Chapter 1
A Primer on Feller Semigroups and Feller Processes

Throughout this chapter, E denotes a locally compact and separable space; later on we will restrict ourselves to the Euclidean space $\mathbb{R}^d$ and its subsets. By $C_\infty(E)$ we denote the space of continuous functions $u : E \to \mathbb{R}$ which **vanish at infinity**, i.e.

$$\forall \epsilon > 0 \quad \exists K \subset E \text{ compact} \quad \forall x \in K^c \ : \ |u(x)| \leqslant \epsilon. \tag{1.1}$$

If $E = \mathbb{R}^d$, then (1.1) is the same as $\lim_{|x|\to\infty} u(x) = 0$; if $E = \mathbb{B}(z, r)$ is an open ball in $\mathbb{R}^d$, then (1.1) entails that (there is an extension of u such that) $u(x) = 0$ on the boundary $|x - z| = r$, and if E is a compact set, then $C_\infty(E) = C(E)$, i.e. the space of all continuous functions on E. Observe that

$$(C_\infty(E), \|\cdot\|_\infty), \quad \|u\|_\infty := \sup_{x\in E} |u(x)|, \tag{1.2}$$

is a Banach space, and the space of compactly supported continuous functions $C_c(E)$ is a dense subspace. If E is not compact, we can use the one-point compactification E_∂ by adding the point ∂. Since the complements of compact sets $K \subset E$ form a neighbourhood base of the point ∂ at infinity, we can identify $C_\infty(E)$ with $\{u \in C(E_\partial) : u(\partial) = 0\}$.

The topological dual $C_\infty^*(E)$ of $C_\infty(E)$ consists of the bounded signed Radon measures $\mathcal{M}_b(E)$, i.e. the signed Borel measures μ on E with finite total mass $|\mu|(E) < \infty$. A sequence $(u_n)_{n\geqslant 1} \subset C_\infty(E)$ **converges weakly** to $u \in C_\infty(E)$, if $\lim_{n\to\infty} \int u_n \, d\mu = \int u \, d\mu$ for all $\mu \in \mathcal{M}_b(E)$. Weak convergence is the same as **bp (bounded pointwise) convergence**

$$\text{bp-}\lim_{n\to\infty} u_n = u \iff \sup_{n\geqslant 1} \|u_n\|_\infty < \infty \quad \text{and} \quad \lim_{n\to\infty} u_n(x) = u(x) \quad \forall x \in E, \tag{1.3}$$

cf. Dunford–Schwartz [93, Corollary IV.6.4, p. 265]. Note that this only holds for sequences, cf. Ethier–Kurtz [100, Appendix 3, pp. 495–496].

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The norm topology on the Banach space $\mathcal{M}_b(E)$ is given by the total variation norm $\|\mu\|_{TV} := \mu^+(E) + \mu^-(E)$ where $\mu = \mu^+ - \mu^-$ is the Hahn–Jordan decomposition. More often, we use on $\mathcal{M}_b(E)$ the **weak-*** or **vague topology**, i.e.

$$\mu_n \xrightarrow[n\to\infty]{\text{vaguely}} \mu \iff \lim_{n\to\infty} \int u\,d\mu_n = \int u\,d\mu \quad \forall u \in C_\infty(E). \tag{1.4}$$

If, in addition, $\lim_{n\to\infty} \mu_n^\pm(E) = \mu^\pm(E)$, then one speaks of **weak convergence (of measures)**, i.e.

$$\mu_n \xrightarrow[n\to\infty]{\text{weakly}} \mu \iff \lim_{n\to\infty} \int u\,d\mu_n = \int u\,d\mu \quad \forall u \in C_b(E). \tag{1.5}$$

Mind that this is *not weak convergence in the topological sense*.

1.1 Feller Semigroups

There is no standard usage of the term **Feller semigroup** in the literature and *every author has his or her own definition of "Feller semigroup"* (Rogers and Williams [255, p. 241]). Therefore we take the opportunity to develop some of the core material in a consistent way.

Definition 1.1. Let $(T_t)_{t\geqslant 0}$ be a family of linear operators defined on the bounded Borel measurable functions $B_b(E)$. If

$$T_0 = \mathrm{id} \quad \text{and} \quad T_t T_s u = T_s T_t u = T_{t+s} u \quad \forall u \in B_b(E),\ s,t \geqslant 0$$

then $(T_t)_{t\geqslant 0}$ is said to be a **(one-parameter operator) semigroup**.

A **sub-Markov semigroup** is an operator semigroup $(T_t)_{t\geqslant 0}$ which is **positivity preserving**

$$T_t u \geqslant 0 \quad \forall u \in B_b(E),\ u \geqslant 0 \tag{1.6}$$

and has the **sub-Markov property**

$$T_t u \leqslant 1 \quad \forall u \in B_b(E),\ u \leqslant 1. \tag{1.7}$$

A **Markov semigroup** is a sub-Markov semigroup which is **conservative**, i.e. $T_t 1 = 1$.

Note that a sub-Markov semigroup is automatically **monotone**

$$T_t v \leqslant T_t w \quad \forall v, w \in B_b(E),\ v \leqslant w \tag{1.8}$$

(take $u = w - v$ in (1.6)), it satisfies **Jensen's inequality**

$$\phi(T_t u) \leqslant T_t \phi(u) \quad \forall u \in B_b(E) \text{ and any convex } \phi : \mathbb{R} \to \mathbb{R},\ \phi(0) = 0 \tag{1.9}$$

(observe that a convex function with $\phi(0) = 0$ is the upper envelope of affine-linear functions $\ell(x) = ax + b$ where $a \in \mathbb{R}$ and $b \leqslant 0$ such that $\ell(x) \leqslant u(x)$ for all x, and use $\ell(T_t u) \leqslant T_t \ell(u) \leqslant T_t \phi(u)$, see e.g. [283, Theorem 12.14, p. 116] for the standard proof) and it is **contractive**

$$\|T_t u\|_\infty \leqslant \|u\|_\infty \quad \forall u \in B_b(E) \tag{1.10}$$

(use $|T_t u| \leqslant T_t |u| \leqslant T_t \|u\|_\infty \leqslant \|u\|_\infty$).

Definition 1.2. A **Feller semigroup** is a sub-Markov semigroup $(T_t)_{t \geqslant 0}$ which satisfies the **Feller property**

$$T_t u \in C_\infty(E) \quad \forall u \in C_\infty(E),\ t > 0 \tag{1.11}$$

and which is **strongly continuous** in the Banach space $C_\infty(E)$

$$\lim_{t \to 0} \|T_t u - u\|_\infty = 0 \quad \forall u \in C_\infty(E). \tag{1.12}$$

One of the reasons to consider semigroups acting on spaces of continuous functions is the fact that such semigroups are integral operators with *pointwise everywhere* defined measure kernels, cf. the Riesz representation theorem, Theorem 1.5 below. This is particularly attractive for the study of stochastic processes where these kernels will serve as transition functions, cf. Sect. 1.2.

Example 1.3. Throughout the text we will use the following standard examples for Feller semigroups. For simplicity we consider only $E \subset \mathbb{R}^d$ and $u \in B_b(E)$.

a) (*Shift semigroup*) Let $\ell \in \mathbb{R}^d$. The shift semigroup is $T_t u(x) := u(x + t\ell)$, $t \geqslant 0$.
b) (*Poisson semigroup*) Let $\ell \in \mathbb{R}^d$ and $\lambda > 0$. The Poisson semigroup is defined as $T_t u(x) = \sum_{j=0}^\infty u(x + j\ell) \frac{(\lambda t)^j}{j!} e^{-t\lambda}$.
c) (*Heat/Brownian semigroup*) Let $g_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$ be the heat kernel or normal distribution (mean zero, variance t) on $\mathbb{R}^d$. The heat or Brownian semigroup is $T_t u(x) = \int_{\mathbb{R}^d} u(y) g_t(y - x)\, dy$.
d) (*Symmetric stable semigroups*) Let $g_{t,\alpha}(x)$, $\alpha \in (0, 2]$ be the symmetric stable probability density. It is implicitly defined through the characteristic function (inverse Fourier transform)[1] $\mathcal{F}^{-1}[g_{t,\alpha}](\xi) := \int_{\mathbb{R}^d} e^{ix\cdot\xi} g_{t,\alpha}(x)\, dx = e^{-t|\xi|^\alpha}$. The symmetric α-stable semigroup is $T_t u(x) := \int_{\mathbb{R}^d} u(y) g_{t,\alpha}(y - x)\, dy$.

[1]See the beginning of Chap. 2 for the conventions for the Fourier transform and characteristic functions.

Only for $\alpha = 1$ and 2 the densities $g_{t,\alpha}$ are explicitly known: $\alpha = 1$ yields the Cauchy density $g_{t,1}(x) = t\Gamma\left(\frac{d+1}{2}\right)\pi^{-\frac{d+1}{2}}\left(t^2 + |x|^2\right)^{-\frac{d+1}{2}}$, and $\alpha = 2$ gives the heat semigroup $g_{t,2}(x) = g_{2t}(x) = (4\pi t)^{-\frac{d}{2}} e^{-|x|^2/4t}$ with twice the normal speed.

e) (*Convolution/Lévy semigroups*) Let $(\mu_t)_{t\geqslant 0}$ be a family of infinitely divisible probability measures on $\mathbb{R}^d$, i.e. for every $n \geqslant 2$ we can write μ_t as an n-fold convolution of the measures $\mu_{t/n}$. Moreover, assume that $t \mapsto \mu_t$ is continuous in the vague topology.

Then $T_t u(x) := \int_{\mathbb{R}^d} u(x+y)\,\mu_t(dy)$ is a semigroup of convolution operators. We will discuss the structure of these semigroups in Sect. 2.1 below.

Note that all previously defined semigroups fall in this category. Because of the structure of these semigroups, the Feller property is easily seen using the dominated convergence theorem. Strong continuity follows from the vague continuity of the family $(\mu_t)_{t\geqslant 0}$, see also Berg–Forst [24, Chap. II.§12, pp. 85–97] or [284, Proposition 7.3, pp. 87–89] for a probabilistic proof for the heat semigroup which carries over to general convolution semigroups.

f) (*Ornstein–Uhlenbeck semigroup*) Let $(\mu_t)_{t\geqslant 0}$ be a family of infinitely divisible probability measures on $\mathbb{R}^d$ such that $t \mapsto \mu_t$ is continuous in the vague topology, and $B \in \mathbb{R}^{d\times d}$.

Then, $T_t u(x) := \int_{\mathbb{R}^d} u(e^{tB}x + y)\,\mu_t(dy)$ defines the so-called Ornstein–Uhlenbeck semigroup. Note that this is a special case of the Mehler semigroup, see e.g. Bogachev et al. [36]. The strong continuity and the Feller property of the Ornstein–Uhlenbeck semigroup was proved in Sato–Yamazato [268, Theorem 3.1].

Further examples are *generalized, Lévy-driven Ornstein–Uhlenbeck semigroups* which have been studied by Behme–Lindner [18], see also Examples 1.17(f) and 3.34(b) below for details.

g) (*One-sided stable semigroups*) On $E = [0,\infty)$ one defines the density $p_{t,\alpha}(x)$, $t, x \geqslant 0$, $0 < \alpha < 1$ through the Laplace transform $\int_0^\infty e^{-sx} p_{t,\alpha}(x)\,dx = e^{-ts^\alpha}$. Then $T_t u(x) := \int_0^\infty u(x+y) p_{t,\alpha}(y)\,dy$ is the one-sided α-stable semigroup.

The Lévy density $p_{t,1/2}(x) = (4\pi)^{-1/2}\, t\, x^{-3/2}\, e^{-t^2/4x}\mathbb{1}_{(0,\infty)}(x)$ is the only density in this family for which a closed-form expression is known.

h) (*Diffusion semigroups*) Consider a second order partial differential operator in divergence form $L = \frac{1}{2}\nabla\cdot(Q(\cdot)\nabla)$ where $Q : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is a measurable, symmetric matrix-valued function which is uniformly elliptic, i.e. there exist constants $0 < c \leqslant C < \infty$ such that

$$c|\xi|^2 \leqslant \langle Q(x)\xi, \xi\rangle \leqslant C|\xi|^2 \quad \forall \xi \in \mathbb{R}^d$$

and (for simplicity) $Q \in C_b^\infty(\mathbb{R}^d)$. It is well known that the initial value problem $\frac{d}{dt}u(t,x) = Lu(t,x)$, $u(0,x) = \phi(x)$ admits a fundamental solution $p(t,x,y)$ satisfying $p \in \bigcup_{n=1}^\infty C_b^\infty\big([1/n, n]\times\mathbb{R}^d\times\mathbb{R}^d; (0,\infty)\big)$. The fundamental solution leads to a Feller semigroup $u(t,x) = T_t\phi(x) = \int_{\mathbb{R}^d}\phi(y)p(t,x,y)\,dy$, cf. Itô [151, Chap. 1] or Stroock [310]. In general, the explicit expression of $p(t,x,y)$

is not known; however, we have Aronson's estimates which allow us to compare $p(t,x,y)$ from above and below with the well-known fundamental solution of the heat equation, i.e. where the differential operator is $\frac{1}{4}q_j\Delta$, $q_j>0$, $j=1,2$:

$$(q_1\pi t)^{-d/2}\exp\left(-\frac{|x-y|^2}{q_1 t}\right)\leqslant p(t,x,y)\leqslant(q_2\pi t)^{-d/2}\exp\left(-\frac{|x-y|^2}{q_2 t}\right)$$

with $q_j=q_j(c,C,d)$. This beautiful result is originally due to Aronson [6], see also Stroock [310].

i) (*Affine semigroups*) Consider on $E=\mathbb{R}_+^m\times\mathbb{R}^{d-m}$, $d\geqslant m\geqslant 0$, the semigroup $(T_t)_{t\geqslant 0}$ given by $T_t u(x)=\int_E u(y)p_t(x,dy)$ (with a suitable transition kernel $p_t(x,dy)$). Then $(T_t)_{t\geqslant 0}$ is called **affine**, if for every $t\in[0,\infty)$ the characteristic function (inverse Fourier transform) of the measure $p_t(x,\cdot)$ has exponential-affine dependence on x. In general, $p_t(x,\cdot)$ is not known explicitly; however, affine semigroups are characterized by the existence of functions

$$\phi:[0,\infty)\times i\mathbb{R}^d\to\mathbb{C}_-\quad\text{and}\quad\psi:[0,\infty)\times i\mathbb{R}^d\to\mathbb{C}_-^m\times i\mathbb{R}^{d-m}$$

(as usual, we write $\mathbb{C}_-=\{z\in\mathbb{C}:\operatorname{Re}z\leqslant 0\}$) such that for every $x\in E$ and for all $(t,\xi)\in[0,\infty)\times\mathbb{R}^d$

$$T_t e_\xi(x)=\int e^{iy\cdot\xi}p_t(x,dy)=e^{\phi(t,i\xi)+\sum_{j=1}^d x_j\psi_j(t,i\xi)}=e^{\phi(t,i\xi)+x\cdot\psi(t,i\xi)}$$

holds; in this generality, affine semigroups have been considered for the first time by Duffie–Filipović–Schachermayer [91, Sect. 2].

If the measures $p_s(x,\cdot)$ converge weakly (in the sense of measures) to $p_t(x,\cdot)$ as $s\to t$ for all $(t,x)\in[0,\infty)\times E$ or, equivalently, if the functions $\phi(t,i\xi)$ and $\psi(t,i\xi)$ are continuous in $t\in[0,\infty)$, for every $\xi\in\mathbb{R}^d$, then $(T_t)_{t\geqslant 0}$ is a Feller semigroup, cf. Keller-Ressel [175, Sect. 1.3, Theorem 1.1, p. 16] or Keller-Ressel–Schachermayer–Teichmann [176, Sect. 3, Theorem 3.5]. □

The Role of Strong Continuity. Using the linearity and contractivity (1.10) of a Feller semigroup, it is not hard to see that (1.12) is equivalent to

$$\lim_{s\to t}\|T_s u-T_t u\|_\infty=0\quad\forall u\in C_\infty(E),\ t\geqslant 0.\tag{1.12'}$$

In fact, we can even replace (1.12) by the notion of pointwise convergence.

Lemma 1.4. *Let $(T_t)_{t\geqslant 0}$ be a sub-Markov semigroup which satisfies the Feller property. Then each of the following conditions is equivalent to the strong continuity (1.12).*

$$\lim_{t\to 0}T_t u(x)=u(x)\quad\forall u\in C_\infty(E),\ x\in E;\tag{1.13}$$

$$[0,\infty)\times E \ni (t,x)\mapsto T_t u(x) \ \textit{ is continuous for each } u\in C_\infty(E); \tag{1.14}$$

$$[0,\infty)\times E\times C_\infty(E) \ni (t,x,u)\mapsto T_t u(x) \ \textit{ is continuous.} \tag{1.15}$$

Proof. It is enough to prove that (1.12) ⇒ (1.15) and (1.13) ⇒ (1.12). The first implication is a standard $\epsilon/3$-argument: Fix $(t,x,u)\in[0,\infty)\times E\times C_\infty(E)$ and pick any (s,y,v) in some $\epsilon/3$-neighbourhood. Then

$$\begin{aligned}|T_t u(x) - T_s v(y)| &\leqslant |T_t u(x) - T_t u(y)| + |T_t u(y) - T_s u(y)| + |T_s u(y) - T_s v(y)|\\ &\leqslant |T_t u(x) - T_t u(y)| + \|T_{|t-s|}u - u\|_\infty + \|u-v\|_\infty.\end{aligned}$$

The second implication is less trivial. We follow the proof given in Dellacherie–Meyer [84, Theorème XIII.19, pp. 98–99], see also Revuz–Yor [250, Proposition III.2.4, p. 89]. Let $u\in C_\infty(E)$. Clearly, (1.13) entails $\lim_{t\to s} T_t u(x) = T_s u(x)$ for all $s>0$ and $x\in E$. Therefore the integral

$$U_\alpha u(x) := \int_0^\infty e^{-\alpha s} T_s u(x)\,ds$$

defines a family of linear operators on $C_\infty(E)$ and, by dominated convergence and a simple change of variables, it is easy to see that $\lim_{\alpha\to\infty}(\alpha U_\alpha u(x) - u(x)) = 0$ for every $x\in E$. In fact, $(U_\alpha)_{\alpha>0}$ is a resolvent satisfying the resolvent equation

$$U_\alpha u - U_\beta u = (\beta-\alpha)U_\alpha U_\beta u \quad \forall \alpha,\beta>0. \tag{1.16}$$

Thus, the range $\mathcal{R} := U_\alpha C_\infty(E)$ does not depend on $\alpha>0$. Once again by dominated convergence, we see for any $\rho\in\mathcal{M}_b^+(E)$

$$\int u(x)\,\rho(dx) = \lim_{\alpha\to\infty}\int \alpha U_\alpha u(x)\,\rho(dx).$$

If ρ is orthogonal to $\mathcal{R}$, this equality shows that $\int u\,d\rho = 0$ for all $u\in C_\infty(E)$, hence $\rho = 0$. This proves that $\mathcal{R}$ is dense in $C_\infty(E)$. Now we can use Fubini's theorem to deduce

$$T_t U_\alpha u(x) = e^{\alpha t}\int_t^\infty e^{-\alpha s} T_s u(x)\,ds$$

as well as

$$\|T_t U_\alpha u - U_\alpha u\|_\infty \leqslant (e^{\alpha t}-1)\|U_\alpha u\|_\infty + t\|u\|_\infty.$$

This shows that $\lim_{t\to 0}\|T_t f - f\|_\infty$ for all $f\in\mathcal{R}$, and a standard density argument proves (1.12). □

Lemma 1.4 may be a bit surprising as it allows to replace uniform convergence by pointwise convergence. This is a variation of a theme from the theory of operator semigroups which says that for contraction semigroups the notions of continuity in the norm topology and in the weak topology coincide, see e.g. [78, Proposition 1.23, p. 15]. For Feller semigroups, weak continuity means that $t \mapsto \int T_t u(x)\, \rho(dx)$ is continuous for all $\rho \in \mathcal{M}_b(E)$. Since (sequential) weak convergence is the same as bounded pointwise convergence, it is indeed enough to check that the function $t \mapsto \int T_t u(x)\, \delta_y(dx) = T_t u(y)$ is continuous for each $y \in E$.

Feller Semigroups Defined on $C_\infty(E)$. Sometimes a strongly continuous, positivity preserving, conservative semigroup $(T_t)_{t \geqslant 0}$ with $T_t : C_\infty(E) \to C_\infty(E)$ is called a Feller semigroup—although it is only defined on $C_\infty(E)$. Using a variant of the Riesz representation theorem, cf. Rudin [258, Theorem 6.19, p. 130], we can extend $(T_t)_{t \geqslant 0}$ onto $B_b(E)$.

Theorem 1.5 (Riesz). *Let $T_t : C_\infty(E) \to C_\infty(E)$, $t > 0$, be a family of positivity preserving linear operators. Then T_t is an integral operator of the form*

$$T_t u(x) = \int u(y)\, p_t(x, dy) \tag{1.17}$$

where $p_t(x, \cdot)$ is a uniquely defined positive Radon measure.

It is not hard to see that $p_t(x, dy)$ is a sub-probability measure, if $T_t u \leqslant 1$ whenever $u \leqslant 1$. Moreover, $(t, x) \mapsto p_t(x, B)$ is for every $B \in \mathscr{B}(E)$ measurable: If $B = U$ is an open set, this follows immediately from (1.15) since we can approximate $\mathbb{1}_U$ by an increasing sequence of positive C_∞-functions. For general $B \in \mathscr{B}(E)$ we use a Dynkin system or monotone class argument. If $(T_t)_{t \geqslant 0}$ is a semigroup, the kernels $p_t(x, dy)$ satisfy the **Chapman–Kolmogorov equations**

$$p_{s+t}(x, B) = \int p_t(y, B)\, p_s(x, dy) \quad \forall B \in \mathscr{B}(E),\ s, t > 0. \tag{1.18}$$

This shows that every Feller semigroup defined on $C_\infty(E)$ can be uniquely extended to a sub-Markov semigroup in the sense of Definition 1.1, i.e. it becomes a Feller semigroup in the sense of Definition 1.2.

Other Feller Properties. As already mentioned, there is no uniform agreement on what a "Feller semigroup" should be. Usually the question is on which space the semigroup should be defined. Let us review some common alternative definitions and give them distinguishing names.

Definition 1.6. A sub-Markov semigroup $(T_t)_{t \geqslant 0}$ is called a **C_b-Feller semigroup** if it enjoys the **C_b-Feller property**, i.e.

$$T_t u \in C_b(E) \quad \forall u \in C_b(E),\ t > 0 \tag{1.19}$$

and if $t \mapsto T_t u$ is continuous in the topology of locally uniform convergence in the space $C_b(E)$.

If E is compact, the notions of Feller- and C_b-Feller semigroups coincide. The correct choice of topology on $C_b(E)$ is a major issue. Although $(C_b(E), \|\cdot\|_\infty)$ is a perfectly good Banach space, the requirement of strong continuity is so strong that only few semigroups enjoy this property.

Example 1.7. a) Let $T_t u(x) = u(x + t\ell)$ be the shift-semigroup on $\mathbb{R}^d$ (Example 1.3(a)). Since $T_t u(x) - u(x) = u(x + t\ell) - u(x)$, strong continuity of the shift semigroup entails that u is uniformly continuous. This means that $(T_t)_{t\geqslant 0}$ is not strongly continuous on $(C_b(\mathbb{R}^d), \|\cdot\|_\infty)$.

b) Let $T_t u(x)$ be the Poisson semigroup on $\mathbb{R}^d$ (Example 1.3(b)). Then

$$|T_t u(x) - u(x)| = \left| \sum_{j=1}^{\infty} \big(u(x + j\ell) - u(x)\big) \frac{(\lambda t)^j}{j!} e^{-\lambda t} \right| \leqslant 2\|u\|_\infty (1 - e^{-\lambda t})$$

shows that $(T_t)_{t\geqslant 0}$ is strongly continuous on $C_b(\mathbb{R}^d)$ (and even on $B_b(\mathbb{R}^d)$).

c) Denote by $(A, \mathfrak{D}(A))$ the generator of the Feller semigroup $(T_t)_{t\geqslant 0}$, cf. Sect. 1.4 below. If A is a bounded operator with respect to $\|\cdot\|_\infty$, then $(T_t)_{t\geqslant 0}$ is strongly continuous on $C_b(\mathbb{R}^d)$, and even on $B_b(\mathbb{R}^d)$. This follows from

$$|T_t u(x) - u(x)| = \left| \int_0^t A T_s u(x)\, ds \right| \leqslant t\|A\| \|u\|_\infty \quad \forall x \in \mathbb{R}^d,\ u \in C_\infty(\mathbb{R}^d)$$

cf. Lemma 1.26, and a standard extension argument for linear operators (the B.L.T. theorem, Reed–Simon [248, Theorem I.7, p. 9]). This shows that a (Feller) semigroup with bounded generator is continuous in the strong operator topology: $\|T_t - 1\| = \sup_{\|u\|_\infty \leqslant 1} \|T_t u - u\|_\infty \leqslant t\|A\|$. Conversely, any semigroup which is continuous in the strong operator topology has a bounded generator, cf. Pazy [236, Theorem 1.2, p. 2].

d) The heat semigroup (Example 1.3(c)) is not strongly continuous on $C_b(\mathbb{R})$. To see this, define a function $u \in C_b(\mathbb{R})$ by

$$u(x) := \sum_{n=2}^{\infty} u_n(x) \quad \text{and} \quad u_n(x) := \begin{cases} 0, & |x - n| \geqslant \frac{1}{n}, \\ n\big(x - n + \frac{1}{n}\big), & n - \frac{1}{n} < x \leqslant n, \\ n\big(n + \frac{1}{n} - x\big), & n \leqslant x < n + \frac{1}{n}. \end{cases}$$

Then we find for $x \in \mathbb{R}$, $t > 0$ and $\delta > 0$

$$|T_t u(x) - u(x)| \geqslant \left| \int_{|y|\leqslant\delta} (u(x + y) - u(x)) g_t(y)\, dy \right| - 2\|u\|_\infty \int_{|y|>\delta} g_t(y)\, dy.$$

Pick $x = n$ and $\delta = n^{-1}$. Then

$$\|T_t u - u\|_\infty \geqslant \int_{\frac{1}{2n} \leqslant |y| \leqslant \frac{1}{n}} (u(n) - u(n+y)) g_t(y)\, dy - 2 \int_{|y| > \frac{1}{n}} g_t(y)\, dy$$
$$\geqslant \frac{1}{2} \int_{\frac{1}{2n} \leqslant |y\sqrt{t}| \leqslant \frac{1}{n}} g_1(y)\, dy - 2 \int_{|y\sqrt{t}| > \frac{1}{n}} g_1(y)\, dy.$$

Now we use $t = t_n := (4n^2)^{-1}$ and write $\Phi(x) = \int_{-\infty}^{x} g_1(y)\, dy$ for the normal cumulative distribution function. Then

$$\overline{\lim_{t \to 0}} \|T_t u - u\|_\infty \geqslant \overline{\lim_{n \to \infty}} \|T_{t_n} u - u\|_\infty \geqslant \tfrac{5}{2}\Phi(2) - \tfrac{1}{2}\Phi(1) - 2 \geqslant 0.01.$$

A similar calculation shows that $(T_t)_{t \geqslant 0}$ is actually strongly continuous for all uniformly continuous functions u.

e) Let $(T_t)_{t \geqslant 0}$ be an affine semigroup (Example 1.3(i)) on $E = \mathbb{R}_+^m \times \mathbb{R}^{d-m}$ which is a Feller semigroup, i.e. $p_s(x, \cdot)$ converges weakly to $p_t(x, \cdot)$ as $s \to t$ for all $(t, x) \in [0, \infty) \times E$. Then $(T_t)_{t \geqslant 0}$ is also a C_b-Feller semigroup. According to Theorem 1.9 below, this follows from

$$T_t 1(x) = T_t e_0(x) = e^{\phi(t,0) + x \cdot \psi(t,0)},$$

which shows that $x \mapsto T_t 1(x)$ is continuous and bounded. □

If we replace uniform convergence by uniform convergence *on compact sets*, the restriction of a Feller semigroup to $C_b(E)$ will be continuous at $t = 0$, cf. [274, Lemma 3.1].

Lemma 1.8. *Let $(T_t)_{t \geqslant 0}$ be a Feller semigroup. Then $\lim_{t \to 0} T_t u(x) = u(x)$ locally uniformly in x for all $u \in C_b(E)$.*

The following criterion for a Feller semigroup to be a C_b-Feller semigroup is again taken from [274, Sect. 3].

Theorem 1.9. *Let $(T_t)_{t \geqslant 0}$ be a sub-Markov semigroup. Then*

$$\Big(T_t : C_\infty(E) \to C_\infty(E) \quad \textit{and} \quad T_t 1 \in C_b(E)\Big) \implies T_t : C_b(E) \to C_b(E).$$

In particular, if $(T_t)_{t \geqslant 0}$ is a Feller semigroup with $T_t 1 \in C_b(E)$, then it is also a C_b-Feller semigroup.

A necessary and sufficient condition that a C_b-Feller semigroup is a Feller semigroup is given in the next theorem.

Theorem 1.10. *Let $(T_t)_{t \geqslant 0}$ be a C_b-Feller semigroup and $(p_t(x, dy))_{t > 0}$ the transition kernels, i.e. for any $t > 0$, $x \in E$ and $u \in C_b(E)$, $T_t u(x) = \int u(y)\, p_t(x, dy)$.*

Then, $(T_t)_{t>0}$ is a Feller semigroup if, and only if, for all $t > 0$ and any increasing sequence of bounded sets $B_n \in \mathscr{B}(E)$ with $\bigcup_{n\geqslant 1} B_n = E$ we have

$$\lim_{|x|\to\infty} p_t(x, B_n) = 0 \quad \forall n \geqslant 1. \tag{1.20}$$

Proof. Since E is locally compact and separable, E is σ-compact, and there exists a sequence of bounded (even compact) sets B_n increasing towards E. Assume that $(T_t)_{t\geqslant 0}$ has the C_b-Feller property.

By the definition of $C_\infty(\mathbb{R}^d)$, there is for every $\epsilon > 0$ some $N(\epsilon)$ such that for all $n \geqslant N(\epsilon)$ we have $|u|\mathbb{1}_{E\setminus B_n} \leqslant \epsilon$. Thus, for $x \in E$,

$$\begin{aligned}|T_t u(x)| &\leqslant \int_{B_n} |u(y)|\, p_t(x, dy) + \int_{E\setminus B_n} |u(y)|\, p_t(x, dy)\\ &\leqslant \|u\|_\infty p_t(x, B_n) + \epsilon.\end{aligned}$$

Hence,

$$\lim_{|x|\to\infty} |T_t u(x)| \leqslant \|u\|_\infty \lim_{|x|\to\infty} p_t(x, B_n) + \epsilon = \epsilon.$$

Letting $\epsilon \to 0$ yields that $T_t u \in C_\infty(E)$. In order to see strong continuity on $C_\infty(E)$, we remark that $t \mapsto T_t u(x)$ is continuous for all $x \in E$ and $u \in C_\infty(E)$. Thus, by Lemma 1.4, we conclude that $(T_t)_{t\geqslant 0}$ is strongly continuous on $C_\infty(E)$, hence a Feller semigroup.

On the other hand, for any bounded set $B \in \mathscr{B}(E)$, there is some $u \in C_\infty(E)$ such that $u \geqslant 0$ and $u|_B \equiv 1$. Therefore,

$$T_t u(x) \geqslant \int_B u(y)\, p_t(x, dy) = p_t(x, B).$$

Since $(T_t)_{t\geqslant 0}$ is a Feller semigroup,

$$0 = \lim_{|x|\to\infty} |T_t u(x)| = \lim_{|x|\to\infty} T_t u(x) \geqslant \lim_{|x|\to\infty} p_t(x, B). \qquad \square$$

The criterion (1.20) ensuring the Feller property in Theorem 1.10 is not easy to check. If $E = \mathbb{R}^d$ we can use the structure of the infinitesimal generator to obtain a simpler condition; we postpone this to Theorem 2.49 in Sect. 2.5.

In potential theory one often requires the following strong Feller property.

Definition 1.11. A sub-Markov semigroup $(T_t)_{t\geqslant 0}$ is said to be a **strong Feller semigroup** if $T_t : B_b(E) \to C_b(E)$ for all $t > 0$.

Among other things, the strong Feller property ensures that α-excessive functions are lower semicontinuous, see Blumenthal–Getoor [33, (2.16), p. 77]; for a detailed discussion we also refer to Bliedtner–Hansen [30, Sect. V.3, pp. 175–184].

If $(T_t)_{t\geqslant 0}$ is a strong Feller semigroup, then the operators $T_t : B_b(E) \to C_b(E)$, $t > 0$, are compact if we equip $B_b(E)$ with the topology of uniform convergence and $C_b(E)$ with the topology of locally uniform convergence, cf. Revuz [249, Proposition 1.5.8, Theorem 1.5.9, p. 37] or [290, Proposition 2.3]. The following result is from Bliedtner–Hansen [30, Proposition 2.10, p. 181].

Lemma 1.12. *Let $(T_t)_{t\geqslant 0}$ be a sub-Markov semigroup on $B_b(E)$. Then the following assertions are equivalent.*

a) *$(T_t)_{t\geqslant 0}$ is a strong Feller semigroup and for every $t > 0$ and $u \in C_c(E)$ it holds that $\lim_{s\downarrow t} T_s u = T_t u$ locally uniformly.*
b) *For every $u \in B_b(E)$ the function $(t,x) \mapsto T_t u(x)$ is continuous on $(0,\infty)\times E$.*

Example 1.13. a) The shift and the Poisson semigroups [Examples 1.3(a) and 1.3(b)] are not strongly Feller.
b) A convolution semigroup (Example 1.3(e)) is strongly Feller if, and only if, the convolution kernel $\mu_t(dy)$ is absolutely continuous with respect to Lebesgue measure. This result is due to Hawkes [132, Lemma 2.1, p. 338], see also Jacob [157, Lemmas 4.8.19, 4.8.20, pp. 438–439]. □

A strong Feller semigroup need not be C_b-Feller nor Feller. Conversely, the strong Feller property does not follow from the (C_b-)Feller property without further conditions. Typically one has to assume some kind of (uniform) absolute continuity property or some ultracontractivity property. The following results are adapted from [290, Sects. 2.1 and 2.2].

Theorem 1.14. *Let $(T_t)_{t\geqslant 0}$ be a C_b-Feller semigroup with kernels $(p_t(x,dy))_{t\geqslant 0}$. Then the following assertions are equivalent.*

a) *$(T_t)_{t\geqslant 0}$ is a strong Feller semigroup.*
b) *There exists a probability measure $\mu \in \mathcal{M}^+(E)$ such that for every $t > 0$ the family $(p_t(x,dy))_{x\in E}$ is locally absolutely continuous with respect to μ, i.e. for any compact set $K \subset E$ it holds $\lim_{\delta\to 0} \sup_{B\in\mathscr{B}(E),\,\mu(B)\leqslant\delta} \sup_{z\in K} p_t(z,B) = 0$.*

In particular, if $(T_t)_{t\geqslant 0}$ is a C_b-Feller semigroup such that the representing kernels are of the form $p_t(x,dy) = p_t(x,y)\,\mu(dy)$ for some Radon measure $\mu \in \mathcal{M}^+(E)$ and a locally bounded density $(x,y) \mapsto p_t(x,y)$, then $(T_t)_{t\geqslant 0}$ is a strong Feller semigroup.

Another criterion is based on ultracontractivity. Hoh remarked in [138, Theorem 8.9, p. 134], see also Jacob–Hoh [140, Theorem 2.1], that a Feller semigroup on $C_\infty(\mathbb{R}^d)$ which is ultracontractive, i.e.

$$\|T_t u\|_\infty \leqslant c_t\,\|u\|_{L^2(dx)} \quad \forall t > 0,\ u \in C_c(\mathbb{R}^d),$$

is already a strong Feller semigroup.

Using Orlicz spaces we can obtain a necessary and sufficient condition. Let us recall some facts about Orlicz space from Rao–Ren [247]. A positive function $\Phi : \mathbb{R} \to [0,\infty]$ is a **Young function** if it is convex, even, satisfies $\Phi(0) = 0$ and

$\lim_{x\to\infty} \Phi(x) = \infty$. Given a Young function Φ and a Radon measure $\mu \in \mathcal{M}^+(E)$, we define the **Orlicz space** as

$$\mathbb{L}^\Phi(\mu) = \left\{ f \;:\; E \to \mathbb{R} \text{ measurable and } \int \Phi(\alpha f)\, d\mu < \infty \text{ for some } \alpha > 0 \right\}.$$

The set $\mathbb{L}^\Phi(\mu)$ is a linear space. If $\Phi(x) = |x|^p$, $p \geqslant 1$, then $\mathbb{L}^\Phi(\mu)$ coincides with the usual Lebesgue space $L^p(\mu)$. The **Orlicz norm**

$$\|f\|_\Phi = \sup\left\{ \int |fg|\, d\mu \;:\; \int \Phi_c(g)\, d\mu \leqslant 1,\; g \in B_b(E) \right\},$$

where Φ_c is the Legendre transform of Φ, i.e. $\Phi_c(y) := \sup_{x\geqslant 0} \big(x|y| - \Phi(x)\big)$, turns $\mathbb{L}^\Phi(\mu)$ into a Banach space. We have, cf. [290, Theorem 2.8],

Theorem 1.15. *Let $(T_t)_{t\geqslant 0}$ be a C_b-Feller semigroup. Then the following assertions are equivalent.*

a) *$(T_t)_{t\geqslant 0}$ is a strong Feller semigroup.*
b) *For every $t > 0$ there exists a Radon measure $\mu_t \in \mathcal{M}^+(E)$ and some Young function $\Phi_t : \mathbb{R} \to [0,\infty)$ which is strictly increasing on $[0,\infty)$ such that for all compact sets $K \subset E$ and $u \in C_c(E)$*

$$\|\mathbb{1}_K T_t u\|_\infty \leqslant C(K,t) \|u\|_{\Phi_t}. \tag{1.21}$$

Proof. This is a variant of [290, Theorem 2.8]. Note that the definition of a C_b-Feller semigroup includes the condition [290, Theorem 2.8 (2)]. In order to see that (b) entails (a), one only needs that $x = 0$ is the only zero of the Young functions Φ_t; this is clearly ensured by the strong monotonicity of Φ_t. The proof of the converse is based on the de la Vallée–Poussin characterization of uniform integrability, cf. [283, Theorem 16.8(vii), p. 170], and the argument in [283] allows us to take $\Phi_t(x)$ strictly increasing on $[0,\infty)$. □

One-Point Compactifications and Sub-Markovianity. The following technique allows us to restrict our attention to Markov semigroups, i.e. sub-Markov semigroups satisfying $T_t 1 = 1$. Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup which is not necessarily conservative. Denote by E_∂ the one-point compactification of E and define

$$T_t^\partial u := u(\partial) + T_t(u - u(\partial)) \quad \forall u \in C_\infty(E_\partial) = C_b(E_\partial). \tag{1.22}$$

Then $(T_t^\partial)_{t\geqslant 0}$ is a *conservative* Feller semigroup. Without problems we see that the new semigroup inherits all relevant properties from $(T_t)_{t\geqslant 0}$. Only the positivity is not so obvious. This can be seen by functional-analytic arguments as in Ethier–Kurtz [100, Lemma 4.2.3, p. 166]; alternatively let $(p_t(x,dy))_{t\geqslant 0}$ be the kernels from the

Riesz representation of $(T_t)_{t\geqslant 0}$, cf. Theorem 1.5. Then the corresponding kernels for $(T_t^\partial)_{t\geqslant 0}$ are given by

$$\begin{aligned} p_t^\partial(x,\{\partial\}) &:= 1 - p_t(x,E), && t \geqslant 0,\ x \in E, \\ p_t^\partial(\partial, B) &:= \delta_\partial(B), && t \geqslant 0,\ B \in \mathscr{B}(E_\partial), \\ p_t^\partial(x,B) &:= p_t(x,B), && t \geqslant 0,\ x \in E,\ B \in \mathscr{B}(E), \end{aligned} \tag{1.23}$$

and the positivity of each T_t^∂ follows.

This means that we can restrict ourselves to conservative semigroups, if needed.

1.2 From Feller Processes to Feller Semigroups—and Back

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and assume that $(X_t, \mathscr{F}_t)_{t\geqslant 0}$ is a time-homogeneous Markov process with state space $(E, \mathscr{B}(E))$. As usual, we denote by $\mathbb{P}^x$ and $\mathbb{E}^x$ the probability measures $\mathbb{P}(\cdot \mid X_0 = x)$ and the corresponding expectation, respectively. From the Markov property one easily sees that

$$\mathbb{E}^x u(X_t) := \int_E u(y)\ \mathbb{P}^x(X_t \in dy) \quad \forall u \in B_b(E),\ x \in E \tag{1.24}$$

defines a Markov semigroup.

We always require that $(X_t)_{t\geqslant 0}$ is **normal**, i.e. $\mathbb{P}^x(X_0 = x) = 1$ for all $x \in E$. Moreover, we assume for simplicity that the process has **infinite life-time**, i.e. $\mathbb{P}^x(X_t \in E) = 1$ for all $t > 0$ and $x \in E$, otherwise we would get a sub-Markov semigroup.

Definition 1.16. A **Feller process** is a time-homogeneous Markov process whose transition semigroup $T_t u(x) = \mathbb{E}^x u(X_t)$ is a Feller semigroup.

A function $p_t(x,B)$ defined on $[0,\infty) \times E \times \mathscr{B}(E)$ is a time-homogeneous **transition function** if

$$\begin{gathered} p_t(x,\cdot) \ \text{ is a (sub-)probability measure on } E,\ t \geqslant 0,\ x \in E, \\ p_0(x,\cdot) = \delta_x(\cdot), \quad x \in E, \\ p_\cdot(\cdot, B) \ \text{ is jointly measurable},\ B \in \mathscr{B}(E), \\ p_{t+s}(x,B) = \int p_s(y,B)\, p_t(x,dy), \quad s,t \geqslant 0,\ B \in \mathscr{B}(E). \end{gathered} \tag{1.25}$$

Clearly, $p_t(x,B) = \mathbb{P}^x(X_t \in B) = \mathbb{P}(X_t \in B \mid X_0 = x)$ is such a transition function.

Conversely, assume that we start with a Feller semigroup $(T_t)_{t\geqslant 0}$. Using the Riesz representation theorem, Theorem 1.5, we can write T_t as an integral operator

$$T_t u(x) = \int u(y)\, p_t(x, dy) \tag{1.26}$$

where the family of kernels $(p_t(x,\cdot))_{t \geqslant 0, x \in E}$ is a uniquely defined transition function.

Using Kolmogorov's standard procedure we can construct a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a Markov process $(X_t)_{t \geqslant 0}$ with state space E such that

$$\mathbb{P}^x(X_t \in B) = \mathbb{P}(X_t \in B \mid X_0 = x) = p_t(x, B) \quad \text{and} \quad \mathbb{E}^x\, u(X_t) = T_t u(x).$$

Example 1.17. The semigroups of Example 1.3 correspond to the following stochastic processes.

a) (*Shift semigroup*) $X_t = t\ell$ is a deterministic movement with speed $\ell \in \mathbb{R}^d$.
b) (*Poisson semigroup*) X_t is a Poisson process with intensity $\lambda > 0$ and jump height $\ell \in \mathbb{R}^d$: $\mathbb{P}(X_t = j\ell) = \frac{(\lambda t)^j}{j!}\, e^{-t\lambda}$, $j = 0, 1, 2, \dots$.
   $(X_t)_{t \geqslant 0}$ is spatially homogeneous, i.e. $\mathbb{P}^x(X_t \in B) = \mathbb{P}(X_t + x \in B)$.
c) (*Heat/Brownian semigroup*) X_t is a d-dimensional standard Brownian motion, $\mathbb{P}^x(X_t \in dy) = g_t(x - y)\, dy$. $(X_t)_{t \geqslant 0}$ is spatially homogeneous.
d) (*Symmetric stable semigroups*) X_t is a rotationally symmetric α-stable Lévy process, $\mathbb{P}^x(X_t \in dy) = g_{t,\alpha}(x - y)\, dy$. If $\alpha = 1$, we get the Cauchy process. $(X_t)_{t \geqslant 0}$ is spatially homogeneous.
e) (*Convolution/Lévy semigroups*) X_t is a **Lévy process**, i.e. a stochastic process with values in $\mathbb{R}^d$ and with the following properties:
   stationary increments: $X_t - X_s \sim X_{t-s}$ for $0 \leqslant s < t$;
   independent increments: $(X_{t_j} - X_{t_{j-1}})_{j=1}^n$, $0 \leqslant t_0 < \dots < t_n$ are independent random variables;
   stochastic continuity: $\lim_{h \to 0} \mathbb{P}(|X_h| > \epsilon) = 0$ for all $\epsilon > 0$.
   Note that the stationary increment property entails that $X_0 \sim \delta_0$ or $X_0 = 0$ a.s. Since the transition semigroup is a convolution operator, $(X_t)_{t \geqslant 0}$ is **spatially homogeneous**:

$$\begin{aligned}\mathbb{E}^x\, u(X_t) = T_t u(x) &= \int u(x + y)\, \mu_t(dy) = \int u(x + y)\, \mathbb{P}^0(X_t \in dy)\\ &= \mathbb{E}^0\, u(X_t + x),\end{aligned}$$

   i.e. $\mathbb{P}^x(X_t \in dy) = \mu_t(dy - x)$. All previously considered examples are Lévy processes.
f) (*Ornstein–Uhlenbeck semigroups*) Let $(Z_t)_{t \geqslant 0}$ be a Lévy process and $B \in \mathbb{R}^{d \times d}$. The process $X_t^x := e^{tB}x + \int_0^t e^{(t-s)B}\, dZ_s$, $x \in \mathbb{R}^d$, is a (Lévy-driven) Ornstein–Uhlenbeck process, which is the unique strong solution to the following stochastic differential equation:

$$dX_t = BX_t\, dt + dZ_t, \qquad X_0 = x \in \mathbb{R}^d.$$

A *generalized Ornstein–Uhlenbeck* process is the strong solution of the SDE

$$dV_t = V_{t-}\, dX_t^{(1)} + dX_t^{(2)}, \qquad V_0 = x \in \mathbb{R},$$

where $X_t = (X_t^{(1)}, X_t^{(2)})$, $t \geqslant 0$, is a two-dimensional Lévy process. Behme–Lindner [18, Theorem 3.1] show that $(V_t)_{t\geqslant 0}$ is a one-dimensional Feller process.

g) (*One-sided stable semigroups*) X_t is an α-stable subordinator, i.e. an increasing Lévy process with values in $[0,\infty)$.
h) (*Affine semigroups*) An affine process $(X_t)_{t\geqslant 0}$ is a Markov process that corresponds to the affine semigroup $(T_t)_{t\geqslant 0}$ of Example 1.3. Well-known examples are the Cox–Ingersoll–Ross process on $E = [0,\infty)$, the Ornstein–Uhlenbeck process on $E = \mathbb{R}^d$, the process of a Heston model on $E = [0,\infty) \times \mathbb{R}^d$ or the Wishart process on the more general state space of positive semidefinite d-dimensional matrices $E = S_d^+$. Note that the condition to be a Feller semigroup in Example 1.3 is equivalent to the stochastic continuity of the affine process, cf. (1.27). □

We have seen that Feller processes and Feller semigroups are in one-to-one correspondence. Clearly, the semigroup property is equivalent to the Chapman–Kolmogorov equations of the transition function, hence the Markov property of the Feller process. The strong continuity is linked to stochastic continuity of the process. Recall that a Markov process $(X_t)_{t\geqslant 0}$ is **stochastically continuous**, if

$$\lim_{t\to 0} \mathbb{P}^x(X_t \in E \setminus U_x) = 0 \quad \forall x \in E,\ \ U_x \text{ open neighbourhood of } x. \tag{1.27}$$

If (1.27) holds uniformly for all x (in compact sets) we speak of **(local) uniform stochastic continuity**. For example, any Lévy process (Example 1.17(e)) is uniformly stochastically continuous, see Dynkin [97, Chap. II.§5, 2.23, p. 77].

Lemma 1.18. *Let $(X_t)_{t\geqslant 0}$ be a (temporally homogeneous) Markov process and $(T_t)_{t\geqslant 0}$ be the corresponding Markov semigroup; assume that each T_t has the Feller property, i.e. $T_t : C_\infty(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)$. Then the strong continuity of $(T_t)_{t\geqslant 0}$ entails that $(X_t)_{t\geqslant 0}$ is (locally uniformly) stochastically continuous. Conversely, if $(X_t)_{t\geqslant 0}$ is stochastically continuous, the semigroup $(T_t)_{t\geqslant 0}$ is on the space $C_\infty(\mathbb{R}^d)$ weakly, hence strongly continuous.*

If $(X_t)_{t\geqslant 0}$ is a Feller process then we denote by $\mathscr{F}_t^X = \sigma(X_s : s \leqslant t)$ its natural filtration. Using some standard martingale regularization arguments one proves, see e.g. Revuz–Yor [250, Theorem III.2.7, p. 81],

Theorem 1.19. *Let $(X_t)_{t\geqslant 0}$ be a Feller process. Then it has a* ***càdlàg modification****, that is there exists a Feller process $(\tilde{X}_t)_{t\geqslant 0}$ such that $\mathbb{P}^x(X_t = \tilde{X}_t) = 1$ for all $t \geqslant 0$ and $x \in E$, and $t \mapsto \tilde{X}_t(\omega)$ is for almost all ω right-continuous with finite left-hand limits (càdlàg[2]).*

[2] càdlàg is the acronym for the French *continue à droite et limitée à gauche*.

In particular, any Lévy process (Example 1.17(e)) has a càdlàg modification.

Often the natural filtration is too small. Usually one considers the right-continuous filtration $\mathscr{F}_t = \mathscr{F}^X_{t+} := \bigcap_{\epsilon>0} \mathscr{F}^X_{t+\epsilon}$, cf. Ethier–Kurtz [100, Theorem 4.2.7, p. 169], or even larger universal augmentations, cf. Revuz–Yor [250, Proposition III.2.10, p. 93], which are automatically right-continuous. For a Lévy process it is enough to augment $\mathscr{F}^X_t$ by all $\mathbb{P}$ null sets to get a right-continuous filtration, see Protter [243, Theorem I.31, p. 22]. We define

$$\tilde{\mathscr{F}}_t := \bigcap_{\mu\in\mathcal{M}^+(E),\, \mu(E)=1} \sigma(\mathscr{F}_t, \mathcal{N}^\mu) \tag{1.28}$$

where $\mathcal{N}^\mu$ is the family of the null sets corresponding to the initial distribution μ. Then $(\tilde{\mathscr{F}}_t)_{t\geqslant 0}$ is a complete and right-continuous filtration [159, Theorem 3.5.10, p. 101] and we have the following extension of Theorem 1.19, see, for example, Jacob [159, Theorem 3.5.14, p. 104].

Theorem 1.20. *Let $(\tilde{X}_t)_{t\geqslant 0}$ be the càdlàg modification of a Feller process and $(\tilde{\mathscr{F}}_t)_{t\geqslant 0}$ be the filtration constructed in (1.28). Then $((\tilde{X}_t)_{t\geqslant 0}, (\tilde{\mathscr{F}}_t)_{t\geqslant 0})$ is a strong Markov process.*

1.3 Resolvents

Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup and denote by $(p_t(x,\cdot))_{t>0,x\in E}$ the transition function. By Fubini's theorem, the integral

$$U_\alpha u(x) := \int_0^\infty e^{-\alpha t}\, T_t u(x)\, dt, = \int_0^\infty\!\!\int e^{-\alpha t} u(y)\, p_t(x,dy)\, dt \tag{1.29}$$

exists for all $\alpha > 0$, $x \in E$, $u \in B_b(E)$, and is a linear map $U_\alpha : B_b(E) \to B_b(E)$.

Definition 1.21. Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup and $\alpha > 0$. The operator U_α given by (1.29) is the α**-potential operator** or **resolvent operator** at $\alpha > 0$.

If we interpret U_α as the (vector-valued) Laplace transform of the Feller semigroup $(T_t)_{t\geqslant 0}$, it is not surprising that there is a one-to-one relationship between $(U_\alpha)_{\alpha>0}$ and $(T_t)_{t\geqslant 0}$.

Theorem 1.22. *Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup. Then $(U_\alpha)_{\alpha>0}$ is a Feller contraction resolvent, i.e. for all $\alpha > 0$ the operators αU_α*

a) *are positivity preserving and sub-Markov: $0 \leqslant u \leqslant 1 \implies 0 \leqslant \alpha U_\alpha u \leqslant 1$;*
b) *satisfy the Feller property: $\alpha U_\alpha : C_\infty(E) \to C_\infty(E)$;*
c) *are strongly continuous on $C_\infty(E)$: $\lim_{\alpha\to\infty} \|\alpha U_\alpha u - u\|_\infty = 0$;*

d) *satisfy the* ***resolvent equation***

$$U_\alpha u - U_\beta u = (\beta - \alpha) U_\beta U_\alpha u \quad \forall \alpha, \beta > 0,\ u \in B_b(E); \tag{1.30}$$

e) *satisfy the* ***inversion formula*** *(or* ***exponential formula****)*

$$T_t u = \lim_{n\to\infty} \left[\tfrac{n}{t} U_{n/t}\right]^n u \quad \forall u \in C_\infty(E),\ \textit{(strong limit in } C_\infty(E)\textit{)}. \tag{1.31}$$

Proof. The properties (a)–(c) and (1.30) follow at once from the integral representation (1.29), see e.g. [284, Proposition 7.13, p. 97]. The inversion formula (1.31) is the vector-valued real Post–Widder inversion formula for the Laplace transform. The (non-trivial) proof can be found in Pazy [236, Theorem 1.8.3, p. 33]. □

Since the formula (1.31) holds for general contraction semigroups, we can use it to deduce the following result.

Corollary 1.23. *Let $(T_t)_{t\geqslant 0}$ be a contraction semigroup on $B_b(E)$ and $(U_\alpha)_{\alpha>0}$ the corresponding family of potential operators.*

a) *$(T_t)_{t\geqslant 0}$ is positivity preserving (sub-Markov, strongly continuous) if, and only if, $(\alpha U_\alpha)_{\alpha>0}$ is positivity preserving (sub-Markov, strongly continuous);*
b) *$T_t : C_\infty(E) \to C_\infty(E)$ for all $t \geqslant 0$ if, and only if, $U_\alpha : C_\infty(E) \to C_\infty(E)$ for all $\alpha > 0$.*

Note that the analogue of property (b) for C_b also holds, but it fails for the strong Feller property: The shift semigroup $T_t u(x) = u(x + t\ell)$ is not a strong Feller semigroup while its resolvent $U_\alpha u(x) = \int_0^\infty e^{-t\alpha} u(x + t\ell)\,dt$ is a convolution operator which maps $B_b(E)$ to $C_b(E)$.

1.4 Generators of Feller Semigroups and Processes

Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup. If we understand the semigroup property

$$T_{t+s} = T_t T_s \quad \text{and} \quad T_0 = \mathrm{id}$$

as an operator-valued functional equation it is an educated guess to expect that T_t is some kind of exponential e^{tA} where A is a suitable operator. For matrix (semi-)groups this is an elementary exercise. Having in mind the classical functional equation, we know that we have to assume some kind of boundedness and continuity; in fact, strong continuity of $(T_t)_{t\geqslant 0}$ will be enough. The key issue is the question how to define T_t as an “exponential” if A is an unbounded operator. This problem was independently solved by Hille and Yosida in 1948, and we refer to any text on operator semigroups for a complete description, for example [78, 236, 354] or [150] for a probabilistic perspective. Here we concentrate on Feller semigroups as in [100] or [284].

Definition 1.24. A **Feller generator** or **(infinitesimal) generator** of a Feller semigroup $(T_t)_{t\geqslant 0}$ or a Feller process $(X_t)_{t\geqslant 0}$ is a linear operator $(A, \mathcal{D}(A))$ defined by

$$\begin{aligned} \mathcal{D}(A) &:= \left\{ u \in C_\infty(E) \; : \; \lim_{t\to 0} \frac{T_t u - u}{t} \quad \text{exists as uniform limit} \right\}, \\ Au &:= \lim_{t\to 0} \frac{T_t u - u}{t} \quad \forall u \in \mathcal{D}(A). \end{aligned} \tag{1.32}$$

In general, $(A, \mathcal{D}(A))$ is an unbounded operator which is densely defined, i.e. $\mathcal{D}(A)$ is dense in $C_\infty(E)$, and **closed**

$$\left. \begin{array}{l} (u_n)_{n\geqslant 1} \subset \mathcal{D}(A), \ \lim_{n\to\infty} u_n = u, \\ (Au_n)_{n\geqslant 1} \text{ is a Cauchy sequence} \end{array} \right\} \implies \left\{ \begin{array}{l} u \in \mathcal{D}(A) \text{ and} \\ Au = \lim_{n\to\infty} Au_n. \end{array} \right. \tag{1.33}$$

Example 1.25. In general, it is difficult to determine the exact domain of the generator. For the semigroups from Example 1.3 we find

a) (*Shift semigroup*) $Au(x) = \ell \cdot \nabla u(x)$ where $\ell \in \mathbb{R}^d$ and ∇ is the d-dimensional gradient. We have $C^1_\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$.
b) (*Poisson semigroup*) $Au(x) = \lambda(u(x+\ell) - u(x))$ with $\lambda > 0, \ell \in \mathbb{R}^d$. Since this is a bounded operator, $\mathcal{D}(A) = C_\infty(\mathbb{R}^d)$.
c) (*Heat/Brownian semigroup*) $Au(x) = \frac{1}{2}\Delta u(x)$ where Δ is the d-dimensional Laplacian on $\mathbb{R}^d$. It is easy to see that $C^2_\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$. If $d = 1$, we have $C^2_\infty(\mathbb{R}) = \mathcal{D}(A)$ (cf. [284, Example 7.20, p. 102]). If $d > 2$, the inclusion is strict, cf. Günter [129, Chap. II.§14, pp. 82–83] or [128, Chap. II.§14, pp. 85–86] for a concrete example and Dautray–Lions [82, Remark 5, pp. 290–291] for an abstract argument. In general, $u \in \mathcal{D}(A)$ if, and only if, $u \in C_\infty(\mathbb{R}^d)$ and Δu exists in the sense of Schwartz' distributions (i.e. as generalized function) and is represented by a C_∞-function, cf. Itô [149, Sect. 3.§2, pp. 92–96].
d) (*Symmetric stable semigroups*) If $\alpha \in (0,2)$ then $Au(x) = -(-\Delta)^{\alpha/2}u(x)$. The fractional power of the Laplacian is, at least for $u \in C^2_\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$, given by

$$Au(x) = c_\alpha \int_{\mathbb{R}^d\setminus\{0\}} \big(u(x+y) - u(x) - \nabla u(x)\cdot y\chi(|y|)\big) \frac{dy}{|y|^{\alpha+d}} \tag{1.34}$$

where $c_\alpha = \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma\big(\frac{\alpha+d}{2}\big)\big/\Gamma\big(1-\frac{\alpha}{2}\big)$ and for some truncation function $\chi \in B_b[0,\infty)$ such that $0 \leqslant 1-\chi(s) \leqslant \kappa\min(s,1)$ (for some $\kappa > 0$) and $s\chi(s)$ is bounded.

Since $\lim_{\epsilon\to 0}\int_{|y|>\epsilon} y\chi(|y|)\,dy = 0$, we can rewrite (1.34) under the integral without $\nabla u(x)\cdot y\chi(|y|)$ as a Cauchy principal value integral, if $\alpha \in [1,2)$, and as a *bona fide* integral, if $\alpha \in (0,1)$.

To get more details on the domain $\mathcal{D}(A)$ we use for $\alpha \in (0,1)$ and a function $u \in C^2_\infty(\mathbb{R}^d)$ an alternative representation of the generator, see Sato [267, Example 32.7, p. 217]

$$Au(x) = c \int_{\mathbb{S}^d} \int_0^\infty \frac{u(x + r\gamma) - u(x)}{r^{1+\alpha}}\, dr\, \sigma_d(d\gamma) \tag{1.35}$$

where σ_d is the uniform measure on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^d$. For non-integer $\beta > 0$, let $C^\beta(\mathbb{R}^d)$ denote the Hölder space of $\lfloor\beta\rfloor$-times differentiable functions whose $\lfloor\beta\rfloor^{\text{th}}$ derivative is Hölder continuous with index $\beta - \lfloor\beta\rfloor$; as usual, $\|\cdot\|_{C^\beta}$ denotes the corresponding norm.

Splitting the inner integral in (1.35) yields for $u \in C^2_\infty(\mathbb{R}^d) \cap C^\beta(\mathbb{R}^d)$, $1 > \beta > \alpha$,

$$\|Au\|_\infty \leqslant c(\|u\|_{C^\beta} + \|u\|_\infty), \tag{1.36}$$

and this shows that we have $C_\infty(\mathbb{R}^d) \cap C^{\alpha+\epsilon}(\mathbb{R}^d) \subset \mathcal{D}(A)$ for all $\epsilon > 0$.

For $\alpha \geqslant 1$ a similar argument yields the same statement. See also [16, Remark 5.3] for an extension to stable-like processes in the sense of Bass.

e) (*Convolution/Lévy semigroups*) The generator of a general Lévy semigroup is, for $u \in C^2_\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$, of the form

$$Au(x) = l \cdot \nabla u(x) + \frac{1}{2}\operatorname{div} Q \nabla u(x) + \int\limits_{\mathbb{R}^d \setminus \{0\}} \big(u(x+y) - u(x) - \nabla u(x) \cdot y\, \chi(|y|)\big)\, \nu(dy) \tag{1.37}$$

where $l \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$ positive semidefinite and $\nu \in \mathcal{M}^+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d \setminus \{0\}} \min(|y|^2, 1)\, \nu(dy) < \infty$; χ is a truncation function as in the previous example. For a proof we refer to Sato [267, Theorem 31.5, p. 208] or to Theorem 2.21 and Corollary 2.22 below.

f) (*Ornstein–Uhlenbeck semigroups*) Let A be the operator given by (1.37), and let $B \in \mathbb{R}^{d \times d}$. Then, the generator of the Ornstein–Uhlenbeck semigroup is, for every $u \in C^2_\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$, of the form

$$Lu(x) = Au(x) + Bx \cdot \nabla u(x).$$

For a proof we refer to [268, Theorem 3.1].

The generator of the generalized Ornstein–Uhlenbeck semigroup, cf. Example 1.17(f), is defined for $u \in C^2_c(\mathbb{R})$ by

$$Au(x) = (\gamma_1 x + \gamma_2) u'(x) + \frac{1}{2}\big(x^2 \sigma_{11}^2 + 2x\sigma_{12} + \sigma_{22}^2\big) u''(x) +$$
$$+ \iint\limits_{\mathbb{R}^2 \setminus \{0\}} \big[u(x + y_1 x + y_2) - u(x) - u'(x)(y_1 x + y_2)\mathbb{1}_{[0,1]}(|x|)\big]\, \nu(dy_1, dy_2)$$

where $\left(\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}, \begin{pmatrix}\sigma_{11}^2 & \sigma_{12}\\ \sigma_{12} & \sigma_{22}^2\end{pmatrix}, \nu(dy_1, dy_2)\right)$ is the Lévy triplet of the driving Lévy process $X_t \in \mathbb{R}^2$.

Moreover, $\{u \in C_\infty^2(\mathbb{R}) : xu'(x),\ x^2u''(x) \in C_\infty(\mathbb{R})\} \subset \mathcal{D}(A)$ and the test functions $C_c^\infty(\mathbb{R}^d)$ are an operator core. For a proof we refer to Behme–Lindner [18, Theorem 3.1]. Using the technique of Theorem 3.8, in particular the remark following its statement, we can get a similar result with a different proof relying on the symbol, cf. Example 3.34(b).

g) (*Affine semigroups*) The generator of an affine semigroup exists if, and only if, $(T_t)_{t\geqslant 0}$ is *regular*, i.e. if $\frac{\partial^+}{\partial t} T_t e^{\xi\cdot x}|_{t=0}$ exists for all $(x,\xi) \in E \times (\mathbb{C}_-^m \times i\mathbb{R}^{d-m})$ and defines a function which is continuous at $\xi = 0$ for all $x \in E$; equivalently, $F(\xi) = \frac{\partial^+}{\partial t}\phi(t,\xi)|_{t=0}$ and $R(\xi) = \frac{\partial^+}{\partial t}\psi(t,\xi)|_{t=0}$ exist for all $\xi \in \mathbb{C}_-^m \times i\mathbb{R}^{d-m}$ and are continuous at $\xi = 0$. Regularity follows from the stochastic continuity of the corresponding affine process or from the condition for being Feller in Example 1.3, cf. Keller-Ressel–Schachermayer–Teichmann [176, Theorem 5.1] or [177, Theorem 3.10] for general state spaces. For $u \in C_c^2(E)$ the generator is given by

$$\begin{aligned}
Au(x) &= l\cdot\nabla u(x) + \frac{1}{2}\operatorname{div} Q\nabla u(x) + \int_{E\setminus\{0\}} \big(u(x+y) - u(x) - \nabla u(x)\cdot\chi(y)\big)\nu(dy)\\
&+ \sum_{j=1}^m x_j \left[\frac{1}{2}\operatorname{div} Q^j\nabla u(x) + \int_{E\setminus\{0\}} \big(u(x+y) - u(x) - \nabla u(x)\cdot\chi^j(y)\big)\,\nu^j(dy)\right]\\
&+ \sum_{j=1}^d x_j l^j\cdot\nabla u(x),
\end{aligned}$$

where $x = (x_1,\dots,x_d) \in E$, $l, l^j \in \mathbb{R}^d$ for $j = 1,\dots,d$, with $Q, Q^j \in \mathbb{R}^{d\times d}$ positive semidefinite matrices, $\nu, \nu^j \in \mathcal{M}^+(E)$ and χ, χ^j are truncation functions for $j = 1,\dots,m$. These parameters are subject to further restrictions, cf. Duffie–Filipović–Schachermayer [91, Definition 2.6, p. 991].

The domain of the generator is strictly larger than $C_c^2(E)$ as it contains all functions $u \in C_\infty^2(E)$ which satisfy certain decay conditions at infinity, cf. Duffie–Filipović–Schachermayer [91, Sect. 8, p. 1026] for details. □

By definition, the generator A is the strong (right-)derivative of T_t at $t = 0$. Using the semigroup property one can show the following analogue of the fundamental theorem of differential calculus.

Lemma 1.26. *Let $(A, \mathcal{D}(A))$ be the generator of the Feller semigroup $(T_t)_{t\geqslant 0}$. Then*

$$\begin{aligned} T_t u - u &= A \int_0^t T_s u \, ds \quad \forall u \in C_\infty(\mathbb{R}^d) \\ &= \int_0^t A T_s u \, ds \quad \forall u \in \mathcal{D}(A) \\ &= \int_0^t T_s A u \, ds \quad \forall u \in \mathcal{D}(A). \end{aligned} \tag{1.38}$$

A straightforward calculation using the formula (1.29) for the α-potential operator (e.g. [284, Theorem 7.13(f), pp. 97–98]) shows

Lemma 1.27. *Let $(A, \mathcal{D}(A))$ be a Feller generator. Then for each $\alpha > 0$ the operator $\alpha - A$ has a bounded inverse which is just the α-potential operator U_α. In particular, $\mathcal{D}(A) = U_\alpha(C_\infty(E))$ independently of $\alpha > 0$.*

In other words, the lemma shows that the equation

$$\alpha u - Au = f$$

has the solution $u = (\alpha - A)^{-1} f = U_\alpha f$. Therefore $(U_\alpha)_{\alpha>0}$ is called the resolvent.

Because of the positivity of a Feller semigroup we find for any $u \in \mathcal{D}(A)$ which admits a global maximum $u_{\max} = u(x_0) = \sup_{y \in E} u(y)$

$$T_t u(x_0) - u(x_0) \leqslant T_t u_{\max} - u(x_0) \leqslant u_{\max} - u(x_0) = 0.$$

This proves the first part of the following lemma. The second part can be found in [284, Lemma 7.18, p. 101] and [100, Lemma 1.2.11, p. 16].

Lemma 1.28. *A Feller generator $(A, \mathcal{D}(A))$ satisfies the* ***positive maximum principle***

$$u \in \mathcal{D}(A), \; u(x_0) = \sup_{y \in E} u(y) \geqslant 0 \implies Au(x_0) \leqslant 0. \tag{1.39}$$

Conversely, if the linear operator $(A, \mathcal{D})$, where $\mathcal{D} \subset C_\infty(E)$ is a dense subspace, satisfies the positive maximum principle, then $(A, \mathcal{D})$ is ***dissipative****, i.e.*

$$\|\lambda u - Au\|_\infty \geqslant \lambda \|u\|_\infty \quad \forall \lambda > 0. \tag{1.40}$$

In particular, $(A, \mathcal{D})$ has a closed extension $(A, \mathcal{D}(A))$, and this extension satisfies again the positive maximum principle.

For the Laplace operator (1.39) is quite familiar: At a (global) maximum the second derivative is negative.

Remark 1.29. The positive maximum principle can also be seen as the limiting case (for $p \to \infty$) of the notion of an $L^p(m)$-Dirichlet operator. Let $(A^{(p)}, \mathcal{D}(A^{(p)}))$ be the generator of a strongly continuous, sub-Markovian semigroup $(T_t^{(p)})_{t\geqslant 0}$ in $L^p(m)$ for some $p > 1$. Then $A^{(p)}$ is an $L^p(m)$-**Dirichlet operator**, i.e.

$$\int_E ((u(x)-1)^+)^{p-1} A^{(p)} u(x)\, m(dx) \leqslant 0 \quad \forall u \in \mathcal{D}(A) \tag{1.41}$$

and this condition is necessary and sufficient for the Markov property of the semigroup, cf. [281, Theorem 2.2]. If $A^{(p)}$ generates for every $p > p_0$ a sub-Markovian semigroup and if there is a sufficiently rich set $\mathcal{D} \subset \mathcal{D}(A^{(p)})$ (for all $p \geqslant p_0$) such that $A^{(p)}(\mathcal{D})$ consists of lower semicontinuous and bounded functions, then (1.41) becomes as $p \to \infty$ the positive maximum principle (1.39), cf. [281, Theorem 2.7]. The notion of a Dirichlet operator in L^2 is due to Bouleau–Hirsch [49], for the spaces L^p it was introduced by Jacob [157, Sect. 4.6, pp. 364–382]. □

If $E \subset \mathbb{R}^d$, the positive maximum principle will have consequences for the structure of the generator, cf. Theorem 2.21 below. For the time being, we are more interested in the consequences the positive maximum principle imposes upon the semigroup: It allows to adapt the classical Hille–Yosida theorem, see e.g. Ethier–Kurtz [100, Theorem 1.2.12, p. 16], to the context of Feller semigroups, cf. [100, Theorem 4.2.2, p. 165].

Theorem 1.30 (Hille–Yosida–Ray). *Let $(A, \mathcal{D})$ be a linear operator on $C_\infty(E)$. $(A, \mathcal{D})$ is closable and the closure $(A, \mathcal{D}(A))$ is the generator of a Feller semigroup if, and only if,*

a) *$\mathcal{D} \subset C_\infty(E)$ is dense;*
b) *$(A, \mathcal{D})$ satisfies the positive maximum principle;*
c) *$(\lambda - A)(\mathcal{D}) \subset C_\infty(E)$ is dense for some (or all) $\lambda > 0$.*

Proof. The necessity of the conditions (a) and (c) follows from the Hille–Yosida theorem, while condition (b) is the first half of Lemma 1.28.

By the second part of Lemma 1.28, the condition (b) shows that $(A, \mathcal{D})$ is dissipative. Then the Hille–Yosida theorem ensures that the closure of $(A, \mathcal{D})$ generates a strongly continuous contraction semigroup on $C_\infty(E)$. Using once again condition (b), we see now that the associated resolvent, hence the semigroup, is positive, cf. [284, Lemma 7.18, p. 101]. □

Usually it is a problem to describe the domain $\mathcal{D}(A)$ of a Feller generator. The following result, due to Reuter and Dynkin, is often helpful if we want to determine the domain of the generator; our formulation follows Rogers–Williams [255, Lemma III.4.17, p. 237] and [284, Theorem 7.15, p. 100].

Lemma 1.31 (Dynkin; Reuter). *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a Feller semigroup, and assume that $(G, \mathcal{D}(G))$, $\mathcal{D}(G) \subset C_\infty(E)$, extends $(A, \mathcal{D}(A))$, i.e. $\mathcal{D}(A) \subset \mathcal{D}(G)$ and $G|_{\mathcal{D}(A)} = A$. If for all $u \in \mathcal{D}(G)$*

$$Gu = u \implies u = 0, \tag{1.42}$$

then $(A, \mathcal{D}(A)) = (G, \mathcal{D}(G))$.

Since the positive maximum principle or dissipativity guarantee (1.42), Lemma 1.31 tells us that a Feller generator is maximally dissipative, i.e. that it has no proper dissipative extension.

Recall that the weak limit of a family $(u_t)_{t\geqslant 0} \subset C_\infty(E)$ is defined by

$$\text{weak-}\lim_{t\to 0} u_t = u \iff \forall \mu \in \mathcal{M}_b(E) : \quad \lim_{t\to 0} \int u_t \, d\mu = \int u \, d\mu.$$

Definition 1.32. Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup. The **pointwise (infinitesimal) generator** is a linear operator $(A_p, \mathcal{D}(A_p))$ defined by

$$\mathcal{D}(A_p) := \left\{ u \in C_\infty(E) \;\middle|\; \exists g \in C_\infty(E)\ \forall x \in E : g(x) = \lim_{t\to 0} \frac{T_t u(x) - u(x)}{t} \right\},$$

$$A_p u(x) := \lim_{t\to 0} \frac{T_t u(x) - u(x)}{t} \quad \forall u \in \mathcal{D}(A_p),\ x \in E. \tag{1.43}$$

The **weak (infinitesimal) generator** is a linear operator $(A_w, \mathcal{D}(A_w))$ defined by

$$\mathcal{D}(A_w) := \left\{ u \in C_\infty(E) \;\middle|\; \exists g \in C_\infty(E) : g = \text{weak-}\lim_{t\to 0} \frac{T_t u - u}{t} \right\},$$

$$A_w u := \text{weak-}\lim_{t\to 0} \frac{T_t u - u}{t} \quad \forall u \in \mathcal{D}(A_w). \tag{1.44}$$

Since strong convergence entails weak convergence and since weak convergence in $C_\infty(E)$ is actually bounded pointwise convergence, it is not hard to see that

$$\mathcal{D}(A) \subset \mathcal{D}(A_w) \subset \mathcal{D}(A_p) \quad \text{and} \quad A = A_w|_{\mathcal{D}(A)} = A_p|_{\mathcal{D}(A)}.$$

From the theory of operator semigroups we know that $(A, \mathcal{D}(A)) = (A_w, \mathcal{D}(A_w))$, cf. Pazy [236, Theorem 2.1.3, p. 43], and we even get $(A, \mathcal{D}(A)) = (A_p, \mathcal{D}(A_p))$ if we use Davies' proof of the "weak equals strong" theorem [78, Theorem 1.24, p. 17] and the fact that finite linear combinations of Dirac measures are vaguely (i.e. weak-*) dense in $\mathcal{M}_b(E)$.

Alternatively, we can use the positive maximum principle. Clearly $(A_p, \mathcal{D}(A_p))$ extends $(A, \mathcal{D}(A))$, and A_p satisfies the positive maximum principle. By (the analogue of) Lemma 1.28 we see that A_p is dissipative and from Lemma 1.31 we conclude that $\mathcal{D}(A) = \mathcal{D}(A_p)$. This proves the following result.

Theorem 1.33. *Let* $(T_t)_{t\geqslant 0}$ *be a Feller semigroup generated by* $(A, \mathcal{D}(A))$. *Then*

$$\mathcal{D}(A) = \left\{ u \in C_\infty(E) \;\middle|\; \exists g \in C_\infty(E)\ \forall x \in E \,:\, g(x) = \lim_{t\to 0} \frac{T_t u(x) - u(x)}{t} \right\}. \tag{1.45}$$

In particular, $(A, \mathcal{D}(A)) = (A_w, \mathcal{D}(A_w)) = (A_p, \mathcal{D}(A_p))$.

Operator Cores. Let $(A, \mathcal{D}(A))$ be a densely defined, closed linear operator and $\mathcal{D} \subset \mathcal{D}(A)$ be a dense subset. If $\mathcal{D}$ determines A in the sense that the closure of $(A, \mathcal{D})$ is $(A, \mathcal{D}(A))$, then $\mathcal{D}$ is called an **(operator) core**. In other words, $\mathcal{D}$ is an operator core if, and only if,

$$\forall u \in \mathcal{D}(A) \quad \exists (u_n)_{n\geqslant 1} \subset \mathcal{D} \,:\, \lim_{n\to\infty} \left(\|u - u_n\|_\infty + \|Au - Au_n\|_\infty \right) = 0. \tag{1.46}$$

Usually it is hard to determine operator cores, and the following abstract criterion often comes in handy.

Lemma 1.34. *Let* $(T_t)_{t\geqslant 0}$ *be a Feller semigroup,* $(U_\alpha)_{\alpha>0}$ *the resolvent,* $(A, \mathcal{D}(A))$ *the generator and* $\mathcal{D}_0 \subset \mathcal{D} \subset \mathcal{D}(A)$ *dense subsets of* $C_\infty(E)$. *Then* $\mathcal{D}$ *is an operator core for* $(A, \mathcal{D}(A))$ *if one of the following conditions is satisfied.*

a) $T_t(\mathcal{D}_0) \subset \mathcal{D}$ *for all* $t > 0$.
b) $U_\alpha(\mathcal{D}_0) \subset \mathcal{D}$ *for some* $\alpha > 0$.

Proof. The first condition is a standard result from semigroup theory, see e.g. [100, Proposition 1.3.3, p. 17] or Davies [78, Theorem 1.9, p. 8]. The following simple proof for the second condition is taken from [9, proof of Theorem 4.4]. Fix any $u \in \mathcal{D}(A)$ and set $g := \alpha u - Au$. Since $\mathcal{D}_0$ is dense in $C_\infty(E)$, there exists a sequence $(g_n)_{n\geqslant 1} \subset \mathcal{D}_0$ converging to g. Then, for $u_n := U_\alpha g_n$,

$$(u_n,\ Au_n) = (U_\alpha g_n,\ \alpha u_n - g_n) \xrightarrow[n\to\infty]{\text{uniformly}} (u,\ \alpha u - g) = (u,\ Au).$$

By assumption, $u_n \in U_\alpha(\mathcal{D}_0) \subset \mathcal{D}$, and this shows that $\mathcal{D}$ is a core. □

The Full Generator. Sometimes it is useful to extend the notion of a generator even further. The starting point is the observation that $\frac{d}{dt} T_t = T_t A$ on $\mathcal{D}(A)$ or

$$T_t u(x) - u(x) = \int_0^t T_s A u(x)\, ds \quad \forall x \in E,\ u \in \mathcal{D}(A) \tag{1.47}$$

see Lemma 1.26. This motivates the following definition, cf. Ethier–Kurtz [100, pp. 23–24].

Definition 1.35. Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup. The **full generator** is the set

$$\hat{A}_b := \left\{(f,g) \in B_b(E)\times B_b(E) \;:\; T_t f - f = \int_0^t T_s g\,ds\right\}. \tag{1.48}$$

By (1.47), $\{(u, Au) : u \in \mathcal{D}(A)\} \subset \hat{A}_b$. Observe that the full generator need not be single-valued, i.e. for any $f \in B_b(E)$ there may be more than one $g \in B_b(E)$ such that $(f,g) \in \hat{A}_b$: For the shift semigroup $T_t f(x) := f(x+t)$ on $B_b(\mathbb{R})$ one has $(0,g) \in \hat{A}_b$ for each $g \in B_b(\mathbb{R})$ which is Lebesgue almost everywhere zero. The full generator is linear, dissipative and closed with respect to bounded pointwise limits $\text{bp-}\lim_{n\to\infty}(f_n, g_n) = (f,g)$. A thorough discussion of the full generator can be found in Ethier–Kurtz [100, Sect. 1.5, pp. 22–28]. The full generator is most useful in connection with the martingale problem. At this point we restrict ourselves to the following fact, cf. [100, Proposition 4.17, p. 162].

Theorem 1.36. *Let $(X_t)_{t\geqslant 0}$ be a Feller process (or a Markov process) with full generator $\hat{A}_b$. Then*

$$M_t := f(X_t) - f(X_0) - \int_0^t g(X_s)\,ds \quad \forall (f,g) \in \hat{A}_b \tag{1.49}$$

is a martingale with respect to the natural filtration $\mathscr{F}_t^X := \sigma(X_s : s \leqslant t)$.

Taking expectations in (1.49), we see that $\mathbb{E}^x M_t = 0$ for all $x \in E$; this is just (1.47), if $f = u \in \mathcal{D}(A)$ and $g = Au$.

Theorem 1.36 allows a stochastic characterization of the full generator $\hat{A}_b$.

Corollary 1.37. *Let $(X_t)_{t\geqslant 0}$ be a Feller process (or a strong Markov process) with full generator $\hat{A}_b$, denote by*

$$M_t^{[f,g]} = f(X_t) - f(X_0) - \int_0^t g(X_s)\,ds, \quad t \geqslant 0,$$

and write $\mathscr{F}_t^X = \sigma(X_s : s \leqslant t)$ for the natural filtration of the process $(X_t)_{t\geqslant 0}$. Then

$$\hat{A}_b = \left\{(f,g) \in B_b(E)\times B_b(E) \;:\; \left(M_t^{[f,g]}, \mathscr{F}_t^X\right)_{t\geqslant 0} \text{ is a martingale}\right\}.$$

Sometimes it is important to consider unbounded measurable functions f, g. While it is, in general, not clear how to define $T_t f = \mathbb{E}^x f(X_t)$ for an *unbounded* function f, the expression $f(X_t)$ is well-defined, and the stochastic version of $\hat{A}_b$ can be extended to this situation. We set

$$\hat{A} = \left\{(f,g) \in B(E)\times B(E) \;:\; \left(M_t^{[f,g]}, \mathscr{F}_t^X\right)_{t\geqslant 0} \text{ is a local martingale}\right\} \tag{1.50}$$

and, by a stopping argument and the strong Markov property of $(X_t)_{t\geqslant 0}$, we see that $\hat{A}_b = \hat{A} \cap (B_b(E)\times B_b(E))$.

Note that the full generator $\hat{A}$ need not be single-valued, i.e. there might be two (or more) functions $g_1 \neq g_2$ such that $(f, g_1), (f, g_2) \in \hat{A}$. Therefore we avoid the notion of **extended generator** which is sometimes found in the literature, e.g. Davis [80, (14.15), p. 32] or Meyn–Tweedie [226]

$$\hat{\mathfrak{D}}(A) := \big\{ f \in B(E) \;:\; \exists!\, g \in B(E),\ (f, g) \in \hat{A} \big\}. \tag{1.51}$$

Dynkin's Characteristic Operator. The following extension is due to Dynkin, see [97, Chap. V.§§3–4, pp. 140–149], our presentation follows [284, Sect. 7.5, pp. 103–109]. Let $(X_t)_{t\geqslant 0}$ be a Feller process, denote by $\mathscr{F}_t^X = \sigma(X_s \;:\; s \leqslant t)$ the natural filtration, and by

$$\tau_r^x := \inf\big\{t > 0 \;:\; X_t \in \overline{\mathbb{B}}^c(x, r)\big\}, \quad r \geqslant 0,\ x \in E \tag{1.52}$$

the first hitting time of the open set $E \setminus \overline{\mathbb{B}}(x, r)$ (this is always a stopping time for $\mathscr{F}_{t+}^X$). Note that

$$\tau_0^x = \inf\{t > 0 \;:\; X_t \neq x\}, \quad x \in E.$$

Using the strong Markov property of a Feller process one can show, cf. [284, Theorem A.26, p. 350], that

$$\mathbb{P}^x(\tau_0^x \geqslant t) = e^{-\lambda(x)t} \quad \text{for some} \quad \lambda(x) \in [0, \infty],\ x \in E.$$

This allows us to characterize points in the state space:

$$x \in E \;\text{ is called }\; \begin{cases} \text{an } \textbf{exponential holding point}, & \text{if } 0 < \lambda(x) < \infty, \\ \text{an } \textbf{instantaneous point}, & \text{if } \lambda(x) = \infty, \\ \text{an } \textbf{absorbing point} \text{ or a } \textbf{trap}, & \text{if } \lambda(x) = 0. \end{cases} \tag{1.53}$$

If x is not absorbing, then there is some $r > 0$ such that $\mathbb{E}^x \tau_r^x < \infty$, and the following definition makes sense.

Definition 1.38. Let $(X_t)_{t\geqslant 0}$ be a Feller process and denote by τ_r^x the first hitting time of the set $\overline{\mathbb{B}}^c(x, r)$. Dynkin's **characteristic operator** is the linear operator defined by

$$\mathfrak{A}u(x) := \begin{cases} \displaystyle\lim_{r\to 0} \frac{\mathbb{E}^x u(X_{\tau_r^x}) - u(x)}{\mathbb{E}^x \tau_r^x}, & \text{if } x \text{ is not absorbing}, \\ 0, & \text{if } x \text{ is absorbing}, \end{cases} \tag{1.54}$$

on the set $\mathfrak{D}(\mathfrak{A})$ consisting of all $u \in B_b(E)$ such that the limit in (1.54) exists for each non-absorbing point $x \in E$.

From (1.49) and the optional stopping theorem for martingales we easily derive **Dynkin's formula**

$$\mathbb{E}^x u(X_\sigma) - u(x) = \mathbb{E}^x \int_0^\sigma Au(X_s)\,ds, \quad u \in \mathcal{D}(A) \tag{1.55}$$

where σ is a stopping time such that $\mathbb{E}^x \sigma < \infty$. If we use $\sigma = \tau_r^x$, this formula allows us to show that the characteristic operator $(\mathfrak{A}, \mathcal{D}(\mathfrak{A}))$ extends the generator $(A, \mathcal{D}(A))$, see Dynkin [97, Chap. V.§3, Theorem 5.5, pp. 142–143] or [284, Theorem 7.26, p. 107].

Theorem 1.39. *Let $(X_t)_{t\geqslant 0}$ be a Feller process with generator $(A, \mathcal{D}(A))$ and characteristic operator $(\mathfrak{A}, \mathcal{D}(\mathfrak{A}))$. Then $\mathfrak{A}$ is an extension of A and $\mathfrak{A}|_{\mathcal{D}} = A$ where $\mathcal{D} = \{u \in \mathcal{D}(\mathfrak{A}) \cap C_\infty(E) : \mathfrak{A}u \in C_\infty(E)\}$.*

As an application of the characteristic operator we can characterize the structure of the generators of Feller processes in $\mathbb{R}^d$ with continuous sample paths. A linear operator $L : \mathcal{D}(L) \subset B_b(\mathbb{R}^d) \to B_b(\mathbb{R}^d)$ is called **local**, if $Lu(x) = Lw(x)$ whenever $u, w \in \mathcal{D}(L)$ coincide in some neighbourhood of the point x, i.e. $u|_{\mathbb{B}(x,\epsilon)} = w|_{\mathbb{B}(x,\epsilon)}$.

Theorem 1.40. *Let $(X_t)_{t\geqslant 0}$ be a Feller process with values in $\mathbb{R}^d$ and continuous sample paths. Then the generator $(A, \mathcal{D}(A))$ is a local operator.*

If the test functions $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$ are in the domain of the generator we can use a result due to Peetre [237] to see that local operators are differential operators.

Theorem 1.41 (Peetre). *Let $L : C_c^\infty(\mathbb{R}^d) \to C_c^k(\mathbb{R}^d)$ be a linear operator where $k \geqslant 0$ is fixed. If $\operatorname{supp} Lu \subset \operatorname{supp} u$, then $Lu = \sum_{\alpha\in\mathbb{N}_0^d} a_\alpha(\cdot)\frac{\partial^{|\alpha|}}{\partial x^\alpha}u$ with finitely many, uniquely determined distributions $a_\alpha \in \mathcal{D}'(\mathbb{R}^d)$ (i.e. the topological dual of $C_c^\infty(\mathbb{R}^d)$) which are locally represented by functions of class $C^k(\mathbb{R}^d)$.*

If we apply this to Feller generators, we get

Corollary 1.42. *Let $(X_t)_{t\geqslant 0}$ be a Feller process with continuous sample paths and generator $(A, \mathcal{D}(A))$. If $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(A)$, then A is a second-order differential operator.*

Proof. By Theorem 1.41, A is a differential operator. Since A has to satisfy the positive maximum principle, A is at most a *second order differential operator*. This follows from the fact that we can find test functions $\phi \in C_c^\infty(\mathbb{R}^d)$ such that x_0 is a global maximum while $\partial_j\partial_k\partial_l\phi(x_0)$ has arbitrary sign and arbitrary modulus. □

Recall the so-called **Dynkin–Kinney criterion** which guarantees the continuity of the trajectories of a stochastic process:

$$\forall\epsilon > 0,\ r > 0 : \lim_{h\to 0}\sup_{t\leqslant h}\sup_{|x|\leqslant r}\frac{1}{h}\mathbb{P}^x(r > |X_t - x| > \epsilon) = 0, \tag{1.56}$$

see Dynkin [96, Kapitel 6.§5, Satz 6.6, p. 139].[3] In fact, the Dynkin–Kinney criterion (1.56) is equivalent to the locality of the generator, cf. [284, Theorem 7.30, p. 108].

Corollary 1.43. *Let* $(X_t)_{t\geqslant 0}$ *be a Feller process such that the test functions* $C_c^\infty(\mathbb{R}^d)$ *are in the domain of the generator* $(A, \mathcal{D}(A))$*. Then* $(A, C_c^\infty(\mathbb{R}^d))$ *is a local operator if, and only if, the Dynkin–Kinney criterion* (1.56) *holds.*

1.5 Feller Semigroups and L^p-Spaces

A Feller semigroup $(T_t)_{t\geqslant 0}$ is, *a priori*, defined on the bounded measurable functions $B_b(E)$ or the continuous functions vanishing at infinity $C_\infty(E)$. We will briefly discuss some standard situations which allow to extend $T_t|_{C_c(E)}$ onto a space of integrable functions. Throughout this section we assume that $(E, \mathscr{B}(E), m)$ is a measure space such that the m is a positive Radon measure with full topological support, i.e. for any open set $U \subset E$ we have $m(U) > 0$.

We assume that the operators T_t are ***m*-symmetric** in the following sense

$$\int_E T_t u(x) \cdot w(x)\, m(dx) = \int_E u(x) \cdot T_t w(x)\, m(dx) \quad \forall u, w \in C_c(E). \tag{1.57}$$

If $|w| \leqslant 1$, we have $|T_t w| \leqslant T_t|w| \leqslant T_t 1 \leqslant 1$, and so

$$\|T_t u\|_{L^1(m)} = \sup_{\substack{w\in C_c(E)\\ |w|\leqslant 1}} \left| \int_E T_t u \cdot w\, dm \right| \leqslant \sup_{\substack{w\in C_c(E)\\ |w|\leqslant 1}} \int_E |u| \cdot |T_t w|\, dm \leqslant \|u\|_{L^1(m)}.$$

Since $C_c(E)$ is dense in $L^1(m)$ this shows that $T_t|_{C_c(E)}$ has an extension $T_t^{(1)}$ such that $T_t^{(1)} : L^1(m) \to L^1(m)$ is a contraction operator. It is easy to see that $(T_t^{(1)})_{t\geqslant 0}$ is a strongly continuous sub-Markovian contraction semigroup on $L^1(m)$.

To proceed, we need a version of the Riesz convexity theorem which we take from Butzer–Berens [57, Sect. 3.3.2, pp. 187–191].

[3] In [96] the criterion reads

$$\forall \epsilon > 0,\ r > 0 : \lim_{h\to 0} \sup_{t\leqslant h} \sup_{|x|\leqslant r} \frac{1}{h}\, \mathbb{P}^x(|X_t - x| > \epsilon) = 0.$$

A careful check of the proof reveals that (1.56) is sufficient. Alternatively, if we already have a càdlàg modification, we can use the simplified argument in [271, Theorem 2]: Just observe in that proof the following identity $\{|X_t - X_s| > \epsilon, \sup_{u\leqslant T}|X_u| \leqslant r\} = \{2r \geqslant |X_t - X_s| > \epsilon, \sup_{u\leqslant T}|X_u| \leqslant r\}$.

Theorem 1.44 (M. Riesz). *Let $(E, \mathscr{B}(E), \mu)$ and $(F, \mathscr{B}(F), \nu)$ be two σ-finite measure spaces, $1 \leqslant p \leqslant q \leqslant \infty$ and assume that*

$$T : L^p(\mu) + L^q(\mu) \to L^p(\nu) + L^q(\nu)$$

is a bounded linear operator. Then $T : L^r(\mu) \to L^r(\nu)$ is bounded for any $r \in [p, q]$, and we have the following estimate for the operator norm

$$\|T\|_{L^r(\mu)\to L^r(\nu)} \leqslant \|T\|^{\theta}_{L^p(\mu)\to L^p(\nu)} \cdot \|T\|^{1-\theta}_{L^q(\mu)\to L^q(\nu)} \quad \textit{if} \quad \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}, \ \theta \in [0,1].$$

Using $p = 1$ and the L^1-semigroup $(T_t^{(1)})_{t\geqslant 0}$ as the left end-point and $q = \infty$ and the Feller semigroup $(T_t)_{t\geqslant 0}$ as the right end-point, we see that $T_t|_{C_c(E)}$ extends to all intermediate spaces $L^p(m)$, $1 < p < \infty$ in such a way that the extensions yield strongly continuous, sub-Markovian contraction semigroups $(T_t^{(p)})_{t\geqslant 0}$ in $L^p(m)$.

The symmetry assumption (1.57) is quite restrictive and we can relax it in the following way. Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup and denote by T_t^* the formal adjoint of T_t with respect to the space $L^2(m)$, i.e. the linear operator defined by

$$\int_E T_t u(x) \cdot w(x)\, m(dx) = \int_E u(x) \cdot (T_t^* w)(dx) \quad \forall u, w \in C_c(E). \tag{1.58}$$

Note that $T_t^* w \in \mathcal{M}_b(E)$ since the bounded Radon measures $\mathcal{M}_b(E)$ are the topological dual of $C_\infty(E)$. If we know that

$$T_t^{\circledast} := T_t^*|_{L^1(m)} \ \text{maps } L^1(m) \text{ into itself,}$$

then the calculations can be modified under the assumption that $T_t^{\circledast}$ is sub-Markovian, i.e. $0 \leqslant T_t^{\circledast} u \leqslant 1$ for all $u \in L^1(m)$ such that $0 \leqslant u \leqslant 1$.

Lemma 1.45. *Let $(T_t)_{t\geqslant 0}$ be a Feller semigroup and assume that the operators T_t are m-symmetric or that the $L^2(m)$-adjoints $T_t^{\circledast} := T_t^*|_{L^1(m)}$ are sub-Markovian. Then $(T_t)_{t\geqslant 0}$ has for every $1 \leqslant p < \infty$ an extension $(T_t^{(p)})_{t\geqslant 0}$ to a strongly continuous, positivity preserving, sub-Markovian contraction semigroup on $L^p(m)$.*

If the domain $\mathcal{D}(A)$ of the Feller generator A contains a subset $\mathcal{D}$ which is dense both in $C_\infty(E)$ and $L^1(m)$, one can show that the $L^p(m)$-generators $(A^{(p)}, \mathcal{D}(A^{(p)}))$ coincide on this set with $A|_{\mathcal{D}}$. A proof for the m-symmetric case and $p = 2$ is given in Proposition 3.15.

Remark 1.46. There are good reasons to consider semigroups in an L^p-setting and not only in L^2. In general, L^p-theories lead to better regularity and embedding results than the corresponding L^2-theory; moreover, one has much better control on capacities. Therefore, L^p-semigroups have been studied by Fukushima [114]; building on earlier work of Malliavin, see [215, Part II] for a survey, (r, p)-capacities were studied by Fukushima and Kaneko [116] in order to get a better grip on exceptional (capacity-zero) sets. These are also discussed in [165, 166]. For further regularity results using Paley–Littlewood theory, we refer to Stein [307]. If we happen to know that $(T_t)_{t\geqslant 0}$ is an analytic semigroup on all spaces L^p, standard results from semigroup theory tell us that $T_t(L^p) \subset \bigcap_{k\geqslant 1} \mathcal{D}((-A)^k)$ and, should we have Sobolev embeddings, then $\bigcap_{k\geqslant 1} \mathcal{D}((-A)^k)$ can be embedded into spaces of continuous and differentiable functions, see [167, Sect. 2]. A concrete application to gradient perturbations is given in [349, Theorem 1.1 and its proof]. Finally, many results on functional inequalities are set in L^p spaces, see e.g. Wang [340, Chap. 5].

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