Chapter 2
The Hahn Quantum Variational Calculus

We introduce the Hahn quantum variational calculus. Necessary and sufficient optimality conditions for the basic, isoperimetric, and Hahn quantum Lagrange problems, are studied. We also show the validity of Leitmann's direct method (Almeida and Torres 2010b; Carlson and Leitmann 2005a,b, 2008; Leitmann 2002, 2003) for the Hahn quantum variational calculus, and give explicit solutions to some concrete problems. Next, we prove a necessary optimality condition of Euler–Lagrange type for quantum variational problems involving Hahn's derivatives of higher-order. Finally, we extend the previous results and obtain optimality conditions for generalized quantum variational problems with a Lagrangian depending on the free endpoints. To illustrate the results, we provide several examples and discuss quantum versions of the Ramsey model and an adjustment model in economics.

2.1 Preliminaries

Let $q \in]0, 1[$ and $\omega \geq 0$. Define $\omega_0 := \dfrac{\omega}{1-q}$ and let I be a real interval containing ω_0. For a function f defined on I, the *Hahn difference operator* of f is given by

$$D_{q,\omega}[f](t) := \begin{cases} \dfrac{f(qt+\omega) - f(t)}{(q-1)t+\omega} & \text{if } t \neq \omega_0 \\ \\ f'(\omega_0) & \text{if } t = \omega_0 \end{cases}$$

provided that f is differentiable at ω_0 (where f' denotes the Fréchet derivative of f). $D_{q,\omega}[f]$ is called the *q, ω-derivative of f*, and f is said to be *q, ω-differentiable on I* if $D_{q,\omega}[f](\omega_0)$ exists.

A. B. Malinowska and D. F. M. Torres, *Quantum Variational Calculus*, SpringerBriefs in Control, Automation and Robotics,
DOI: 10.1007/978-3-319-02747-0_2,

Remark 2.1 Note that when $q \to 1$ we obtain the forward h-difference operator

$$\Delta_h [f](t) := \frac{f(t+h) - f(t)}{h},$$

and when $\omega = 0$ we obtain the Jackson q-difference operator

$$D_{q,0}[f](t) := \begin{cases} \dfrac{f(qt) - f(t)}{(q-1)t} & \text{if } t \neq 0 \\ \\ f'(0) & \text{if } t = 0 \end{cases}$$

provided $f'(0)$ exists. Hence, we can state that the $D_{q,\omega}$ operator generalizes the forward h-difference and the Jackson q-difference operators (Ernst 2008; Kac and Cheung 2002; Koornwinder 1994). Notice also that, under appropriate conditions,

$$\lim_{q \to 1} D_{q,0}[f](t) = f'(t).$$

Example 2.2 Let $q = \omega = 1/2$. In this case $\omega_0 = 1$. It is easy to see that f:$[-1, 1] \to \mathbb{R}$ given by

$$f(t) = \begin{cases} -t & \text{if } t \in]-1, 0[\cup]0, 1] \\ 0 & \text{if } t = -1 \\ 1 & \text{if } t = 0 \end{cases}$$

is not a continuous function but is q, ω-differentiable in $[-1, 1]$ with

$$D_{q,\omega}[f](t) = \begin{cases} -1 & \text{if } t \in]-1, 0[\cup]0, 1] \\ 1 & \text{if } t = -1 \\ -3 & \text{if } t = 0. \end{cases}$$

Example 2.3 Let $q \in]0, 1[$, $\omega = 0$, and

$$f(t) = \begin{cases} t^2 & \text{if } t \in \mathbb{Q} \\ -t^2 & \text{if } t \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Note that f is only Fréchet differentiable in zero, but since $\omega_0 = 0$, f is q, ω-differentiable on the entire real line.

The Hahn difference operator has the following properties:

Theorem 2.4 (Aldwoah 2009; Aldwoah and Hamza 2011) *If $f, g : I \to \mathbb{R}$ are q, ω-differentiable and $t \in I$, then:*

1. $D_{q,\omega}[f](t) \equiv 0$ *on* I *if and only if* f *is constant;*
2. $D_{q,\omega}[f+g](t) = D_{q,\omega}[f](t) + D_{q,\omega}[g](t)$*;*
3. $D_{q,\omega}[fg](t) = D_{q,\omega}[f](t)\,g(t) + f(qt+\omega)\,D_{q,\omega}[g](t)$*;*
4. $D_{q,\omega}\left[\dfrac{f}{g}\right](t) = \dfrac{D_{q,\omega}[f](t)\,g(t) - f(t)\,D_{q,\omega}[g](t)}{g(t)\,g(qt+\omega)}$ *if* $g(t)\,g(qt+\omega) \neq 0$*;*
5. $f(qt+\omega) = f(t) + (t(q-1)+\omega)\,D_{q,\omega}[f](t)$*.*

Proposition 2.5 (Aldwoah 2009) *Let* $a, b \in \mathbb{R}$*. We have*

$$D_{q,\omega}(at+b)^n = a\sum_{k=0}^{n-1}(a(qt+\omega)+b)^k(at+b)^{n-k-1},$$

for $n \in \mathbb{N}$ *and* $t \neq \omega_0$*.*

Let $\sigma(t) = qt+\omega$, for all $t \in I$. Note that σ is a contraction, $\sigma(I) \subseteq I$, $\sigma(t) < t$ for $t > \omega_0$, $\sigma(t) > t$ for $t < \omega_0$, and $\sigma(\omega_0) = \omega_0$.

We use the following standard notation of q-calculus: for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $[k]_q := \dfrac{1-q^k}{1-q}$.

Lemma 2.6 (Aldwoah 2009) *Let* $k \in \mathbb{N}$ *and* $t \in I$*. Then,*

1. $\sigma^k(t) = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{k\text{-times}}(t) = q^k t + \omega[k]_q$*;*
2. $\left(\sigma^k(t)\right)^{-1} = \sigma^{-k}(t) = \dfrac{t-\omega[k]_q}{q^k}$.

Following (Aldwoah 2009; Aldwoah and Hamza 2011) we define the notion of q,ω-*integral* (also known as the *Jackson–Nörlund integral*) as follows:

Definition 2.7 Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ the q,ω-*integral of* f from a to b is given by

$$\int_a^b f(t)\,d_{q,\omega}t := \int_{\omega_0}^b f(t)\,d_{q,\omega}t - \int_{\omega_0}^a f(t)\,d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)\,d_{q,\omega}t := (x(1-q)-\omega)\sum_{k=0}^{+\infty} q^k f\left(xq^k + \omega[k]_q\right), x \in I,$$

provided that the series converges at $x = a$ and $x = b$. In that case, f is called q,ω-*integrable on* $[a, b]$. We say that f is q,ω-*integrable over* I if it is q,ω-*integrable* over $[a, b]$ for all $a, b \in I$.

Remark 2.8 The q,ω-*integral* generalizes the Jackson q-integral and the Nörlund sum (Kac and Cheung 2002). When $\omega = 0$, we obtain the Jackson q-integral

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t,$$

where

$$\int_0^x f(t)\, d_q t := x(1-q) \sum_{k=0}^{+\infty} q^k f\left(xq^k\right).$$

When $q \to 1$, we obtain the Nörlund sum

$$\int_a^b f(t)\, \Delta_\omega t := \int_{+\infty}^b f(t)\, \Delta_\omega t - \int_{+\infty}^a f(t)\, \Delta_\omega t,$$

where

$$\int_{+\infty}^x f(t)\, \Delta_\omega t := -\omega \sum_{k=0}^{+\infty} f(x + k\omega).$$

It can be shown that if $f : I \to \mathbb{R}$ is continuous at ω_0, then f is *q, ω-integrable over I* (see Aldwoah (2009); Aldwoah and Hamza (2011) for the proof).

Theorem 2.9 (Aldwoah 2009) ***(Fundamental Theorem of Hahn's Calculus)*** *Assume that $f : I \to \mathbb{R}$ is continuous at ω_0 and, for each $x \in I$, define*

$$F(x) := \int_{\omega_0}^x f(t)\, d_{q,\omega} t.$$

Then F is continuous at ω_0. Furthermore, $D_{q,\omega}[F](x)$ exists for every $x \in I$ and $D_{q,\omega}[F](x) = f(x)$. Conversely, $\int_a^b D_{q,\omega}[f](t)\, d_{q,\omega} t = f(b) - f(a)$ for all $a, b \in I$.

Aldwoah proved that the *q, ω-integral* has the following properties:

Theorem 2.10 (Aldwoah 2009; Aldwoah and Hamza 2011) *Let $f, g : I \to \mathbb{R}$ be q, ω-integrable on I, $a, b, c \in I$ and $k \in \mathbb{R}$. Then,*

1. $\int_a^a f(t)\, d_{q,\omega} t = 0;$
2. $\int_a^b kf(t)\, d_{q,\omega} t = k \int_a^b f(t)\, d_{q,\omega} t;$
3. $\int_a^b f(t)\, d_{q,\omega} t = -\int_b^a f(t)\, d_{q,\omega} t;$
4. $\int_a^b f(t)\, d_{q,\omega} t = \int_a^c f(t)\, d_{q,\omega} t + \int_c^b f(t)\, d_{q,\omega} t;$
5. $\int_a^b (f(t) + g(t))\, d_{q,\omega} t = \int_a^b f(t)\, d_{q,\omega} t + \int_a^b g(t)\, d_{q,\omega} t;$
6. *Every Riemann integrable function f on I is* q, ω-integrable on I*;*

7. *If* $f, g : I \to \mathbb{R}$ *are* q, ω*-differentiable and* $a, b \in I$*, then*

$$\int_a^b f(t)\, D_{q,\omega}[g](t)\, d_{q,\omega}t = \Big[f(t)\, g(t)\Big]_a^b - \int_a^b D_{q,\omega}[f](t)\, g(qt+\omega)\, d_{q,\omega}t.$$

Property 7 of Theorem 2.10 is known as q, ω*-integration by parts formula.*

Lemma 2.11 (Annaby et al. 2012) *Let* $s \in I$ *and* f *and* g *be* q, ω*-integrable over* I*. Suppose that*

$$|f(t)| \leq g(t), \quad \forall t \in \left\{q^n s + \omega\, [n]_q : n \in \mathbb{N}_0\right\}.$$

If $\omega_0 \leq s$*, then for* $b \in \left\{q^n s + \omega\, [n]_q : n \in \mathbb{N}_0\right\}$

$$\left|\int_{\omega_0}^b f(t) d_{q,\omega}t\right| \leq \int_{\omega_0}^b g(t) d_{q,\omega}t.$$

Remark 2.12 Note that there is an inconsistency in Aldwoah (2009). Indeed, Lemma 6.2.7 of Aldwoah (2009) is only valid if $b \geq \omega_0$ and $a \leq b$.

Remark 2.13 In general, the Jackson–Nörlund integral does not satisfies the following inequality (for a counterexample see Aldwoah (2009)):

$$\left|\int_a^b f(t)\, d_{q,\omega}t\right| \leq \int_a^b |f(t)|\, d_{q,\omega}t, \quad a, b \in I.$$

For $s \in I$ we define

$$[s]_{q,\omega} := \left\{q^n s + \omega\, [n]_q : n \in \mathbb{N}_0\right\} \cup \{\omega_0\}.$$

The following definition and lemma are important for our purposes.

Definition 2.14 Let $s \in I$ and $g : I \times]-\bar{\theta}, \bar{\theta}[\to \mathbb{R}$. We say that $g(t, \cdot)$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |\theta - \theta_0| < \delta \Rightarrow \left|\frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t, \theta_0)\right| < \varepsilon$$

for all $t \in [s]_{q,\omega}$, where $\partial_2 g = \dfrac{\partial g}{\partial \theta}$.

Lemma 2.15 *Let* $s \in I$ *and assume that* $g : I \times]-\bar{\theta}, \bar{\theta}[\to \mathbb{R}$ *is differentiable at* θ_0 *uniformly in* $[s]_{q,\omega}$*,* $G(\theta) := \int_{\omega_0}^s g(t,\theta)\, d_{q,\omega}t$ *for* θ *near* θ_0*, and*

$\int_{\omega_0}^{s} \partial_2 g(t, \theta_0)\, d_{q,\omega}t$ *exist. Then,* $G(\theta)$ *is differentiable at* θ_0 *with* $G'(\theta_0)$ $= \int_{\omega_0}^{s} \partial_2 g(t, \theta_0)\, d_{q,\omega}t$.

Proof For $s = \omega_0$ the result is clear. Let $s \neq \omega_0$ and $\varepsilon > 0$ be arbitrary. Since $g(t, \cdot)$ is differentiable at θ_0, uniformly in t, there exists $\delta > 0$, such that, for all $t \in [s]_{q,\omega}$, and for $0 < |\theta - \theta_0| < \delta$, the following inequality holds:

$$\left| \frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0) \right| < \frac{\varepsilon}{s - \omega_0}.$$

Applying Theorem 2.10 and Lemma 2.11, for $0 < |\theta - \theta_0| < \delta$, we have

$$\begin{aligned}
&\left| \frac{G(\theta) - G(\theta_0)}{\theta - \theta_0} - G'(\theta_0) \right| \\
&= \left| \frac{\int_{\omega_0}^{s} g(t,\theta) d_{q,\omega}t - \int_{\omega_0}^{s} g(t,\theta_0) d_{q,\omega}t}{\theta - \theta_0} - \int_{\omega_0}^{s} \partial_2 g(t,\theta_0) d_{q,\omega}t \right| \\
&= \left| \int_{\omega_0}^{s} \left[\frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0) \right] d_{q,\omega}t \right| \\
&< \int_{\omega_0}^{s} \frac{\varepsilon}{s - \omega_0} d_{q,\omega}t = \frac{\varepsilon}{s - \omega_0} \int_{\omega_0}^{s} 1 d_{q,\omega}t = \varepsilon.
\end{aligned}$$

Hence, $G(\cdot)$ is differentiable at θ_0 and $G'(\theta_0) = \int_{\omega_0}^{s} \partial_2 g(t,\theta_0) d_{q,\omega}t$.

Let $a, b \in I$ with $a < b$. Recall that I is an interval containing ω_0. We define the q, ω-interval by

$$[a,b]_{q,\omega} := \{q^n a + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{q^n b + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\},$$

that is, $[a,b]_{q,\omega} = [a]_{q,\omega} \cup [b]_{q,\omega}$. For $r \in \mathbb{N}$ we introduce the linear space $\mathcal{Y}^r = \mathcal{Y}^r([a,b], \mathbb{R})$ by

$$\mathcal{Y}^r := \Big\{ y : [a,b] \to \mathbb{R} \mid D^i_{q,\omega}[y],\ i = 0, \ldots, r, \\
\text{are bounded on } [a,b] \text{ and continuous at } \omega_0 \Big\}$$

endowed with the norm

$$\|y\|_{r,\infty} := \sum_{i=0}^{r} \left\| D^i_{q,\omega}[y] \right\|_{\infty},$$

where $\|y\|_{\infty} := \sup_{t \in [a,b]} |y(t)|$.

2.2 The Hahn Quantum Euler–Lagrange Equation

In this section we obtain the Euler–Lagrange equation for the basic problem of the Hahn quantum variational calculus. As in the classical case, we need the following lemma.

Lemma 2.16 (Fundamental Lemma of the Hahn quantum variational calculus) *Let $f \in \mathcal{Y}^0$. One has $\int_a^b f(t)h(qt+\omega)d_{q,\omega}t = 0$ for all functions $h \in \mathcal{Y}^0$ with $h(a) = h(b) = 0$ if and only if $f(t) = 0$ for all $t \in [a, b]_{q,\omega}$.*

Proof The implication "$\Leftarrow$" is obvious. Let us prove the implication "$\Rightarrow$". Suppose, by contradiction, that $f(p) \neq 0$ for some $p \in [a, b]_{q,\omega}$.
Case I If $p \neq \omega_0$, then $p = q^k a + \omega[k]_q$ or $p = q^k b + \omega[k]_q$ for some $k \in \mathbb{N}_0$. Observe that $a(1-q)-\omega$ and $b(1-q)-\omega$ cannot vanish simultaneously. Therefore, without loss of generality, we can assume $a(1-q)-\omega \neq 0$ and $p = q^k a + \omega[k]_q$. Define

$$h(t) = \begin{cases} f(q^k a + \omega[k]_q), & \text{if } t = q^{k+1}a + \omega[k+1]_q \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\int_a^b f(t)h(qt+\omega)d_{q,\omega}t$$
$$= -(a(1-q)-\omega)q^k f(q^k a + \omega[k]_q)h(q^{k+1}a + \omega[k+1]_q) \neq 0,$$

which is a contradiction.
Case II If $p = \omega_0$, then without loss of generality we can assume $f(\omega_0) > 0$. We know that (see Aldwoah (2009); Annaby et al. (2012) for more details)

$$\lim_{n\to\infty} q^n a + [n]_{q,\omega} = \lim_{n\to\infty} q^n b + \omega[n]_q = \omega_0.$$

As f is continuous at ω_0, we have

$$\lim_{n\to\infty} f(q^n a + \omega[n]_q) = \lim_{n\to\infty} f(q^n b + \omega[n]_q) = f(\omega_0).$$

Therefore, there exists $N \in \mathbb{N}$, such that for all $n > N$ the inequalities

$$f(q^n a + \omega[n]_q) > 0 \quad \text{and} \quad f(q^n b + \omega[n]_q) > 0$$

hold. If $\omega_0 \neq a, b$, then we define

$$h(t) = \begin{cases} f(q^n b + \omega[n]_q), & \text{if } t = q^{n+1}a + \omega[n+1]_q, \quad \text{for all} \quad n > N \\ f(q^n a + \omega[n]_q), & \text{if } t = q^{n+1}b + \omega[n+1]_q, \quad \text{for all} \quad n > N \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\int_a^b f(t)h(qt+\omega)d_{q,\omega}t = (b-a)(1-q)\sum_{n=N}^{\infty} q^n f(q^n a+\omega[n]_q)f(q^n b+\omega[n]_q) \neq 0,$$

which is a contradiction. If $\omega_0 = b$, then we define

$$h(t) = \begin{cases} f(\omega_0), & \text{if } t = q^{n+1}a + \omega[n+1]_q, \quad \text{for all} \quad n > N \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \int_a^b f(t)h(qt+\omega)d_{q,\omega}t &= -\int_{\omega_0}^a f(t)h(qt+\omega)d_{q,\omega}t \\ &= -(a(1-q)-\omega)\sum_{n=N}^{\infty} q^n f(q^n a+\omega[n]_q)f(\omega_0) \neq 0, \end{aligned}$$

which is a contradiction. Similarly, we show the case when $\omega_0 = a$.

Consider the following q, ω-variational problem

$$\mathcal{L}[y] = \int_a^b L\left(t, y(qt+\omega), D_{q,\omega}[y](t)\right) d_{q,\omega}t \longrightarrow \text{extr} \tag{2.1}$$

where "extr" denotes "extremize", in the class of functions $y \in \mathcal{Y}^1$ satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta \tag{2.2}$$

for some fixed $\alpha, \beta \in \mathbb{R}$.

Definition 2.17 A function $y \in \mathcal{Y}^1$ is said to be admissible for (2.1)–(2.2) if it satisfies the endpoint conditions (2.2). We say that $h \in \mathcal{Y}^1$ is an admissible variation for (2.1)–(2.2) if $h(a) = h(b) = 0$.

In the sequel we assume that the Lagrangian L satisfies the following hypotheses:

(H1) $(u_0, u_1) \to L(t, u_0, u_1)$ is a $C^1(\mathbb{R}^2, \mathbb{R})$ function for any $t \in [a, b]$;
(H2) $t \to L(t, y(qt+\omega), D_{q,\omega}[y](t))$ is continuous at ω_0 for any $y \in \mathcal{Y}^1$;
(H3) functions $t \to \partial_i L(t, y(qt+\omega), D_{q,\omega}[y](t))$, $i = 2, 3$, belong to $\mathcal{Y}^1$ for all $y \in \mathcal{Y}^1$.

Definition 2.18 We say that y_* is a local minimizer (resp. local maximizer) for problem (2.1)–(2.2) if y_* is an admissible function and there exists $\delta > 0$ such that

$$\mathcal{L}[y_*] \leq \mathcal{L}[y] \quad (\text{resp. } \mathcal{L}[y_*] \geq \mathcal{L}[y])$$

for all admissible y with $\|y_* - y\|_{1,\infty} < \delta$.

For fixed $y, h \in \mathcal{Y}^1$, we define the real function ϕ by

$$\phi(\varepsilon) := \mathcal{L}[y + \varepsilon h].$$

The first variation for problem (2.1) is defined by

$$\delta\mathcal{L}[y, h] := \phi'(0).$$

Observe that,

$$\begin{aligned}\mathcal{L}[y+\varepsilon h] &= \int_a^b L(t, y(qt+\omega) + \varepsilon h(qt+\omega), D_{q,\omega}[y](t) + \varepsilon D_{q,\omega}[h](t)) d_{q,\omega}t \\ &= \int_{\omega_0}^b L(t, y(qt+\omega) + \varepsilon h(qt+\omega), D_{q,\omega}[y](t) + \varepsilon D_{q,\omega}[h](t)) d_{q,\omega}t \\ &\quad - \int_{\omega_0}^a L(t, y(qt+\omega) + \varepsilon h(qt+\omega), D_{q,\omega}[y](t) + \varepsilon D_{q,\omega}[h](t)) d_{q,\omega}t.\end{aligned}$$

Writing

$$\mathcal{L}_b[y+\varepsilon h] = \int_{\omega_0}^b L(t, y(qt+\omega) + \varepsilon h(qt+\omega), D_{q,\omega}[y](t) + \varepsilon D_{q,\omega}[h](t)) d_{q,\omega}t$$

and

$$\mathcal{L}_a[y+\varepsilon h] = \int_{\omega_0}^a L(t, y(qt+\omega) + \varepsilon h(qt+\omega), D_{q,\omega}[y](t) + \varepsilon D_{q,\omega}[h](t)) d_{q,\omega}t,$$

we have

$$\mathcal{L}[y+\varepsilon h] = \mathcal{L}_b[y+\varepsilon h] - \mathcal{L}_a[y+\varepsilon h].$$

Therefore,

$$\delta\mathcal{L}[y, h] = \delta\mathcal{L}_b[y, h] - \delta\mathcal{L}_a[y, h]. \tag{2.3}$$

In order to simplify expressions, we introduce the operator $\{\cdot\}$ defined in the following way:

$$\{y\}(t) := (t, y(qt+\omega), D_{q,\omega}[y](t)),$$

where $y \in \mathcal{Y}^1$.

Knowing (2.3), the following lemma is a direct consequence of Lemma 2.15.

Lemma 2.19 *For fixed $y, h \in \mathcal{Y}^1$ let*

$$g(t, \varepsilon) = L(t, y(qt + \omega) + \varepsilon h(qt + \omega), D_{q,\omega}[y](t) + \varepsilon D_{q,\omega}[h](t))$$

for $\varepsilon \in]-\bar{\varepsilon}, \bar{\varepsilon}[$, for some $\bar{\varepsilon} > 0$, i.e.,

$$g(t, \varepsilon) = L\{y + \varepsilon h\}(t).$$

Assume that:

(i) *$g(t, \cdot)$ is differentiable at 0 uniformly in $t \in [a, b]_{q,\omega}$;*

(ii) *$\mathcal{L}_a[y + \varepsilon h] = \int_{\omega_0}^{a} g(t, \epsilon)\, d_{q,\omega}t$ and $\mathcal{L}_b[y + \varepsilon h] = \int_{\omega_0}^{b} g(t, \epsilon)\, d_{q,\omega}t$ exist for $\varepsilon \approx 0$;*

(iii) *$\int_{\omega_0}^{a} \partial_2 g(t, 0) d_{q,\omega}t$ and $\int_{\omega_0}^{b} \partial_2 g(t, 0) d_{q,\omega}t$ exist.*

Then,

$$\delta\mathcal{L}[y, h] = \int_a^b \Big(\partial_2 L\{y\}(t) \cdot h(qt + \omega) + \partial_3 L\{y\}(t) \cdot D_{q,\omega}[h](t)\Big) d_{q,\omega}t.$$

The following result offers a necessary condition for local extremizer.

Theorem 2.20 (A necessary optimality condition for problem (2.1)–(2.2)) *Suppose that the optimal path to problem (2.1)–(2.2) exists and is given by $\tilde{y}$. Then, $\delta\mathcal{L}[\tilde{y}, h] = 0$.*

Proof Without loss of generality, we can assume $\tilde{y}$ to be a local minimizer. Let h be any admissible variation and define a function $\phi :]-\bar{\varepsilon}, \bar{\varepsilon}[\to \mathbb{R}$ by $\phi(\varepsilon) = \mathcal{L}[\tilde{y} + \varepsilon h]$. Since $\tilde{y}$ is a local minimizer, there exists $\delta > 0$, such that $\mathcal{L}[\tilde{y}] \leq \mathcal{L}[y]$ for all admissible y with $\|y - \tilde{y}\|_{1,\infty} < \delta$. Therefore, $\phi(\varepsilon) = \mathcal{L}[\tilde{y} + \varepsilon h] \geq \mathcal{L}[\tilde{y}] = \phi(0)$ for all $\varepsilon < \frac{\delta}{\|h\|_{1,\infty}}$. Hence, ϕ has a local minimum at $\varepsilon = 0$, and thus our assertion follows.

Theorem 2.21 (The Hahn quantum Euler–Lagrange equation for problem (2.1)–(2.2)) *Under hypotheses (H1)–(H3) and conditions (i)–(iii) of Lemma 2.19 on the Lagrangian L, if $\tilde{y}$ is a local minimizer or local maximizer to problem (2.1)–(2.2), then $\tilde{y}$ satisfies the Euler–Lagrange equation*

$$\partial_2 L\{y\}(t) - D_{q,\omega}[\partial_3 L]\{y\}(t) = 0 \tag{2.4}$$

for all $t \in [a, b]_{q,\omega}$.

Proof Suppose that $\mathcal{L}$ has a local extremum at $\tilde{y}$. Let h be any admissible variation and define a function $\phi :]-\bar{\varepsilon}, \bar{\varepsilon}[\to \mathbb{R}$ by $\phi(\varepsilon) = \mathcal{L}[\tilde{y} + \varepsilon h]$. A necessary condition for $\tilde{y}$ to be an extremizer is given by $\phi'(0) = 0$. Note that

$$\phi'(0) = \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t) \cdot h(qt + \omega) + \partial_3 L\{\tilde{y}\}(t) \cdot D_{q,\omega}[h](t)\Big) d_{q,\omega}t.$$

Since $h(a) = h(b) = 0$, then

$$\phi'(0) = \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t) \cdot h(qt + \omega) + \partial_3 L\{\tilde{y}\}(t) \cdot D_{q,\omega}[h](t)\Big) d_{q,\omega}t.$$

Integration by parts gives

$$\begin{aligned}\int_a^b \partial_3 L\{\tilde{y}\}(t) \cdot D_{q,\omega}[h](t) d_{q,\omega}t &= \Big[\partial_3 L\{\tilde{y}\}(t) \cdot h(t)\Big]_a^b \\ &\quad - \int_a^b D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t) \cdot h(qt + \omega) d_{q,\omega}t\end{aligned}$$

and since $h(a) = h(b) = 0$, then

$$\phi'(0) = 0 \Leftrightarrow \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t) - D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t)\Big) \cdot h(qt + \omega) d_{q,\omega}t = 0.$$

Thus, by Lemma 2.16, we have

$$\partial_2 L\{\tilde{y}\}(t) - D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t) = 0$$

for all $t \in [a, b]_{q,\omega}$.

Remark 2.22 Under appropriate conditions, when $(\omega, q) \to (0, 1)$, we obtain a corresponding result in the classical context of the calculus of variations (1.4):

$$\frac{d}{dt}\partial_3 L(t, y(t), y'(t)) = \partial_2 L(t, y(t), y'(t)).$$

Remark 2.23 In practical terms the hypotheses of Theorem 2.21 are not easy to verify *a priori*. However, we can assume that all hypotheses are satisfied and apply the q, ω-Euler–Lagrange equation (2.4) heuristically to obtain a *candidate*. If such a candidate is, or not, a solution to the variational problem is a different question that require further analysis (see Sects. 2.4 and 2.8.5).

2.3 The Hahn Quantum Isoperimetric Problem

We now study the isoperimetric problem with an integral constraint. Both normal and abnormal extremizers are considered. Isoperimetric problems have found a broad class of important applications throughout the centuries. Areas of application include also economy (see, e.g., Almeida and Torres (2009b); Caputo (2005) and the references given there).

The isoperimetric problem consists of minimizing or maximizing the functional (2.1) in the class of functions $y \in \mathcal{Y}^1$ satisfying the boundary conditions (2.2), and the integral constraint

$$\mathcal{J}[y] = \int_a^b F\left(t, y(qt+\omega), D_{q,\omega}[y](t)\right) d_{q,\omega}t = \gamma \tag{2.5}$$

for some $\gamma \in \mathbb{R}$.

Definition 2.24 We say that $\tilde{y} \in \mathcal{Y}^1$ is a local minimizer (resp. local maximizer) for the isoperimetric problem (2.1), (2.2) and (2.5) if there exists $\delta > 0$ such that $\mathcal{L}[\tilde{y}] \leq \mathcal{L}[y]$ (resp. $\mathcal{L}[\tilde{y}] \geq \mathcal{L}[y]$) for all $y \in \mathcal{Y}^1$ satisfying the boundary conditions (2.2) and the isoperimetric constraint (2.5) and $\|\widetilde{y} - y\|_{1,\infty} < \delta$.

Definition 2.25 We say that $y \in \mathcal{Y}^1$ is an extremal to $\mathcal{J}$ if y satisfies the Euler–Lagrange equation (2.4) relatively to $\mathcal{J}$. An extremizer (i.e., a local minimizer or a local maximizer) to problem (2.1), (2.2) and (2.5) that is not an extremal to $\mathcal{J}$ is said to be a normal extremizer; otherwise, the extremizer is said to be abnormal.

Theorem 2.26 (Necessary optimality condition for normal extremizers to (2.1), (2.2) **and** (2.5)) *Suppose that L and F satisfy hypotheses (H1)–(H3) and conditions (i)–(iii) of* Lemma 2.19*, and suppose that $\widetilde{y} \in \mathcal{Y}^1$ gives a local minimum or a local maximum to the functional $\mathcal{L}$ subject to the integral constraint* (2.5)*. If $\widetilde{y}$ is not an extremal to $\mathcal{J}$, then there exists a real number λ such that $\widetilde{y}$ satisfies the equation*

$$\partial_2 H\{y\}(t) - D_{q,\omega}[\partial_3 H]\{y\}(t) = 0 \tag{2.6}$$

for all $t \in [a, b]_{q,\omega}$, where $H = L - \lambda F$.

Proof Suppose that $\widetilde{y} \in \mathcal{Y}^1$ is a normal extremizer to problem (2.1), (2.2) and (2.5). Define the real functions $\phi, \psi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi(\epsilon_1, \epsilon_2) = \mathcal{L}[\widetilde{y} + \epsilon_1 h_1 + \epsilon_2 h_2],$$
$$\psi(\epsilon_1, \epsilon_2) = \mathcal{J}[\widetilde{y} + \epsilon_1 h_1 + \epsilon_2 h_2] - \gamma,$$

where $h_2 \in \mathcal{Y}^1$ is fixed (that we will choose later) and $h_1 \in \mathcal{Y}^1$ is an arbitrary function. Note that

$$\frac{\partial \psi}{\partial \epsilon_2}(0,0) = \int_a^b \Big(\partial_2 F\{\widetilde{y}\}(t) \cdot h_2(qt+\omega) + \partial_3 F\{\widetilde{y}\}(t) \cdot D_{q,\omega}[h_2](t)\Big) d_{q,\omega}t.$$

Using integration by parts formula we get

$$\begin{aligned}\frac{\partial \psi}{\partial \epsilon_2}(0,0) = &\int_a^b \Big(\partial_2 F\{\widetilde{y}\}(t) - D_{q,\omega}[\partial_3 F]\{\widetilde{y}\}(t)\Big) \cdot h_2(qt+\omega) d_{q,\omega}t \\ &+ \Big[\partial_3 F\{\widetilde{y}\}(t) \cdot h_2(t)\Big]_a^b.\end{aligned}$$

Restricting h_2 to those such that $h_2(a) = h_2(b) = 0$ we obtain

$$\frac{\partial \psi}{\partial \epsilon_2}(0,0) = \int_a^b \Big(\partial_2 F\{\widetilde{y}\}(t) - D_{q,\omega}[\partial_3 F]\{\widetilde{y}\}(t)\Big) \cdot h_2(qt+\omega) d_{q,\omega}t.$$

Since $\widetilde{y}$ is not an extremal to $\mathcal{J}$, then we can choose h_2 such that $\dfrac{\partial \psi}{\partial \epsilon_2}(0,0) \neq 0$. We keep h_2 fixed. Since $\psi(0,0) = 0$, by the Implicit Function Theorem there exists a function g defined in a neighborhood V of zero, such that $g(0) = 0$ and $\psi(\epsilon_1, g(\epsilon_1)) = 0$, for any $\epsilon_1 \in V$, that is, there exists a subset of variation curves $y = \widetilde{y} + \epsilon_1 h_1 + g(\epsilon_1) h_2$ satisfying the isoperimetric constraint. Note that $(0,0)$ is an extremizer of ϕ subject to the constraint $\psi = 0$ and

$$\nabla \psi(0,0) \neq (0,0).$$

By the Lagrange multiplier rule, there exists some constant $\lambda \in \mathbb{R}$ such that

$$\nabla \phi(0,0) = \lambda \nabla \psi(0,0). \tag{2.7}$$

Restricting h_1 to those such that $h_1(a) = h_1(b) = 0$ we get

$$\frac{\partial \phi}{\partial \epsilon_1}(0,0) = \int_a^b \Big(\partial_2 L\{\widetilde{y}\}(t) - D_{q,\omega}[\partial_3 L]\{\widetilde{y}\}(t)\Big) \cdot h_1(qt+\omega) d_{q,\omega}t$$

and

$$\frac{\partial \psi}{\partial \epsilon_1}(0,0) = \int_a^b \Big(\partial_2 F\{\widetilde{y}\}(t) - D_{q,\omega}[\partial_3 F]\{\widetilde{y}\}(t)\Big) \cdot h_1(qt+\omega) d_{q,\omega}t.$$

Using (2.7) it follows that

$$\begin{aligned}\int_a^b \Big(&\partial_2 L\{\widetilde{y}\}(t) - D_{q,\omega}[\partial_3 L]\{\widetilde{y}\}(t) \\ &- \lambda\Big(\partial_2 F\{\widetilde{y}\}(t) - D_{q,\omega}[\partial_3 F]\{\widetilde{y}\}(t)\Big)\Big) \cdot h_1(qt+\omega) d_{q,\omega}t = 0.\end{aligned}$$

Using the fundamental lemma of the Hahn quantum variational calculus (Lemma 2.16), and recalling that h_1 is arbitrary, we conclude that

$$\partial_2 L\{\widetilde{y}\}(t) - D_{q,\omega}[\partial_3 L]\{\widetilde{y}\}(t) - \lambda\Big(\partial_2 F\{\widetilde{y}\}(t) - D_{q,\omega}[\partial_3 F]\{\widetilde{y}\}(t)\Big) = 0$$

for all $t \in [a, b]_{q,\omega}$, proving that $H = L - \lambda F$ satisfies the Euler–Lagrange condition (2.6).

Introducing an extra multiplier λ_0 we can also deal with abnormal extremizers to the isoperimetric problem (2.1), (2.2) and (2.5).

Theorem 2.27 (Necessary optimality condition for normal and abnormal extremizers to (2.1)*,* (2.2) **and** (2.5)**)** *Suppose that L and F satisfy hypotheses (H1)–(H3) and conditions (i)–(iii) of Lemma 2.19, and suppose that $\widetilde{y} \in \mathcal{Y}^1$ gives a local minimum or a local maximum to the functional $\mathcal{L}$ subject to the integral constraint* (2.5)*. Then there exist two constants λ_0 and λ, not both zero, such that $\widetilde{y}$ satisfies the equation*

$$\partial_2 H\{y\}(t) - D_{q,\omega}[\partial_3 H]\{y\}(t) = 0 \tag{2.8}$$

for all $t \in [a, b]_{q,\omega}$, where $H = \lambda_0 L - \lambda F$.

Proof The proof is similar to the proof of Theorem 2.26. Since $(0, 0)$ is an extremizer of ϕ subject to the constraint $\psi = 0$, the abnormal Lagrange multiplier rule (cf., e.g., van Brunt (2004)) guarantees the existence of two reals λ_0 and λ, not both zero, such that

$$\lambda_0 \nabla \phi = \lambda \nabla \psi.$$

Remark 2.28 Note that if $\widetilde{y}$ is a normal extremizer then, by Theorem 2.26, one can choose $\lambda_0 = 1$ in Theorem 2.27. The condition $(\lambda_0, \lambda) \neq (0, 0)$ guarantees that Theorem 2.27 is a useful necessary condition. In general we cannot guarantee, a priori, that λ_0 be different from zero. The interested reader about abnormality is referred to the book (Arutyunov 2000).

Suppose now that it is required to find functions y_1 and y_2 for which the functional

$$\mathcal{L}[y_1, y_2] = \int_a^b f(t, y_1(qt+\omega), y_2(qt+\omega), D_{q,\omega}[y_1](t), D_{q,\omega}[y_2](t)) d_{q,\omega}t \tag{2.9}$$

has an extremum, where the admissible functions satisfy the boundary conditions

$$(y_1(a), y_2(a)) = (y_1^a, y_2^a) \text{ and } (y_1(b), y_2(b)) = (y_1^b, y_2^b), \tag{2.10}$$

and the subsidiary nonholonomic condition

$$g(t, y_1(qt + \omega), y_2(qt + \omega), D_{q,\omega}[y_1](t), D_{q,\omega}[y_2](t)) = 0. \tag{2.11}$$

The problem (2.9)–(2.11) can be reduced to the isoperimetric one by transforming (2.11) into a constraint of the type (2.5). For that, we multiply both sides of (2.11) by an arbitrary function $\lambda(t)$, and then take the q,ω-integral from a to b. We obtain the new constraint

$$\mathcal{K}[y_1, y_2] = \int_a^b \lambda(t)g(t, y_1(qt+\omega), y_2(qt+\omega), D_{q,\omega}[y_1](t), D_{q,\omega}[y_2](t))d_{q,\omega}t = 0. \tag{2.12}$$

Under the conditions of Theorem 2.26, the solutions (y_1, y_2) of the isoperimetric problem (2.9) and (2.12) satisfy the Euler–Lagrange equation for the functional

$$\int_a^b (f - \tilde{\lambda}(t)g)d_{q,\omega}t, \tag{2.13}$$

$\tilde{\lambda}(t) = \bar{\lambda}\lambda(t)$ for some constant $\bar{\lambda}$. Since (2.12) follows from (2.11), the solutions of problem (2.9)–(2.11) satisfy the Euler–Lagrange equation for functional (2.13) as well.

2.4 Sufficient Condition for Optimality

In this subsection we prove sufficient optimality conditions for problem (2.1)–(2.2). Similar to the classical calculus of variations we assume the Lagrangian function to be convex (or concave).

Theorem 2.29 *Let $L(t, u_0, u_1)$ be jointly convex (resp. concave) in (u_0, u_1). If $\tilde{y}$ satisfies condition (2.4), then $\tilde{y}$ is a global minimizer (resp. maximizer) to problem (2.1)–(2.2).*

Proof We give the proof for the convex case. Since L is jointly convex in (u_0, u_1), then for any $h \in \mathcal{Y}^1$,

$$\begin{aligned}\mathcal{L}[\tilde{y}+h] - \mathcal{L}[\tilde{y}] &= \int_a^b (L\{\tilde{y}+h\}(t) - L\{\tilde{y}\}(t))\, d_{q,\omega}t \\ &\geq \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t)\cdot h(qt+\omega) + \partial_3 L\{\tilde{y}\}(t)\cdot D_{q,\omega}[h](t)\Big)d_{q,\omega}t.\end{aligned}$$

Proceeding analogously as in the proof of Theorem 2.21 and since $\tilde{y}$ satisfies condition (2.4) we obtain $\mathcal{L}(\tilde{y}+h) - \mathcal{L}(\tilde{y}) \geq 0$, proving the desired result.

2.5 Leitmann's Direct Method

Leitmann's direct method permits to compute global solutions to some problems that are variationally invariant under a family of transformations (Leitmann 1967, 2001a,b; Silva and Torres 2006; Torres and Leitmann 2008; Wagener 2009). It should be mentioned that such invariance transformations are useful not only in connection with Leitmann's method but also to apply Noether's Theorem (Torres 2002, 2004a). Moreover, the invariance transformations are related with the notion of Carathéodory equivalence (Carlson 2002; Torres 2004b).

Recently, it has been noticed in Malinowska and Torres (2010a) that the invariance transformations, that keep the Lagrangian invariant, do not depend on the time scale. This is also true for the generalized Hahn quantum setting that we are considering in this work: given a Lagrangian $L : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the invariance transformations, that keep it invariant up to a gauge term, are exactly the same if the Lagrangian L is used to define a Hahn quantum functional (2.1) or a classical functional $\mathcal{L}[y] = \int_a^b L(t, y(t), y'(t))dt$ of the calculus of variations. Thus, if the quantum problem we want to solve admits an enough rich family of invariance transformations, that keep it invariant up to a gauge term, then one does not need to solve a Hahn quantum Euler–Lagrange equation to find its minimizer: instead, we can try to use Leitmann's direct method. The question of how to find the invariance transformations is addressed in Gouveia and Torres (2005); Gouveia et al. (2006).

Let $\bar{L} : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We assume that $\bar{L}$ satisfies hypotheses (H1)–(H3). Consider the integral

$$\bar{\mathcal{L}}[\bar{y}] = \int_a^b \bar{L}\{\bar{y}\}(t)d_{q,\omega}t.$$

Lemma 2.30 (Leitmann's fundamental lemma via Hahn's quantum operator) *Let $y = z(t, \bar{y})$ be a transformation having an unique inverse $\bar{y} = \bar{z}(t, y)$ for all $t \in [a, b]$, such that there is a one-to-one correspondence*

$$y(t) \Leftrightarrow \bar{y}(t)$$

for all functions $y \in \mathcal{Y}^1$ satisfying (2.2) and all functions $\bar{y} \in \mathcal{Y}^1$ satisfying

$$\bar{y}(a) = \bar{z}(a, \alpha), \quad \bar{y}(b) = \bar{z}(b, \beta). \tag{2.14}$$

If the transformation $y = z(t, \bar{y})$ is such that there exists a function $G : [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfying the functional identity

$$L\{y\}(t) - \bar{L}\{\bar{y}\}(t) = D_{q,\omega}G(t, \bar{y}(t)), \tag{2.15}$$

then if $\bar{y}^*$ yields the extremum of $\bar{\mathcal{L}}$ with $\bar{y}^*$ satisfying (2.14), $y^* = z(t, \bar{y}^*)$ yields the extremum of $\mathcal{L}$ for y^* satisfying (2.2).

Remark 2.31 The functional identity (2.15) is exactly the definition of variationally invariance when we do not consider transformations of the time variable t (cf. (4) and (5) of Torres and Leitmann (2008)). Function G that appears in (2.15) is sometimes called a gauge term (Torres 2004a).

Proof The proof is similar in spirit to Leitmann's proof (Leitmann 1967, 2001a,b, 2004). Let $y \in \mathcal{Y}^1$ satisfy (2.2), and define functions $\bar{y} \in \mathcal{Y}^1$ through the formula $\bar{y} = \bar{z}(t, y)$, $t \in [a, b]$. Then $\bar{y} \in \mathcal{Y}^1$ and satisfies (2.14). Moreover, as a result of (2.15), it follows that

$$\begin{aligned}\mathcal{L}[y] - \bar{\mathcal{L}}[\bar{y}] &= \int_a^b L\{y\}(t)d_{q,\omega}t - \int_a^b \bar{L}\{\bar{y}\}(t)d_{q,\omega}t = \int_a^b D_{q,\omega}G(t, \bar{y}(t))d_{q,\omega}t \\ &= G(b, \bar{y}(b)) - G(a, \bar{y}(a)) = G(b, \bar{z}(b, \beta)) - G(a, \bar{z}(a, \alpha)),\end{aligned}$$

from which the desired conclusion follows immediately since the right-hand side of the above equality is a constant, depending only on the fixed-endpoint conditions (2.2).

Examples 2.33, 2.34 and 2.35 in the next section illustrate the applicability of Lemma 2.30. The procedure is as follows: (i) we use the computer algebra package described in Gouveia and Torres (2005) and available from the *Maple Application Center* at http://www.maplesoft.com/applications/view.aspx?SID=4805 to find the transformations that keep the problem of the calculus of variations or optimal control invariant; (ii) we use such invariance transformations to solve the Hahn quantum variational problem by applying Leitmann's fundamental lemma (Lemma 2.30).

2.6 Illustrative Examples

We provide some examples in order to illustrate our main results.

Example 2.32 Let q, ω be fixed real numbers, and I be a closed interval of $\mathbb{R}$ such that $\omega_0, 0, 1 \in I$. Consider the problem

$$\mathcal{L}[y] = \int_0^1 \left(y(qt + \omega) + \frac{1}{2}(D_{q,\omega}[y](t))^2\right) d_{q,\omega}t \longrightarrow \min \tag{2.16}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1. \tag{2.17}$$

If y is a local minimizer to problem (2.16)–(2.17), then by Theorem 2.21 it satisfies the Euler–Lagrange equation

$$D_{q,\omega}D_{q,\omega}[y](t) = 1 \tag{2.18}$$

for all $t \in \{\omega[n]_q : n \in \mathbb{N}_0\} \cup \{q^n + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\}$. By direct substitution it can be verified that $y(t) = \frac{1}{q+1}t^2 + \frac{q}{q+1}t$ is a candidate solution to problem (2.16)–(2.17).

In next examples we solve quantum variational problems using Leitmann's direct method (see Sect. 2.5).

Example 2.33 Let q, ω, and a, b $(a < b)$ be fixed real numbers, and I be a closed interval of $\mathbb{R}$ such that $\omega_0 \in I$ and $a, b \in \{q^n s + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\}$ for some $s \in I$. Let α and β be two given real numbers, $\alpha \neq \beta$. We consider the following problem:

$$\begin{gathered}\mathcal{L}[y] = \int_a^b \left((D_{q,\omega}[y](t))^2 + y(qt+\omega) + tD_{q,\omega}[y](t)\right) d_{q,\omega}t \longrightarrow \min \\ y(a) = \alpha, \quad y(b) = \beta.\end{gathered} \tag{2.19}$$

We transform problem (2.19) into the trivial problem

$$\begin{gathered}\bar{\mathcal{L}}[\bar{y}] = \int_a^b (D_{q,\omega}[\bar{y}](t))^2 d_{q,\omega}t \longrightarrow \min \\ \bar{y}(a) = 0, \quad \bar{y}(b) = 0,\end{gathered}$$

which has solution $\bar{y} \equiv 0$. For that we consider the transformation

$$y(t) = \bar{y}(t) + ct + d, \quad c, d \in \mathbb{R},$$

where constants c and d will be chosen later. According to the above, we have

$$D_{q,\omega}[y](t) = D_{q,\omega}[\bar{y}](t) + c, \quad y(qt+\omega) = \bar{y}(qt+\omega) + c(qt+\omega) + d,$$

and

$$\begin{aligned}&(D_{q,\omega}[y](t))^2 + y(qt+\omega) + tD_{q,\omega}[y](t) \\ &\quad = (D_{q,\omega}[\bar{y}](t))^2 + 2cD_{q,\omega}[\bar{y}](t) + c^2 + \bar{y}(qt+\omega) + c(qt+\omega) + d \\ &\qquad + tD_{q,\omega}[\bar{y}](t) + ct \\ &\quad = (D_{q,\omega}[\bar{y}](t))^2 + D_{q,\omega}[2c\bar{y}(t) + t\bar{y}(t) + ct^2 + (c^2+d)t].\end{aligned}$$

In order to obtain the solution to the original problem, it suffices to chose c and d so that

$$ca + d = \alpha, \quad cb + d = \beta. \tag{2.20}$$

Solving the system of equations (2.20) we obtain $c = \frac{\alpha-\beta}{a-b}$ and $d = \frac{\beta a - b\alpha}{a-b}$. Hence, the global minimizer for problem (2.19) is

$$y(t) = \frac{\alpha - \beta}{a - b} t + \frac{\beta a - b\alpha}{a - b}.$$

Example 2.34 Let q, ω, and a, b ($a < b$) be fixed real numbers, and I be a closed interval of $\mathbb{R}$ such that $\omega_0 \in I$ and $a, b \in \{q^n s + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\}$ for some $s \in I$. Let α and β be two given real numbers, $\alpha \neq \beta$. We consider the following problem:

$$\begin{array}{c} \mathcal{L}[y] = \int_a^b \left(D_{q,\omega}[yg](t)\right)^2 d_{q,\omega}t \longrightarrow \min \\ y(a) = \alpha, \quad y(b) = \beta, \end{array} \tag{2.21}$$

where g does not vanish on the interval $[a, b]_{q,\omega}$. Observe that $\bar{y}(t) = g^{-1}(t)$ minimizes $\mathcal{L}$ with end conditions $\bar{y}(a) = g^{-1}(a)$ and $\bar{y}(b) = g^{-1}(b)$. Let $y(t) = \bar{y}(t) + p(t)$. Then

$$\left(D_{q,\omega}[yg](t)\right)^2 = \left(D_{q,\omega}[\bar{y}g](t)\right)^2 + D_{q,\omega}[pg](t) D_{q,\omega}[2\bar{y}g + pg](t). \tag{2.22}$$

Consequently, if $p(t) = (At + B)g^{-1}(t)$, where A and B are constants, then (2.22) is of the form (2.15), since $D_{q,\omega}[pg](t)$ is constant. Thus, the function

$$y(t) = (At + C)g^{-1}(t)$$

with

$$A = [\alpha g(a) - \beta g(b)]\,(a - b)^{-1}, \quad C = [a\beta g(b) - b\alpha g(a)]\,(a - b)^{-1},$$

minimizes (2.21).

Using the idea of Leitmann, we can also solve quantum optimal control problems defined in terms of Hahn's operators.

Example 2.35 Let q, ω be real numbers on a closed interval I of $\mathbb{R}$ such that $\omega_0 \in I$ and $0, 1 \in \{q^n s + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\}$ for some $s \in I$. Consider the global minimum problem

$$\mathcal{L}[u_1, u_2] = \int_0^1 \Big((u_1(t))^2 + u_2(t))^2\Big)\, d_{q,\omega}t \longrightarrow \min \tag{2.23}$$

subject to the control system

$$D_{q,\omega}[y_1](t) = \exp(u_1(t)) + u_1(t) + u_2(t), \quad D_{q,\omega}[y_2](t) = u_2(t), \tag{2.24}$$

and conditions

$$\begin{array}{c} y_1(0) = 0, \quad y_1(1) = 2, \quad y_2(0) = 0, \quad y_2(1) = 1, \\ u_1(t), u_2(t) \in \Omega = [-1, 1]. \end{array} \tag{2.25}$$

This example is inspired from Torres and Leitmann (2008). It is worth mentioning that due to the constraints on the values of the controls, $(u_1(t), u_2(t)) \in \Omega = [-1, 1]$, a theory based on necessary optimality conditions to solve problem (2.23)–(2.25) does not exist at the moment.

We begin noticing that problem (2.23)–(2.25) is variationally invariant according to Gouveia and Torres (2005) under the one-parameter family of transformations

$$y_1^s = y_1 + st, \quad y_2^s = y_2 + st, \quad u_2^s = u_2 + s \quad (t^s = t \text{ and } u_1^s = u_1). \tag{2.26}$$

To prove this, we need to show that both the functional integral $\mathcal{L}$ and the control system stay invariant under the s-parameter transformations (2.26). This is easily seen by direct calculations:

$$\begin{aligned} \mathcal{L}^s[u_1^s, u_2^s] &= \int_0^1 \left(u_1^s(t)\right)^2 + \left(u_2^s(t)\right)^2 d_{q,\omega}t \\ &= \int_0^1 u_1^2(t) + (u_2(t) + s)^2 d_{q,\omega}t \\ &= \int_0^1 \left(u_1^2(t) + u_2^2(t) + D_{q,\omega}[s^2 t + 2sy_2(t)]\right) d_{q,\omega}t \\ &= \mathcal{L}[u_1, u_2] + s^2 + 2s. \end{aligned} \tag{2.27}$$

We remark that $\mathcal{L}^s$ and $\mathcal{L}$ have the same minimizers: adding a constant $s^2 + 2s$ to the functional $\mathcal{L}$ does not change the minimizer of $\mathcal{L}$. It remains to prove that the control system also remains invariant under transformations (2.26):

$$\begin{aligned} D_{q,\omega}[y_1^s](t) &= D_{q,\omega}[y_1](t) + s = \exp(u_1(t)) + u_1(t) + u_2(t) + s \\ &= \exp(u_1^s(t)) + u_1^s(t) + u_2^s(t), \\ D_{q,\omega}[y_2^s](t) &= D_{q,\omega}[y_2](t) + s = u_2(t) + s = u_2^s(t). \end{aligned} \tag{2.28}$$

Conditions (2.27) and (2.28) prove that problem (2.23)–(2.25) is invariant under the s-parameter transformations (2.26) up to $D_{q,\omega}\left(s^2 t + 2sy_2(t)\right)$. Using the invariance transformations (2.26), we generalize problem (2.23)–(2.25) to a s-parameter family of problems, $s \in \mathbb{R}$, which include the original problem for $s = 0$:

$$\mathcal{L}^s[u_1, u_2] = \int_0^1 (u_1^s(t))^2 + (u_2^s(t))^2 d_{q,\omega}t \longrightarrow \min$$

subject to the control system

$$D_{q,\omega}[y_1^s](t) = \exp(u_1^s(t)) + u_1^s(t) + u_2^s(t), \quad D_{q,\omega}[y_2^s(t)] = u_2^s(t),$$

and conditions

$$y_1^s(0) = 0, \quad y_1^s(1) = 2 + s, \quad y_2^s(0) = 0, \quad y_2^s(1) = 1 + s,$$
$$u_1^s(t) \in [-1, 1], \quad u_2^s(t) \in [-1 + s, 1 + s].$$

It is clear that $\mathcal{L}^s \geq 0$ and that $\mathcal{L}^s = 0$ if $u_1^s(t) = u_2^s(t) \equiv 0$. The control equations, the boundary conditions and the constraints on the values of the controls imply that $u_1^s(t) = u_2^s(t) \equiv 0$ is admissible only if $s = -1$: $y_1^{s=-1}(t) = t, y_2^{s=-1}(t) \equiv 0$. Hence, for $s = -1$ the global minimum to $\mathcal{L}^s$ is 0 and the minimizing trajectory is given by

$$\tilde{u}_1^s(t) \equiv 0, \quad \tilde{u}_2^s(t) \equiv 0, \quad \tilde{y}_1^s(t) = t, \quad \tilde{y}_2^s(t) \equiv 0.$$

Since for any s one has by (2.27) that $\mathcal{L}[u_1, u_2] = \mathcal{L}^s[u_1^s, u_2^s] - s^2 - 2s$, we conclude that the global minimum for problem $\mathcal{L}[u_1, u_2]$ is 1. Thus, using the inverse functions of the variational symmetries (2.26),

$$u_1(t) = u_1^s(t), \quad u_2(t) = u_2^s(t) - s, \quad y_1(t) = y_1^s(t) - st, \quad y_2(t) = y_2^s(t) - st,$$

and the absolute minimizer for problem (2.23)–(2.25) is

$$\tilde{u}_1(t) = 0, \quad \tilde{u}_2(t) = 1, \quad \tilde{y}_1(t) = 2t, \quad \tilde{y}_2(t) = t.$$

2.7 Higher-order Hahn's Quantum Variational Calculus

We define the q, ω-derivatives of higher-order in the usual way: the rth q, ω-derivative ($r \in \mathbb{N}$) of $f : I \to \mathbb{R}$ is the function $D_{q,\omega}^r[f] : I \to \mathbb{R}$ given by $D_{q,\omega}^r[f] := D_{q,\omega}[D_{q,\omega}^{r-1}[f]]$, provided $D_{q,\omega}^{r-1}[f]$ is q, ω-differentiable on I and where $D_{q,\omega}^0[f] := f$. The following notations are in order: $\sigma(t) = qt + \omega$, $y^\sigma(t) = y^{\sigma^1}(t) = (y \circ \sigma)(t) = y(qt + \omega)$, and $y^{\sigma^k} = y \circ y^{\sigma^{k-1}}$, $k = 2, 3, \ldots$

Our main goal is to establish necessary optimality conditions for the higher-order q,ω-variational problem

$$\begin{gathered}
\mathcal{L}[y] = \int_a^b L\left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t), \ldots, D_{q,\omega}^r[y](t)\right) d_{q,\omega}t \longrightarrow \text{extr} \\
y \in \mathcal{Y}^r([a, b], \mathbb{R}) \\
y(a) = \alpha_0, \qquad y(b) = \beta_0, \\
\vdots \\
D_{q,\omega}^{r-1}[y](a) = \alpha_{r-1}, \qquad D_{q,\omega}^{r-1}[y](b) = \beta_{r-1},
\end{gathered} \tag{P}$$

where $r \in \mathbb{N}$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 0, \ldots, r-1$, are given.

Definition 2.36 We say that y is an admissible function for (2.7) if $y \in \mathcal{Y}^r([a,b],\mathbb{R})$ and y satisfies the boundary conditions $D^i_{q,\omega}[y](a) = \alpha_i$ and $D^i_{q,\omega}[y](b) = \beta_i$ of problem (2.7), $i = 0, \ldots, r-1$.

The Lagrangian L is assumed to satisfy the following hypotheses:

(H1) $(u_0, \ldots, u_r) \to L(t, u_0, \ldots, u_r)$ is a $C^1(\mathbb{R}^{r+1}, \mathbb{R})$ function for any $t \in [a,b]$;
(H2) $t \to L(t, y(t), D_{q,\omega}[y](t), \ldots, D^r_{q,\omega}[y](t))$ is continuous at ω_0 for any admissible y;
(H3) functions $t \to \partial_{i+2} L(t, y(t), D_{q,\omega}[y](t), \cdots, D^r_{q,\omega}[y](t))$, $i = 0, 1, \cdots, r$, belong to $\mathcal{Y}^1([a,b],\mathbb{R})$ for all admissible y.

Definition 2.37 We say that y_* is a local minimizer (resp. local maximizer) for problem (2.7) if y_* is an admissible function and there exists $\delta > 0$ such that

$$\mathcal{L}[y_*] \leq \mathcal{L}[y] \quad (\text{resp. } \mathcal{L}[y_*] \geq \mathcal{L}[y])$$

for all admissible y with $\|y_* - y\|_{r,\infty} < \delta$.

Definition 2.38 We say that $\eta \in \mathcal{Y}^r([a,b],\mathbb{R})$ is a *variation* if $\eta(a) = \eta(b) = 0$, …, $D^{r-1}_{q,\omega}[\eta](a) = D^{r-1}_{q,\omega}[\eta](b) = 0$.

2.7.1 Higher-order Fundamental Lemma

The chain rule, as known from classical calculus, does not hold in Hahn's quantum context (see a counterexample in Aldwoah (2009); Annaby et al. (2012)). However, we can prove the following.

Lemma 2.39 *If f is q,ω-differentiable on I, then the following equality holds:*

$$D_{q,\omega}\left[f^\sigma\right](t) = q\left(D_{q,\omega}[f]\right)^\sigma(t), \, t \in I.$$

Proof For $t \neq \omega_0$ we have

$$\begin{aligned}\left(D_{q,\omega}[f]\right)^\sigma(t) &= \frac{f(q(qt+\omega)+\omega) - f(qt+\omega)}{(q-1)(qt+\omega)+\omega} \\ &= \frac{f(q(qt+\omega)+\omega) - f(qt+\omega)}{q((q-1)t+\omega)}\end{aligned}$$

and

$$D_{q,\omega}\left[f^\sigma\right](t) = \frac{f^\sigma(qt+\omega) - f^\sigma(t)}{(q-1)t+\omega} = \frac{f(q(qt+\omega)+\omega) - f(qt+\omega)}{(q-1)t+\omega}.$$

Therefore, $D_{q,\omega}[f^\sigma](t) = q\left(D_{q,\omega}[f]\right)^\sigma(t)$. If $t = \omega_0$, then $\sigma(\omega_0) = \omega_0$. Thus,

$$\left(D_{q,\omega}\left[f\right]\right)^{\sigma}\left(\omega_0\right) = \left(D_{q,\omega}\left[f\right]\right)\left(\sigma\left(\omega_0\right)\right) = \left(D_{q,\omega}\left[f\right]\right)\left(\omega_0\right) = f'\left(\omega_0\right)$$

and $D_{q,\omega}\left[f^{\sigma}\right]\left(\omega_0\right) = \left[f^{\sigma}\right]'\left(\omega_0\right) = f'\left(\sigma\left(\omega_0\right)\right)\sigma'\left(\omega_0\right) = qf'\left(\omega_0\right)$.

Lemma 2.40 *If $\eta \in \mathcal{Y}^r\left(\left[a,b\right],\mathbb{R}\right)$ is such that $D^i_{q,\omega}\left[\eta\right]\left(a\right) = 0$ (resp. $D^i_{q,\omega}\left[\eta\right]\left(b\right) = 0$) for all $i \in \{0, 1, \dots, r\}$, then $D^{i-1}_{q,\omega}\left[\eta^{\sigma}\right]\left(a\right) = 0$ (resp. $D^{i-1}_{q,\omega}\left[\eta^{\sigma}\right]\left(b\right) = 0$) for all $i \in \{1, \dots, r\}$.*

Proof If $a = \omega_0$ the result is trivial (because $\sigma\left(\omega_0\right) = \omega_0$). Suppose now that $a \neq \omega_0$ and fix $i \in \{1, \dots, r\}$. Note that

$$D^i_{q,\omega}\left[\eta\right]\left(a\right) = \frac{\left(D^{i-1}_{q,\omega}\left[\eta\right]\right)^{\sigma}\left(a\right) - D^{i-1}_{q,\omega}\left[\eta\right]\left(a\right)}{\left(q-1\right)a + \omega}.$$

Because, by hypothesis, $D^i_{q,\omega}\left[\eta\right]\left(a\right) = 0$ and $D^{i-1}_{q,\omega}\left[\eta\right]\left(a\right) = 0$, then

$$\left(D^{i-1}_{q,\omega}\left[\eta\right]\right)^{\sigma}\left(a\right) = 0.$$

Lemma 2.39 shows that

$$\left(D^{i-1}_{q,\omega}\left[\eta\right]\right)^{\sigma}\left(a\right) = \left(\frac{1}{q}\right)^{i-1} D^{i-1}_{q,\omega}\left[\eta^{\sigma}\right]\left(a\right).$$

We conclude that $D^{i-1}_{q,\omega}\left[\eta^{\sigma}\right]\left(a\right) = 0$. The case $t = b$ is proved in the same way.

Lemma 2.41 *Suppose that $f \in \mathcal{Y}^1\left(\left[a,b\right],\mathbb{R}\right)$. One has*

$$\int_a^b f\left(t\right) D_{q,\omega}\left[\eta\right]\left(t\right) d_{q,\omega}t = 0$$

for all functions $\eta \in \mathcal{Y}^1\left(\left[a,b\right],\mathbb{R}\right)$ such that $\eta\left(a\right) = \eta\left(b\right) = 0$ if and only if $f\left(t\right) = c$, $c \in \mathbb{R}$, for all $t \in \left[a,b\right]_{q,\omega}$.

Proof The implication "$\Leftarrow$" is obvious. We prove "$\Rightarrow$". We begin noting that

$$\underbrace{\int_a^b f\left(t\right) D_{q,\omega}\left[\eta\right]\left(t\right) d_{q,\omega}t}_{=0} = \underbrace{f\left(t\right)\eta\left(t\right)\Big|_a^b}_{=0} - \int_a^b D_{q,\omega}\left[f\right]\left(t\right)\eta^{\sigma}\left(t\right) d_{q,\omega}t.$$

Hence,

$$\int_a^b D_{q,\omega}\left[f\right]\left(t\right)\eta\left(qt+\omega\right) d_{q,\omega}t = 0$$

for any $\eta \in \mathcal{Y}^1([a,b],\mathbb{R})$ such that $\eta(a) = \eta(b) = 0$. We need to prove that, for some $c \in \mathbb{R}$, $f(t) = c$ for all $t \in [a,b]_{q,\omega}$, that is, $D_{q,\omega}[f](t) = 0$ for all $t \in [a,b]_{q,\omega}$. Suppose, by contradiction, that there exists $p \in [a,b]_{q,\omega}$ such that $D_{q,\omega}[f](p) \neq 0$.

(1) If $p \neq \omega_0$, then $p = q^k a + \omega[k]_q$ or $p = q^k b + \omega[k]_q$ for some $k \in \mathbb{N}_0$. Observe that $a(1-q)-\omega$ and $b(1-q)-\omega$ cannot vanish simultaneously.

(a) Suppose that $a(1-q)-\omega \neq 0$ and $b(1-q)-\omega \neq 0$. In this case we can assume, without loss of generality, that $p = q^k a + \omega[k]_q$ and we can define

$$\eta(t) = \begin{cases} D_{q,\omega}[f]\left(q^k a + \omega[k]_q\right) & \text{if } t = q^{k+1}a + \omega[k+1]_q \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned}&\int_a^b D_{q,\omega}[f](t)\cdot\eta(qt+\omega)\,d_{q,\omega}t \\ &= -(a(1-q)-\omega)\,q^k D_{q,\omega}[f]\left(q^k a+\omega[k]_q\right)\cdot D_{q,\omega}[f]\left(q^k a+\omega[k]_q\right) \neq 0,\end{aligned}$$

which is a contradiction.

(b) If $a(1-q)-\omega \neq 0$ and $b(1-q)-\omega = 0$, then $b = \omega_0$. Since $q^k\omega_0 + \omega[k]_q = \omega_0$ for all $k \in \mathbb{N}_0$, then $p \neq q^k b + \omega[k]_q\ \forall k \in \mathbb{N}_0$ and, therefore,

$$p = q^k a + \omega[k]_{q,\omega} \text{ for some } k \in \mathbb{N}_0.$$

Repeating the proof of (a) we obtain again a contradiction.

(c) If $a(1-q)-\omega = 0$ and $b(1-q)-\omega \neq 0$, then the proof is similar to (b).

(2) If $p = \omega_0$ then, without loss of generality, we can assume $D_{q,\omega}[f](\omega_0) > 0$. Since

$$\lim_{n\to+\infty}\left(q^n a + \omega[k]_q\right) = \lim_{n\to+\infty}\left(q^n b + \omega[k]_q\right) = \omega_0$$

(see Aldwoah (2009)) and $D_{q,\omega}[f]$ is continuous at ω_0, then

$$\begin{aligned}\lim_{n\to+\infty} D_{q,\omega}[f]\left(q^n a+\omega[k]_q\right) &= \lim_{n\to+\infty} D_{q,\omega}[f]\left(q^n b+\omega[k]_q\right) \\ &= D_{q,\omega}[f](\omega_0) > 0.\end{aligned}$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ one has

$$D_{q,\omega}[f]\left(q^n a+\omega[k]_q\right) > 0 \text{ and } D_{q,\omega}[f]\left(q^n b+\omega[k]_q\right) > 0.$$

(a) If $\omega_0 \neq a$ and $\omega_0 \neq b$, then we can define

$$\eta(t) = \begin{cases} D_{q,\omega}[f]\left(q^N b + \omega [N]_q\right) \; if \;\; t = q^{N+1}a + \omega [N+1]_q \\ D_{q,\omega}[f]\left(q^N a + \omega [N]_q\right) \; if \;\; t = q^{N+1}b + \omega [N+1]_q \\ 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Hence,

$$\int_a^b D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t$$
$$= (b-a)(1-q)\,q^N D_{q,\omega}[f]\left(q^N b + \omega[N]_q\right) \cdot D_{q\omega}[f]\left(q^N a + \omega[N]_q\right) \neq 0,$$

which is a contradiction.

(b) If $\omega_0 = b$, then we define

$$\eta(t) = \begin{cases} D_{q,\omega}[f](\omega_0) \; if \;\; t = q^{N+1}a + \omega[N+1]_q \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} &\int_a^b D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t \\ &\quad = -\int_{\omega_0}^a D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t \\ &\quad = -\left(a(1-q)-\omega\right) q^N D_{q,\omega}[f]\left(q^N a + \omega[k]_q\right) \cdot D_{q,\omega}[f](\omega_0) \neq 0, \end{aligned}$$

which is a contradiction.

(c) When $\omega_0 = a$, the proof is similar to (b).

Lemma 2.42 (Fundamental lemma of Hahn's variational calculus) *Let* $f, g \in \mathcal{Y}^1([a,b],\mathbb{R})$.

If

$$\int_a^b \left(f(t)\,\eta^\sigma(t) + g(t)\,D_{q,\omega}[\eta](t)\right) d_{q,\omega}t = 0$$

for all $\eta \in \mathcal{Y}^1([a,b],\mathbb{R})$ *such that* $\eta(a) = \eta(b) = 0$*, then*

$$D_{q,\omega}[g](t) = f(t) \;\; \forall t \in [a,b]_{q,\omega}.$$

Proof Define the function A by

$$A(t) := \int_{\omega_0}^{t} f(\tau)\, d_{q,\omega}\tau.$$

Then $D_{q,\omega}[A](t) = f(t)$ for all $t \in [a, b]$ and

$$\begin{aligned}
\int_a^b A(t)\, D_{q,\omega}[\eta](t)\, d_{q,\omega}t &= A(t)\,\eta(t)\Big|_a^b - \int_a^b D_{q,\omega}[A](t)\,\eta^{\sigma}(t)\, d_{q,\omega}t \\
&= -\int_a^b D_{q,\omega}[A](t)\,\eta^{\sigma}(t)\, d_{q,\omega}t \\
&= -\int_a^b f(t)\,\eta^{\sigma}(t)\, d_{q,\omega}t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_a^b \big(f(t)\,\eta^{\sigma}(t) + g(t)\, D_{q,\omega}[\eta](t)\big)\, d_{q,\omega}t = 0 \\
&\qquad \Leftrightarrow \int_a^b (-A(t) + g(t))\, D_{q,\omega}[\eta](t)\, d_{q,\omega}t = 0.
\end{aligned}$$

By Lemma 2.41 there is a $c \in \mathbb{R}$ such that $-A(t) + g(t) = c$ for all $t \in [a, b]_{q,\omega}$. Hence $D_{q,\omega}[A](t) = D_{q,\omega}[g](t)$ for $t \in [a, b]_{q,\omega}$, which provides the desired result: $D_{q,\omega}[g](t) = f(t)\ \forall t \in [a, b]_{q,\omega}$.

We are now in conditions to deduce the higher-order fundamental Lemma of Hahn's quantum variational calculus.

Lemma 2.43 (Higher-order fundamental lemma of Hahn's variational calculus) *Let* $f_0, f_1, \ldots, f_r \in \mathcal{Y}^1([a, b], \mathbb{R})$. *If*

$$\int_a^b \left(\sum_{i=0}^{r} f_i(t)\, D_{q,\omega}^{i}\left[\eta^{\sigma^{r-i}}\right](t) \right) d_{q,\omega}t = 0$$

for any variation η, *then*

$$\sum_{i=0}^{r} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^{i}[f_i](t) = 0$$

for all $t \in [a, b]_{q,\omega}$.

Proof We proceed by mathematical induction. If $r = 1$ the result is true by Lemma 2.42. Assume that

$$\int_a^b \left(\sum_{i=0}^{r+1} f_i(t)\, D_{q,\omega}^i \left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t = 0$$

for all functions η such that $\eta(a) = \eta(b) = 0, \ldots, D_{q,\omega}^r[\eta](a) = D_{q,\omega}^r[\eta](b) = 0$. Note that

$$\begin{aligned}
&\int_a^b f_{r+1}(t)\, D_{q,\omega}^{r+1}[\eta](t)\, d_{q,\omega}t \\
&\qquad = f_{r+1}(t)\, D_{q,\omega}^r[\eta](t)\Big|_a^b - \int_a^b D_{q,\omega}\left[f_{r+1}\right](t)\left(D_{q,\omega}^r[\eta]\right)^\sigma(t)\, d_{q,\omega}t \\
&\qquad = -\int_a^b D_{q,\omega}\left[f_{r+1}\right](t)\left(D_{q,\omega}^r[\eta]\right)^\sigma(t)\, d_{q,\omega}t
\end{aligned}$$

and, by Lemma 2.39,

$$\int_a^b f_{r+1}(t)\, D_{q,\omega}^{r+1}[\eta](t)\, d_{q,\omega}t = -\int_a^b D_{q,\omega}\left[f_{r+1}\right](t)\left(\frac{1}{q}\right)^r D_{q,\omega}^r\left[\eta^\sigma\right](t)\, d_{q,\omega}t.$$

Therefore,

$$\begin{aligned}
&\int_a^b \left(\sum_{i=0}^{r+1} f_i(t)\, D_{q,\omega}^i \left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t \\
&\quad = \int_a^b \left(\sum_{i=0}^{r} f_i(t)\, D_{q,\omega}^i \left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t \\
&\qquad - \int_a^b D_{q,\omega}\left[f_{r+1}\right](t)\left(\frac{1}{q}\right)^r D_{q,\omega}^r\left[\eta^\sigma\right](t)\, d_{q,\omega}t \\
&\quad = \int_a^b \Bigg[\sum_{i=0}^{r-1} f_i(t)\, D_{q,\omega}^i \left[\left(\eta^\sigma\right)^{\sigma^{r-i}}\right](t)\, d_{q,\omega}t \\
&\qquad + \left(f_r - \left(\frac{1}{q}\right)^r D_{q,\omega}\left[f_{r+1}\right]\right)(t)\, D_{q,\omega}^r\left[\eta^\sigma\right](t)\Bigg] d_{q,\omega}t.
\end{aligned}$$

By Lemma 2.40, η^σ is a variation. Hence, using the induction hypothesis,

$$\begin{aligned}
&\sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i\left[f_i\right](t) \\
&\qquad + (-1)^r \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} D_{q,\omega}^r\left[\left(f_r - \frac{1}{q^r} D_{q,\omega}\left[f_{r+1}\right]\right)\right](t)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D^i_{q,\omega}\left[f_i\right](t) + (-1)^r \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} D^r_{q,\omega}\left[f_r\right](t) \\
&\quad + (-1)^{r+1} \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} \frac{1}{q^r} D^r_{q,\omega}\left[D_{q,\omega}\left[f_{r+1}\right]\right](t) \\
&= 0
\end{aligned}
$$

for all $t \in [a, b]_{q,\omega}$, which leads to

$$
\sum_{i=0}^{r+1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D^i_{q,\omega}\left[f_i\right](t) = 0, \ \ t \in [a, b]_{q,\omega}.
$$

2.7.2 *Higher-order Hahn's Quantum Euler–Lagrange Equation*

For a variation η and an admissible function y, we define the function $\phi : (-\bar{\epsilon}, \bar{\epsilon}) \to \mathbb{R}$ by $\phi(\epsilon) = \phi(\epsilon, y, \eta) := \mathcal{L}[y + \epsilon\eta]$. The first variation of the variational problem (2.7) is defined by $\delta\mathcal{L}[y, \eta] := \phi'(0)$. Observe that

$$
\begin{aligned}
\mathcal{L}[y + \epsilon\eta] &= \int_a^b L\Bigg(t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t), \\
&\qquad \ldots, D^r_{q,\omega}[y](t) + \epsilon D^r_{q,\omega}[\eta](t)\Bigg) d_{q,\omega}t \\
&= \mathcal{L}_b[y + \epsilon\eta] - \mathcal{L}_a[y + \epsilon\eta]
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{L}_\xi[y + \epsilon\eta] = \int_{\omega_0}^{\xi} L\Bigg(t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t), \\
\ldots, D^r_{q,\omega}[y](t) + \epsilon D^r_{q,\omega}[\eta](t)\Bigg) d_{q,\omega}t,
\end{aligned}
$$

$\xi \in \{a, b\}$. Therefore,

$$
\delta\mathcal{L}[y, \eta] = \delta\mathcal{L}_b[y, \eta] - \delta\mathcal{L}_a[y, \eta]. \tag{2.29}
$$

Considering (2.29), the following is a direct consequence of Lemma 2.15:

Lemma 2.44 *For a variation η and an admissible function y, let*

$$g(t,\epsilon) := L\Bigg(t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t),\\ \dots, D^r_{q,\omega}[y](t) + \epsilon D^r_{q,\omega}[\eta](t)\Bigg),$$

$\epsilon \in (-\bar{\epsilon}, \bar{\epsilon})$. *Assume that:*
(1) $g(t,\cdot)$ *is differentiable at* 0 *uniformly in* $t \in [a,b]_{q,\omega}$*;*
(2) $\mathcal{L}_a[y+\epsilon\eta] = \int_{\omega_0}^{a} g(t,\epsilon)\, d_{q,\omega}t$ *and* $\mathcal{L}_b[y+\epsilon\eta] = \int_{\omega_0}^{b} g(t,\epsilon)\, d_{q,\omega}t$ *exist for* $\epsilon \approx 0$*;*
(3) $\int_{\omega_0}^{a} \partial_2 g(t,0)\, d_{q,\omega}t$ *and* $\int_{\omega_0}^{b} \partial_2 g(t,0)\, d_{q,\omega}t$ *exist.*
Then

$$\begin{aligned}\phi'(0) &= \delta\mathcal{L}[y,\eta]\\ &= \int_a^b \Bigg(\sum_{i=0}^{r} \partial_{i+2}L\left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t), \dots, D^r_{q,\omega}[y](t)\right)\\ &\qquad \cdot D^i_{q,\omega}\left[\eta^{\sigma^{r-i}}\right](t)\Bigg) d_{q,\omega}t,\end{aligned}$$

where $\partial_i L$ denotes the partial derivative of L with respect to its ith argument.

The following result gives a necessary condition of Euler–Lagrange type for an admissible function to be a local extremizer for (2.7).

Theorem 2.45 (The higher-order Hahn quantum Euler–Lagrange equation) *Under hypotheses (H1)–(H3) and conditions (1)–(3) of Lemma 2.44 on the Lagrangian L, if $y_* \in \mathcal{Y}^r$ is a local extremizer for problem (2.7), then y_* satisfies the q,ω-Euler–Lagrange equation*

$$\sum_{i=0}^{r} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D^i_{q,\omega}\left[\partial_{i+2}L\right]\Big(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t),\\ \dots, D^r_{q,\omega}[y](t)\Big) = 0 \tag{2.30}$$

for all $t \in [a,b]_{q,\omega}$.

Proof Let y_* be a local extremizer for problem (2.7) and η a variation. Define $\phi : (-\bar{\epsilon}, \bar{\epsilon}) \to \mathbb{R}$ by $\phi(\epsilon) := \mathcal{L}[y_* + \epsilon\eta]$. A necessary condition for y_* to be an extremizer is given by $\phi'(0) = 0$. By Lemma 2.44 we conclude that

$$\int_a^b \left(\sum_{i=0}^{r} \partial_{i+2} L \left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}} \right](t), \ldots, D_{q,\omega}^r [y](t) \right) \right.$$
$$\left. \cdot D_{q,\omega}^i \left[\eta^{\sigma^{r-i}} \right](t) \right) d_{q,\omega} t = 0$$

and (2.30) follows from Lemma 2.43.

Remark 2.46 In practical terms the hypotheses of Theorem 2.45 are not so easy to verify *a priori*. One can, however, assume that all hypotheses are satisfied and apply the q, ω-Euler–Lagrange equation (2.30) heuristically to obtain a *candidate*. If such a candidate is, or not, a solution to problem (2.7) is a different question that always requires further analysis (see an example in Sect. 2.7.3).

When $\omega = 0$ one obtains from (2.30) the higher-order q-Euler–Lagrange equation:

$$\sum_{i=0}^{r} (-1)^i \left(\frac{1}{q} \right)^{\frac{(i-1)i}{2}} D_q^i \left[\partial_{i+2} L \right] \left(t, y^{\sigma^r}(t), D_q \left[y^{\sigma^{r-1}} \right](t), \ldots, D_q^r [y](t) \right) = 0$$

for all $t \in \{aq^n : n \in \mathbb{N}_0\} \cup \{bq^n : n \in \mathbb{N}_0\} \cup \{0\}$. The higher-order h-Euler–Lagrange equation is obtained from (2.30) taking the limit $q \to 1$:

$$\sum_{i=0}^{r} (-1)^i \Delta_h^i \left[\partial_{i+2} L \right] \left(t, y^{\sigma^r}(t), \Delta_h \left[y^{\sigma^{r-1}} \right](t), \ldots, \Delta_h^r [y](t) \right) = 0$$

for all $t \in \{a + nh : n \in \mathbb{N}_0\} \cup \{b + nh : n \in \mathbb{N}_0\}$. The classical Euler–Lagrange equation (van Brunt 2004) is recovered when $(\omega, q) \to (0, 1)$:

$$\sum_{i=0}^{r} (-1)^i \frac{d^i}{dt^i} \partial_{i+2} L \left(t, y(t), y'(t), \ldots, y^{(r)}(t) \right) = 0$$

for all $t \in [a, b]$.

We now illustrate the usefulness of our Theorem 2.45 by means of an example that is not covered by previous available results in the literature.

2.7.3 An Example

Let $q = \frac{1}{2}$ and $\omega = \frac{1}{2}$. Consider the following problem:

$$\mathcal{L}[y] = \int_{-1}^{1} \left(y^\sigma(t) + \frac{1}{2} \right)^2 \left(\left(D_{q,\omega}[y](t) \right)^2 - 1 \right)^2 d_{q,\omega} t \longrightarrow \min \tag{2.31}$$

over all $y \in \mathcal{Y}^1$ satisfying the boundary conditions

$$y(-1) = 0 \quad \text{and} \quad y(1) = -1. \tag{2.32}$$

This is an example of problem (2.7) with $r = 1$. Our q, ω-Euler–Lagrange equation (2.30) takes the form

$$D_{q,\omega} [\partial_3 L] \left(t, y^\sigma (t), D_{q,\omega} [y] (t)\right) = \partial_2 L \left(t, y^\sigma (t), D_{q,\omega} [y] (t)\right).$$

Therefore, we look for an admissible function y_* of (2.31)–(2.32) satisfying

$$\begin{aligned} D_{q,\omega} &\left[4 \left(y^\sigma + \frac{1}{2} \right)^2 \left(\left(D_{q,\omega} [y] \right)^2 - 1 \right) D_{q,\omega} [y] \right] (t) \\ &= 2 \left(y^\sigma (t) + \frac{1}{2} \right) \left(\left(D_{q,\omega} [y] (t) \right)^2 - 1 \right) \end{aligned} \tag{2.33}$$

for all $t \in [-1, 1]_{q,\omega}$. It is easy to see that

$$y_*(t) = \begin{cases} -t & \text{if } t \in (-1, 0) \cup (0, 1] \\ 0 & \text{if } t = -1 \\ 1 & \text{if } t = 0 \end{cases}$$

is an admissible function for (2.31)–(2.32) with

$$D_{q,\omega} [y_*] (t) = \begin{cases} -1 & \text{if } t \in (-1, 0) \cup (0, 1] \\ 1 & \text{if } t = -1 \\ -3 & \text{if } t = 0, \end{cases}$$

satisfying the q, ω-Euler–Lagrange equation (2.33). We now prove that the *candidate* y_* is indeed a minimizer for (2.31)–(2.32). Note that here $\omega_0 = 1$ and, by Lemma 2.11 and item (3) of Theorem 2.10,

$$\mathcal{L} [y] = \int_{-1}^{1} \left(y^\sigma (t) + \frac{1}{2} \right)^2 \left(\left(D_{q,\omega} [y] (t) \right)^2 - 1 \right)^2 d_{q,\omega} t \geq 0 \tag{2.34}$$

for all admissible functions $y \in \mathcal{Y}^1 ([-1, 1], \mathbb{R})$. Since $\mathcal{L} [y_*] = 0$, we conclude that y_* is a minimizer for problem (2.31)–(2.32).

It is worth mentioning that the minimizer y_* of (2.31)–(2.32) is not continuous while the classical calculus of variations (van Brunt 2004), the calculus of variations on time scales (Ferreira and Torres 2008; Malinowska and Torres 2009; Martins and Torres 2009), or the nondifferentiable scale variational calculus (Almeida and Torres 2009a, 2010a; Cresson et al. 2009), deal with functions which are necessarily continuous. As an open question, we pose the problem of determining conditions on the data of problem (2.7) assuring, *a priori*, the minimizer to be regular.

2.8 Generalized Transversality Conditions

The main purpose of this section is to generalize the Hahn calculus of variations (Malinowska and Torres 2010c) by considering the following q, ω-variational problem:

$$\mathcal{L}[y] = \int_a^b L\left(t, y(qt+\omega), D_{q,\omega}[y](t), y(a), y(b)\right) d_{q,\omega}t \longrightarrow \text{extr.} \tag{2.35}$$

In Sect. 2.8.1 we obtain the Euler–Lagrange equation for problem (2.35) in the class of functions $y \in \mathcal{Y}^1$ satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta \tag{2.36}$$

for some fixed $\alpha, \beta \in \mathbb{R}$. The transversality conditions for problem (2.35) are obtained in Sect. 2.8.2. In Sect. 2.8.3 we prove necessary optimality conditions for isoperimetric problems. A sufficient optimality condition under an appropriate convexity assumption is given in Sect. 2.8.4.

In the sequel we assume that the Lagrangian L satisfies the following hypotheses:

(H1) $(u_0, \ldots, u_3) \to L(t, u_0, \ldots, u_3)$ is a $C^1(\mathbb{R}^4, \mathbb{R})$ function for any $t \in [a, b]$;
(H2) $t \to L(t, y(qt+\omega), D_{q,\omega}[y](t), y(a), y(b))$ is continuous at ω_0 for any $y \in \mathcal{Y}^1$;
(H3) functions $t \to \partial_{i+2}L(t, y(qt+\omega), D_{q,\omega}[y](t), y(a), y(b)), i = 0, \ldots, 3$ belong to $\mathcal{Y}^1$ for all $y \in \mathcal{Y}^1$.

In order to simplify expressions, we introduce the operator $\{\cdot\}$ defined in the following way:

$$\{y\}(t, a, b) := (t, y(qt+\omega), D_{q,\omega}[y](t), y(a), y(b))$$

where $y \in \mathcal{Y}^1$.

The following lemma can be obtained similar to Lemma 2.15.

Lemma 2.47 *For fixed $y, h \in \mathcal{Y}^1$ let*

$$\begin{aligned} g(t,\varepsilon) = L(t, &y(qt+\omega) + \varepsilon h(qt+\omega), D_{q,\omega}[y](t) \\ &+ \varepsilon D_{q,\omega}[h](t), y(a) + \varepsilon h(a), y(b) + \varepsilon h(b)) \end{aligned}$$

for $\varepsilon \in]-\bar{\varepsilon}, \bar{\varepsilon}[$, for some $\bar{\varepsilon} > 0$, i.e., $g(t,\varepsilon) = L\{y + \varepsilon h\}(t, a, b)$. Assume that:

(i) *$g(t, \cdot)$ is differentiable at 0 uniformly in $t \in [a, b]_{q,\omega}$;*

(ii) $\mathcal{L}_a[y+\varepsilon h] = \int_{\omega_0}^{a} g(t,\epsilon)\, d_{q,\omega}t$ *and* $\mathcal{L}_b[y+\varepsilon h] = \int_{\omega_0}^{b} g(t,\epsilon)\, d_{q,\omega}t$ *exist for* $\varepsilon \approx 0$*;*

(iii) $\int_{\omega_0}^{a} \partial_2 g(t,0) d_{q,\omega}t$ *and* $\int_{\omega_0}^{b} \partial_2 g(t,0) d_{q,\omega}t$ *exist.*

Then,

$$\delta\mathcal{L}[y,h] = \int_a^b \Big(\partial_2 L\{y\}(t,a,b)\cdot h(qt+\omega) + \partial_3 L\{y\}(t,a,b)\cdot D_{q,\omega}[h](t) \\ + \partial_4 L\{y\}(t,a,b)\cdot h(a) + \partial_5 L\{y\}(t,a,b)\cdot h(b)\Big) d_{q,\omega}t.$$

2.8.1 The Hahn Quantum Euler–Lagrange Equation

In the following theorem, we give the Euler–Lagrange equation for problem (2.35)–(2.36).

Theorem 2.48 (Necessary optimality condition to (2.35)–(2.36)**)** *Under hypotheses (H1)–(H3) and conditions (i)–(iii) of Lemma 2.47 on the Lagrangian L, if* $\tilde{y}$ *is a local minimizer or local maximizer to problem* (2.35)–(2.36)*, then* $\tilde{y}$ *satisfies the Euler–Lagrange equation*

$$\partial_2 L\{y\}(t,a,b) - D_{q,\omega}[\partial_3 L]\{y\}(t,a,b) = 0 \tag{2.37}$$

for all $t \in [a,b]_{q,\omega}$.

Proof Suppose that $\mathcal{L}$ has a local extremum at $\tilde{y}$. Let h be any admissible variation and define a function $\phi :]-\bar{\varepsilon},\bar{\varepsilon}[\to \mathbb{R}$ by $\phi(\varepsilon) = \mathcal{L}[\tilde{y}+\varepsilon h]$. A necessary condition for $\tilde{y}$ to be an extremizer is given by $\phi'(0) = 0$. Note that

$$\phi'(0) = \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t,a,b)\cdot h(qt+\omega) + \partial_3 L\{\tilde{y}\}(t,a,b)\cdot D_{q,\omega}[h](t) \\ + \partial_4 L\{\tilde{y}\}(t,a,b)\cdot h(a) + \partial_5 L\{\tilde{y}\}(t,a,b)\cdot h(b)\Big) d_{q,\omega}t.$$

Since $h(a) = h(b) = 0$, then

$$\phi'(0) = \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t,a,b)\cdot h(qt+\omega) + \partial_3 L\{\tilde{y}\}(t,a,b)\cdot D_{q,\omega}[h](t)\Big) d_{q,\omega}t.$$

Integration by parts gives

$$\int_a^b \partial_3 L\{\tilde{y}\}(t,a,b)\cdot D_{q,\omega}[h](t)d_{q,\omega}t = \Big[\partial_3 L\{\tilde{y}\}(t,a,b)\cdot h(t)\Big]_a^b - \int_a^b D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t,a,b)\cdot h(qt+\omega)d_{q,\omega}t$$

and since $h(a) = h(b) = 0$, then

$$\phi'(0) = 0 \Leftrightarrow \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t,a,b) - D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t,a,b)\Big)\cdot h(qt+\omega)d_{q,\omega}t = 0.$$

Thus, by Lemma 2.16, we have

$$\partial_2 L\{\tilde{y}\}(t,a,b) - D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t,a,b) = 0$$

for all $t \in [a,b]_{q,\omega}$.

Remark 2.49 Under appropriate conditions, when $(\omega, q) \to (0,1)$, we obtain a corresponding result in the classical context of the calculus of variations (Cruz et al. 2010) (see also Malinowska and Torres (2010b)):

$$\frac{d}{dt}\partial_3 L(t, y(t), y'(t), y(a), y(b)) = \partial_2 L(t, y(t), y'(t), y(a), y(b)).$$

Remark 2.50 In the basic problem of the calculus of variations, L does not depend on $y(a)$ and $y(b)$, and equation (2.37) reduces to the Hahn quantum Euler–Lagrange equation (2.4).

2.8.2 Natural Boundary Conditions

The following theorem provides necessary optimality conditions for problem (2.35).

Theorem 2.51 (Natural boundary conditions to (2.35)**)** *Under hypotheses (H1)–(H3) and conditions (i)–(iii) of Lemma 2.47 on the Lagrangian L, if $\tilde{y}$ is a local minimizer or local maximizer to problem* (2.35)*, then $\tilde{y}$ satisfies the Euler–Lagrange equation* (2.37) *and*

1. *if $y(a)$ is free, then the natural boundary condition*

$$\partial_3 L\{\tilde{y}\}(a,a,b) = \int_a^b \partial_4 L\{\tilde{y}\}(t,a,b)d_{q,\omega}t \tag{2.38}$$

holds;

2. *if $y(b)$ is free, then the natural boundary condition*

$$\partial_3 L\{\tilde{y}\}(b,a,b) = -\int_a^b \partial_5 L\{\tilde{y}\}(t,a,b)d_{q,\omega}t \tag{2.39}$$

holds.

Proof Suppose that $\tilde{y}$ is a local minimizer (resp. maximizer) to problem (2.35). Let h be any $\mathcal{Y}^1$ function. Define a function $\phi :]-\bar{\varepsilon}, \bar{\varepsilon}[\to \mathbb{R}$ by $\phi(\varepsilon) = \mathcal{L}[\tilde{y} + \varepsilon h]$. It is clear that a necessary condition for $\tilde{y}$ to be an extremizer is given by $\phi'(0) = 0$. From the arbitrariness of h and using similar arguments as the ones used in the proof of Theorem 2.48, it can be proved that $\tilde{y}$ satisfies the Euler–Lagrange equation (2.37).

1. Suppose now that $y(a)$ is free. If $y(b) = \beta$ is given, then $h(b) = 0$; if $y(b)$ is free, then we restrict ourselves to those h for which $h(b) = 0$. Therefore,

$$\begin{aligned}
0 &= \phi'(0)\\
&= \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t,a,b) - D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t,a,b)\Big)\cdot h(qt+\omega)d_{q,\omega}t \qquad (2.40)\\
&\quad + \Big(\int_a^b \partial_4 L\{\tilde{y}\}(t,a,b)d_{q,\omega}t - \partial_3 L\{\tilde{y}\}(a,a,b)\Big)\cdot h(a) = 0.
\end{aligned}$$

Using the Euler–Lagrange equation (2.37) into (2.40) we obtain

$$\Big(\int_a^b \partial_4 L\{\tilde{y}\}(t,a,b)d_{q,\omega}t - \partial_3 L\{\tilde{y}\}(a,a,b)\Big)\cdot h(a) = 0.$$

From the arbitrariness of h it follows that

$$\partial_3 L\{\tilde{y}\}(a,a,b) = \int_a^b \partial_4 L\{\tilde{y}\}(t,a,b)d_{q,\omega}t.$$

2. Suppose now that $y(b)$ is free. If $y(a) = \alpha$, then $h(a) = 0$; if $y(a)$ is free, then we restrict ourselves to those h for which $h(a) = 0$. Thus,

$$\begin{aligned}
0 &= \phi'(0)\\
&= \int_a^b \Big(\partial_2 L\{\tilde{y}\}(t,a,b) - D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t,a,b)\Big)\cdot h(qt+\omega)d_{q,\omega}t \qquad (2.41)\\
&\quad + \Big(\int_a^b \partial_5 L\{\tilde{y}\}(t,a,b)d_{q,\omega}t + \partial_3 L\{\tilde{y}\}(b,a,b)\Big)\cdot h(b) = 0.
\end{aligned}$$

Using the Euler–Lagrange equation (2.37) into (2.41), and from the arbitrariness of h, it follows that

$$\partial_3 L\{\tilde{y}\}(b,a,b) = -\int_a^b \partial_5 L\{\tilde{y}\}(t,a,b)d_{q,\omega}t.$$

In the case where L does not depend on $y(a)$ and $y(b)$, under appropriate assumptions on the Lagrangian L, we obtain the following result.

Corollary 2.52 *If $\tilde{y}$ is a local minimizer or local maximizer to problem*

$$\mathcal{L}[y] = \int_a^b L\{\tilde{y}\}(t) d_{q,\omega}t \longrightarrow extr$$

then $\tilde{y}$ satisfies the Euler–Lagrange equation

$$\partial_2 L\{\tilde{y}\}(t) - D_{q,\omega}[\partial_3 L]\{\tilde{y}\}(t) = 0$$

for all $t \in [a, b]_{q,\omega}$, and

1. *if $y(a)$ is free, then the natural boundary condition*

$$\partial_3 L\{\tilde{y}\}(a) = 0 \tag{2.42}$$

holds;

2. *if $y(b)$ is free, then the natural boundary condition*

$$\partial_3 L\{\tilde{y}\}(b) = 0 \tag{2.43}$$

holds.

Remark 2.53 Under appropriate conditions, when $(\omega, q) \to (0, 1)$ equations (2.42) and (2.43) reduce to the well-known natural boundary conditions for the basic problem of the calculus of variations

$$\partial_3 L(a, \tilde{y}(a), \tilde{y}'(a)) = 0 \quad \text{and} \quad \partial_3 L(b, \tilde{y}(b), \tilde{y}'(b)) = 0,$$

respectively.

2.8.3 Isoperimetric Problem

We now study the general Hahn quantum isoperimetric problem with an integral constraint. Both normal and abnormal extremizers are considered. The isoperimetric problem consists of minimizing or maximizing the functional

$$\mathcal{L}[y] = \int_a^b L\left(t, y(qt+\omega), D_{q,\omega}[y](t), y(a), y(b)\right) d_{q,\omega}t \tag{2.44}$$

in the class of functions $y \in \mathcal{Y}^1$ satisfying the integral constraint

$$\mathcal{J}[y] = \int_a^b F\left(t, y(qt+\omega), D_{q,\omega}[y](t), y(a), y(b)\right) d_{q,\omega}t = \gamma \tag{2.45}$$

for some $\gamma \in \mathbb{R}$.

Theorem 2.54 (Necessary optimality condition for normal extremizers to (2.44)–(2.45)) *Suppose that L and F satisfy hypotheses (H1)–(H3) and conditions (i)–(iii) of Lemma 2.47, and suppose that $\widetilde{y} \in \mathcal{Y}^1$ gives a local minimum or a local maximum to the functional $\mathcal{L}$ subject to the integral constraint (2.45). If $\widetilde{y}$ is not an extremal to $\mathcal{J}$, then there exists a real λ such that $\widetilde{y}$ satisfies the equation*

$$\partial_2 H\{y\}(t,a,b) - D_{q,\omega}[\partial_3 H]\{y\}(t,a,b) = 0 \tag{2.46}$$

for all $t \in [a,b]_{q,\omega}$, where $H = L - \lambda F$ and

1. *if $y(a)$ is free, then the natural boundary condition*

$$\partial_3 H\{\tilde{y}\}(a,a,b) = \int_a^b \partial_4 H\{\tilde{y}\}(t,a,b) d_{q,\omega}t \tag{2.47}$$

   holds;

2. *if $y(b)$ is free, then the natural boundary condition*

$$\partial_3 H\{\tilde{y}\}(b,a,b) = -\int_a^b \partial_5 H\{\tilde{y}\}(t,a,b) d_{q,\omega}t \tag{2.48}$$

   holds.

Proof The proof is left to the reader. Hint: recall proofs of Theorem 2.26 and Theorem 2.51.

Introducing an extra multiplier λ_0 we can also deal with abnormal extremizers to the isoperimetric problem (2.44)–(2.45).

Theorem 2.55 (Necessary optimality condition for normal and abnormal extremizers to (2.44)–(2.45)) *Suppose that L and F satisfy hypotheses (H1)–(H3) and conditions (i)–(iii) of Lemma 2.47, and suppose that $\widetilde{y} \in \mathcal{Y}^1$ gives a local minimum or a local maximum to the functional $\mathcal{L}$ subject to the integral constraint (2.45). Then there exist two constants λ_0 and λ, not both zero, such that $\widetilde{y}$ satisfies the equation*

$$\partial_2 H\{y\}(t,a,b) - D_{q,\omega}[\partial_3 H]\{y\}(t,a,b) = 0 \tag{2.49}$$

for all $t \in [a,b]_{q,\omega}$, where $H = \lambda_0 L - \lambda F$ and

1. *if $y(a)$ is free, then the natural boundary condition*

$$\partial_3 H\{\tilde{y}\}(a,a,b) = \int_a^b \partial_4 H\{\tilde{y}\}(t,a,b) d_{q,\omega}t \tag{2.50}$$

holds;

2. *if $y(b)$ is free, then the natural boundary condition*

$$\partial_3 H\{\tilde{y}\}(b, a, b) = -\int_a^b \partial_5 H\{\tilde{y}\}(t, a, b) d_{q,\omega} t \tag{2.51}$$

holds.

In the case where L and F do not depend on $y(a)$ and $y(b)$, under appropriate assumptions on Lagrangians L and F, we obtain the following result.

Corollary 2.56 *If $\tilde{y}$ is a local minimizer or local maximizer to the problem*

$$\mathcal{L}[y] = \int_a^b L\{y\}(t) d_{q,\omega} t \longrightarrow extr$$

subject to the integral constraint

$$\mathcal{J}[y] = \int_a^b F\{y\}(t) d_{q,\omega} t = \gamma$$

for some $\gamma \in \mathbb{R}$, then there exist two constants λ_0 and λ, not both zero, such that $\widetilde{y}$ satisfies the following equation

$$\partial_2 H\{y\}(t) - D_{q,\omega}[\partial_3 H]\{y\}(t) = 0$$

for all $t \in [a, b]_{q,\omega}$, where $H = \lambda_0 L - \lambda F$ and

1. *if $y(a)$ is free, then the natural boundary condition*

$$\partial_3 H\{\tilde{y}\}(a) = 0$$

holds;

2. *if $y(b)$ is free, then the natural boundary condition*

$$\partial_3 H\{\tilde{y}\}(b) = 0$$

holds.

2.8.4 Sufficient Condition for Optimality

The following theorem gives sufficient optimality conditions for problem (2.35).

Theorem 2.57 *Let $L(t, u_1, \ldots, u_4)$ be jointly convex (respectively concave) in $(u_1, \ldots, u_4)$. If $\tilde{y}$ satisfies conditions (2.37), (2.38) and (2.39), then $\tilde{y}$ is a global minimizer (respectively maximizer) to problem (2.35).*

Proof The proof can be adapted from the proof of Theorem 2.29.

2.8.5 Illustrative Examples

We provide some examples in order to illustrate our results.

Example 2.58 Let $q \in]0, 1[$ and $\omega \geq 0$ be fixed real numbers, and I be an interval of $\mathbb{R}$ such that $\omega_0, 0, 1 \in I$. Consider the problem

$$\mathcal{L}[y] = \int_0^1 \left(y(qt + \omega) + \frac{1}{2}(D_{q,\omega}[y](t))^2 \right) d_{q,\omega}t \longrightarrow \min \tag{2.52}$$

over all $y \in \mathcal{Y}^1$ satisfying the boundary condition $y(1) = 1$. If $\widetilde{y}$ is a local minimizer to problem (2.52), then by Corollary 2.52 it satisfies the following conditions:

$$D_{q,\omega}D_{q,\omega}[\widetilde{y}](t) = 1, \tag{2.53}$$

for all $t \in \{\omega[n]_q : n \in \mathbb{N}_0\} \cup \{q^n + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\}$ and

$$D_{q,\omega}[\widetilde{y}](0) = 0. \tag{2.54}$$

It is easy to verify that $\widetilde{y}(t) = \frac{1}{q+1}t^2 - (\frac{\omega}{q+1} - c)t + d$, where $c, d \in \mathbb{R}$, is a solution to equation (2.53). Using the natural boundary condition (2.54) we obtain that $c = 0$. In order to determine d we use the fixed boundary condition $y(1) = 1$, and obtain that $d = \frac{q+\omega}{q+1}$. Hence

$$\widetilde{y}(t) = \frac{1}{q+1}t^2 - \frac{\omega}{q+1}t + \frac{q+\omega}{q+1}$$

is a candidate to be a minimizer to problem (2.52). Moreover, since L is jointly convex, by Theorem 2.57, $\widetilde{y}$ is a global minimizer to problem (2.52).

Example 2.59 Let $q \in]0, 1[$ and $\omega \geq 0$ be fixed real numbers, and I be an interval of $\mathbb{R}$ such that $\omega_0, 0, 1 \in I$. Consider the problem of minimizing

$$\mathcal{L}[y] = \int_0^1 \left(y(qt + \omega) + \frac{1}{2}(D_{q,\omega}[y](t))^2 + \gamma\frac{1}{2}(y(1) - 1)^2 + \lambda\frac{1}{2}y^2(0) \right) d_{q,\omega}t, \tag{2.55}$$

where $\gamma, \lambda \in \mathbb{R}^+$. If $\widetilde{y}$ is a local minimizer to (2.55), then by Theorem 2.51 it satisfies the following conditions:

$$D_{q,\omega} D_{q,\omega}[\widetilde{y}](t) = 1, \tag{2.56}$$

for all $t \in \{\omega[n]_q : n \in \mathbb{N}_0\} \cup \{q^n + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\}$, and

$$D_{q,\omega}[\widetilde{y}](0) = \int_0^1 \lambda \widetilde{y}(0) d_{q,\omega} t, \tag{2.57}$$

$$D_{q,\omega}[\widetilde{y}](1) = -\int_0^1 \gamma(\widetilde{y}(1) - 1) d_{q,\omega} t. \tag{2.58}$$

As in Example 2.58, $\widetilde{y}(t) = \frac{1}{q+1}t^2 - (\frac{\omega}{q+1} - c)t + d$, where $c, d \in \mathbb{R}$, is a solution to equation (2.56). In order to determine c and d we use the natural boundary conditions (2.57) and (2.58). This gives

$$\begin{aligned} \widetilde{y}(t) = &\frac{1}{q+1}t^2 - \frac{\omega(\lambda+\gamma) - \lambda(\gamma-1)(q+1) + \gamma\lambda}{(q+1)(\gamma+\lambda\gamma+\lambda)}t \\ &+ \frac{(\gamma-1)(q+1) - \gamma(1-\omega)}{(q+1)(\gamma+\lambda\gamma+\lambda)} \end{aligned} \tag{2.59}$$

as a candidate to be a minimizer to (2.55). Moreover, since L is jointly convex, by Theorem 2.57 it is a global minimizer. The minimizer (2.59) is represented in Fig. 2.1 for fixed $\gamma = \lambda = 2$, $q = 0.99$ and different values of ω.

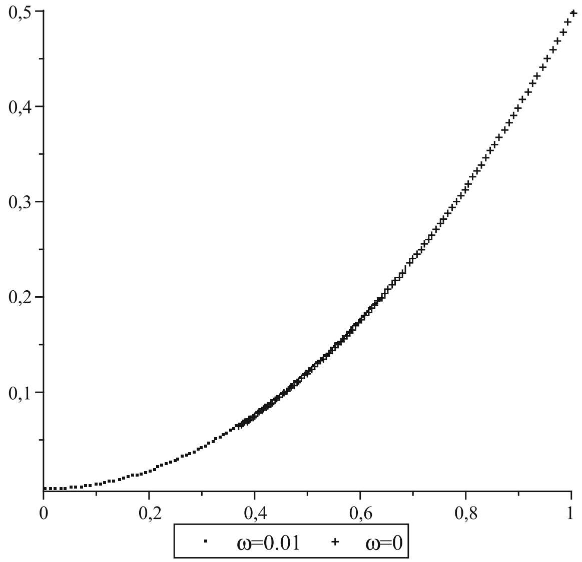

Fig. 2.1 The minimizer (2.59) of Example 2.59 for fixed $\gamma = \lambda = 2$, $q = 0.99$ and different values of ω

We note that in the limit, when $\gamma, \lambda \to +\infty$, $\widetilde{y}(t) = \frac{1}{q+1}t^2 + \frac{q}{q+1}t$ and coincides with the solution of the following problem with fixed initial and terminal points (see Example 2.32):

$$\mathcal{L}[y] = \int_0^1 \left(y(qt+\omega) + \frac{1}{2}(D_{q,\omega}[y](t))^2 \right) d_{q,\omega}t \longrightarrow \min$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1.$$

Expression $\gamma\frac{1}{2}(y(1) - 1)^2 + \lambda\frac{1}{2}y^2(0)$ added to the Lagrangian $y(qt + \omega) + \frac{1}{2}(D_{q,\omega}[y](t))^2$ works like a penalty function when γ and λ go to infinity. The penalty function itself grows, and forces the merit function (2.55) to increase in value when the constraints $y(0) = 0$ and $y(1) = 1$ are violated, and causes no growth when constraints are fulfilled. The minimizer (2.59) is represented in Fig. 2.2 for fixed $q = 0.5$, $\omega = 1$ and different values of γ and λ.

Remark 2.60 Let

$$\mathcal{L}[y] = \int_0^1 \left(y(qt+\omega) + \frac{1}{2}(D_{q,\omega}[y](t))^2 \right) d_{q,\omega}t$$

and

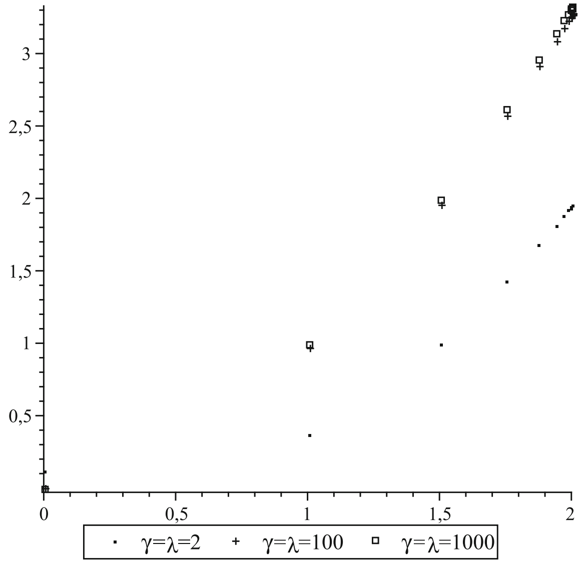

Fig. 2.2 The minimizer (2.59) of Example 2.59 for fixed $q = 0.5$, $\omega = 1$ and different values of γ and λ

$$\widetilde{y}_1(t) = \frac{1}{q+1}t^2 - \frac{\omega}{q+1}t + \frac{q+\omega}{q+1} \quad \text{and} \quad \widetilde{y}_2(t) = \frac{1}{q+1}t^2 + \frac{q}{q+1}t.$$

Comparing Example 2.58 and Example 2.59, we can conclude that

$$\mathcal{L}[\widetilde{y}_1] < \mathcal{L}[\widetilde{y}_2].$$

2.9 An Application Towards Economics

As the variables, that are usually considered and observed by the economist, are the outcome of a great number of decisions, taken by different operators at different points of time, it seems natural to look for new kinds of models which are more flexible and realistic. Hahn's approach allows for more complex applications than the discrete or the continuous models. A consumer might have income from work at unequal time intervals and/or make expenditures at unequal time intervals. Therefore, it is possible to obtain more rigorous and more accurate solutions with the approach here proposed.

In the first example we discuss the application of the Hahn quantum variational calculus to the Ramsey model, which determines the behavior of saving/consumption as the result of optimal inter-temporal choices by individual households (Atici and McMahan 2009). For a complete treatment of the classical Ramsey model we refer the reader to Barro and Sala-i-Martin (1999).

Example 2.61 Before writing the quantum model in terms of the Hahn operators we will present its discrete and continuous versions. The discrete-time Ramsey model is

$$\max_{[W_t]} \sum_{t=0}^{T-1} (1+p)^{-t} U\left[W_t - \frac{W_{t+1}}{1+r}\right], \quad C_t = W_t - \frac{W_{t+1}}{1+r},$$

while the continuous Ramsey model is

$$\max_{W(\cdot)} \int_0^T e^{-pt} U\left[rW(t) - W'(t)\right] dt, \quad C(t) = rW(t) - W'(t), \tag{2.60}$$

where the quantities are defined as

- W – production function,
- C – consumption,
- p – discount rate,
- r – rate of yield,
- U – instantaneous utility function.

One may assume, due to some constraints of economical nature, that the dynamics do not depend on the usual derivative or the forward difference operator, but on the

Hahn quantum difference operator $D_{q,\omega}$. In this condition, one is entitled to assume again that the constraint $C(t)$ has the form

$$C(t) = -\left[E\left(-r, \frac{t-\omega}{q}\right)\right]^{-1} D_{q,\omega}\left[E\left(-r, \frac{t-\omega}{q}\right) W(t)\right],$$

where $E(z, \cdot)$ is the q, ω-exponential function defined by

$$E(z, t) := \prod_{k=0}^{\infty}(1 + zq^k(t(1-q)-\omega))$$

for $z \in \mathbb{C}$. Several nice properties of the q, ω-exponential function can be found in Aldwoah (2009); Annaby et al. (2012). By taking the q, ω-derivative of $\left[E\left(-r, \frac{t-\omega}{q}\right) W(t)\right]$ the following is obtained:

$$C(t) = -\left[E\left(-r, \frac{t-\omega}{q}\right)\right]^{-1}\left[E\left(-r, \frac{t-\omega}{q}\right) D_{q,\omega} W(t)\right.$$
$$\left. + E\left(-r, \frac{t-\omega}{q}\right) W(qt+\omega)\frac{r\left(1-\frac{1}{q}\right) - r\left(1 + r\left(t - \frac{t-\omega}{q}\right)\right)}{\left(1 + r\left(t - \frac{t-\omega}{q}\right)\right)(1 - r(t(1-q)-\omega))}\right].$$

The quantum Ramsey model with the Hahn difference operator consists to maximize

$$\int_0^T E(-p,t) U\left[W(qt+\omega)\frac{r\left(1 + r\left(t - \frac{t-\omega}{q}\right)\right) - r\left(1 - \frac{1}{q}\right)}{\left(1 + r\left(t - \frac{t-\omega}{q}\right)\right)(1 - r(t(1-q)-\omega))}\right.$$
$$\left. - D_{q,\omega} W(t)\right] d_{q,\omega} \tag{2.61}$$

subject to the constraint

$$C(t) = W(qt+\omega)\frac{r\left(1 + r\left(t - \frac{t-\omega}{q}\right)\right) - r\left(1 - \frac{1}{q}\right)}{\left(1 + r\left(t - \frac{t-\omega}{q}\right)\right)(1 - r(t(1-q)-\omega))} - D_{q,\omega} W(t). \tag{2.62}$$

The quantum Euler–Lagrange equation is, by Theorem 2.21, given by

$$E(-p,t)U'[C(t)]\frac{r\left(1 + r\left(t - \frac{t-\omega}{q}\right)\right) - r\left(1 - \frac{1}{q}\right)}{\left(1 + r\left(t - \frac{t-\omega}{q}\right)\right)(1 - r(t(1-q)-\omega))}$$
$$+ D_{q,\omega}\left[E(-p,t)U'[C(t)]\right] = 0. \tag{2.63}$$

Note that for $q \uparrow 1$ and $\omega \downarrow 0$ problem (2.61)–(2.62) reduces to (2.60), and (2.63) to the classical Ramsey's Euler–Lagrange differential equation.

In the next example we analyze an adjustment model in economics. For a deeper discussion of this model we refer the reader to Sengupta (1997).

Example 2.62 Consider the dynamic model of adjustment

$$\mathcal{J}[y] = \sum_{t=1}^{T} r^t \left[\alpha(y(t) - \bar{y}(t))^2 + (y(t) - y(t-1))^2)\right] \longrightarrow \min,$$

where $y(t)$ is the output (state) variable, $r > 1$ is the exogenous rate of discount and $\bar{y}(t)$ is the desired target level, and T is the horizon. The first component of the loss function above is the disequilibrium cost due to deviations from desired target and the second component characterizes the agent's aversion to output fluctuations. In the continuous case the objective function has the form

$$\mathcal{J}[y] = \int_1^T e^{(r-1)t} \left[\alpha(y(t) - \bar{y}(t))^2 + (y'(t))^2\right] \longrightarrow \min.$$

Let $q \in]0, 1[$ and $\omega \geq 0$ be fixed real numbers, and I be an interval of $\mathbb{R}$ such that $\omega_0, 0, T \in I$. The quantum model in terms of the Hahn operators which we wish to minimize is

$$\mathcal{J}[y] = \int_0^T E(1-r, t)\left[\alpha(y(qt+\omega) - \bar{y}(qt+\omega))^2 + (D_{q,\omega}[y](t))^2\right] d_{q,\omega}t, \tag{2.64}$$

where $E(z, \cdot)$ is the q, ω-exponential function. By Theorem 2.51, a minimizer to (2.64) should satisfy the conditions

$$E(1-r, t)\left[\alpha(y(qt+\omega) - \bar{y}(qt+\omega))\right] = D_{q,\omega}\left[E(1-r, t)D_{q,\omega}[y](t)\right], \tag{2.65}$$

for all $t \in \{\omega[n]_q : n \in \mathbb{N}_0\} \cup \{Tq^n + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\}$; and

$$E(1-r, t)D_{q,\omega}[y](t)\Big|_{t=0} = 0, \quad E(1-r, t)D_{q,\omega}[y](t)\Big|_{t=T} = 0. \tag{2.66}$$

Taking the q, ω-derivative of the right side of (2.65) and applying properties of the q, ω-exponential function, for t such that $|t - \omega_0| < \frac{1}{(r-1)(1-q)}$, we can rewrite (2.65) and (2.66) as

$$\begin{aligned}[1 - (r-1)(t(1-q) - \omega)]\,\alpha(y(qt+\omega) - \bar{y}(qt+\omega))& \\ = (r-1)D_{q,\omega}[y](t) + D_{q,\omega}D_{q,\omega}[y](t),&\end{aligned} \tag{2.67}$$

$$D_{q,\omega}[y](t)\Big|_{t=0} = 0, \quad D_{q,\omega}[y](t)\Big|_{t=T} = 0. \tag{2.68}$$

Note that for $(q, \omega) \to (1, 0)$ equations (2.67) and (2.68) reduce to

$$\alpha(y(t) - \bar{y}(t)) = (r - 1)y'(t) + y''(t),$$

$$y'(t)\big|_{t=0} = 0, \quad y'(t)\big|_{t=T} = 0,$$

which are necessary optimality conditions for the continuous model.

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http://www.springer.com/978-3-319-02746-3

Quantum Variational Calculus
Malinowska, A.B.; Torres, D.F.M.
2014, XVIII, 84 p. 3 illus., Softcover
ISBN: 978-3-319-02746-3