Linear Toric Fibrations

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1 Introduction

These notes are based on three lectures given at the 2013 CIME/CIRM summer school *Combinatorial Algebraic Geometry*.

The purpose of this series of lectures is to introduce the notion of a *toric fibration* and to give its geometrical and combinatorial characterizations.

Toric fibrations $f : X \to Y$, together with a choice of an ample line bundle L on X are associated to convex polytopes called *Cayley sums*. Such a polytope is a convex polytope $P \subset \mathbb{R}^n$ obtained by assembling a number of lower dimensional polytopes R_i, whose normal fan defines the same toric variety Y. Let $\mathbb{R}^n = M \otimes \mathbb{R}$, for a lattice M. The building-blocks R_i are glued together following their image via a surjective map of lattices $\pi : M \to \Lambda$, see Definition 3.7. In particular the normal fan of the polytope $\pi(P)$ defines the generic fiber of the map f. We will denote Cayley sums by $\mathrm{Cayley}(R_0, \dots, R_t)_{\pi,Y}$. Our aim is to illustrate how classical notions in projective geometry are captured by certain properties of the associated Cayley sum.

When the image polytope $\pi(P)$ is a unimodular simplex Δ_k the generic fiber of the fibration f is a projective space $\mathbb{P}^k$ embedded linearly, i.e. $L|_F = \mathcal{O}_{\mathbb{P}^k}(1)$. For this reason the fibration is called a *linear toric fibration*. The following picture illustrates a linear toric fibration and the representation of the associated polytope as a Cayley sum.

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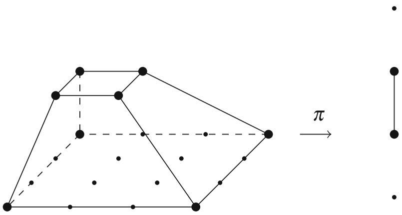
$f: \mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)) \to \mathbb{P}^2$



Section 3 will be devoted to define these concepts and to give the most relevant examples. In the following two sections we will present two characterizations of Cayley sums corresponding to linear toric fibrations. In both cases there are rich and interesting connections with classical projective geometry.

Section 4 discusses discriminants of polynomials. A polynomial supported on a subset $\mathscr{A} \subset \mathbb{Z}^n$ is a polynomial in n variables $x = (x_1, \ldots, x_n)$ of the form $p_{\mathscr{A}} = \sum_{a\in\mathscr{A}} c_a x^a$. The $\mathscr{A}$-discriminant is again a polynomial in $|\mathscr{A}|$ variables, $\Delta_{\mathscr{A}}(c_a)$, vanishing whenever the corresponding polynomial has at least one singularity in the torus $(\mathbb{C}^*)^n$. Understanding the existence and in that case the degree of the discriminant polynomial, for given classes of point-configurations $\mathscr{A}$, is highly desirable. Finite subsets $\mathscr{A} \subset \mathbb{Z}^n$ define toric projective varieties, $X_{\mathscr{A}} \subset \mathbb{P}^{|\mathscr{A}|-1}$. It is classical in Algebraic Geometry to associate to a given embedding, $X \subset \mathbb{P}^m$, the variety parametrizing hyperplanes singular along X. This variety is called the dual variety and it is denoted by $X^\vee$. Understanding when the codimension of the dual variety is higher that one and giving efficient formulas for its degree is a long standing problem. We will see that projective duality is a useful tool for describing the discriminants $\Delta_{\mathscr{A}}$ when the associated polytope $\mathrm{Conv}(\mathscr{A})$ is smooth or simple. In fact the case when $\Delta_{\mathscr{A}} = 1$ is completely characterized by Cayley sums and thus by toric fibrations.

In the non singular case the following holds.

Characterization 1. If $P_{\mathscr{A}} = \mathrm{Conv}(\mathscr{A})$ is a smooth polytope then the following assertions are equivalent:

(a) $P_{\mathscr{A}} = \mathrm{Cayley}_{\pi,Y}(R_0, \ldots, R_t)$ with $t \geqslant max(2, \frac{n+1}{2})$.
(b) $\mathrm{codim}(X^\vee_{\mathscr{A}}) > 1$.
(c) $\Delta_{\mathscr{A}} = 1$.

When the codimension of $X^\vee_{\mathscr{A}}$ is one then its degree is given by an alternating sum of volumes of the faces of the polytope $P_{\mathscr{A}}$. We will see that this formula corresponds to the top Chern class of the so called first jet bundle. This interpretation has a useful consequence. When the codimension of $X^\vee_{\mathscr{A}}$ is higher than one this Chern class has to vanish. This leads to another characterization of Cayley sums.

Characterization 2. If $P_{\mathscr{A}} = \mathrm{Conv}(\mathscr{A})$ is a smooth polytope then the following assertions are equivalent:

(a) $P_{\mathscr{A}} = \mathrm{Cayley}_{\pi,Y}(R_0, \ldots, R_t)$ with $t \geqslant max(2, \frac{n+1}{2})$.
(b) $\sum_{\emptyset\neq F\prec P_{\mathscr{A}}} (-1)^{\mathrm{codim}(F)}(\dim(F) + 1)!\,\mathrm{Vol}(F) = 0$.

In Sect. 5 we discuss the problem of classifying convex polytopes and algebraic varieties. A classification is typically done via invariants. In recent years much attention has been concentrated on the notion of codegree of a convex polytope.

$$\mathrm{codeg}(P) = min_{\mathbb{Z}}\{t \,|\, tP \text{ has interior lattice points}\}$$

The unimodular simplex for example has $\mathrm{codeg}(\Delta_n) = n + 1$. Batyrev and Nill conjectured that imposing this invariant to be large should force the polytope to be a Cayley sum.

It turned out that a $\mathbb{Q}$-version of this invariant, what we denote by $\mu(P)$, corresponds to a classical invariant in classification theory of algebraic varieties, called the log-canonical threshold. Let $(X_P, \mathscr{L}_P)$ be the toric variety and ample line bundle associated to the polytope P. The canonical threshold $\mu(\mathscr{L}_P)$ and the nef-value $\tau(\mathscr{L}_P)$ are the invariants used heavily in the classification theory of Gorenstein algebraic varieties. In particular Beltrametti-Sommese-Wisniewski conjectured that imposing $\mu(\mathscr{L}_P)$ to be large should force the variety to have the structure of a fibration.

Again in the toric setting we will see that these two stories intersect making it possible to prove the above conjectures, at least in the smooth case, and leading to yet another characterization of Cayley sums.

Characterization 3. Let P be a smooth polytope. The following assertions are equivalent:

(a) $\mathrm{codeg}(P) \geqslant (n + 3)/2$.
(b) P is isomorphic to a Cayley sum $\mathrm{Cayley}(R_0, \ldots, R_t)_{\pi,Y}$ where $t + 1 = \mathrm{codeg}(P)$ with $k > \frac{n}{2}$.
(c) $\mu(\mathscr{L}_P) = \tau(\mathscr{L}_P) \geqslant (n + 3)/2$.

In fact the characterizations above extend to more general classes of polytopes, not necessarily smooth, as we explain in Sects. 4 and 5. Section 6 is devoted to give a complete proof of these characterizations.

2 Conventions and Notation

We assume basic knowledge of toric geometry and refer to [EW, FU, ODA] for the necessary background on toric varieties. We will moreover assume some knowledge of projective algebraic geometry. We refer the reader to [HA, FUb] for further details. Throughout this paper, we work over the field of complex numbers $\mathbb{C}$. By a polarized variety we mean a pair (X, L) where X is an algebraic variety and L is an ample line bundle on X.

2.1 Toric Geometry

In this note a toric variety, X, is always assumed to be normal and thus defined by a fan $\Sigma_X \subset N \otimes \mathbb{R}$ for a lattice N. By $\Sigma_X(t)$ we will denote the collection of t-dimensional cones of Σ_X. The invariant sub-variety of codimension t associated to a cone $\sigma \in \Sigma(t)$ will be denoted by $V(\sigma)$.

For a lattice Δ we set $\Delta_{\mathbb{R}} = \Delta \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $\Delta^{\vee} = Hom(\Delta, \mathbb{Z})$ the dual lattice. If $\pi : \Delta \to \Gamma$ is a morphism of lattices we denote by $\pi_{\mathbb{R}} : \Delta_{\mathbb{R}} \to \Gamma_{\mathbb{R}}$ the induced $\mathbb{R}$-homomorphism. By a lattice polytope $P \subset \Delta_{\mathbb{R}}$ we mean a polytope with vertices in Δ.

Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n. Consider the graded semigroup Π_P generated by $(\{1\} \times P) \cap (\mathbb{N} \times \mathbb{Z}^n)$. The polarized variety $(Proj(\mathbb{C}[\Pi_P]), \mathscr{O}(1))$ is a toric variety associated to the polytope P. It will be sometimes denoted by (X_P, L_P). Notice that the toric variety X_P is defined by the (inner) normal fan of P. Vice versa the symbol $P_{(X,L)}$ will denote the lattice polytope associated to a polarized toric variety (X, L).

Two polytopes are said to be *normally equivalent* if their normal fans are isomorphic.

The symbol Δ_n denotes the smooth (unimodular) simplex of dimension n. Recall that an n-dimensional polytope is *simple* if through every vertex pass exactly n edges. A lattice polytope is *smooth* if it is simple and the primitive vectors of the edges through every vertex form a lattice basis. Smooth polytopes are associated to smooth projective toric varieties. Simple polytopes are associated to $\mathbb{Q}$-factorial projective toric varieties.

When the toric variety is defined via a point configuration $\mathscr{A} \subset \mathbb{Z}^n$ we will use the symbol $(X_{\mathscr{A}}, \mathscr{L}_{\mathscr{A}})$ for the associated polarized toric variety and $P_{\mathscr{A}} = \mathrm{Conv}(\mathscr{A})$ for the associated polytope. The corresponding fan is denoted by $\Sigma_{\mathscr{A}}$.

2.2 Vector Bundles

The notion of Chern classes of a vector bundle is an essential tool in some of the proofs. Let E be a vector bundle of rank k over an n-dimensional algebraic variety X. Recall that the i-th Chern class of E, $c_i(E)$, is the class of a codimension i cycle on X modulo rational equivalence. The top Chern class of a rank $k \geqslant n$ vector bundle is $c_n(E)$. The same symbol $c_n(E)$ will be used to denote the degree of the associated zero-dimensional subvariety.

The projectivization of a vector bundle plays a fundamental role throughout these notes. Let $S^l(E)$ denote the l-th symmetric power of a rank $r+1$ vector bundle E. The projectivization of E is $\mathbb{P}(E) = \mathrm{Proj}(\oplus_{l=0}^{\infty} S^l(E))$. It is a projective bundle with fiber $F = \mathbb{P}(E)_x = \mathbb{P}(E_x) = \mathbb{P}^r$. Let $\pi : \mathbb{P}(E) \to Y$ be the bundle map. There is a line bundle ξ on $\mathbb{P}(E)$, called the tautological line bundle, defined by the property that $\xi_F \cong \mathscr{O}_{\mathbb{P}^r}(1)$. When E is a vector bundle on a toric variety Y then the projective bundle $\mathbb{P}(E)$ has the structure of a toric variety if and only if $E = L_1 \oplus \ldots \oplus L_k$, [DRS04, Lemma 1.1.]. When the line bundles L_i are ample then the tautological line bundle ξ is also ample.

We refer to [FU] for the necessary background on vector bundles and their characteristic classes.

3 Toric Fibrations

Definition 3.1. A *toric fibration* is a surjective flat map $f : X \to Y$ with connected fibers where

(a) X is a toric variety
(b) Y is a normal algebraic variety
(c) $\dim(Y) < \dim(X)$.

Remark 3.2. A surjective morphism $f : X \to Y$, with connected fibers between normal projective varieties, induces a homomorphism from the connected component of the identity of the automorphism group of X to the connected component of the identity of the automorphism group of Y, with respect to which f is equivariant. It follows that if $f : X \to Y$ is a toric fibration then Y and a general fiber F admit a toric structure with respect to which f becomes an equivariant morphism. Moreover if X is smooth, respectively $\mathbb{Q}$-factorial, then Y and F are also smooth, respectively $\mathbb{Q}$-factorial.

Example 3.3. Let $L_0, \ldots, L_k$ be line bundles over a toric variety Y. The total space $\mathbb{P}(L_0 \oplus \ldots \oplus L_k)$ is a toric variety, Lemma 1.1, and the projective bundle $\pi : \mathbb{P}(L_0 \oplus \ldots \oplus L_k) \to Y$ is a toric fibration.

3.1 Combinatorial Characterization

A toric fibration has the following combinatorial characterization, see [EW, Chapter VI] for further details. Let $N \cong \mathbb{Z}^n$ be a lattice, $\Sigma \subset N \otimes \mathbb{R}$ be a fan and $X = X_\Sigma$, the associated toric variety. Let $i : \Delta \hookrightarrow N$ be a sub-lattice.

Proposition 3.4 ([EW]). *The inclusion i induces a toric fibration, $f : X \to Y$ if and only if:*

(a) *Δ is a primitive lattice, i.e. $(\Delta \otimes \mathbb{R}) \cap N = \Delta$.*
(b) *For every $\sigma \in \Sigma(n)$, $\sigma = \tau + \eta$, where $\tau \in \Delta$ and $\eta \cap \Delta = \{0\}$ (i.e. Σ is a split fan).*

We briefly outline the construction. The projection $\pi : N \to N/\Delta$ induces a map of fans $\Sigma \to \pi(\Sigma)$ and thus a map of toric varieties $f : X \to Y$. The general fiber

F is a toric variety defined by the fan $\Sigma_F = \{\sigma \in \Sigma \cap \Delta\}$. The invariant varieties $V(\tau)$ in X, where $\tau \in \Sigma$ is a maximal-dimensional cone in Σ_F, are called invariant sections of the fibration. The subvariety $V(\tau)$ is the invariant section passing through the fixed point of F corresponding to the cone $\tau \in \Sigma_F$. Observe that they are all isomorphic, as toric varieties, to Y.

Example 3.5. In Example 3.3 let $\Gamma \subset \mathbb{R}^n$ be the fan defining Y, and let $D_1, \dots, D_s$ be the generators of $Pic(Y)$ associated to the rays $\eta_i, \dots, \eta_s \subset \Gamma$. The line bundle L_i can be written as $L_i = \sum \phi_i(\eta_j) D_j$ where $\phi_i : \Gamma_{\mathbb{R}} \to \mathbb{R}$ are piecewise linear functions. Let $e_1, \dots, e_k \in \mathbb{Z}^k$ be a lattice basis and let $e_0 = -e_1 - \dots - e_k$. One can define a map:

$$\psi : \mathbb{R}^n \to \mathbb{R}^{n+k} \text{ as } \psi(v) = (v, \sum \phi_i(v) e_i).$$

Consider now the fan $\Sigma' \subset \mathbb{R}^{n+k}$ given by the image of Γ under ψ, $\Sigma' = \{\psi(\sigma), \sigma \subset \Gamma\}$. Let $\Pi \subset \mathbb{Z}^k$ be the fan defining $\mathbb{P}^k$. The fan $\Sigma = \{\sigma' + \tau | \sigma' \in \Sigma', \tau \in \Pi\}$ is a split fan, defining the toric fibration $\pi : \mathbb{P}(L_0 \oplus \dots \oplus L_k) \to Y$. See also [ODAb, Proposition 1.33].

Definition 3.6. A *polarized toric fibration* is a pair $(f : X \to Y, L)$, where f is a toric fibration and L is an ample line bundle on X.

Observe that for a general fiber F, the pair $(F, L|_F)$ is also a polarized toric variety. It follow that both pairs (X, L) and $(F, L|_F)$ define lattice polytopes $P_{(X,L)}, P_{(F,L|_F)}$. The polytope $P_{(X,L)}$ is in fact a "twisted sum" of a finite number of lattice polytopes fibering over $P_{(F,L|_F)}$.

Definition 3.7. Let $R_0, \dots, R_k \subset M_{\mathbb{R}}$ be lattice polytopes and let $k \geqslant 1$. Let $\pi : M \to \Lambda$ be a surjective map of lattices such that $\pi_{\mathbb{R}}(R_i) = v_i$ and such that $v_0, \cdots, v_k$ are distinct vertices of $\mathrm{Conv}(v_0, \dots, v_k)$. We will call a *Cayley π-twisted sum (or simply a Cayley sum) of* $R_0, \dots, R_k$ a polytope which is affinely isomorphic to $\mathrm{Conv}(R_0, \dots, R_k)$.

$$\text{We will denote it by: } [R_0 \star \dots \star R_k]_\pi.$$

If the polytopes R_i are additionally normally equivalent, i.e. they define the same normal fan Σ_Y, we will denote the Cayley sum by:

$$\mathrm{Cayley}(R_0, \dots, R_k)_{(\pi,Y)}.$$

We will see that these are the polytopes that are associated to polarized toric fibrations.

Proposition 3.8 ([CDR08]). *Let $X = X_\Sigma$ be a toric variety of dimension n, where $\Sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^n$, and let $i : \Delta \hookrightarrow N$ be a sublattice. Let L be an ample line bundle* *on X. Then the inclusion i induces a polarized toric fibration $(f : X \to Y, L)$ if and only if $P_{(X,L)} = \mathrm{Cayley}(R_0, \ldots, R_k)_{(\pi,Y)}$, where $R_0, \ldots, R_k$ are normally equivalent polytopes on Y and $\pi : M \to \Lambda$ is the lattice map dual to i.*

Proof. We first prove the implication "$\Rightarrow$".

Assume that $i : \Delta \hookrightarrow N$ induces a toric fibration $f : X \to Y$ and consider the polarization L on X. We will prove that $P_{(X,L)} = \mathrm{Cayley}(R_0, \ldots, R_k)_{(\pi,Y)}$ for some normally equivalent polytopes $R_0, \ldots, R_k$.

Notice first that the fact that Δ is a primitive sub-lattice of N implies that the dual map $\pi : M \to \Lambda$ is a surjection. Let F be a general fiber of f, and let $S := P_{(F,L_{|F})} \subset \Lambda_{\mathbb{R}}$. Denote by $v_0, \ldots, v_k$ the vertices of S. Every v_i corresponds to a fixed point of F; call $Y_i = V(\tau_i)$ the invariant section of f passing through that point. Note that $\tau_i \in \Sigma_X$, $\dim \tau_i = \dim F$ and $\tau_i \subset \Delta_{\mathbb{R}}$. Let R_i be the face of $P_{(X,L)}$ corresponding to Y_i.

Observe that $\mathit{Aff}(\tau_i) = \Delta_{\mathbb{R}}$, so that $\mathit{Aff}(\tau_i^{\perp}) = \Delta_{\mathbb{R}}^{\perp} = \ker \pi_{\mathbb{R}}$. Then there exists $u_i \in M$ such that:

- $\mathit{Aff}(R_i) + u_i = \ker(\pi_{\mathbb{R}})$;
- $R_i + u_i = P_{(Y_i, L_{|Y_i})}$.

This says that $R_0, \ldots, R_k$ are normally equivalent (because every Y_i is isomorphic to Y), and that $\pi_{\mathbb{R}}(R_i)$ is a point. Since the Y_i's are pairwise disjoint, the same holds for the R_i's. If s is the number of fixed points of Y, then each R_i has s vertices. On the other hand, we know that F has $(k+1)$ fixed points, and therefore X must have $s(k+1)$ fixed points. So $P_{(X,L)}$ has $s(k+1)$ vertices, namely the union of all vertices of the R_i's. We can conclude that

$$P_{(X,L)} = \mathrm{Conv}(R_0, \ldots, R_k)$$

Let $D = \sum_{x \in \Sigma(1)} a_x D_x$ be an invariant Cartier divisor on X such that $L = \mathcal{O}_X(D)$. Since F is a general fiber, we have $D_x \cap F \neq \emptyset$ if and only if $x \in \Delta$, and $D_{|F} = \sum_{x \in \Delta} a_x D_{x|F}$. This implies that $\pi_{\mathbb{R}}(R_i) = v_i$ and $\pi_{\mathbb{R}}(P_{(X,L)}) = S$. We conclude that $P_{(X,L)} = \mathrm{Cayley}(R_0, \ldots, R_k)_{(\pi,Y)}$.

We now show the other direction: "$\Leftarrow$".

Assume that $P_{(X,L)} = \mathrm{Cayley}(R_0, \ldots, R_k)_{(\pi,Y)}$. We will prove that the associated polarized toric variety is a polarized toric fibration. First observe that the fact that the dual map π is a surjection implies that the sublattice Δ is primitive. Since v_i is a vertex of $\pi_{\mathbb{R}}(P_{(X,L)})$, R_i is a face of $P_{(X,L)}$ for every $i = 0, \ldots, k$. Let Y be the projective toric variety defined by the polytopes R_i. Observe that $\mathit{Aff}(R_i)$ is a translate of $\ker \pi_{\mathbb{R}}$, and $(\ker \pi)^{\vee} = N/\Delta$. So the fan Σ_Y is contained in $(N/\Delta)_{\mathbb{R}}$.

Let $\gamma \in \Sigma_Y(\dim(Y))$ and for every $i = 0, \ldots, k$ let w_i be the vertex of R_i corresponding to γ. We will show that $Q := \mathrm{Conv}(w_0, \ldots, w_k)$ is a face of $P_{(X,L)}$.

Observe first that $(\pi_{\mathbb{R}})_{\mathit{Aff}(Q)} : \mathit{Aff}(Q) \to \Lambda_{\mathbb{R}}$ is bijective. Let H be the linear subspace of $M_{\mathbb{R}}$ which is a translate of $\mathit{Aff}(Q)$. Then we have $M_{\mathbb{R}} = H \oplus \ker \pi_{\mathbb{R}}$. Dually $N_{\mathbb{R}} = \Delta_{\mathbb{R}} \oplus H^{\perp}$, where $H^{\perp}$ projects isomorphically onto $(N/\Delta)_{\mathbb{R}}$.

Let $u \in H^{\perp}$ be such that its image in $(N/\Delta)_{\mathbb{Q}}$ is contained in the interior of γ. Then for every $i = 0, \ldots, k$ we have that (see [FU, §1.5]):

$$\begin{aligned} (u, x) &\geqslant (u, w_i) \quad \text{for every } x \in R_i, \\ (u, x) &= (u, w_i) \quad \text{if and only if } x = w_i. \end{aligned}$$

Moreover u is constant on $\mathit{Aff}(Q)$, namely there exists $m_0 \in \mathbb{Q}$ such that $(u, z) = m_0$ for every $z \in Q$.

Any $z \in P$ can be written as $z = \sum_{i=1}^{l} \lambda_i z_i$, with $z_i \in R_i$, $\lambda_i \geqslant 0$ and $\sum_{i=0}^{l} \lambda_i = 1$. Then

$$(u, z) = \sum_{i=1}^{l} \lambda_i (u, z_i) \geqslant \sum_{i=1}^{l} \lambda_i (u, w_i) = \sum_{i=1}^{l} \lambda_i m_0 = m_0.$$

Moreover, $(u, z) = m_0$ if and only if $\lambda_i > 0$ for every i such that $(u, z_i) = (u, w_i)$, and $\lambda_i = 0$ otherwise. This happens if and only if $z \in Q$, implying that Q is a face of $P_{(X,L)}$.

Let $\sigma \in \Sigma_X$ be a cone of maximal dimension, and let w be the corresponding vertex of $P_{(X,L)}$. Then $\pi(w)$ is a vertex, say v_1, of $\pi_{\mathbb{Q}}(P_{(X,L)})$ and hence w lies in R_1. Since R_1 is also a face of $P_{(X,L)}$, w is a vertex of R_1 and hence it corresponds to a maximal dimensional cone in Σ_Y. In each R_i, consider the vertex w_i corresponding to the same cone of Σ_Y. We set $w_1 = w$. We have shown that $Q := \mathrm{Conv}(w_0, \ldots, w_k)$ is a face of $P_{(X,L)}$, and $w = Q \cap R_1$. Now call τ and η the cones of Σ_X corresponding respectively to R_1 and Q. It follows that $\sigma = \tau + \eta$, $\tau \subset \Delta_{\mathbb{Q}}$, and $\eta \cap \Delta_{\mathbb{Q}} = \{0\}$. This concludes the proof. □

The previous proof shows the following corollary.

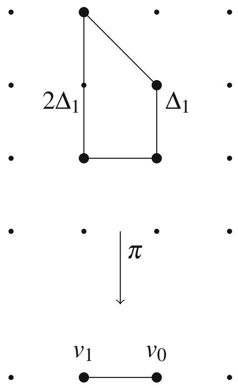
Corollary 3.9. *Let $(f : X \to Y, L)$ be a polarized toric fibration and let $P_{(X,L)} =$ $\mathrm{Cayley}(R_0, \ldots, R_k)_{(\pi,Y)}$ be the associated polytope. Let F be a general fiber of the fibration, $Y_0, \ldots, Y_k$ be the invariant sections and $\pi(R_i) = v_i$. The following holds.*

- **(a)** *The polarized toric variety $(F, L|_F)$ corresponds to the polytope $P_{(F,L|_F)} =$ $\mathrm{Conv}(v_0, \ldots, v_k)$.*
- **(b)** *The polarized toric varieties (Y_i, L_{Y_i}) correspond to the polytopes*

$$R_0 - u_0, \cdots, R_k - u_k,$$

where $u_i \in M$ is a point such that $\pi(u_i) = \pi(R_i)$.

Example 3.10. The toric surface obtained by blowing up $\mathbb{P}^2$ at a fixed point has the structure of a toric fibration, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$. It is often referred to as the Hirzebruch surface $\mathbb{F}_1$. Consider the polarization given by the tautological line bundle $\xi = 2\phi^*(\mathcal{O}_{\mathbb{P}^2}(1)) - E$ where ϕ is the blow-up map and E is the exceptional divisor. The associated polytope is $P = \mathrm{Cayley}(\Delta_1, 2\Delta_1)$, see the figure below.



Remark 3.11. The following are important classes of polarized toric fibrations, relevant both in Combinatorics and Algebraic Geometry.

Projective Bundles. When $\pi(P) = \Delta_t$ the polytope $\text{Cayley}(R_0, \ldots, R_t)_{(\pi,Y)}$ defines the polarized toric fibration $(\mathbb{P}(L_0 \oplus \ldots \oplus L_t) \to Y, \xi)$, where the L_i are ample line bundles on the toric variety Y and ξ is the tautological line bundle. In particular $L|_F = \mathscr{O}_{\mathbb{P}^t}(1)$. These fibrations play an important role in the theory of discriminants and resultants of polynomial systems. See Sect. 4 for more details.

Mori Fibrations. When $\pi(P)$ is a simplex (not necessarily smooth) the Cayley polytope $\text{Cayley}(R_0, \ldots, R_k)_{(\pi,Y)}$ defines a Mori fibration, i.e. a surjective flat map onto a $\mathbb{Q}$-factorial toric variety whose generic fiber is reduced and has Picard number one. This type of fibrations are important blocks in the Minimal Model Program for toric varieties. See [CDR08] and [Re83] for more details.

$\mathbb{P}^k$-Bundles. When $\pi(P) = k\Delta_t$ then again the variety has the structure of a $\mathbb{P}^t$-fibration whose general fiber F is embedded as an k-Veronese variety: $(F, L|_F) = (\mathbb{P}^t, \mathscr{O}_{\mathbb{P}^t}(k))$. These fibrations arise in the study of k-th toric duality, see [DDRP12].

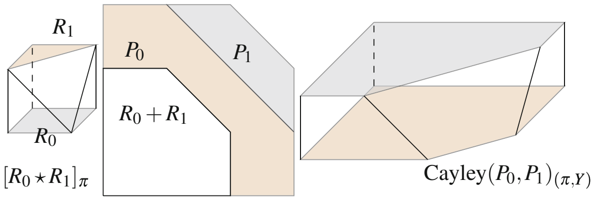
In the polarized toric fibration $(\mathbb{P}(L_0 \oplus \ldots \oplus L_t), \xi)$ the fibers are embedded as linear spaces. For this reason the associated Cayley polytopes $\text{Cayley}(R_0, \ldots, R_t)_{(\pi,Y)}$ can be referred to as *linear toric fibrations*.

Remark 3.12. For general Cayley sums, $[R_0 \star \ldots \star R_k]_\pi$, one has the following geometrical interpretation. Let (X, L) be the associated polarized toric variety and let Y be the toric variety defined by the Minkowski sum $R_0 + \ldots + R_k$. The fan defining Y is a refinement of the normal fans of the R_i for $i = 0, \ldots, k$. Consider the associated birational maps $\phi_i : Y \to Y_i$, where (Y_i, L_i) is the polarized toric variety defined by the polytope R_i. The line bundles $H_i = \phi_i^*(L_i)$ are nef line bundles on Y and the polytopes $P_{(Y,H_i)}$ are affinely isomorphic to R_i. In particular $[R_0 \star \ldots \star R_k]_\pi$ is the polytope defined by the tautological line bundle on the toric fibration $\mathbb{P}(H_0 \oplus \ldots \oplus H_k) \to Y$. Notice that in this case the line bundle ξ may not be ample.

If we want to relate $[R_0 \star \ldots \star R_k]_\pi$ to a polarized toric fibration we need to enlarge the polytopes R_i is order to get an ample tautological line bundle. Consider the polytopes $P_i = P_{(Y,H_i)} + \sum_0^k R_j$. The normal fan of P_i is isomorphic to the fan defining the common resolution, Y, for $i = 0, \ldots, k$. Hence the polytopes P_i are normally equivalent. Let (Y, M_i) be the polarized toric variety associated to the polytope P_i. One can then define the Cayley sum $\mathrm{Cayley}(P_0, \ldots, P_k)_{(\pi,Y)}$, whose normal fan is in fact a refinement of the one defining $[R_0 \star \ldots \star R_k]_\pi$. Let $(\mathbb{P}(M_0 \oplus \ldots \oplus M_k) \to Y, \xi)$ be the polarized toric fibration associated to $\mathrm{Cayley}(P_0, \ldots, P_k)_{(\pi,Y)}$. There is a birational morphism $\phi : \mathbb{P}(M_0 \oplus \ldots \oplus M_k) \to X$.

Example 3.13. Consider the polytopes $R_0 = \Delta_2$, $R_1 = \Delta_1 \times \Delta_1$ in $\mathbb{Q}^2$. Consider the projection onto the first component $\pi : \mathbb{Z}^3 \to \mathbb{Z}$ and $P = \mathrm{Conv}(R_0 \times \{0\}, R_1 \times \{1\})$. The polytope P is then isomorphic to $[R_0 \star R_1]_\pi$, and $\pi_{\mathbb{Q}}(P) = \Delta_1$. The common refinement defined by $R_0 + R_1$ is the fan of the blow up of $\mathbb{P}^2$ at two fixed points, $\phi : Y \to \mathbb{P}^2$. The polytopes P_0 defines the polarized toric variety $(Y, \phi^*(\mathscr{O}_{\mathbb{P}^2}(4)) - E_1 - E_2)$ and the polytope P_1 the pair $(Y, \phi^*(\mathscr{O}_{\mathbb{P}^2}(5)) - 2E_1 - 2E_2)$, where E_i are the exceptional divisors. The polarized toric fibration $(\mathbb{P}(M_0 \oplus M_1) \to Y, \xi)$ is then

$$(\mathbb{P}([\phi^*(\mathscr{O}_{\mathbb{P}^2}(4)) - E_1 - E_2] \oplus [\phi^*(\mathscr{O}_{\mathbb{P}^2}(5)) - 2E_1 - 2E_2]) \to Y, \xi).$$



3.2 Historical Remark

The definition of a Cayley polytope originated from what is "classically" referred to as the *Cayley trick*, in connection with the Resultant and Discriminant of a system of polynomials. A system of n polynomials in n variables $x = (x_1, \ldots, x_n)$, $f_1(x), \ldots, f_n(x)$, is supported on $(\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n)$, where $\mathscr{A}_i \subset \mathbb{Z}^n$ if $f_i = \Pi_{a_j \in \mathscr{A}_i} c_j x^{a_j}$.

The $(\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n)$-*resultant* is a polynomial, $R(\ldots, c_j, \ldots)$, in the coefficients c_j, which vanishes whenever the corresponding polynomials have a common zero.

The discriminant of a finite subset $\mathscr{A} \subset \mathbb{Z}^n$, $\Delta_{\mathscr{A}}$, is also a polynomial $\Delta_{\mathscr{A}}(\ldots, c_j, \ldots)$ in the variables c_j, which vanishes whenever the corresponding polynomial supported on $\mathscr{A}$, $f = \Pi_{a_j \in \mathscr{A}} c_j x^{a_j}$, has a singularity in the torus $(\mathbb{C}^*)^n$.

Theorem 3.14 ([GKZ] Cayley Trick). *The $(\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n)$-resultant equals the $\mathscr{A}$-discriminant where*

$$\mathscr{A} = (\mathscr{A}_1 \times \{0\}) \cup (\mathscr{A}_2 \times \{e_1\}) \cup \ldots \cup (\mathscr{A}_n \times \{e_{n-1}\}) \subset \mathbb{Z}^{2n-1}$$

where $(e_1, \ldots, e_{n-1})$ is a lattice basis for $\mathbb{Z}^{n-1}$.

Let $R_i = N(f_i) \subset \mathbb{R}^n$ be the Newton polytopes of the polynomials f_i supported on $\mathscr{A}_i$. The Newton polytope of the polynomial f supported on $\mathscr{A}$ is the Cayley sum

$$N(f) = [R_1 \star \ldots \star R_n]_\pi,$$

where $\pi : \mathbb{Z}^{2n-1} \to \mathbb{Z}^{n-1}$ is the natural projection such that $\pi_{\mathbb{R}}([R_1 \star \ldots \star R_n]_\pi) = \Delta_{n-1}$.

4 Toric Discriminants and Toric Fibrations

The term "discriminant" is well known in relation with low degree equations or ordinary differential equations. We will study discriminants of polynomials in n variables with prescribed monomials, i.e. polynomials whose exponents are given by lattice points in $\mathbb{Z}^n$.

Polynomials in n-variables describe locally the hyperplane sections of a projective n-dimensional algebraic variety, $\phi : X \hookrightarrow \mathbb{P}^m$. The monomials are prescribed by the local representation of a basis of the vector space of global sections $H^0(X, \phi^*(\mathscr{O}_{\mathbb{P}^m}(1)))$. For this reason the term discriminant has also been classically used in Algebraic Geometry.

In what follows we will describe discriminants from a combinatorial and an algebraic geometric prospective. The two points of view coincide when the projective embedding is toric.

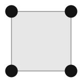
4.1 The $\mathscr{A}$ Discriminant

Let $\mathscr{A} = \{a_0, \ldots, a_m\}$ be a subset of $\mathbb{Z}^n$. The discriminant of $\mathscr{A}$ (when it exists) is an irreducible homogeneous polynomial $\Delta_{\mathscr{A}}(c_0, \ldots, c_m)$ vanishing when the corresponding Laurent polynomial supported on $\mathscr{A}$, $f(x) = \sum_{a_i \in \mathscr{A}} c_i x^{a_i}$, has at least one singularity in the torus $(\mathbb{C}^*)^n$. Geometrically, the zero-locus of the discriminant is an irreducible algebraic variety of codimension one in the dual projective space $\mathbb{P}^{m\vee}$, called the *dual variety* of the embedding $X_{\mathscr{A}} \hookrightarrow \mathbb{P}^m$.

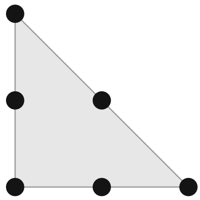
Example 4.1. Consider the point configuration

$$\mathscr{A} = \{(0,0),(1,0),(0,1),(1,1)\} \subset \mathbb{Z}^2.$$

The discriminant is given by an homogeneous polynomial $\Delta_{\mathscr{A}}(a_0,a_1,a_2,a_3)$ vanishing whenever the quadric $a_0+a_1x+a_2y+a_3xy$ has a singular point in $(\mathbb{C}^*)^2$. It is well known that this locus correspond to singular 2×2 matrices and it is thus described by the vanishing of the determinant: $\Delta_{\mathscr{A}}(a_0,a_1,a_2,a_3) = a_0a_3 - a_1a_2$. Similarly, one can associate the polynomials supported on $\mathscr{A}$ with local expansions of global sections in $H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathscr{O}(1,1))$ defining the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^1$ in $\mathbb{P}^3$.



Example 4.2. The 2-Segre embedding $\nu_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ defined by the global sections of the line bundle $\mathscr{O}_{\mathbb{P}^2}(2)$ can be associated to the point configuration $\mathscr{A} = \{a_0,a_1,a_2,a_3,a_4,a_5\} = \{(0,0),(0,1),(1,0),(0,2),(1,1),(2,0)\}$



A simple computation shows that $\Delta_{\mathscr{A}} = \det \begin{bmatrix} c_0 & c_1 & c_2 \\ c_1 & c_3 & c_4 \\ c_2 & c_4 & c_5 \end{bmatrix}$

Projective duality is a classical subject in algebraic geometry. Given en embedding $i : X \hookrightarrow \mathbb{P}^m$ of an n-dimensional algebraic variety, the dual variety, $X^\vee \subset (\mathbb{P}^m)^\vee$ is defined as the Zariski-closure of all the hyperplanes $H \subset \mathbb{P}^m$ tangent to X at some non singular point. We can speak of a defining homogeneous polynomial $\Delta(c_0,\ldots,c_m)$, and thus of a discriminant, only when the dual variety has codimension one. Embeddings whose dual variety has higher codimension are called *dually defective* and the discriminant is set to be 1. Finding formulas for the discriminant Δ and giving a classification of the embeddings with discriminant 1 is a long standing problem in algebraic geometry. In the case of a toric embedding defined by a point-configuration, $X_{\mathscr{A}} \hookrightarrow \mathbb{P}^{|\mathscr{A}|-1}$, the problem is equivalent to finding formulas for the discriminant $\Delta_{\mathscr{A}}$ and giving a classification of the *dually defective point-configurations,* i.e. the point-configurations with discriminant $\Delta_{\mathscr{A}} = 1$.

Dickenstein-Sturmfels [DS02] characterized the case when $m = n + 2$, Cattani-Curran [CC07] extended the classification to $m = n + 3, n + 4$. In these cases the corresponding embedding is possibly very singular and the methods used are purely combinatorial. In [DiR06] and [CDR08] we completely characterize the case when $P_{\mathscr{A}} = \mathrm{Conv}(\mathscr{A})$ is smooth or simple. The latter characterisation relies on tools from Algebraic Geometry which will be explained in the next paragraph.

4.2 The Dual Variety of a Projective Variety

The dual variety corresponds to the locus of singular hyperplane sections of a given embedding. By requiring the singularity to be of a given order, one can define more general dual varieties. Singularities of fixed multiplicity k correspond to hyperplanes tangent "to the order k." Consider an embedding $i : X \hookrightarrow \mathbb{P}^m$ of an n-dimensional variety, defined by the global sections of the line bundle $\mathscr{L} = i^*(\mathscr{O}_{\mathbb{P}^m}(1))$. For any smooth point x of the embedded variety let:

$$jet_x^k : H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L} \otimes \mathscr{O}_X/\mathfrak{m}_x^{k+1})$$

be the map assigning to a global section s in $H^0(X, \mathscr{L})$ the tuple

$$jet_x^k(s) = (s(x), \ldots, (\partial^t s/\partial \underline{x}^t)(x), \ldots)_{t \leqslant k}$$

where $\underline{x} = (x_1, \ldots, x_n)$ is a local system of coordinates around x. The k*-th osculating space* at x is defined as $\mathbb{O}sc_x^k = \mathbb{P}(Im(jet_x^k))$. As the map jet_x^1 is surjective, the first osculating space is always isomorphic to $\mathbb{P}^n$ and it is classically called the *projective tangent space*. The jet maps of higher order do not necessarily have maximal rank and thus the dimension of the osculating spaces of order bigger than 1 can vary. The embeddings admitting osculating space of maximal dimension at every point are called k-jet spanned.

Definition 4.3. A line bundle $\mathscr{L}$ on X is called k*-jet spanned* at x if the map jet_x^k is surjective. It is called k-jet spanned if it is k-jet spanned at every smooth point $x \in X$.

Example 4.4. A line bundle $\mathscr{L} = \mathscr{O}_{\mathbb{P}^n}(a)$ on $\mathbb{P}^n$ is k-jet spanned for all $a \geqslant k$. In fact the map

$$jet_x^k : H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(a)) \to J_k(\mathscr{O}_{\mathbb{P}^n}(a))_x$$

is surjective for all $x \in \mathbb{P}^2$, as a local basis of the global sections of $\mathscr{O}_{\mathbb{P}^n}(a)$ consists of all the monomials in n variables of degree up to a and we are assuming $a \geqslant k$.

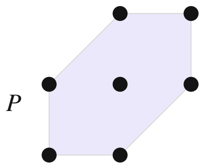
Example 4.5. Let $\mathscr{L}$ be a line bundle on a non singular toric variety X. Then the following statements are equivalent, see [DiR01]:

(a) $\mathscr{L}$ is k-jet spanned.
(b) all the edges of $P_{\mathscr{L}}$ have length at least k.
(c) $\mathscr{L} \cdot C \geqslant k$ for every invariant curve C on X.

As an example consider the polytope P in figure below. The associated torc embedding is the embedding of the blow up of $\mathbb{P}^2$ at the three fixed points, $\phi : X \to \mathbb{P}^2$, defined by the anticanonical line bundle $\phi^*(\mathscr{O}_{\mathbb{P}^2}(3)) - E_1 - E_2 - E_3$. Here E_i denote the exceptional divisors. The embedded variety is a Del Pezzo surface of degree 6. Let F be the set of the 6 fixed points on X and $E = \{\phi^*(\mathscr{O}_{\mathbb{P}^2}(3)) - E_i - E_j, i \neq i\} \cup \{E_1, E_2, E_3\}$ be the set of invariant curves. The osculating spaces can easily seen to be:

$$\mathbb{O}sc^2_p = \begin{cases} \mathbb{P}^3 = & < jet^2_p(1), jet^2_p(x), jet^2_p(y), jet^2_p(xy) > \\ & \text{if } x \in F. \\ \mathbb{P}^4 = & < jet^2_p(1), jet^2_p(x), jet^2_p(y), jet^2_p(xy), jet^2_p(x^2y) > \\ & \text{if } x \in E \setminus F. \\ \mathbb{P}^5 = & < jet^2_p(1), jet^2_p(x), jet^2_p(y), jet^2_p(xy), jet^2_p(x^2y), jet^2_p(xy^2) > \\ & \text{at a general point } p \in X \setminus E. \end{cases}$$

The embedding defined by P is not 2-jet spanned on the whole X. It is 2-jet spanned at every point in $X \setminus E$.



Definition 4.6. A hyperplane $H \subset \mathbb{P}^m$ is tangent at x to the order k if it contains the k-th osculating space at x: $\mathbb{O}sc^k_x \subset H$.

Definition 4.7. The *k-th order dual variety* X^k is:

$$X^k = \overline{\{H \in \mathbb{P}^{m*} \text{ tangent to the order } k \text{ to } X \text{ at some non singular point}\}}.$$

Notice that $X^1 = X^\vee$ and that X^2 is contained in the singular locus of $X^\vee$. General properties of the higher order dual variety have been studied by S. Kleiman and R. Piene. Because the definition is related to local osculating properties and generation of jets, it is useful to introduce the sheaf of jets, $J_k(\mathscr{L})$, associated to a polarized variety $(X, \mathscr{L})$. In the classical literature it is sometime referred to as the *sheaf of principal parts.*

Consider the projections $\pi_i : X \times X \to X$ and let $\mathscr{I}_{\Delta_X}$ be the ideal sheaf of the diagonal in $X \times X$. The sheaf of k-th order jets of the line bundle $\mathscr{L}$ is defined as

$$J_k(\mathscr{L}) = \pi_{2*}(\pi_1^*(\mathscr{L}) \otimes (\mathscr{O}_{X\times X}/\mathscr{I}_{\Delta_X}^{k+1})).$$

When the variety X is smooth $J_k(\mathscr{L})$ is a vector bundle of rank $\binom{n+k}{n}$, called the *k-jet bundle.*

Example 4.8. If $\mathscr{L} \neq \mathscr{O}_X$ is a globally generated line bundle then $J_k(\mathscr{L})$ splits as a sum of line bundles only if $X = \mathbb{P}^n$ and $\mathscr{L} = \mathscr{O}_{\mathbb{P}^n}(a)$. In fact:

$$J_k(\mathscr{O}_{\mathbb{P}^n}(a)) = \bigoplus_1^{\binom{n+k}{k}} \mathscr{O}_{\mathbb{P}^n}(a-k)$$

See [DRS01] for more details.

It is important to note that when the map jet_x^k is surjective for all smooth points x, properties of the higher dual variety X^k can be related to vanishing of Chern classes of the associated k-th jet bundle, $J_k(\mathscr{L})$. We start by identifying the k-th dual variety with a projection of the conormal bundle. Let X be a smooth algebraic variety and let $\mathscr{L}$ be a k-jet spanned line bundle on X. Consider the following commutative diagram.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & S^{k+1}\Omega_X^1 \otimes \mathscr{L} \arrow[d] & \\
 & & & J_{k+1}(\mathscr{L}) \arrow[d] & \\
0 \arrow[r] & K_k \arrow[r, "\beta_k"] \arrow[uurr, "II_k^*"] & X \times H^0(X,\mathscr{L}) \arrow[r, "jet^k"] \arrow[ur, "jet^{k+1}"] & J_k(\mathscr{L}) \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd} \tag{1}$$

The vertical exact sequence is often called the k-jet sequence. The map jet^k is defined as $jet^k(s,x) = jet_x^k(s)$. The vector bundle K_k is the kernel of the map jet^k (which has maximal rank!). The induced map $II_k^\vee$ can be identified with the dual of the k-th fundamental form. See [L94, GH79] for more details. By dualizing the map β_k and projectivizing the corresponding vector bundles one gets the following maps:

$$\begin{tikzcd}
& \mathbb{P}(K_k^\vee) \arrow[dl, bend right, "\gamma_k"'] \arrow[d, hook] \arrow[dr, bend left, "\alpha_k"] & \\
X & X \times \mathbb{P}(H^0(X,L)^\vee) \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & (\mathbb{P}^m)^\vee
\end{tikzcd} \tag{2}$$

It is straightforward to see that $X^k = Im(\alpha_k)$. A simple dimension count shows that when the map jet^k has maximal rank one expects the codimension of the k-th dual variety to be $\operatorname{codim}(X^k) = \binom{n+k}{k} - n$. Notice that this is equivalent to requiring that the map α_k is generically finite. When the codimension is higher than the expected one the embedding is said to be *k-th dually defective.*

The commutativity in diagram (1) has the following useful consequence.

Lemma 4.9. *Let $(X, \mathscr{L})$ be a polarized variety, where X is smooth and the line bundle $\mathscr{L}$ is $(k+1)$-jet spanned. Then the dual variety X^k has the expected dimension.*

Proof. We follow diagram (1). Because the line bundle $\mathscr{L}$ is $(k+1)$-jet ample the map II_k^* is surjective. This means that for every $x \in X$ and for every monomial $\Pi_{\sum t_i = k+1} x_1^{t_1} \cdots x_n^{t_n}$ there is an hyperplane section that locally around x is defined as

$$C \cdot \Pi_{\sum t_i = k+1} x_1^{t_1} \cdots x_n^{t_n} + \text{ higher order terms } = 0, \text{ where } C \neq 0$$

In other words, hyperplanes tangent at a point x to the order k are in one-to-one correspondence with elements of the linear system $|\mathscr{O}_{\mathbb{P}^{n-1}}(k+1)|$. The map α_k having positive dimensional fibers is equivalent to saying that hyperplanes tangent at a point x to the order k are also tangent to nearby points $y \neq x$, which in turn implies that the linear system $|\mathscr{O}_{\mathbb{P}^{n-1}}(k+1)|$ has base points. This is a contradiction as the linear system is $k+1$-jet spanned and thus base-point free. $\square$

When $k = 1$ the contact locus of a general singular hyperplane H, $\gamma_k(\alpha_k^{-1}(H))$ is always a linear subspace. This implies that if finite then $\deg(\alpha_1) = 1$. For higher order tangencies, $k > 1$, the degree can be higher. When the map α_k is finite we set $n_k = \deg(\alpha_k)$.

Lemma 4.10 ([LM00, DDRP12]). *Let X be a smooth variety and let $\mathscr{L}$ be a k-jet spanned line bundle. Then $\operatorname{codim}(X^k) > \binom{n+k}{k} - n$ if and only if $c_n(J_k(\mathscr{L})) = 0$. Moreover when $\operatorname{codim}(X^k) = \binom{n+k}{k} - n$ the degree of the k-dual variety is given by:*

$$n_k \deg(X^k) = c_n((J_k(\mathscr{L})).$$

Proof. Observe first that because the map jet_k is of maximal rank the vector bundle $J_k(\mathscr{L})$ is spanned by the global sections of the line bundle $\mathscr{L}$. This implies that, after fixing a basis $\{s_1, \ldots, s_{m+1}\}$ of $H^0(X, \mathscr{L}) \cong \mathbb{C}^{m+1}$, the Chern class $c_n(J_k(\mathscr{L}))$ is represented by the set:

$$\{x \in X \mid \dim(\mathrm{Span}(jet_x^k(s_1), jet_x^k(s_2))) \leqslant 1\}$$

Notice that an hyperplane in the linear span $\mathbb{P}^t = \langle s_1, \ldots, s_{t+1}\rangle$ is tangent at a point x to the order k exactly when $\dim(\mathrm{Span}(jet_x^k(s_1), \ldots, jet_x^k(s_{t+1}))) = t + 1$. The map γ_k in diagram (2) defines a projective bundle of rank $m - \binom{n+k}{k}$. The statement $c_n(J_k(\mathscr{L})) = 0$ is then equivalent to $\alpha_k(\gamma^{-1}(x)) \cap \mathbb{P}^1 = \emptyset$ for every $x \in X$ and for a general $\mathbb{P}^1 = \langle s_1, s_2\rangle$. By Bertini this is equivalent to $\mathrm{codim}(X^k) > \binom{n+k}{k} - n$. Assume now that $c_n(J_k(\mathscr{L})) \neq 0$ and thus that the generic fiber of the map α_k is finite. The degree of $X^k = im(\alpha_k)$ times the degree of the map α_k is given by the degree of the line bundle $\alpha_k^*(\mathscr{O}_{\mathbb{P}^{m\vee}}(1))$ which corresponds to the tautological line bundle $\mathscr{O}_{\mathbb{P}(K_k^\vee)}(1)$.

$$n_k \deg(X^k) = c_1(\alpha_k^*(\mathscr{O}_{(\mathbb{P}^m)^\vee} 1))^{m+n-\binom{n+k}{k}} = c_1(\mathscr{O}_{\mathbb{P}(K_k^\vee)}(1))^{m+n-\binom{n+k}{k}}.$$

From diagram (1) we see that

$$c_n(J_k(\mathscr{L})) = c_n(K_k^\vee)^{-1} = s_n(K_k^\vee)$$

Finally let $\pi : \mathbb{P}(K_k^\vee) \to X$ be the bundle map. By relating the Segre class $s_n(K_k^\vee)$ to the tautological bundle [FU, 3.1] $s_n(K_k^\vee) = \pi_*(c_1(\mathscr{O}_{\mathbb{P}(K_k^\vee)}(1))^{m+n-\binom{n+k}{k}}) = c_1(\mathscr{O}_{\mathbb{P}(K_k^\vee)}(1))^{m+n-\binom{n+k}{k}}$ we conclude that: $n_k \deg(X^k) = c_n(J_k(\mathscr{L}))$. □

The case of $k = 1$ is referred to as classical projective duality. When the codimension of the dual variety is one, the homogeneous polynomial in $m + 1$ variables defining it is called the *discriminant of the embedding*. For a polarized variety the discriminant, when it exists, parametrizes the singular hyperplane sections.

4.3 The Toric Discriminant

In the case of singular varieties the sheaves of k-jets are not necessarily locally free and thus it is not possible to use Chern-classes techniques.

For toric varieties however estimates of the degree of the dual varieties are possible, even in the singular case, and rely on properties of the associated polytope. In the classical case $k = 1$ there is a precise characterization in any dimension. For higher order duality, results in dimension 3 and for $k = 2$ can be found in [DDRP12]. A generalization to higher dimension and higher order is an open problem.

Proposition 4.11 ([GKZ, DiR06, MT11]). *Let $(X_{\mathscr{A}}, L_{\mathscr{A}})$ be a polarized toric variety associated to the polytope $P_{\mathscr{A}}$. Set*

$$\delta_i = \sum_{\emptyset \neq F \prec P} (-1)^{\mathrm{codim}(F)} \{ \binom{\dim(F)+1}{i} + ((-1)^{i-1}(i-1)\} \,\mathrm{Vol}(F)\,\mathrm{Eu}(V(F)).$$

Then $\mathrm{codim}(X_{\mathscr{A}}^{\vee}) = r = min\{i, \delta_i \neq 0\}$ *and* $\deg(X_{\mathscr{A}}^{\vee}) = \delta_r$.

The function Eu : {invariant subvarieties of X_A} $\to \mathbb{Z}$ in the above proposition assigns an integer to all invariant subvarieties. Its value is different from 1 only when the variety is singular. In particular, when $X_{\mathscr{A}}$ is smooth we have:

$$\mathrm{codim}(X_A^{\vee}) > 1 \Leftrightarrow \sum_{\emptyset \neq F \prec P} (-1)^{\mathrm{codim}(F)} (\dim(F)+1)!\,\mathrm{Vol}(F) = 0$$

In fact in the smooth case one can prove this characterization using the vector bundle of 1-jets.

Proposition 4.12. *Let* $(X_{\mathscr{A}}, L_{\mathscr{A}})$ *be an* n*-dimensional non singular polarized toric variety associated to the polytope* $P_{\mathscr{A}}$*. Assume* $\mathscr{A} = P_{\mathscr{A}} \cap \mathbb{Z}^n$*. Then*

$$c_n(J_1(\mathscr{L}_{\mathscr{A}})) = \sum_{\emptyset \neq F \prec P} (-1)^{\mathrm{codim}(F)} (\dim(F)+1)!\,\mathrm{Vol}(F)$$

Proof. Chasing the diagram (1) one sees:

$$c_n(J_1(\mathscr{L}_{\mathscr{A}})) = \sum_{i=0}^{n} (n+1-i) c_i(\Omega^1_{X_{\mathscr{A}}}) \cdot \mathscr{L}^i_{\mathscr{A}}$$

Consider now the generalized Euler sequence for smooth toric varieties [BC94, 12.1]:

$$0 \to \Omega^1_{X_{\mathscr{A}}} \to \bigoplus_{\xi \in \Sigma_{\mathscr{A}}(1)} \mathscr{O}_{X_{\mathscr{A}}}(V(\xi)) \to \mathscr{O}_{X_{\mathscr{A}}}^{|\Sigma_{\mathscr{A}}(1)|-n} \to 0$$

Where $V(\xi)$ is the invariant divisor associated to the ray $\xi \in \Sigma_{\mathscr{A}}(1)$. It follows that: $c_i(\Omega^1_{X_{\mathscr{A}}})) = (-1)^i \sum_{\xi_1 \neq \xi_2 \neq \ldots \neq \xi_i} [V(\xi_1)] \cdot \ldots \cdot [V(\xi_i)]$. Recall that the intersection products $[V(\xi_1)] \cdot \ldots \cdot [V(\xi_i)]$ correspond to codimension i invariant subvarieties and thus faces of $P_{\mathscr{A}}$ of dimension $n-i$. Moreover the degree of the embedded subvariety $[V(\xi_1)] \cdot \ldots \cdot [V(\xi_i)]$ is equal to $\mathscr{L}^{n-i} \cdot ([V(\xi_1)] \cdot \ldots \cdot [V(\xi_i)]) = (n-i)!\,\mathrm{Vol}(F)$, where F is the corresponding face. We can then conclude:

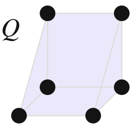
$$\begin{aligned} c_n(J_1(\mathscr{L}_{\mathscr{A}})) &= \textstyle\sum_{\emptyset \neq F \prec P_{\mathscr{A}}} (n+1-i)(n-i)!(-1)^i\,\mathrm{Vol}(F) = \\ &= \textstyle\sum_{\emptyset \neq F \prec P_{\mathscr{A}}} (-1)^{\mathrm{codim}(F)} (\dim(F)+1)!\,\mathrm{Vol}(F) \end{aligned}$$

$\square$

Example 4.13. Consider the simplex $2\Delta_2$ in Example 4.2. All the edges have length equal to two and therefore the toric embedding is 2-jet spanned. The dual variety is then an hypersurface and the degree of the discriminant is given by $c_2(J_1(\mathscr{O}_{\mathbb{P}^2}(2)) = c_2(\mathscr{O}_{\mathbb{P}^2}(1) \oplus \mathscr{O}_{\mathbb{P}^2}(1) \oplus \mathscr{O}_{\mathbb{P}^2}(1)) = 3$. The volume formula gives in fact:

$$c_2(J_1(\mathscr{O}_{\mathbb{P}^2}(2)) = 6Vol(2\delta_2) - 2\sum_1^3 Vol(2\Delta_1) + 3 = 12 - 12 + 3 = 3.$$

Example 4.14. Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, associated to the polytope Q. Then $c_3(J_1(\mathscr{L})) = 4!\frac{1}{2} - 3!(1 + 1 + 1 + \frac{1}{2} + \frac{1}{2}) + 2(9) - 6 = 0$. This embedding is therefore dually defective.



The following is an amusing observation, which is a simple consequence of the previous characterization.

Corollary 4.15. *Let $P_{\mathscr{A}}$ be a smooth polytope such that $\mathscr{A} = P_{\mathscr{A}} \cap \mathbb{Z}^n$. Then*

$$\sum_{\emptyset \neq F \prec P_{\mathscr{A}}} (-1)^{\mathrm{codim}(F)} (\dim(F) + 1)!\, \mathrm{Vol}(F) \geqslant 0$$

Proof. Because the associated line bundle $\mathscr{L}_{\mathscr{A}}$ defines an embedding of the variety $X_{\mathscr{A}}$, the map jet^1 has maximal rank and thus the vector bundle $J_1(\mathscr{L}_{\mathscr{A}})$ is spanned (by the global sections of $\mathscr{L}_{\mathscr{A}}$). It follows that the degree of its Chern classes must be non negative which implies the assertion. □

Now we can state the characterization of $\mathbb{Q}$-factorial and non singular toric embeddings admitting discriminant $\Delta_{\mathscr{A}} = 1$. The theorem will include the combinatorial characterization and the equivalent algebraic geometry description. The proof in the non singular case will be given in Sect. 6.

Theorem 4.16 ([DiR06, CDR08]). *Let $\mathscr{A} = P_{\mathscr{A}} \cap \mathbb{Z}^n$ and assume that $X_{\mathscr{A}}$ is $\mathbb{Q}$-factorial. Then the following equivalent statements hold.*

(a) *The point-configuration $\mathscr{A}$ is dually defective if and only if $P_{\mathscr{A}}$ is a Cayley sum of the form $P_{\mathscr{A}} \cong \mathrm{Cayley}(R_0, \ldots, R_t)_{(\pi,Y)}$, where $\pi(P)$ is a simplex (not necessarily unimodular) in $\mathbb{R}^t$ and $R_0, \ldots, R_t$ are normally equivalent polytopes with $t > \frac{n}{2}$. If moreover $P_{\mathscr{A}}$ is smooth then $\pi(P)$ is a unimodular simplex.*

(b) *The projective dual variety of the toric embedding* $X_{\mathscr{A}} \hookrightarrow \mathbb{P}^{|\mathscr{A}\cap\mathbb{Z}^n|-1}$ *has codimension* $s \geqslant 2$ *if and only if* $X_{\mathscr{A}}$ *is a Mori-fibration,* $X_{\mathscr{A}} \to Y$ *and* $\dim(Y) < \dim(X)/2$. *If moreover* $X_{\mathscr{A}}$ *is non singular then* $(X_{\mathscr{A}}, L_{\mathscr{A}}) = (\mathbb{P}(L_0 \oplus \cdots \oplus L_t), \xi)$, *where* L_i *are line bundles on a toric variety* Y *of dimension* $m < t$.

Proposition 4.16 provides a characterization of the class of smooth polytopes achieving the minimal value 0.

Corollary 4.17. *Let* P *be a convex smooth lattice polytope. Then*

$$\sum_{\emptyset \neq F \prec P_{\mathscr{A}}} (\dim(F) + 1)!(-1)^{\operatorname{codim}(F)} \operatorname{Vol}(F) = 0$$

If and only if $P = \operatorname{Cayley}(R_0, \ldots, R_t)$ *for normally equivalent smooth lattice polytopes* R_i *with* $\dim(R_i) < t$.

5 Toric Fibrations and Adjunction Theory

The classification of projective algebraic varieties is a central problem in Algebraic Geometry dating back to early nineteenth century. The way one can realistically carry out a classification theory is through invariants, such as the degree, genus, Hilbert polynomial. Modern adjunction theory and Mori theory are the basis for major advances in this area.

Let $(X, \mathscr{L})$ be a polarized n-dimensional variety. Assume that X is Gorenstein (i.e. the canonical class K_X is a Cartier divisor). The two key invariants occurring in classification theory, see [Fuj90], are the *effective log threshold* $\mu(\mathscr{L})$ and the *nef value* $\tau(\mathscr{L})$:

$$\mu(\mathscr{L}) := \sup_{\mathbb{R}}\{s \in \mathbb{Q} : \dim(H^0(K_X + s\mathscr{L})) = 0\}$$
$$\tau(\mathscr{L}) := \min_{\mathbb{R}}\{s \in \mathbb{R} : K_X + s\mathscr{L} \text{ is nef}\}.$$

Both invariants are at most equal to $n + 1$. Kawamata proved that $\mu(\mathscr{L})$ is indeed a rational number and recent advances in the minimal model program establish the same for $\mu(\mathscr{L})$. They can be visualized as follows.

Traveling from $\mathscr{L}$ in the direction of the vector K_X in the Neron-Severi space $\operatorname{NS}(X) \otimes \mathbb{R}$ of divisors, $\mathscr{L} + (1/\mu(\mathscr{L}))K_X$ is the meeting point with the cone of effective divisors $\operatorname{Eff}(X)$ and $\mathscr{L} + (1/\tau(\mathscr{L}))K_X$ is the meeting point with the cone of nef-divisors $\operatorname{Nef}(X)$, see Fig. 1.

A multiple of the nef line bundle $K_X + \tau\mathscr{L}$ defines a morphism $X \to \mathbb{P}^M$ which can be decomposed (Remmert-Stein factorization) as a composition of a morphism $\phi_\tau : X \to Y$ with connected fibers onto a normal variety Y and finite-to-one morphism $Y \to \mathbb{P}^M$. The map ϕ_τ is called the *nef-value morphism*. Kawamata showed that if one writes $r\tau = u/v$ for coprime integers u, v, then:

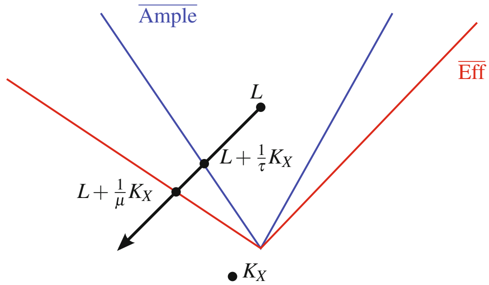


Fig. 1 Illustrating $\mu(\mathscr{L})$ and $\tau(\mathscr{L})$

$$u \leqslant r(1 + \max_{y\in Y}(\dim(\phi_\tau^{-1}(y))).$$

Corollary 5.1. *Let $(X, \mathscr{L})$ be a polarized variety. Then the nef-value achieves the maximum value $\tau(\mathscr{L}) = n + 1$ if and only if $(X, \mathscr{L}) = (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$.*

Proof. Consider the nef value morphism $\phi_\tau : X \to Y$ and observe that

$$(n + 1) \leqslant (1 + \max_{y\in Y}(\dim(\phi_\tau^{-1}(y))).$$

This implies that the dimension of a fiber of ϕ_τ must be n and thus that the morphism contracts the whole space X to a point. By construction, the fact that ϕ_τ contracts the whole space implies that $K_X + (n+1)\mathscr{L} = \mathscr{O}_X$. A celebrated criterion in projective geometry, called the Kobayashi-Ochiai theorem, asserts that if L is an ample line bundle such that $K_X + (n + 1)\mathscr{L} = \mathscr{O}_X$ then $(X, \mathscr{L}) = (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$. □

Remark 5.2. Recall that the interior of the closure of the effective cone is the cone of big divisors, $(\overline{\text{Eff}(X)})^\circ = \text{Big}(X)$, and that the closure of the ample cone is the nef cone, $\overline{\text{Ample}(X)} = \text{Nef}(X)$. In particular the equality $\tau(\mathscr{L}) = \mu(\mathscr{L})$ occurs if and only if the line bundle $K_X + \tau(\mathscr{L})\mathscr{L}$ is nef and *not* big, which implies that ϕ_τ defines a fibration structure on X.

A fibration structure on an algebraic variety is a powerful geometrical tool as many invariants are induced by corresponding invariants on the (lower dimensional) basis and generic fiber. Criteria for a space to be a fibration are therefore highly desirable. Beltrametti, Sommese and Wisniewski conjectured the if the effective log threshold is strictly bigger than half the dimension then the nef-value morphism should be a fibration.

Conjecture 5.3 ([BS94]). If X is non singular and $\mu(\mathscr{L}) > (n + 1)/2$ then $\mu(\mathscr{L}) = \tau(\mathscr{L})$.

Let us now assume that the algebraic variety is toric. In this case it is immediate to see that the defined invariants are rational numbers as the cones $\text{Eff}(X), \text{Big}(X), \text{Ample}(X), \text{Nef}(X)$ are all rational cones.

We have seen in Sect. 3 that toric fibrations are associated to certain Cayley polytopes. Analogously to the classification theory of projective algebraic varieties it is important to find invariants of polytopes that would characterize a Cayley structure. One invariant which has attracted increasing attention in recent years is the *codegree* of a lattice polytope:

$$\mathrm{codeg}(P) = min\{t \in \mathbb{Z}_{>0} \text{ such that } tP \text{ contains interior lattice points}\}.$$

Via Ehrhart theory one can conclude that $\mathrm{codeg}(P) \leqslant n+1$ and that $\mathrm{codeg}(P) = n+1$ if and only if $P = \Delta_n$. This is in fact a simple consequence of our previous observations.

Corollary 5.4. *Let P be a Gorenstein lattice polytope. Then* $\mathrm{codeg}(P) = n+1$ *if and only if $P = \Delta_n$.*

Proof. Let $(X, \mathscr{L})$ be the Gorenstein toric variety associated to P. Notice that, because $K_X = -\sum D_i$ where the D_i are the invariant divisors, the polytope defined by the line bundle $K_X + t\mathscr{L}$ is the convex hull of the interior points of tP. The equality $\mathrm{codeg}(P) = n+1$ is equivalent to $H^0(K_X + t\mathscr{L}) = 0$ for $t \leqslant n$. Because nef line bundles must have sections (in particular being nef is equivalent to being globally generated on toric varieties) we have $\tau(\mathscr{L}) \geqslant \mathrm{codeg}(P) = n+1$. It follows from Corollary 5.1 that $(X, \mathscr{L}) = (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$ and thus $P = \Delta_n$. □

Let us now examine the class of Cayley polytopes we encountered in the characterization of dually defective toric embeddings. We will see that this is a class of polytopes satisfying the strong lower bound $\mathrm{codeg}(P) \geqslant \frac{\dim(P)}{2} + 1$ and the equality $\mathrm{codeg}(P) = \mu(\mathscr{L})$.

Lemma 5.5. *Let $P = \mathrm{Cayley}_{h,Y}(R_0, \ldots, R_t)$ with $t > \frac{n}{2}$, then:*

$$\tau(\mathscr{L}_P) = \mu(\mathscr{L}_P) = \mathrm{codeg}(P) = t+1 \geqslant \frac{n+3}{2}.$$

Proof. Observe that $X_P = \mathbb{P}(L_0 \oplus \ldots \oplus L_t)$ for ample line bundles L_i on the toric variety Y and $\mathscr{L} = \xi$ is the tautological line bundle. Consider the projective bundle map $\pi : X_P \to Y$. The Picard group of X_P is generated by the pull back of generators of $\mathrm{Pic}(Y)$ and by the tautological line bundle ξ. Moreover the canonical line bundle is given by the following expression:

$$K_{X_P} = \pi^*(K_Y + L_0 + \ldots + L_t) - (t+1)\xi.$$

The toric nefness criterion says that a line bundle on a toric variety is nef if and only if the intersection with all the invariant curves is non-negative, see for example [ODA]. On the toric variety $\mathbb{P}(L_0 \oplus \ldots \oplus L_t)$ there are two types of rational invariant curves. The ones contained in the fibers $F \cong \mathbb{P}^t$ and the pull back of rational invariant curves in Y which will be denoted by $\pi^*(C)_i$ when contained in the invariant section defined by the polytope R_i. For any rational invariant curve

$C \subset F$, it holds that $\xi|_C = \mathscr{O}_{\mathbb{P}^1}(1)$ and $\pi^*(D) \cdot C = 0$, for all divisors D on Y. For every curve of the form $\pi^*(C)_i$ it holds that $\pi^*(C)_i \cdot \pi^*(D) = C \cdot D$ and $\xi \cdot \pi^*(C)_i = L_i \cdot C$. See [DiR06, Remark 3] for more details. We conclude that $K_{X_P} + s\mathscr{L}$ is nef if the following is satisfied:

$$\begin{array}{ll} [\pi^*(K_Y + L_0 + \ldots + L_t) + (s-t-1)\xi]C = s-t-1 \geqslant 0 & \text{if } C \subset F \\ (K_Y + L_0 + \ldots + (s-t)L_i + \ldots + L_t) \cdot C \geqslant 0 & \text{if } C = \pi^*(C)_i \end{array}$$

In [MU02] Mustata proved a toric-Fujita conjecture showing that if for a line bundle H on an n-dimensional toric variety, $H \cdot C \geqslant n$ for every invariant curve C, then the adjoint bundle $K + H$ is globally generated, unless $H = \mathscr{O}_{\mathbb{P}^n}(n)$. Because

$$[L_0 + \ldots + (s-t)L_i + \ldots + L_t] \cdot C \geqslant (s-t) + t$$

it follows that $(K_Y + L_0 + \ldots + (s-t)L_i + \ldots + L_t) \cdot C \geqslant 0$ for all invariant curves $C = \pi^*(C)_i$ if $s \geqslant t$. This implies that $K_{X_P} + s\mathscr{L}$ is nef if and only if $s \geqslant t+1$ and thus $\tau(\xi) = t+1$.

Consider now the projection $h : \mathbb{R}^n \to \mathbb{R}^t$ such that $h(P) = \Delta_t$. Under this projection interior points of a dilation sP are mapped to interior points of the corresponding dilation $s\Delta_t$. This implies that $\operatorname{codeg}(P) = t+1$. Notice that $\mu(\mathscr{L}) \leqslant \operatorname{codeg}(P) = t+1$ as interior points of sP correspond to global sections of $K_{X_P} + s\mathscr{L}$. On the other hand, see [HA, Ex. 8.4]:

$$\begin{aligned} &H^0(u(\pi^*(K_Y + L_0 + \ldots + L_t)) + (v - u(t+1))\xi) = \\ &= H^0(\pi_*(u(\pi^*(K_Y + L_0 + \ldots + L_t)) + (v - u(t+1))\xi)) = \\ &= H^0(u(K_Y + L_0 + \ldots + L_t) + \pi_*((u - v(t+1))\xi) = 0 \text{ if } v - u(t+1) < 0. \end{aligned}$$

This implies that $\mu(\mathscr{L}) \geqslant t+1$, which proves the assertion. □

Recently Batyrev and Nill in [BN08] classified polytopes with $\operatorname{codeg}(P) = n$ and conjectured the following.

Conjecture 5.6 ([BN08]). There is a function $f(n)$ such that any n-dimensional polytope P with $\operatorname{codeg}(P) \geqslant f(n)$ decomposes as a Cayley sum of lattice polytopes.

The above conjecture was proven by Haase, Nill and Payne in [HNP09]. They showed that $f(n)$ is at most quadratic in n. It is important to observe that, as interior lattice points of tP correspond to global sections of $K_X + t\mathscr{L}$ for the associated toric embedding, $\operatorname{codeg}(P)$ can be considered as the *integral variant* of $\mu(\mathscr{L})$. This observation, techniques from toric Mori theory and adjunction theory led to prove a stronger version of Conjectures 5.3 and 5.6 for smooth polytopes giving yet another characterization of Cayley sums.

Theorem 5.7 ([DDRP09, DN10]). *Let $P \subset \mathbb{R}^n$ be a smooth n-dimensional polytope. Then the following statements are equivalent.*

(a) $\operatorname{codeg}(P) \geqslant (n+3)/2$.
(b) *P is affinely isomorphic to a Cayley sum* $\operatorname{Cayley}(R_0, \ldots, R_t)_{\pi,Y}$ *where* $t+1 = \operatorname{codeg}(P)$ *with* $t > \frac{n}{2}$.
(c) $\mu(\mathscr{L}_P) = \tau(\mathscr{L}_P) = t+1$ *and* $t > \frac{n}{2}$.
(d) $(X_P, \mathscr{L}_P) = (\mathbb{P}(L_0 \oplus \cdots \oplus L_t), \xi)$ *for ample line bundles* L_i *on a non singular toric variety* Y.

Notice that Theorem 5.7 proves the reverse statement of Lemma 5.5.

Conjectures 5.3 and 5.6, made independently in two apparently unrelated fields, constitute a beautiful example of the interplay between classical projective (toric) geometry and convex geometry. In view of the results above one could hope that in the toric setting the conjectures should hold in more generally.

Conjecture 5.8. Let $(X, \mathscr{L})$ be an n-dimensional toric polarized variety (not necessarily smooth or even Gorenstein), then $\mu(\mathscr{L}) > (n+1)/2$ implies that $\mu(\mathscr{L}) = \tau(\mathscr{L})$.

The invariants $\mu(\mathscr{L}), \tau(\mathscr{L})$ in the non Gorenstein case can be defined using corresponding invariants, $\mu(P), \tau(P)$ of the associated polytope, see below for a definition.

Conjecture 5.9. If an n-dimensional lattice polytope P satisfies $\operatorname{codeg}(P) > (n+2)/2$, then it decomposes as a Cayley sum of lattice polytopes of dimension at most $2(n+1-\operatorname{codeg}(P))$.

Conjecture 5.8 is a toric version of Conjecture 5.3, extending the statement to possibly singular and non Gorenstein varieties. Conjecture 5.9 states that the function $f(n)$ in Conjecture 5.6 should be equal to $(n+2)/2$. An important step to prove these conjectures is to define the convex analog of $\mu(\mathscr{L}_P)$.

Let $P \subseteq \mathbb{R}^n$ be a rational polytope of dimension n. Any such polytope P can be described in a unique minimal way as

$$P = \{x \in \mathbb{R}^n \ : \ \langle a_i, x\rangle \geqslant b_i, \ i = 1, \ldots, m\}$$

where the a_i are the rows of an $m \times n$ integer matrix A, and $b \in \mathbb{Q}^m$.

For any $s \geqslant 0$ we define the *adjoint polytope* $P^{(s)}$ as

$$P^{(s)} := \{x \in \mathbb{R}^n \ : \ Ax \geqslant b + s\mathbf{1}\},$$

where $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}}$.

We call the study of such polytopes $P^{(s)}$ *polyhedral adjunction theory* (Fig. 2).

Definition 5.10. We define the $\mathbb{Q}$-*codegree* of P as

$$\mu(P) := (\sup\{s > 0 \ : \ P^{(s)} \neq \emptyset\})^{-1},$$

and the *core* of P to be $\operatorname{core}(P) := P^{(1/\mu(P))}$.

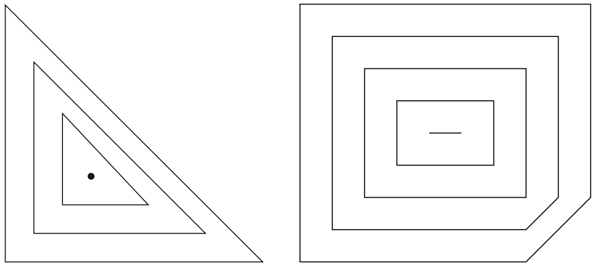


Fig. 2 Two examples of polyhedral adjunction

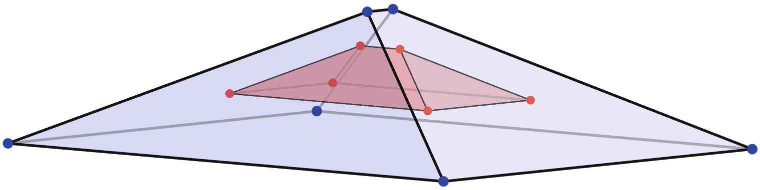


Fig. 3 $P^{(4)} \subseteq P$ for a three-dimensional lattice polytope P

Notice that in this case the supremum is actually a maximum. Moreover, since P is a rational polytope, $\mu(P)$ is a positive rational number.

One sees that for a *lattice* polytope P

$$\mu(P) \leqslant \text{codeg}(P) \leqslant n + 1$$

Definition 5.11. The *nef value* of P is given as

$$\tau(P) := (\sup\{s > 0 : \mathcal{N}(P^{(s)}) = \mathcal{N}(P)\})^{-1} \in \mathbb{R}_{>0} \cup \{\infty\}$$

where $\mathcal{N}(P)$ denotes the normal fan of the polytope P.

Note that in contrast to the definition of the $\mathbb{Q}$-codegree, here the supremum is never a maximum.

Figure 3 illustrates a polytope P with $\tau(P)^{-1} = 2$ and $\mu(P)^{-1} = 6$. In this case core(P) is an interval.

In [DRHNP13] the precise analogue of Conjecture 5.9 for the $\mathbb{Q}$-codegree is proven.

Theorem 5.12 ([DRHNP13]). *Let P be an n-dimensional lattice polytope. If n is odd and $\mu(P) > (n + 1)/2$, or if n is even and $\mu(P) \geqslant (n + 1)/2$, then P is a Cayley polytope.*

Results from [DRHNP13] show Conjecture 5.9 in two interesting cases: when $\lceil \mu(P) \rceil = \text{codeg}(P)$ and when the normal fan of P is Gorenstein and $\mu(P) = \tau(P)$.

6 Connecting the Three Characterizations

In Sect. 4 we have seen that a certain class of Cayley polytopes characterizes dually defective configuration points. Moreover this class corresponds to the polytopes achieving the equality in Corollary 4.15. In Sect. 5 the same class of Cayley polytopes was characterized as corresponding to smooth configurations with codegree larger than slightly more that half the dimension. We will here assemble the three characterizations and provide proofs in the non singular case.

Theorem 6.1. *Let $\mathscr{A} \subset \mathbb{Z}^n$ be a point configuration such that $P_{\mathscr{A}} \cap \mathbb{Z}^n = \mathscr{A}$, $\dim(P_{\mathscr{A}}) = n$ and such that $P_{\mathscr{A}}$ is a smooth polytope. Then the following statements are equivalent.*

(a) *$P_{\mathscr{A}}$ is affinely isomorphic to a Cayley sum* $\mathrm{Cayley}(R_0, \dots, R_t)_{\pi,Y}$ *where $t + 1 = \mathrm{codeg}(P_{\mathscr{A}})$ and $t > \frac{n}{2}$.*
(b) $\mathrm{codeg}(P_{\mathscr{A}}) \geqslant \frac{n+1}{2} + 1$ *and $\tau(P_{\mathscr{A}}) = \mu(P_{\mathscr{A}})$.*
(c) *The discriminant $\Delta_{\mathscr{A}} = 1$.*
(d) $\sum_{\emptyset \neq F \prec P_{\mathscr{A}}} (\dim(F) + 1)!(-1)^{\mathrm{codim}(F)} \mathrm{Vol}(F) = 0$

Proof. $[(d) \Leftrightarrow (c)]$. The implication $(d) \rightsquigarrow (c)$ follows from Lemma 4.10 and Proposition 4.12. The reverse implication follows from Corollary 4.17.
$[(c) \Rightarrow (b)$.] Assume now (c), i.e. assume that the configuration is dually defective. Consider the associated polarized toric manifold $(X_{\mathscr{A}}, \mathscr{L}_{\mathscr{A}})$. It is a classical result that the generic tangent hyperplane is in fact tangent along a linear space in $X_{\mathscr{A}}$. Therefore if $\mathrm{codim}(X_{\mathscr{A}}^{\vee}) = k > 1$ then there is a linear $\mathbb{P}^k$ through a general point of $X_{\mathscr{A}}$. By linear $\mathbb{P}^k$ we mean a subspace $Y \cong \mathbb{P}^k$ such that $\mathscr{L}_{\mathscr{A}}|_Y = \mathscr{O}_{\mathbb{P}^k}(1)$. Moreover, by a result of Ein [E86] $N_{\mathbb{P}^k/X} = (\oplus_1^{\frac{n-k}{2}} \mathscr{O}_{\mathbb{P}^k}) \oplus (\oplus_1^{\frac{n-k}{2}} \mathscr{O}_{\mathbb{P}^k}(1))$. Observe that if we fix a point $x \in X_{\mathscr{A}}$, a sequence $\{F_j\}$ of general linear subspaces $F_J \cong \mathbb{P}^k$ can be chosen so that $x \in \lim(F_j)$. Since the F_i are all linear the limit space has to be also a linear $\mathbb{P}^k$. We can then assume that there is a linear $\mathbb{P}^k$ through every point of $X_{\mathscr{A}}$. Let L now be an invariant line in one of the $\mathbb{P}^k$ through a fixed point. Then:

$$[K_X + t\mathscr{L}]_L = \mathscr{O}_{\mathbb{P}^1}((-n-2-k)/2 + t)$$

which implies $\tau(\mathscr{L}_{\mathscr{A}}) \geqslant \frac{n+k}{2} + 1$. Assume now that $\tau(\mathscr{L}_{\mathscr{A}}) > \frac{n+k}{2} + 1$ and let L be again a line in the family of linear spaces covering X. The quantity $-K_X \cdot L - 2 = \nu$ is called the normal degree of the family. In our case $\nu = (n+k)/2 - 1 > n/2$. By a result of Beltrametti-Sommese-Wisniewski [BSW92], this assumption implies $\nu = \tau - 2$, proving $\tau(\mathscr{L}_A) = \frac{n+k}{2} + 1$. Notice that the nef-morphism ϕ_τ contracts all the linear $\mathbb{P}^k$ of the covering family and thus it is a fibration. As a consequence the line bundle $K_{X_{\mathscr{A}}} + \tau\mathscr{L}_{\mathscr{A}}$ is not big and thus $\tau(\mathscr{L}_{\mathscr{A}}) = \mu(\mathscr{L}_{\mathscr{A}})$. The inequality

$$\mathrm{codeg}(P_{\mathscr{A}}) \geqslant \mu(\mathscr{L}_{\mathscr{A}}) = \tau(\mathscr{L}_{\mathscr{A}}) = \frac{n+k}{2} + 1$$

shows the implication $(c) \rightsquigarrow (b)$.

$[(b) \Rightarrow (a)]$. Assume now (b). The nef-value morphism is then a fibration and

$$\tau(\mathscr{L}_{\mathscr{A}}) > \operatorname{codeg}(P_{\mathscr{A}}) - 1 > \frac{n}{2}.$$

Notice that the nef-morphism ϕ_τ contracts a face of the Mori-cone and thus faces of the lattice polytope $P_{\mathscr{A}}$, i.e. all the invariant curves with 0-intersection with the line bundle $K_{X_{\mathscr{A}}} + \tau\mathscr{L}_{\mathscr{A}}$. Let now C be a generator of an extremal ray contracted by the morphism ϕ_τ. If $\mathscr{L}_{\mathscr{A}} \cdot C \geqslant 2$, then $-K_X \cdot C > n + 1$ which is impossible. We can conclude that C is a line and $\tau(\mathscr{L}_{\mathscr{A}}) = -K_{X_{\mathscr{A}}} \cdot C$ is an integer. It follows that $\tau(\mathscr{L}_{\mathscr{A}}) \geqslant \frac{n+1}{2} + 1$. This inequality implies that ϕ_τ is the contraction of one extremal ray, by [BSW92, Cor. 2.5]. These morphisms are analyzed in detail in [Re83]. Because $X_{\mathscr{A}}$ is smooth and toric and this contraction has connected fibres, the general fiber F of the contraction is a smooth toric variety with Picard number one. It follows that F is a projective space and thus ϕ_τ is a $\mathbb{P}^t$ bundle. Let $L|_F = \mathscr{O}_{\mathbb{P}^t}(a)$. Observe that by construction $K_{X_{\mathscr{A}}}|_F = K_F$. Consider a line $l \subset F$. It follows that

$$0 = (K_{X_{\mathscr{A}}} + \tau\mathscr{L}_{\mathscr{A}}) \cdot l = K_F \cdot l + \tau\mathscr{L}_{\mathscr{A}} \cdot l = -t - 1 + a\tau$$

and thus $\tau = \frac{t+1}{a} > \frac{n+1}{2} + 1$ which implies $a = 1$ and $t > \frac{n+1}{2}$. Since $a = 1$ the fibers are embedded linearly and thus $(X_{\mathscr{A}}, \mathscr{L}_{\mathscr{A}}) = (\mathbb{P}(L_0 \oplus \ldots \oplus L_t), \xi)$, for ample line bundles L_i on a smooth toric variety Y. This proves the implication $(b) \rightsquigarrow (a)$.
$[(a) \Rightarrow (c)]$. Assume now (a). Using notation as in (2), consider the commutative diagram:

$$\begin{tikzcd}
& \alpha^{-1}(\mathbb{P}(E)^\vee_{reg}) \arrow[dl, "\gamma"'] \arrow[dr, "\alpha"] & \\
\mathbb{P}(E) \arrow[dr, "\pi"'] & & \mathbb{P}(E)^\vee_{reg} \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

where $E = L_0 \oplus \ldots \oplus L_t$ and (Y, L_i) is the smooth polarized variety associated to the polytope R_i. The commutativity of the diagram and the existence of f follows from [DeB01, Lemma 1.15]. Let $y \in Y$ and let $F \cong \mathbb{P}^t \subset \mathbb{P}^{|\mathscr{A}|-1}$ be the fiber $\pi^{-1}(y)$. Commutativity of the diagram implies that the contact locus $\gamma(\alpha^{-1}(H))$ is included in F for all $H \in f^{-1}(y)$. Moreover $\mathbb{O}sc_{F,y} \subset \mathbb{O}sc_{\mathbb{P}(E),y} \subset H$ implies that H belongs to the dual variety $F^\vee$, with contact locus at least of the same dimension. Because the map f is dominant we can conclude that: $\dim(F^\vee) \geqslant \dim(\mathbb{P}(E)^\vee) - \dim(Y)$, which implies

$$\operatorname{codim}(\mathbb{P}(E)^\vee) \geqslant \operatorname{codim}(F^\vee) - \dim(Y).$$

Recall that the fibers F are embedded linearly and thus $\operatorname{codim}(F^\vee) = \dim(F) + 1$. It follows that $\operatorname{codim}(\mathbb{P}(E)^\vee) \geqslant \dim(F) + 1 - \dim(Y) > 1$ and thus $\Delta_{\mathscr{A}} = 1$. This proves $(a) \rightsquigarrow (c)$. □

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