditions of Stabilizability

This section presents the main results that consist of two sufficient conditions of asymptotic stability of the T–S system in closed-loop, under the form of two sets of LMIs.

2.7.3.1 Direct Method

In this subsection, a direct method is used to derive sufficient conditions of asymptotic stability based on a common quadratic Lyapunv function candidate.

Theorem 2.11 *For a given fuzzy system* (2.79)*, suppose that local state feedback control matrices* $F_j, \ \ j = 1, ..., r$*, are given. Ellipsoid* $\varepsilon(P, \rho)$ *is a contractively invariant set of the closed-loop system under the fuzzy control law* (2.81) *if there exist matrices* $H_j \in \mathbb{R}^{m\times n}$, $j \in [1, r]$ *such that*

$$A_{ijs}^{\mathrm{T}} P A_{ijs} - P < 0, \quad \forall i, j \in [1, r], \ \forall s \in [1, 2^m] \tag{2.82}$$

$$\varepsilon(P, \rho) \subset \bigcap_{j=1}^{r} \mathscr{L}(H_j), \tag{2.83}$$

where

$$A_{ijs} = A_i + B_i[D_s F_j + D_s^- H_j].$$

Proof For any $x \in \bigcap_{j=1}^{r} \mathscr{L}(H_j)$, since $\sum_{i=1}^{r} h_i(k) = 1$ and $0 \le h_i(k) \le 1$ we have that:

$$x(k) \in \mathscr{L}\left(\sum_{j=1}^{r} h_j(k) H_j\right).$$

Then by Lemma 2.3,

$$\mathrm{sat}(u(k)) = \sum_{s=1}^{2^m} \delta_s(k) \left[D_s \sum_{j=1}^{r} h_j(k) F_j + D_s^- \sum_{j=1}^{r} h_j(k) H_j \right] x(k),$$

with $u(k) = \sum_{j=1}^{r} h_j(k) F_j x(k)$, hence, one can have the system in closed-loop as follows:

$$x(k+1) = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{s=1}^{2^m} \nu_{ijs}(k) A_{ijs} x(k),$$

with $A_{ijs} = A_i + B_i \left[D_s F_j + D_s^- H_j \right]$, and $\nu_{ijs}(k) = h_i(k) h_j(k) \delta_s(k)$.

Then, (2.79) becomes

$$x(k+1) = A(z)x(k),$$

where

$$A(z) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{2^m} \nu_{ijs}(k)A_{ijs}. \tag{2.84}$$

Select Lyapunov function candidate

$$V(x(k)) = x^{\mathrm{T}}(k)Px(k).$$

Computing its rate of increase gives

$$\begin{aligned}
\Delta V(x(k)) &= x^{\mathrm{T}}(k)\left[A(z)^{\mathrm{T}}PA(z) - P\right]x(k) \\
&= x^{\mathrm{T}}(k)\left[\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{2^m} \nu_{ijs}(k)A_{ijs}^{\mathrm{T}}\right)P\right. \\
&\quad \times \left.\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{2^m} \nu_{ijs}(k)A_{ijs}\right) - P\right]x(k) \\
&= x^{T}(k)\left[\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{2^m} h_i(k)h_j(k)\delta_s(k)A_{ijs}^{\mathrm{T}}\right)P\right. \\
&\quad \times \left.\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{2^m} h_i(k)h_j(k)\delta_s(k)A_{ijs}\right) - P\right]x(k)
\end{aligned}$$

for all

$$x(k) \in \bigcap_{j=1}^{r} \mathscr{L}(H_j).$$

By applying Lemma 2.4:

$$\Delta V(x(k)) \le x^{\mathrm{T}}(k)\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{2^m} \nu_{ijs}(k)A_{ijs}^{\mathrm{T}}PA_{ijs} - P\right]x(k).$$

This inequality is equivalent to

$$\Delta V(x(k)) \leq x^{\mathrm{T}}(k)\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{2^m} \nu_{ijs}(k)\left(A_{ijs}^{\mathrm{T}} P A_{ijs} - P\right)\right] x(k).$$

It is easy to see that $\Delta V(x(k)) < 0$ if

$$A_{ijs}^{\mathrm{T}} P A_{ijs} - P < 0, \forall i, j \in [1, r], \ \forall s \in [1, 2^m]$$

and

$$\varepsilon(P, \rho) \subset \bigcap_{j=1}^{r} \mathscr{L}(H_j). \qquad \square$$

In order to synthesize the controller, we give the following result:

Corollary 2.2 *For a given fuzzy system* (2.79)*, if there exist a symmetric positive definite matrix* $Q \in \mathbb{R}^{n\times n}$ *and matrices* $Y_j \in \mathbb{R}^{m\times n}$, $Z_j \in \mathbb{R}^{m\times n}$, $j \in [1, r]$ *and* $X \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{bmatrix} X^{\mathrm{T}} + X - Q & \left[A_i + B_i(D_s Y_j + D_s^- Z_j)\right]^{\mathrm{T}} \\ * & Q \end{bmatrix} > 0 \tag{2.85}$$

$$\forall i, j \in [1, r], \ \forall s \in [1, 2^m]$$

and

$$\begin{bmatrix} \frac{1}{\rho} & Z_{jl} \\ * & X^{\mathrm{T}} + X - Q \end{bmatrix} \geq 0 \tag{2.86}$$

$$\forall j \in [1, r], \ \forall l \in [1, m],$$

where * *denotes the transpose of the off-diagonal element,* Z_{jl} *stands for the* l*th row of matrix* Z_j*, then ellipsoid* $\varepsilon(P, \rho)$ *is a contractively invariant set of closed-loop system* (2.79)*, with*

$$F_i = Y_i X^{-1}, \ H_i = Z_i X^{-1} \textit{ and } P = Q^{-1}$$

Proof Assume that conditions (2.85)–(2.86) hold. Then the inequality (2.82) in 2.11 is equivalent to:

$$X^{\mathrm{T}} A_{ijs}^{\mathrm{T}} P A_{ijs} X - X^{\mathrm{T}} P X < 0,$$

$\forall i, j \in [1, r], \ \forall s \in [1, 2^m]$, and for all nonsingular matrix $X \in \mathbb{R}^{n\times n}$.

Let $Q = P^{-1}$ then we have

$$X^{\mathrm{T}} A_{ijs}^{\mathrm{T}} Q^{-1} A_{ijs} X - X^{\mathrm{T}} Q^{-1} X < 0,$$

$$\forall i, j \in [1, r], \ \forall s \in [1, 2^m].$$

By Schur complement, it is equivalent to:

$$\begin{bmatrix} X^{\mathrm{T}} Q^{-1} X & X^{\mathrm{T}} A_{ijs}^{\mathrm{T}} \\ * & Q \end{bmatrix} > 0,$$
$$\forall i, j \in [1, r], \ \forall s \in [1, 2^m]. \tag{2.87}$$

Since $(X - Q)^{\mathrm{T}} Q^{-1} (X - Q) > 0$, it follows that $X^{\mathrm{T}} Q^{-1} X \geq X^{\mathrm{T}} + X - Q$. Then $\Delta V(x(k)) < 0$ if

$$\begin{bmatrix} X^{\mathrm{T}} + X - Q & X^{\mathrm{T}} \left[A_i + B_i (E_s F_j + E_s^- H_j)\right]^{\mathrm{T}} \\ * & Q \end{bmatrix} > 0,$$

$$\forall i, j \in [1, r], \ \forall s \in [1, 2^m]. \tag{2.88}$$

To obtain an LMI, let $Y_j = F_j X \ and \ Z_j = H_j X$. Then condition (2.88) will be equivalent to

$$\begin{bmatrix} X^{\mathrm{T}} + X - Q & \left[A_i X + B_i (E_s Y_j + E_s^- Z_j)\right]^{\mathrm{T}} \\ * & Q \end{bmatrix} > 0,$$
$$\forall i, j \in [1, r], \ \forall s \in [1, 2^m].$$

Now, consider the condition (2.83) in Theorem 2.11, which is equivalent to [27]:

$$H_{jl} P^{-1} H_{jl}^{\mathrm{T}} \leqslant \frac{1}{\rho}, \ \ \forall j \in [1, r], \ \forall l \in [1, m],$$

where H_{jl} is the lth row of H_j. This inequality is equivalent to
$H_{jl} X X^{-1} P^{-1} X^{-\mathrm{T}} X^{\mathrm{T}} H_{jl}^{\mathrm{T}} \leqslant \frac{1}{\rho}$, for any nonsingular matrix $X \in \mathbb{R}^{n \times n}$.
By Schur complement, one obtains equivalently

$$\begin{bmatrix} \frac{1}{\rho} & H_{jl} X \\ * & X^{\mathrm{T}} P X \end{bmatrix} \geq 0.$$

As $Q = P^{-1}$, then one can have

$$\begin{bmatrix} \frac{1}{\rho} & H_{jl} X \\ * & X^{\mathrm{T}} Q^{-1} X \end{bmatrix} \geq 0.$$

Thus, if

$$\begin{bmatrix} \frac{1}{\rho} & H_{jl}X \\ * & X^{\mathrm{T}} + X - Q \end{bmatrix} \geq 0,$$

then, inequality (2.83) of Theorem 2.11 is satisfied, hence the result is obtained. Note that condition (2.88) implies $X^{\mathrm{T}} + X > 0$, that is, X is nonsingular. □

2.7.3.2 Indirect Method

In this subsection, an indirect method is used to derive sufficient conditions of asymptotic stability by using a common quadratic Lyapunov function.

Theorem 2.12 *For given fuzzy system (2.79), suppose that local state feedback control matrices F_j, $j = 1, ..., r$, are given. Ellipsoid $\varepsilon(P, \rho)$ is a contractively invariant set of the closed-loop system under fuzzy control law (2.81) if there exist matrices $H_j \in \mathbb{R}^{m\times n}$, $j \in [1, r]$, N_1 and $N_2 \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} N_1 A_{ijs} + A_{ijs}^{\mathrm{T}} N_1^{\mathrm{T}} - P & A_{ijs}^{\mathrm{T}} N_2^{\mathrm{T}} - N_1 \\ * & P - N_1 - N_2 \end{bmatrix} < 0 \tag{2.89}$$

$$\forall i, j \in [1, r], \ \forall s \in [1, 2^m]$$

and

$$\varepsilon(P, \rho) \subset \bigcap_{j=1}^{r} \mathscr{L}(H_j), \tag{2.90}$$

where $A_{ijs} = A_i + B_i \left[D_s F_j + D_s^- H_j\right]$.

Proof Let $V(x(k)) = x^{\mathrm{T}}(k)Px(k)$. Then

$$\Delta V(x(k)) = x^{\mathrm{T}}(k+1)Px(k+1) - x^{\mathrm{T}}(k)Px(k) < 0$$

is equivalent to

$$\begin{cases} x^{\mathrm{T}}(k+1)Px(k+1) - x^{\mathrm{T}}(k)Px(k) < 0 \\ x(k+1) = A(z)x(k) \end{cases},$$

which is also equivalent to

$$(\Sigma) \begin{cases} \begin{bmatrix} x^{\mathrm{T}}(k) & x^{\mathrm{T}}(k+1) \end{bmatrix} \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} < 0 \\ \begin{bmatrix} A(z) & -\mathrm{i} \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} = 0 \end{cases}$$

By virtue of Lemma 2.5, (Σ) is also equivalent to:
there exists a matrix X such that:

$$\begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} + X \begin{bmatrix} A^{\mathrm{T}}(z) & -\mathrm{i} \end{bmatrix} + \begin{bmatrix} A(z) & -\mathrm{i} \end{bmatrix}^{\mathrm{T}} X^{T} < 0$$

Let $X = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ then $\Delta V(x(k)) < 0$ if:
$\begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} + \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} A(z) & -\mathrm{i} \end{bmatrix} + \begin{bmatrix} A^{\mathrm{T}}(z) \\ -\mathrm{i} \end{bmatrix} \begin{bmatrix} N_1^{\mathrm{T}} & N_2^{\mathrm{T}} \end{bmatrix} < 0.$
By using (2.84) one can get

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{s=1}^{2^m} \nu_{ijs}(k) \begin{bmatrix} -P + N_1 A_{ijs} + A_{ijs}^{\mathrm{T}} N_1^{\mathrm{T}} & -N_1 + A_{ijs}^{\mathrm{T}} N_2^{\mathrm{T}} \\ N_2 A_{ijs} - N_1^{\mathrm{T}} & P - N_2 - N_2^{\mathrm{T}} \end{bmatrix} < 0,$$

where $\nu_{ijs}(k) = h_i(k) h_j(k) \delta_s(k)$. A sufficient condition to have $\Delta V(x(k)) < 0$ is

$$\begin{bmatrix} -P + N_1 A_{ijs} + A_{ijs}^{\mathrm{T}} N_1^{\mathrm{T}} & A_{ijs}^{\mathrm{T}} N_2^{\mathrm{T}} - N_1 \\ * & P - N_2 - N_2^{\mathrm{T}} \end{bmatrix} < 0 \qquad \square$$

In order to synthesize the controller, we give the following result:

Corollary 2.3 *For a given fuzzy system* (2.84) *and a given* $\sigma \in \mathbb{R}$, *if there exist a symmetric positive definite matrix* $Q \in \mathbb{R}^{n \times n}$ *and matrices* $Y_j,\ Z_j \in \mathbb{R}^{m \times n}$, $j \in [1, r]$ *and* $X \in \mathbb{R}^{n \times n}$ *such that*

$$\begin{bmatrix} \Gamma_{ijs} + \Gamma_{ijs}^{\mathrm{T}} - Q & -\sigma X + \Gamma_{ijs}^{\mathrm{T}} \\ * & \sigma^2 Q - \sigma X - \sigma X^{\mathrm{T}} \end{bmatrix} < 0 \tag{2.91}$$

$$\forall i, j \in [1, r],\ \forall s \in [1, 2^m]$$

and

$$\begin{bmatrix} \frac{1}{\rho} & (Z_j)_l \\ * & Q \end{bmatrix} \geq 0\,, \quad \forall j \in [1, r],\ \forall l \in [1, m], \tag{2.92}$$

where $\Gamma_{ijs} = A_i X + B_i \left[D_s Y_j + D_s^- Z_j \right]$,
then the ellipsoid $\varepsilon(Q^{-1}, \rho)$ *is a contractively invariant set of the closed-loop system* (2.79), *with*

$$F_i = Y_i X^{-1},\ H_i = Z_i X^{-1} \text{ and } P = X^{-\mathrm{T}} Q X^{-1}.$$

Proof In (2.89) let $N_i = X_i^{-\mathrm{T}},\ i = 1, 2$. Pre and post -multiplying inequality (2.89) by $\mathrm{diag}(X_1^{\mathrm{T}},\ X_2^{\mathrm{T}})$ and $\mathrm{diag}(X_1,\ X_2)$, respectively, one can get equivalently:

$$\begin{bmatrix} \Phi_{ijs} & X_1^{\mathrm{T}} A_{ijs}^{\mathrm{T}} - X_2 \\ * & X_2^{\mathrm{T}} P X_2 - X_2 - X_2^{\mathrm{T}} \end{bmatrix} < 0,$$

$$\forall i, j \in [1, r], \quad \forall s \in [1, 2^m]$$

where $\Phi_{ijs} = A_{ijs} X_1 + X_1^{\mathrm{T}} A_{ijs}^{\mathrm{T}} - X_1^{\mathrm{T}} P X_1$. Let $X_2 = \sigma X_1$ and $Q = X_1^{\mathrm{T}} P X_1$. Then,

$$\begin{bmatrix} \Phi_{ijs} & \sigma X_1 + X_1^{\mathrm{T}} A_{ijs}^{\mathrm{T}} \\ A_{ijs} X_1 - \beta X_1^{\mathrm{T}} & \sigma^2 Q - \sigma X_1 - \sigma X_1^{\mathrm{T}} \end{bmatrix} < 0,$$
$$\forall i, j \in [1, r], \quad \forall s \in [1, 2^m].$$

Let $X_1 = X$ then (2.91) is obtained. Note that (2.91) implies that matrix X is nonsingular.
Further, one can show that the inequality (2.90) is equivalent to [30]:
$H_{jl} P^{-1} H_{jl}^{\mathrm{T}} \leqslant \frac{1}{\rho}$ where H_{jl} is the lth row of H_j. That is,

$$H_{jl} X X^{-1} P^{-1} X^{-\mathrm{T}} X^{\mathrm{T}} H_{jl}^{\mathrm{T}} \leqslant \frac{1}{\rho}.$$

By Schur complement, one gets

$$\begin{bmatrix} \frac{1}{\rho} & H_{jl} X \\ * & X^{\mathrm{T}} P X \end{bmatrix} \geq 0.$$

Since $Q = X^{\mathrm{T}} P X$ and $Z_j = H_j X$ then

$$\begin{bmatrix} \frac{1}{\rho} & Z_{jl} \\ * & Q \end{bmatrix} \geq 0,$$

where Z_{jl} is the lth row of Z_j. □

2.7.4 Study of a Real Plant Model

In order to illustrate the obtained results, consider the balancing-up control of a simulated truck trailer proposed in [31] and given by Fig. 2.12.

The discrete-time state space model of the truck trailer is given by:

$$\begin{aligned} x_1(k+1) &= (1 - (v \cdot T_e/L)) x_1(k) + (v \cdot T_e/L) u(k) \\ x_2(k+1) &= x_2(k) + (v \cdot T_e/L) x_1(k) \\ x_3(k+1) &= x_3(k) + v \cdot T_e \cdot sin[x_2(k) + (v \cdot T_e/2L) x_1(k)], \end{aligned} \qquad (2.93)$$

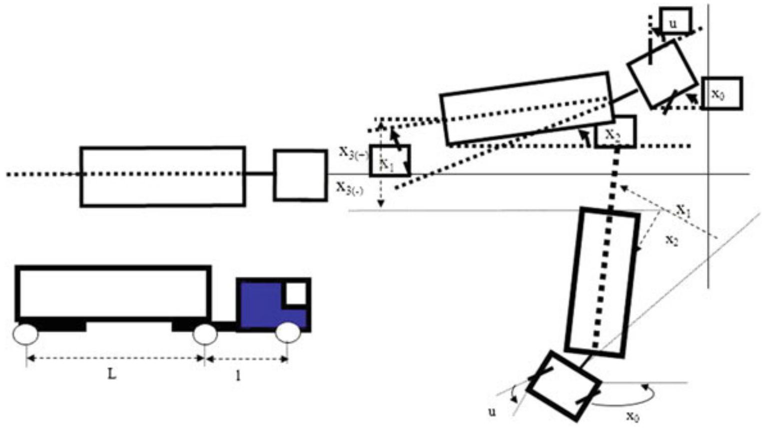


Fig. 2.12 Truck trailer system

where T_e stands for the sampling time, v the speed of bucking up of the engine, L and l are indicated in Fig. 2.12. The membership functions of this model are represented as

$$M_1^1(z(k)) = \frac{\sin(z(k))}{z(k)}, \;\; M_1^2(z(k)) = 1 - M_1^1(z(k)).$$

The nonlinear model of the vehicle can be described by the two following rules as described in [32]:

Rule 1:

if

$$z(k) = x_2(k) + \frac{v \cdot T_e.x_1(k)}{2L}$$

is about 0, then

$$x(k+1) = A_1 x(k) + B_1 u(k)$$

Rule 2:

if

$$z(k) = x_2(k) + \frac{v \cdot T_e.x_1(k)}{2L}$$

is about π: or $-\pi$, then

$$x(k+1) = A_2 x(k) + B_2 u(k)$$

Where:

$$A_1 = \begin{bmatrix} 1 - v \cdot T_e/L & 0 & 0 \\ v \cdot T_e/L & 1 & 0 \\ v^2 \cdot T_e^2/2L & v \cdot T_e & 1 \end{bmatrix}, B_1 = \begin{bmatrix} v \cdot T_e/l \\ 0 \\ 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1 - v \cdot T_e/L & 0 & 0 \\ v \cdot T_e/L & 1 & 0 \\ d \cdot v^2 \cdot T_e^2/2L & d \cdot v \cdot T_e & 1 \end{bmatrix}, B_2 = \begin{bmatrix} v \cdot T_e/l \\ 0 \\ 0 \end{bmatrix},$$

with $x = [x_1 \quad x_2 \quad x_3]^{\mathrm{T}}$, $l = 2.8\,\mathrm{m}$, $L = 5.5\,\mathrm{m}$, $v = -1\,\mathrm{m/s}$, $T_e = 2\,\mathrm{s}$, $d = 0.01/\pi$. The use of Corollary 2.2, leads to the following results obtained with the LMI toolbox of Matlab:

$$H_1 = H_2 = \begin{bmatrix} 1.2609 & -0.6759 & 0.0711 \end{bmatrix}$$

$$F_1 = F_2 = \begin{bmatrix} 2.4052 & -1.3751 & 0.1456 \end{bmatrix}$$

$$P = \begin{bmatrix} 4.5391 & -4.0483 & 0.4266 \\ -4.0483 & 6.2350 & -0.6541 \\ 0.4266 & -0.6541 & 0.1545 \end{bmatrix}.$$

The use of Corollary 2.3, leads to the following results obtained with the LMI toolbox of Matlab:

$$H_1 = H_2 = \begin{bmatrix} -0.0832 & -0.0900 & -0.0335 \end{bmatrix}$$

$$F_1 = F_2 = \begin{bmatrix} -4.5087 & -5.5010 & -0.4179 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.0232 & 0.0107 & 0.0034 \\ 0.0107 & 0.0424 & -0.0029 \\ 0.0034 & -0.0029 & 0.0091 \end{bmatrix}.$$

The results of both corollaries are shown in Figure 2.13.

Figure 2.13 plots the inclusion of the ellipsoid set inside the polyhedral set given by direct and indirect method.

Comment 2.1 *It is obvious that the use of Lemma 2.5 introduces an additional degree of freedom with the parameter h leading to less conservative LMIs as reported by* [28] *for linear systems. The obtained ellipsoid sets of asymptotic stability obtained with these two methods, for the studied example, are presented in Fig. 2.13. It is clear that the one corresponding to the second method is less conservative than expected, even the LMIs are applied without any optimization program.*

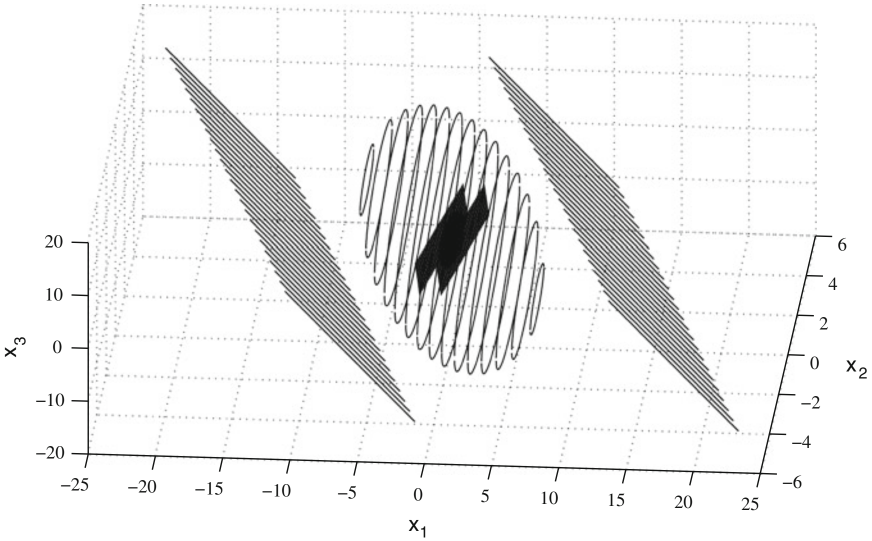


Fig. 2.13 Inclusion of the ellipsoid inside the polyhedral set given by direct (in *black*) and indirect method (in *gray*)

2.8 Conclusion

In this chapter, the problem of constrained nonlinear systems represented by fuzzy systems has been studied. The positive invariance tool has been used. Sufficient conditions of asymptotic stability have been obtained despite the presence of saturations on the control by using a common Lyapunov function and a piecewise Lyapunov function successively. The used approach is the one followed in [19] with uncertain subsystems and upper bound subsystems. The obtained results are successfully applied to two nonlinear systems with constrained control, represented by T–S fuzzy models. This leads to the characterization of a symmetric ellipsoid and a polyhedral common region of positive invariance and asymptotic stability successively. It is also shown that a common Lyapunov function, when it exists, leads to a less conservative region of positive invariance and asymptotic stability when the system is with constrained control. Improved conditions of stabilizability are also presented. It was shown that even a piecewise Lyapunov function is used, no common region is needed at all to guarantee the asymptotic stability of the fuzzy system despite the presence of constraints on the control. Hence, a set of Linear Matrix Inequalities (LMIs) is proposed to built stabilizing controllers.

This chapter also presents stability analysis and design methods for nonlinear systems with actuator saturation. T–S fuzzy models with actuator saturation are used to describe the nonlinear system. Two different methods, one direct and one indirect are used to derive sufficient conditions of asymptotic stability of T–S fuzzy systems with saturated control. Finally, these design methodologies are illustrated by their application to the stabilization of a balancing-up truck trailer. It is shown that the indirect method leads to less conservative LMIs since it leads to more larger stability domains.

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http://www.springer.com/978-3-319-05638-8

Advanced Takagi–Sugeno Fuzzy Systems
Delay and Saturation
Benzaouia, A.; El Hajjaji, A.
2014, XXXII, 294 p. 71 illus., 66 illus. in color., Hardcover
ISBN: 978-3-319-05638-8