Chapter 2
Weighted Graphs and the Associated Markov Chains

In this chapter, we discuss general potential theory for symmetric (reversible) Markov chains on weighted graphs. Note that there are many nice books and lecture notes that treat potential theory and/or Markov chains on graphs, for example [7, 20, 93, 118, 125, 175, 195, 204, 211]. While writing this chapter, we are largely influenced by the lecture notes by Barlow [20].

2.1 Weighted Graphs

Let X be a finite or a countably infinite set, and E is a subset of $\{\{x, y\} : x, y \in X, x \neq y\}$. A graph is a pair (X, E). For $x, y \in X$, we write $x \sim y$ if $\{x, y\} \in E$. A sequence $x_0, x_1, \cdots, x_n$ is called a path with length n if $x_i \in X$ for $i = 0, 1, 2, \cdots, n$ and $x_j \sim x_{j+1}$ for $j = 0, 1, 2, \cdots, n-1$. For $x \neq y$, define $d(x, y)$ to be the length of the shortest path from x to y. If there is no such path, we set $d(x, y) = \infty$ and we set $d(x, x) = 0$. $d(\cdot, \cdot)$ is a metric on X and it is called a graph distance. (X, E) is connected if $d(x, y) < \infty$ for all $x, y \in X$, and it is locally finite if $|\{y : \{x, y\} \in E\}| < \infty$ for all $x \in X$. Throughout the lectures, we will consider connected locally finite graphs (except when we consider the trace of them in Sect. 2.3).

Assume that the graph (X, E) is endowed with a weight (conductance) μ_{xy}, which is a symmetric nonnegative function on $X \times X$ such that $\mu_{xy} > 0$ if and only if $x \sim y$. We call the pair (X, μ) a weighted graph.

Let $\mu_x = \mu(x) = \sum_{y \in X} \mu_{xy}$ and define a measure μ on X by setting $\mu(A) = \sum_{x \in A} \mu_x$ for $A \subset X$. Also, we define $B(x, r) = \{y \in X : d(x, y) < r\}$ for each $x \in X$ and $r \geq 1$.

Definition 2.1.1. We say that (X, μ) has controlled weights (or (X, μ) satisfies p_0-condition) if there exists $p_0 > 0$ such that

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$$\frac{\mu_{xy}}{\mu_x} \geq p_0 \qquad \forall x \sim y.$$

If (X, μ) has controlled weights, then clearly $|\{y \in X : x \sim y\}| \leq p_0^{-1}$.

Once the weighted graph (X, μ) is given, we can define the corresponding quadratic form, Markov chain and the discrete Laplace operator.

Quadratic Form. We define a quadratic form on (X, μ) as follows.

$$H^2(X, \mu) = H^2 = \{f : X \to \mathbb{R} : \mathcal{E}(f, f) = \frac{1}{2} \sum_{\substack{x,y \in X \\ x \sim y}} (f(x) - f(y))^2 \mu_{xy} < \infty\},$$

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{\substack{x,y \in X \\ x \sim y}} (f(x) - f(y))(g(x) - g(y))\mu_{xy} \qquad \forall f, g \in H^2.$$

Physically, $\mathcal{E}(f, f)$ is the energy (per unit time) of the electrical network for an (electric) potential f.

Since the graph is connected, one can easily see that $\mathcal{E}(f, f) = 0$ if and only if f is a constant function. We fix a base point $0 \in X$ and define

$$\|f\|_{H^2}^2 = \mathcal{E}(f, f) + f(0)^2 \qquad \forall f \in H^2.$$

Note that

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x \sim y} (f(x) - f(y))^2 \mu_{xy} \leq \sum_x \sum_y (f(x)^2 + f(y)^2) \mu_{xy} = 2\|f\|_2^2, \tag{2.1}$$

for all $f \in \mathbb{L}^2$ where $\|f\|_2 := (\sum_x f(x)^2 \mu_x)^{1/2}$ is the $\mathbb{L}^2$-norm of f. So $\mathbb{L}^2 \subset H^2$. We give basic facts in the next lemma.

Lemma 2.1.2. *(i) Convergence in H^2 implies the pointwise convergence.*
(ii) H^2 is a Hilbert space.

Proof. (i) Suppose $f_n \to f$ in H^2 and let $g_n = f_n - f$. Then $\mathcal{E}(g_n, g_n) + g_n(0)^2 \to 0$ so $g_n(0) \to 0$. For any $x \in X \setminus \{0\}$, there is a sequence $\{x_i\}_{i=0}^l \subset X$ such that $x_0 = 0, x_l = x, x_i \sim x_{i+1}$ for $i = 0, 1, \cdots, l-1$ and $x_i \neq x_j$ for $i \neq j$. Then

$$|g_n(x) - g_n(0)|^2 \leq l \sum_{i=0}^{l-1} |g_n(x_i) - g_n(x_{i+1})|^2 \leq 2l (\min_{i=0}^{l-1} \mu_{x_i x_{i+1}})^{-1} \mathcal{E}(g_n, g_n) \to 0 \tag{2.2}$$

as $n \to \infty$ so we have $g_n(x) \to 0$.

(ii) Assume that $\{f_n\}_n \subset H^2$ is a Cauchy sequence in H^2. Then $f_n(0)$ is a Cauchy sequence in $\mathbb{R}$ so converges. Thus, similarly to (2.2) f_n converges pointwise to f, say. Now using Fatou's lemma, we have $\| f_n - f\|_{H^2}^2 \le \liminf_m \| f_n - f_m\|_{H^2}^2$, so that $\| f_n - f\|_{H^2}^2 \to 0$. □

Markov Chain. Let $Y = \{Y_n\}$ be a Markov chain on X whose transition probabilities are given by

$$\mathbb{P}(Y_{n+1} = y | Y_n = x) = \frac{\mu_{xy}}{\mu_x} =: P(x, y) \qquad \forall x, y \in X.$$

We write $\mathbb{P}^x$ when the initial distribution of Y is concentrated on x (i.e. $Y_0 = x$, $\mathbb{P}$-a.s.). $(P(x, y))_{x,y\in X}$ is the transition matrix for Y. Y is called a simple random walk when $\mu_{xy} = 1$ whenever $x \sim y$. Y is μ-symmetric since for each $x, y \in X$,

$$\mu_x P(x, y) = \mu_{xy} = \mu_{yx} = \mu_y P(y, x).$$

We define the *heat kernel* of Y by

$$p_n(x, y) = \mathbb{P}^x(Y_n = y)/\mu_y \qquad \forall x, y \in X. \tag{2.3}$$

Using the Markov property, we can easily show the Chapman-Kolmogorov equation:

$$p_{n+m}(x, y) = \sum_z p_n(x, z) p_m(z, y)\mu_z, \qquad \forall x, y \in X. \tag{2.4}$$

Using this and the fact $p_1(x, y) = \mu_{xy}/(\mu_x\mu_y) = p_1(y, x)$, one can verify the following inductively

$$p_n(x, y) = p_n(y, x), \qquad \forall x, y \in X.$$

For $n \ge 1$ and $f : X \to \mathbb{R}$, let

$$P_n f(x) = \sum_y p_n(x, y) f(y)\mu_y = \sum_y \mathbb{P}^x(Y_n = y) f(y) = \mathbb{E}^x[f(Y_n)].$$

We sometimes consider a continuous time Markov chain $\{Y_t\}_{t\ge 0}$ with respect to μ which is defined as follows: each particle stays at a point, say x for (independent) exponential time with parameter 1, and then jumps to another point, say y with probability $P(x, y)$. The heat kernel for the continuous time Markov chain can be expressed as follows.

$$p_t(x, y) = \mathbb{P}^x(Y_t = y)/\mu_y = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!} p_n(x, y), \qquad \forall x, y \in X.$$

Discrete Laplace Operator. For $f : X \to \mathbb{R}$, the discrete Laplace operator is defined by

$$\begin{aligned}\mathcal{L}f(x) &= \sum_y P(x,y)f(y) - f(x) = \frac{1}{\mu_x}\sum_y (f(y)-f(x))\mu_{xy} \\ &= \mathbb{E}^x[f(Y_1)] - f(x) = (P_1 - I)f(x),\end{aligned} \tag{2.5}$$

where Y_1 is the (discrete time) Markov chain on X at time 1. Note that according to Ohm's law "$I = V/R$", $\sum_y (f(y) - f(x))\mu_{xy}$ is the total flux flowing into x, given the potential f.

Definition 2.1.3. Let $A \subset X$. A function $f : X \to \mathbb{R}$ is harmonic on A if

$$\mathcal{L}f(x) = 0, \qquad \forall x \in A.$$

f is sub-harmonic (resp. super-harmonic) on A if $\mathcal{L}f(x) \geq 0$ (resp. $\mathcal{L}f(x) \leq 0$) for $x \in A$.

$\mathcal{L}f(x) = 0$ means that the total flux flowing into x is 0 for the given potential f. This is the behavior of the currents in a network called Kirchhoff's (first) law.

For $A \subset X$, we define the (exterior) boundary of A by

$$\partial A = \{x \in A^c : \exists z \in A \text{ such that } z \sim x\}. \tag{2.6}$$

Proposition 2.1.4 (Maximum Principle). *Let A be a connected subset of X and $h : A \cup \partial A \to \mathbb{R}$ be sub-harmonic on A. If the maximum of h over $A \cup \partial A$ is attained in A, then h is constant on $A \cup \partial A$.*

Proof. Let $x_0 \in A$ be the point where h attains the maximum and let $H = \{z \in A \cup \partial A : h(z) = h(x_0)\}$. If $y \in H \cap A$, then since $h(y) \geq h(x)$ for all $x \in A \cup \partial A$, we have

$$0 \leq \mu_y \mathcal{L}h(y) = \sum_x (h(x) - h(y))\mu_{xy} \leq 0.$$

Thus, $h(x) = h(y)$ (i.e. $x \in H$) for all $x \sim y$. Since A is connected, this implies $H = A \cup \partial A$. □

We can prove the minimum principle for a super-harmonic function h by applying the maximum principle to $-h$.

For $f, g \in \mathbb{L}^2$, denote their $\mathbb{L}^2$-inner product as (f, g), namely $(f, g) = \sum_x f(x)g(x)\mu_x$.

Lemma 2.1.5. *(i) $\mathcal{L} : H^2 \to \mathbb{L}^2$ and $\|\mathcal{L}f\|_2^2 \leq 2\|f\|_{H^2}^2$.*
(ii) For $f \in H^2$ and $g \in \mathbb{L}^2$, we have $(-\mathcal{L}f, g) = \mathcal{E}(f, g)$.
(iii) $\mathcal{L}$ is a self-adjoint operator on $\mathbb{L}^2(X, \mu)$ and the following holds:

$$(-\mathcal{L}f, g) = (f, -\mathcal{L}g) = \mathcal{E}(f, g), \qquad \forall f, g \in \mathbb{L}^2. \tag{2.7}$$

Proof. (i) Using the Schwarz inequality, we have

$$\begin{aligned}\|\mathcal{L}f\|_2^2 &= \sum_x \frac{1}{\mu_x}\Big(\sum_y (f(y)-f(x))\mu_{xy}\Big)^2 \\ &\le \sum_x \frac{1}{\mu_x}\Big(\sum_y (f(y)-f(x))^2\mu_{xy}\Big)\Big(\sum_y \mu_{xy}\Big) = 2\mathcal{E}(f,f) \le 2\|f\|_{H^2}^2.\end{aligned}$$

(ii) Using (i), both sides of the equality are well-defined. Further, using the Schwarz inequality,

$$\begin{aligned}\sum_{x,y}|\mu_{xy}(f(y)-f(x))g(x)| &\le \Big(\sum_{x,y}\mu_{xy}(f(y)-f(x))^2\Big)^{1/2}\Big(\sum_{x,y}\mu_{xy}g(x)^2\Big)^{1/2} \\ &= \sqrt{2}\mathcal{E}(f,f)^{1/2}\|g\|_2 < \infty.\end{aligned}$$

So we can use Fubini's theorem, and we have

$$\begin{aligned}(-\mathcal{L}f,g) &= -\sum_x\Big(\sum_y \mu_{xy}(f(y)-f(x))\Big)g(x) \\ &= \frac{1}{2}\sum_x\sum_y \mu_{xy}(f(y)-f(x))(g(y)-g(x)) = \mathcal{E}(f,g).\end{aligned}$$

(iii) We can prove $(f,-\mathcal{L}g) = \mathcal{E}(f,g)$ similarly and obtain (2.7). □

Equation (2.7) is the discrete Gauss-Green formula.

Lemma 2.1.6. *Set $p_n^x(\cdot) = p_n(x,\cdot)$. Then, the following hold for all $x,y \in X$.*

$$p_{n+m}(x,y) = (p_n^x, p_m^y),\ \ P_1 p_n^x(y) = p_{n+1}^x(y), \tag{2.8}$$

$$\mathcal{L}p_n^x(y) = p_{n+1}^x(y) - p_n^x(y),\ \ \mathcal{E}(p_n^x, p_m^y) = p_{n+m}^x(y) - p_{n+m+1}^x(y), \tag{2.9}$$

$$p_{2n}(x,y) \le \sqrt{p_{2n}(x,x)p_{2n}(y,y)}. \tag{2.10}$$

Proof. The two equations in (2.8) are due to the Chapman-Kolmogorov equation (2.4). The first equation in (2.9) is then clear since $\mathcal{L} = P_1 - I$. The second equation in (2.9) can be obtained by these equations and (2.7). Using (2.8) and the Schwarz inequality, we have

$$p_{2n}(x,y)^2 = (p_n^x, p_n^y)^2 \le (p_n^x, p_n^x)(p_n^y, p_n^y) = p_{2n}(x,x)p_{2n}(y,y),$$

which gives (2.10). □

It can be easily shown that $(\mathcal{E}, \mathbb{L}^2)$ is a regular Dirichlet form on $\mathbb{L}^2(X,\mu)$ (cf. [108]). Then the corresponding Hunt process is the continuous time Markov chain $\{Y_t\}_{t\ge 0}$ with respect to μ and the corresponding self-adjoint operator on $\mathbb{L}^2$ is $\mathcal{L}$ in (2.5).

Remark 2.1.7. Note that $\{Y_t\}_{t\geq 0}$ has the transition probability $P(x,y) = \mu_{xy}/\mu_x$ and it waits at x for an exponential time with mean 1 for each $x \in X$. Since the "speed" of $\{Y_t\}_{t\geq 0}$ is independent of the location, it is sometimes called constant speed random walk (CSRW for short). We can also consider a continuous time Markov chain with the same transition probability $P(x,y)$ and wait at x for an exponential time with mean μ_x^{-1} for each $x \in X$. This Markov chain is called variable speed random walk (VSRW for short). We will discuss VSRW in Chap. 8. The corresponding discrete Laplace operator is

$$\mathcal{L}_V f(x) = \sum_y (f(y) - f(x))\mu_{xy}. \tag{2.11}$$

For each f, g that have finite support, we have

$$\mathcal{E}(f,g) = -(\mathcal{L}_V f, g)_\nu = -(\mathcal{L} f, g)_\mu,$$

where ν is a measure on X such that $\nu(A) = |A|$ for all $A \subset X$. So VSRW is the Markov process associated with the Dirichlet form $(\mathcal{E}, \mathbb{L}^2)$ on $\mathbb{L}^2(X,\nu)$ and CSRW is the Markov process associated with the Dirichlet form $(\mathcal{E}, \mathbb{L}^2)$ on $\mathbb{L}^2(X,\mu)$. VSRW is a time changed process of CSRW and vice versa.

We now introduce the notion of rough isometry.

Definition 2.1.8. Let $(X_1,\mu_1),(X_2,\mu_2)$ be weighted graphs that have controlled weights.

(i) A map $T : X_1 \to X_2$ is called a rough isometry if the following holds. There exist constants $c_1, c_2, c_3 > 0$ such that

$$c_1^{-1} d_1(x,y) - c_2 \leq d_2(T(x),T(y)) \leq c_1 d_1(x,y) + c_2 \quad \forall x,y \in X_1, \tag{2.12}$$

$$d_2(T(X_1), y') \leq c_2 \quad \forall y' \in X_2, \tag{2.13}$$

$$c_3^{-1}\mu_1(x) \leq \mu_2(T(x)) \leq c_3\mu_1(x) \quad \forall x \in X_1, \tag{2.14}$$

where $d_i(\cdot,\cdot)$ is the graph distance of (X_i,μ_i), for $i = 1,2$.

(ii) $(X_1,\mu_1),(X_2,\mu_2)$ are said to be rough isometric if there is a rough isometry between them.

It is easy to see that rough isometry is an equivalence relation. One can easily prove that $\mathbb{Z}^2$, the triangular lattice, and the hexagonal lattice are all roughly isometric if there exists $M > 0$ such that $\mu_{xy} \in [M^{-1}, M]$ whenever $x \sim y$. It can be proved that $\mathbb{Z}^1$ and $\mathbb{Z}^2$ are not roughly isometric.

The notion of rough isometry was first introduced by M. Kanai [145, 146]. Since his work was mainly concerned with Riemannian manifolds, definition of rough isometry included only (2.12), (2.13). The definition equivalent to Definition 2.1.8 is given in [77] (see also [130]). Note that rough isometry corresponds to (coarse) quasi-isometry in the field of geometric group theory, which was introduced by Gromov already in 1981 (see [124]).

When we discuss various properties of Markov chains/Laplace operators, it is important to think about their "stability". In the following, we introduce two types of stability.

Definition 2.1.9. (i) We say a property is stable under bounded perturbation if whenever (X, μ) satisfies the property and (X, μ') satisfies $c^{-1}\mu_{xy} \le \mu'_{xy} \le c\mu_{xy}$ for all $x, y \in X$, then (X, μ') satisfies the property.
(ii) We say a property is stable under rough isometry if whenever (X, μ) satisfies the property and (X', μ') is rough isometric to (X, μ), then (X', μ') satisfies the property.

If a property is stable under rough isometry, then it is clearly stable under bounded perturbation.

It is known that the following properties of weighted graphs are stable under rough isometry.

(i) Transience and recurrence
(ii) The Nash inequality, i.e. $p_n(x, y) \le c_1 n^{-\alpha}$ for all $n \ge 1, x, y \in X$ (for some $\alpha > 0$)
(iii) Parabolic Harnack inequality (see Definition 3.3.4 (2))

We will see (i) later in this chapter, and (ii) and (iii) in Chap. 3. One of the important open problems is to show if the elliptic Harnack inequality, i.e. the Harnack inequality for harmonic functions, is stable under these perturbations or not. (In fact, recently this has been affirmatively solved in [38] under some assumption. Yet, the assumption contains some regularity of the growth of capacities and occupation times. It would be desirable to prove (or disprove) the stability assuming only the volume growth condition.)

Definition 2.1.10. (X, μ) has the Liouville property if there is no bounded non-constant harmonic functions. (X, μ) has the strong Liouville property if there is no positive non-constant harmonic functions.

It is known that both Liouville and strong Liouville properties are not stable under bounded perturbation (see [176], also [45] for a counterexample in the framework of graphs/manifolds with polynomial volume growth).

2.2 Harmonic Functions and Effective Resistances

For $A \subset X$, define

$$\sigma_A = \inf\{n \ge 0 : Y_n \in A\}, \quad \sigma_A^+ = \inf\{n > 0 : Y_n \in A\},$$
$$\tau_A = \inf\{n \ge 0 : Y_n \notin A\}.$$

For $A \subset X$ and $f : A \to \mathbb{R}$, consider the following *Dirichlet problem*.

$$\begin{cases} \mathcal{L}v(x) = 0 \ \ \forall x \in A^c, \\ \quad v|_A = f. \end{cases} \tag{2.15}$$

Proposition 2.2.1. *Assume that $f : A \to \mathbb{R}$ is bounded and set*

$$\varphi(x) = \mathbb{E}^x[f(Y_{\sigma_A}) : \sigma_A < \infty].$$

(i) φ is a solution of (2.15).
(ii) If $\mathbb{P}^x(\sigma_A < \infty) = 1$ for all $x \in X$, then φ is the unique bounded solution of (2.15).

Proof. (i) $\varphi|_A = f$ is clear. For $x \in A^c$, using the Markov property of Y, we have

$$\varphi(x) = \sum_y P(x, y)\varphi(y),$$

so $\mathcal{L}\varphi(x) = 0$.
(ii) Let φ' be another bounded solution and let $H_n = \varphi(Y_n) - \varphi'(Y_n)$. Then H_n is a bounded martingale up to σ_A, so using the optional stopping theorem, we have

$$\begin{aligned} \varphi(x) - \varphi'(x) = \mathbb{E}^x H_0 = \mathbb{E}^x H_{\sigma_A} &= \mathbb{E}^x[\varphi(Y_{\sigma_A}) - \varphi'(Y_{\sigma_A})] \\ &= \mathbb{E}^x[f(Y_{\sigma_A}) - f(Y_{\sigma_A})] = 0 \end{aligned}$$

since $\sigma_A < \infty$ a.s. and $\varphi(x) = \varphi'(x)$ for $x \in A$. □

Remark 2.2.2. (i) In particular, we see that φ is the unique solution of (2.15) when A^c is finite. In this case, we have another proof of the uniqueness of the solution of (2.15): let $u(x) = \varphi(x) - \varphi'(x)$, then $u|_A = 0$ and $\mathcal{L}u(x) = 0$ for $x \in A^c$. So, noting $u \in \mathbb{L}^2$ and using Lemma 2.1.5, $\mathcal{E}(u, u) = (-\mathcal{L}u, u) = 0$ which implies that u is constant on X (so it is 0 since $u|_A = 0$).
(ii) If $h_A(x) := \mathbb{P}^x(\sigma_A = \infty) > 0$ for some $x \in X$, then the function $\varphi + \lambda h_A$ is also a solution of (2.15) for all $\lambda \in \mathbb{R}$, so the uniqueness of the Dirichlet problem fails.

For $A, B \subset X$ such that $A \cap B = \emptyset$, define

$$R_{\text{eff}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f \in H^2, f|_A = 1, f|_B = 0\}. \tag{2.16}$$

(We define $R_{\text{eff}}(A, B) = \infty$ when the right hand side is 0, and $R_{\text{eff}}(A, B) = 0$ when there is no $f \in H^2$ that satisfies $f|_A = 1$ and $f|_B = 0$.) We call $R_{\text{eff}}(A, B)$ the *effective resistance* between A and B. It is easy to see that $R_{\text{eff}}(A, B) = R_{\text{eff}}(B, A)$. If $A \subset A'$, $B \subset B'$ with $A' \cap B' = \emptyset$, then $R_{\text{eff}}(A', B') \le R_{\text{eff}}(A, B)$.

Take a bond $e = \{x, y\}$, $x \sim y$ in a weighted graph (X, μ). We say *cutting the bond e* when we take the conductance μ_{xy} to be 0, and we say *shorting the bond e* when we identify $x = y$ and take the conductance μ_{xy} to be ∞. Clearly, shorting decreases the effective resistance (*shorting law*), and cutting increases the effective resistance (*cutting law*).

The following proposition (*Dirichlet's principle*) shows that among feasible potentials whose voltage is 1 on A and 0 on B, it is a harmonic function on $(A \cup B)^c$ that minimizes the energy.

Proposition 2.2.3. *Assume* $R_{\mathrm{eff}}(A, B) \neq 0$.

(i) The right hand side of (2.16) is attained by a unique minimizer φ.
(ii) φ *in (i) is a solution of the following Dirichlet problem*

$$\begin{cases} \mathcal{L}\varphi(x) = 0, \ \ \forall x \in X \setminus (A \cup B), \\ \quad \varphi|_A = 1, \, \varphi|_B = 0. \end{cases} \tag{2.17}$$

Proof. (i) We fix a base point $x_0 \in B$ and recall that H^2 is a Hilbert space with $\|f\|_{H^2} = \mathcal{E}(f, f) + f(x_0)^2$ (Lemma 2.1.2 (ii)). Since $\mathcal{V} := \{f \in H^2 : f|_A = 1, f|_B = 0\}$ is a non-void (because $R_{\mathrm{eff}}(A, B) \neq 0$) closed convex subset of H^2, a general theorem shows that $\mathcal{V}$ has a unique minimizer for $\|\cdot\|_{H^2}$ (which is equal to $\mathcal{E}(\cdot, \cdot)$ on $\mathcal{V}$).

(ii) Let g be a function on X whose support is finite and is contained in $X \setminus (A \cup B)$. Then, for any $\lambda \in \mathbb{R}$, $\varphi + \lambda g \in \mathcal{V}$, so $\mathcal{E}(\varphi + \lambda g, \varphi + \lambda g) \geq \mathcal{E}(\varphi, \varphi)$. Thus $\mathcal{E}(\varphi, g) = 0$. Applying Lemma 2.1.5 (ii), we have $(\mathcal{L}\varphi, g) = 0$. For each $x \in X \setminus (A \cup B)$, by choosing $g(z) = \delta_x(z)$, we obtain $\mathcal{L}\varphi(x)\mu_x = 0$. □

As we mentioned in Remark 2.2.2 (ii), we do not have uniqueness of the Dirichlet problem in general. So in the following of this section, we will assume that A^c is finite in order to guarantee uniqueness of the Dirichlet problem.

Remark 2.2.4. There is a dual characterization of resistance using flows of the network. It is called *Thompson's principle* (see for example, [20, 93]).

The next theorem gives a probabilistic interpretation of the effective resistance.

Theorem 2.2.5. *If* A^c *is finite, then for each* $x_0 \in A^c$,

$$R_{\mathrm{eff}}(x_0, A)^{-1} = \mu_{x_0} \mathbb{P}^{x_0}(\sigma_A < \sigma_{x_0}^+). \tag{2.18}$$

Proof. Let $v(x) = \mathbb{P}^x(\sigma_A < \sigma_{x_0})$. Then, by Proposition 2.2.1, v is the unique solution of Dirichlet problem with $v(x_0) = 0$, $v|_A = 1$. By Proposition 2.2.3 and Lemma 2.1.5 (noting that $1 - v \in \mathbb{L}^2$),

$$\begin{aligned} R_{\mathrm{eff}}(x_0, A)^{-1} &= \mathcal{E}(v, v) = \mathcal{E}(-v, 1 - v) = (\mathcal{L}v, 1 - v) \\ &= \mathcal{L}v(x_0)\mu_{x_0} = \mathbb{E}^{x_0}[v(Y_1)]\mu_{x_0}. \end{aligned}$$

By definition of v, one can see $\mathbb{E}^{x_0}[v(Y_1)] = \mathbb{P}^{x_0}(\sigma_A < \sigma_{x_0}^+)$ so the result follows. □

Similarly, if A^c is finite one can prove

$$R_{\mathrm{eff}}(B,A)^{-1} = \sum_{x\in B} \mu_x \mathbb{P}^x(\sigma_A < \sigma_B^+).$$

Note that by Ohm's law, the right hand side of (2.18) is the current flowing from x_0 to A.

The following lemma is useful and will be used later in Proposition 4.4.3.

Lemma 2.2.6. *Let $A, B \subset X$ and assume that both A^c, B^c are finite. Then the following holds for all $x \notin A \cup B$.*

$$(R_{\mathrm{eff}}(x, A\cup B))(R_{\mathrm{eff}}(x,A)^{-1} - R_{\mathrm{eff}}(x,B)^{-1}) \le \mathbb{P}^x(\sigma_A < \sigma_B) \le \frac{R_{\mathrm{eff}}(x, A\cup B)}{R_{\mathrm{eff}}(x,A)}.$$

Proof. Using the strong Markov property, we have

$$\begin{aligned}\mathbb{P}^x(\sigma_A < \sigma_B) &= \mathbb{P}^x(\sigma_A < \sigma_B, \sigma_{A\cup B} < \sigma_x^+) + \mathbb{P}^x(\sigma_A < \sigma_B, \sigma_{A\cup B} > \sigma_x^+)\\ &= \mathbb{P}^x(\sigma_A < \sigma_B, \sigma_{A\cup B} < \sigma_x^+) + \mathbb{P}^x(\sigma_{A\cup B} > \sigma_x^+)\mathbb{P}^x(\sigma_A < \sigma_B).\end{aligned}$$

So

$$\mathbb{P}^x(\sigma_A < \sigma_B) = \frac{\mathbb{P}^x(\sigma_A < \sigma_B, \sigma_{A\cup B} < \sigma_x^+)}{\mathbb{P}^x(\sigma_{A\cup B} < \sigma_x^+)} \le \frac{\mathbb{P}^x(\sigma_A < \sigma_x^+)}{\mathbb{P}^x(\sigma_{A\cup B} < \sigma_x^+)}.$$

Using (2.18), the upper bound is obtained. For the lower bound,

$$\begin{aligned}\mathbb{P}^x(\sigma_A < \sigma_B, \sigma_{A\cup B} < \sigma_x^+) &\ge \mathbb{P}^x(\sigma_A < \sigma_x^+ < \sigma_B)\\ &\ge \mathbb{P}^x(\sigma_A < \sigma_x^+) - \mathbb{P}^x(\sigma_B < \sigma_x^+),\end{aligned}$$

so using (2.18) again, the proof is complete. □

As we see in the proof, we only need to assume that A^c is finite for the upper bound.

Now let (X,μ) be an infinite weighted graph. Let $\{A_n\}_{n=1}^\infty$ be a family of finite sets such that $A_n \subset A_{n+1}$ for $n \in \mathbb{N}$ and $\cup_{n\ge1} A_n = X$. Let $x_0 \in A_1$. By the shorting law, $R_{\mathrm{eff}}(x_0, A_n^c) \le R_{\mathrm{eff}}(x_0, A_{n+1}^c)$, so the following limit exists.

$$R_{\mathrm{eff}}(x_0) := \lim_{n\to\infty} R_{\mathrm{eff}}(x_0, A_n^c). \tag{2.19}$$

Further, the limit $R_{\mathrm{eff}}(x_0)$ is independent of the choice of the sequence $\{A_n\}$ mentioned above. (Indeed, if $\{B_n\}$ is another such family, then for each n there exists N_n such that $A_n \subset B_{N_n}$, so $\lim_{n\to\infty} R_{\mathrm{eff}}(x_0, A_n^c) \le \lim_{n\to\infty} R_{\mathrm{eff}}(x_0, B_n^c)$. By changing the role of A_n and B_n, we have the opposite inequality.)

Theorem 2.2.7. *Let (X, μ) be an infinite weighted graph. For each $x \in X$, the following holds*

$$\mathbb{P}^x(\sigma_x^+ = \infty) = (\mu_x R_{\mathrm{eff}}(x))^{-1}.$$

Proof. By Theorem 2.2.5, we have

$$\mathbb{P}^x(\sigma_{A_n^c} < \sigma_x^+) = (\mu_x R_{\mathrm{eff}}(x, A_n^c))^{-1}.$$

Taking $n \to \infty$ and using (2.19), we have the desired equality. □

Definition 2.2.8. We say a Markov chain is recurrent at $x \in X$ if $\mathbb{P}^x(\sigma_x^+ = \infty) = 0$. We say a Markov chain is transient at $x \in X$ if $\mathbb{P}^x(\sigma_x^+ = \infty) > 0$.

The following is well-known for irreducible Markov chains (so in particular it holds for Markov chains corresponding to weighted graphs). See for example [188].

Proposition 2.2.9. *(1) $\{Y_n\}_n$ is recurrent at $x \in X$ if and only if $m := \sum_{n=0}^\infty \mathbb{P}^x(Y_n = x) = \infty$. Further, $m^{-1} = \mathbb{P}^x(\sigma_x^+ = \infty)$.*
(2) If $\{Y_n\}_n$ is recurrent (resp. transient) at some $x \in X$, then it is recurrent (resp. transient) for all $x \in X$.
(3) $\{Y_n\}_n$ is recurrent if and only if $\mathbb{P}^x(\{Y \text{ hits } y \text{ infinitely often}\}) = 1$ for all $x, y \in X$. $\{Y_n\}_n$ is transient if and only if $\mathbb{P}^x(\{Y \text{ hits } y \text{ finitely often}\}) = 1$ for all $x, y \in X$.

From Theorem 2.2.7 and Proposition 2.2.9, we have the following.

$$\begin{aligned} &\{Y_n\} \text{ is transient (resp. recurrent)} \\ &\Leftrightarrow R_{\mathrm{eff}}(x) < \infty \text{ (resp. } R_{\mathrm{eff}}(x) = \infty), \exists x \in X \\ &\Leftrightarrow R_{\mathrm{eff}}(x) < \infty \text{ (resp. } R_{\mathrm{eff}}(x) = \infty), \forall x \in X. \end{aligned} \tag{2.20}$$

Example 2.2.10. Consider $\mathbb{Z}^2$ with weight 1 on each nearest neighbor bond. Let $\partial B_n = \{(x, y) \in \mathbb{Z}^2 : \text{either } |x| \text{ or } |y| \text{ is } n\}$. By shorting ∂B_n for all $n \in \mathbb{N}$, one can obtain

$$R_{\mathrm{eff}}(0) \geq \sum_{n=0}^\infty \frac{1}{4(2n+1)} = \infty.$$

So the simple random walk on $\mathbb{Z}^2$ is recurrent.

Let us recall the following fact.

Theorem 2.2.11 (Pólya 1921). *Simple random walk on $\mathbb{Z}^d$ is recurrent if $d = 1, 2$ and transient if $d \geq 3$.*

The combinatorial proof of this theorem is well-known. For example, for $d = 1$, by counting the total number of paths of length $2n$ that moves both right and left n times,

$$\mathbb{P}^0(Y_{2n}=0)=2^{-2n}\binom{2n}{n}=\frac{(2n)!}{2^{2n}n!n!}\sim(\pi n)^{-1/2},$$

where Stirling's formula is used in the end. Thus

$$m=\sum_{n=0}^{\infty}\mathbb{P}^0(Y_n=0)\sim\sum_{n=1}^{\infty}(\pi n)^{-1/2}+1=\infty,$$

so $\{Y_n\}$ is recurrent.

This argument is not robust. For example, if one changes the weight on $\mathbb{Z}^d$ so that $c_1\le\mu_{xy}\le c_2$ for $x\sim y$, one cannot apply the argument at all. The advantage of the characterization of transience/recurrence using the effective resistance is that one can make a robust argument. Indeed, by (2.20) we can see that transience/recurrence is stable under bounded perturbation. This is because, if $c_1\mu'_{xy}\le\mu_{xy}\le c_2\mu'_{xy}$ for all $x,y\in X$, then $c_1R_{\mathrm{eff}}(x)\le R'_{\mathrm{eff}}(x)\le c_2R_{\mathrm{eff}}(x)$. We can further prove that transience/recurrence is stable under rough isometry.

Finally in this section, we will give more equivalence condition for the transience and discuss some decomposition of H^2. Let H_0^2 be the closure of $C_0(X)$ in H^2, where $C_0(X)$ is the space of compactly supported functions on X. For a finite set $B\subset X$, define the capacity of B by

$$\mathrm{Cap}\,(B)=\inf\{\mathcal{E}(f,f):f\in H_0^2,\,f|_B=1\}.$$

We first give a lemma.

Lemma 2.2.12. *If a sequence of non-negative functions $v_n\in H^2$, $n\in\mathbb{N}$ satisfies $\lim_{n\to\infty}v_n(x)=\infty$ for all $x\in X$ and $\lim_{n\to\infty}\mathcal{E}(v_n,v_n)=0$, then*

$$\lim_{n\to\infty}\|u-(u\wedge v_n)\|_{H^2}=0,\qquad\forall u\in H^2,u\ge 0.$$

Proof. Let $u_n=u\wedge v_n$ and define $U_n=\{x\in X:u(x)>v_n(x)\}$. By the assumption, for each $N\in\mathbb{N}$, there exists $N_0=N_0(N)$ such that $U_n\subset B(0,N)^c$ for all $n\ge N_0$. For $A\subset X$, denote $\mathcal{E}_A(u)=\frac{1}{2}\sum_{x,y\in A}(u(x)-u(y))^2\mu_{xy}$. Since $\mathcal{E}_{U_n^c}(u-u_n)=0$, we have

$$\begin{aligned}\mathcal{E}(u-u_n,u-u_n)&\le 2\cdot\frac{1}{2}\sum_{x\in U_n}\sum_{y:\,y\sim x}\Big(u(x)-u_n(x)-(u(y)-u_n(y))\Big)^2\mu_{xy}\\&\le 2\mathcal{E}_{B(0,N-1)^c}(u-u_n)\le 4\Big(\mathcal{E}_{B(0,N-1)^c}(u)+\mathcal{E}_{B(0,N-1)^c}(u_n)\Big)\end{aligned}\tag{2.21}$$

for all $n\ge N_0$. As $u_n=(u+v_n-|u-v_n|)/2$, we have

$$\begin{aligned}\mathcal{E}_{B(0,N-1)^c}(u_n)&\le c_1\Big(\mathcal{E}_{B(0,N-1)^c}(u)+\mathcal{E}_{B(0,N-1)^c}(v_n)+\mathcal{E}_{B(0,N-1)^c}(|u-v_n|)\Big)\\&\le c_2\Big(\mathcal{E}_{B(0,N-1)^c}(u)+\mathcal{E}_{B(0,N-1)^c}(v_n)\Big).\end{aligned}$$

Thus, together with (2.21), we have

$$\mathcal{E}(u-u_n,u-u_n) \le c_3\Big(\mathcal{E}_{B(0,N-1)^c}(u)+\mathcal{E}_{B(0,N-1)^c}(v_n)\Big)$$
$$\le c_3\Big(\mathcal{E}_{B(0,N-1)^c}(u)+\mathcal{E}(v_n,v_n)\Big).$$

Since $u \in H^2$, $\mathcal{E}_{B(0,N-1)^c}(u) \to 0$ as $N \to \infty$ and by the assumption, $\mathcal{E}(v_n,v_n) \to 0$ as $n \to \infty$. So we obtain $\mathcal{E}(u-u_n,u-u_n) \to 0$ as $n \to \infty$. By the assumption, it is clear that $u-u_n \to 0$ pointwise, so we obtain $\|u-u_n\|_{H^2} \to 0$. □

We say that a quadratic form $(\mathcal{E},\mathcal{F})$ is Markovian if $u \in \mathcal{F}$ and $v=(0 \vee u) \wedge 1$, then $v \in \mathcal{F}$ and $\mathcal{E}(v,v) \le \mathcal{E}(u,u)$. It is easy to see that quadratic forms determined by weighted graphs are Markovian.

Proposition 2.2.13. *The following are equivalent.*

(i) *The Markov chain corresponding to* (X,μ) *is transient.*
(ii) $1 \notin H_0^2$
(iii) *Cap* $(\{x\}) > 0$ *for some* $x \in X$.
(iii)′ *Cap* $(\{x\}) > 0$ *for all* $x \in X$.
(iv) $H_0^2 \ne H^2$
(v) *There exists a non-negative super-harmonic function which is not a constant function.*
(vi) *For each* $x \in X$, *there exists* $c_1(x) > 0$ *such that*

$$|f(x)|^2 \le c_1(x)\mathcal{E}(f,f) \qquad \forall f \in C_0(X). \tag{2.22}$$

Proof. For fixed $x \in X$, define $\varphi(z)=\mathbb{P}^z(\sigma_x<\infty)$. We first show the following: $\varphi \in H_0^2$ and

$$\mathcal{E}(\varphi,\varphi)=(-\mathcal{L}\varphi,1_{\{x\}})=R_{\text{eff}}(x)^{-1}=\text{Cap}\,(\{x\}). \tag{2.23}$$

Indeed, let $\{A_n\}_{n=1}^{\infty}$ be a family of finite sets such that $A_n \subset A_{n+1}$ for $n \in \mathbb{N}$, $x \in A_1$, and $\cup_{n\ge 1}A_n=X$. Then $R_{\text{eff}}(x,A_n^c)^{-1} \downarrow R_{\text{eff}}(x)^{-1}$. Let $\varphi_n(z)=\mathbb{P}^z(\sigma_x<\tau_{A_n})$. Using Lemma 2.1.5 (ii), and noting $\varphi_n \in C_0(X)$, we have, for $m \le n$,

$$\mathcal{E}(\varphi_m,\varphi_n)=(\varphi_m,-\mathcal{L}\varphi_n)=(1_{\{x\}},-\mathcal{L}\varphi_n)=\mathcal{E}(\varphi_n,\varphi_n)=R_{\text{eff}}(x,A_n^c)^{-1}. \tag{2.24}$$

This implies

$$\mathcal{E}(\varphi_m-\varphi_n,\varphi_m-\varphi_n)=R_{\text{eff}}(x,A_m^c)^{-1}-R_{\text{eff}}(x,A_n^c)^{-1}.$$

Hence $\{\varphi_m\}$ is a $\mathcal{E}$-Cauchy sequence. Noting that $\varphi_n \to \varphi$ pointwise, we see that $\varphi_n \to \varphi$ in H^2 as well and $\varphi \in H_0^2$. Taking $n=m$ and $n \to \infty$ in (2.24), we obtain (2.23) except the last equality. To prove the last equality of (2.23), take any $f \in H_0^2$

with $f(x) = 1$. Then $g := f - \varphi \in H_0^2$ and $g(x) = 0$. Let $g_n \in C_0(X)$ with $g_n \to g$ in H_0^2. Then, by Lemma 2.1.5 (ii), $\mathcal{E}(\varphi, g_n) = (-\mathcal{L}\varphi, g_n)$. Noting that φ is harmonic except at x, we see that $\mathcal{L}\varphi \in C_0(X)$. So, letting $n \to \infty$, we have

$$\mathcal{E}(\varphi, g) = (-\mathcal{L}\varphi, g) = -\mathcal{L}\varphi(x)g(x)\mu_x = 0.$$

Thus,

$$\mathcal{E}(f, f) = \mathcal{E}(\varphi + g, \varphi + g) = \mathcal{E}(\varphi, \varphi) + \mathcal{E}(g, g) \geq \mathcal{E}(\varphi, \varphi),$$

which means that φ is the unique minimizer in the definition of Cap $(\{x\})$. So the last equality of (2.23) is obtained.

Given (2.23), we now prove the equivalence.

$(i) \Longrightarrow (iii)'$: This is a direct consequence of (2.20) and (2.23).

$(iii) \Longleftrightarrow (ii) \Longleftrightarrow (iii)'$: This is easy. Indeed, Cap $(\{x\}) = 0$ if and only if there is $f \in H_0^2$ with $f(x) = 1$ and $\mathcal{E}(f, f) = 0$, that is f is identically 1.

$(iii)' \Longrightarrow (vi)$: Let $f \in C_0(X) \subset H_0^2$ with $f(x) \neq 0$, and define $g = f/f(x)$. Then

$$\text{Cap}\,(\{x\}) \leq \mathcal{E}(g, g) = \mathcal{E}(f, f)/f(x)^2.$$

So, letting $c_1(x) = 1/\text{Cap}\,(\{x\}) > 0$, we obtain (vi).

$(vi) \Longrightarrow (i)$: As before, let $\varphi_n(z) = \mathbb{P}^z(\sigma_x < \tau_{A_n})$. Then by (2.22), $\mathcal{E}(\varphi_n, \varphi_n) \geq c_1(x)^{-1}$. So, using the fact $\varphi_n \to \varphi$ in H^2 and (2.23), $R_{\text{eff}}(x)^{-1} = \mathcal{E}(\varphi, \varphi) = \lim_n \mathcal{E}(\varphi_n, \varphi_n) \geq c_1(x)^{-1}$. This means the transience by (2.20).

$(ii) \Longleftrightarrow (iv)$: $(ii) \Longrightarrow (iv)$ is clear since $1 \in H^2$, so we will prove the opposite direction. Suppose $1 \in H_0^2$. Then there exists $\{f_n\}_n \subset C_0(X)$ such that $\|1 - f_n\|_{H^2} < n^{-2}$. Since $\mathcal{E}$ is Markovian, we have $\|1 - f_n\|_{H^2} \geq \|1 - (f_n \vee 0) \wedge 1\|_{H^2}$, so without loss of generality we may assume $f_n \geq 0$. Let $v_n = nf_n \geq 0$. Then $\lim_n v_n(x) = \infty$ for all $x \in X$ and $\mathcal{E}(v_n, v_n) = n^2\mathcal{E}(f_n, f_n) \leq n^{-2} \to 0$ so by Lemma 2.2.12, $\|u - (u \wedge v_n)\|_{H^2} \to 0$ for all $u \in H^2$ with $u \geq 0$. Since $u \wedge v_n \in C_0(X)$, this implies $u \in H_0^2$. For general $u \in H^2$, we can decompose it into $u_+ - u_-$ where $u_+, u_- \geq 0$ are in H^2. So applying the above, we have $u_+, u_- \in H_0^2$ and conclude $u \in H_0^2$.

$(i) \Longrightarrow (v)$: If the corresponding Markov chain is transient, then $\psi(z) = \mathbb{P}^z(\sigma_x^+ < \infty)$ is the non-constant super-harmonic function.

$(i) \Longleftarrow (v)$: Suppose the corresponding Markov chain $\{Y_n\}_n$ is recurrent. For a super-harmonic function $\psi \geq 0$, $M_n = \psi(Y_n) \geq 0$ is a supermartingale, so it converges $\mathbb{P}^x$-a.s. Let M_∞ be the limiting random variable. Since the set $\{n \in \mathbb{N} : Y_n = y\}$ is unbounded $\mathbb{P}^x$-a.s. for all $y \in X$ (due to the recurrence), we have $\mathbb{P}^x(\psi(y) = M_\infty) = 1$ for all $y \in X$, so ψ is constant. □

Remark 2.2.14. $(v) \Longrightarrow (i)$ implies that if the Markov chain corresponding to (X, μ) is recurrent, then it has the strong Liouville property.

For A, B which are subspaces of H^2, we write $A \oplus B = \{f+g : f \in A, g \in B\}$ if $\mathcal{E}(f, g) = 0$ for all $f \in A$ and $g \in B$.

As we see above, the Markov chain corresponding to (X, μ) is recurrent if and only if $H^2 = H_0^2$. When the Markov chain is transient, we have the following decomposition of H^2, which is called the Royden decomposition (see [204, Theorem 3.69]).

Proposition 2.2.15. *Suppose that the Markov chain corresponding to (X, μ) is transient. Then*

$$H^2 = \mathcal{H} \oplus H_0^2,$$

where $\mathcal{H} := \{h \in H^2 : h$ is a harmonic functions on $X\}$. Further the decomposition is unique.

Proof. For each $f \in H^2$, let $a_f = \inf_{h \in H_0^2} \mathcal{E}(f-h, f-h)$. Then, similarly to the proof of Proposition 2.2.3, we can show that there is a unique minimizer $v_f \in H_0^2$ such that $a_f = \mathcal{E}(f - v_f, f - v_f)$, $\mathcal{E}(f - v_f, g) = 0$ for all $g \in H_0^2$, and in particular $f - v_f$ is harmonic on X. For the uniqueness of the decomposition, suppose $f = u+v = u'+v'$ where $u, u' \in \mathcal{H}$ and $v, v' \in H_0^2$. Then, $w := u-u' = v'-v \in \mathcal{H} \cap H_0^2$, so $\mathcal{E}(w, w) = 0$, which implies w is constant. Since $w \in H_0^2$ and the Markov chain is transient, by Proposition 2.2.13 we have $w \equiv 0$. $\square$

2.3 Trace of Weighted Graphs

Finally in this chapter, we briefly mention the trace of weighted graphs, which will be used in Chaps. 4 and 8. Note that there is a general theory on traces for Dirichlet forms (see [74, 108]). Also note that a trace to infinite subset of X may not satisfy locally finiteness, but one can consider quadratic forms on them similarly.

Proposition 2.3.1 (Trace of the Weighted Graph). *Let $V \subset X$ be a non-void set such that $\mathbb{P}^x(\sigma_V < \infty) = 1$ for all $x \in X$ and let $f \in H^2(V) := \{u|_V : u \in H^2\}$. Then there exists a unique $u \in H^2$ which attains the following infimum:*

$$\inf\{\mathcal{E}(v, v) : v \in H^2, v|_V = f\}. \tag{2.25}$$

Moreover, the map $f \mapsto u =: H_V f$ is a linear map and there exist weights $\{\hat{\mu}_{xy}\}_{x,y \in V}$ such that the corresponding quadratic form $\hat{\mathcal{E}}_V(\cdot, \cdot)$ satisfies the following:

$$\hat{\mathcal{E}}_V(f, f) = \mathcal{E}(H_V f, H_V f) \qquad \forall f \in H^2(V).$$

Proof. The proof here is inspired by [153, Sect. 8].

The fact that there exists a unique $u \in H^2$ that attains the infimum of (2.25) can be proved similarly to Proposition 2.2.3 (i). So the map $H_V : H^2(V) \to H^2$ where $f \mapsto H_V f$ is well-defined. Let $\mathcal{H}_V := \{u \in H^2 : \mathcal{E}(u, v) = 0, \text{ for all } v \in H^2 \text{ such that } v|_V = 0\}$; a space of harmonic functions on $X \setminus V$. We claim the following:

$$\mathcal{H}_V = H_V(H^2(V)) \text{ and } R_V : \mathcal{H}_V \to H^2(V), \tag{2.26}$$

where $R_V u = u|_V$ is an inverse operator of H_V. Once this is proved, then we have the linearity of H_V and furthermore we have

$$H^2 = \mathcal{H}_V \oplus \{v \in H^2 : v|_V = 0\}.$$

So let us prove (2.26). If $f \in H^2(V)$ and $u = H_V f$, then for any $v \in H^2$ with $v|_V = f$, we have

$$\mathcal{E}(\lambda(v-u)+u, \lambda(v-u)+u) \geq \mathcal{E}(u,u), \qquad \forall \lambda \in \mathbb{R},$$

because u attains the infimum in (2.25). This implies $\mathcal{E}(v-u, u) = 0$, namely $u \in \mathcal{H}_V$. Clearly $u|_V = f$, so we obtain $\mathcal{H}_V \supset H_V(H^2(V))$ and $R_V \circ H_V$ is an identity map. Next, if $u \in \mathcal{H}_V$ and $u|_V = f \in H^2(V)$, then for any $v \in H^2$ with $v|_V = f$, we have

$$\mathcal{E}(v,v) = \mathcal{E}(v-u+u, v-u+u) = \mathcal{E}(v-u, v-u) + \mathcal{E}(u,u) \geq \mathcal{E}(u,u)$$

because $\mathcal{E}(v-u, u) = 0$ (since $u \in \mathcal{H}_V$). This implies $u = H_V f$, since the infimum in (2.25) is attained uniquely by $H_V f$. So we obtain $\mathcal{H}_V \subset H_V(H^2(V))$ and $H_V \circ R_V$ is an identity map.

Set $\hat{\mathcal{E}}(f, f) = \mathcal{E}(H_V f, H_V f)$. Clearly, $\hat{\mathcal{E}}$ is a non-negative definite symmetric bilinear form and $\hat{\mathcal{E}}(f, f) = 0$ if any only if f is a constant function. So, there exists $\{a_{xy}\}_{x,y \in V}$ with $a_{xy} = a_{yx}$ such that $\hat{\mathcal{E}}(f, f) = \frac{1}{2}\sum_{x,y\in V} a_{xy}(f(x) - f(y))^2$.

Next, we show that $\hat{\mathcal{E}}$ is Markovian. Indeed, writing $\bar{u} = (0 \vee u) \wedge 1$ for a function u, since $\overline{H_V u}|_V = \bar{u}$, we have

$$\hat{\mathcal{E}}(\bar{u}, \bar{u}) = \mathcal{E}(H_V \bar{u}, H_V \bar{u}) \leq \mathcal{E}(\overline{H_V u}, \overline{H_V u}) \leq \mathcal{E}(H_V u, H_V u) = \hat{\mathcal{E}}(u, u),$$

for all $u \in H^2(V)$, where the fact that $\mathcal{E}$ is Markovian is used in the second inequality. Now take $p, q \in V$ with $p \neq q$ arbitrary, and consider a function h such that $h(p) = 1, h(q) = -\alpha < 0$ and $h(z) = 0$ for $z \in V \setminus \{p, q\}$. Then, there exist c_1, c_2 such that

$$\begin{aligned}\hat{\mathcal{E}}(h,h) &= a_{pq}(h(p) - h(q))^2 + c_1 h(p)^2 + c_2 h(q)^2 = a_{pq}(1+\alpha)^2 + c_1 + c_2\alpha^2 \\ &\geq \hat{\mathcal{E}}(\bar{h}, \bar{h}) = a_{pq}(\bar{h}(p) - \bar{h}(q))^2 + c_1 \bar{h}(p)^2 + c_2 \bar{h}(q)^2 = a_{pq} + c_1.\end{aligned}$$

So $(a_{pq} + c_2)\alpha^2 + 2a_{pq}\alpha \geq 0$. Since this holds for all $\alpha > 0$, we have $a_{pq} \geq 0$. Putting $\hat{\mu}_{pq} = a_{pq}$ for each $p, q \in V$ with $p \neq q$, we have $\hat{\mathcal{E}}_V = \hat{\mathcal{E}}$, that is $\hat{\mathcal{E}}$ is associated with the weighted graph $(V, \hat{\mu})$. $\square$

We call the induced weights $\{\hat{\mu}_{xy}\}_{x,y\in V}$ as the *trace of* $\{\mu_{xy}\}_{x,y\in X}$ *to* V. From this proposition, we see that for $x, y \in V$, $R_{\text{eff}}(x, y) = R^V_{\text{eff}}(x, y)$ where $R^V_{\text{eff}}(\cdot, \cdot)$ is the effective resistance for $(V, \hat{\mu})$.

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