Chapter 6
Random Planar Geometry

What is a typical random surface? This question has arisen in the theory of two-dimensional quantum gravity where discrete triangulations have been considered as a discretization of a random continuum Riemann surface. As we will see the typical random surface has a geometry which is very different from the one of the Euclidean plane.

6.1 Uniform Infinite Planar Triangulation (UIPT)

A *planar map* is an embedding of a finite connected planar graph into the two-dimensional sphere up to continuous deformations that preserve the orientation. We deal with planar maps because the little additional structure they bear compared to planar graphs enable us to do combinatorics with them more easily. A planar map is called a triangulation if all its faces have degree three and is called rooted if it has a distinguished oriented edge. We denote $\mathcal{T}_n$ the set of all rooted triangulations with n faces.

The following theorem defines the model of Uniform Infinite Planar Triangulation (UIPT):

Theorem 6.1 ([AS03]). *Let T_n be uniformly distributed over $\mathcal{T}_n$ and let (T_n, ρ) (with a slight abuse of notation) be its associated graph rooted at the origin of the root edge of T_n then we have the following convergence in distribution with respect to* $\mathrm{d}_{\mathrm{loc}}$

$$(T_n, \rho) \underset{n\to\infty}{\longrightarrow} (T_\infty, \rho), \tag{6.1}$$

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where (T_∞, ρ) is a random infinite rooted planar graph called the Uniform Infinite Planar Triangulation (UIPT).[1]

The geometry of UIPT is very interesting and far from the Euclidean one. For examples, Angel showed [Ang03] that the typical volume of a ball of radius r in UIPT is of order r^4. This random graph (and its family) has been extensively studied over the last ten years, see the works of Angel and Schramm, Chassaing and Durhuus, Krikun, Le Gall and Ménard... See also impressive work of Le Gall and Miermont on a different but related point of view: Scaling limits of random maps.

Remark 6.2. UIPT is in fact a stationary and reversible random graph, hence its biased version by $\deg(\rho)^{-1}$ is unimodular. See [AS03].

Unfortunately (or perhaps fortunately?) basic questions about UIPT are still open. Here is the most basic one:

Conjecture 6.3 ([AS03]). The simple random walk on UIPT is recurrent.

Added in proofs: just solved by Gurel-Gurevich and Nachmias [GG13].

In [BC13] it is shown that the simple random walk on the related Uniform Infinite Planar Quadrangulation (UIPQ) is subdiffusive with exponent less that $1/3$.

Conjecture 6.4 ([BC13]). The simple random walk $\{X_n\}_{n\geq 0}$ on the UIPT is subdiffusive with exponent $1/4$, i.e.

$$\mathrm{d_{gr}}(X_0, X_n) \asymp n^{1/4}.$$

6.2 Circle Packing

Since random triangulations and UIPT are planer graphs, it is very tempting to try and understand their conformal structures. The theory of Circle Packing is well-suited for this purpose.

A circle packing on the sphere is an arrangement of circles on a given surface (in our case the sphere) such that no overlapping occurs and so that all circles touch another. The most standard question regarding circle packing is there density, i.e., the portion of surface covered by them. The contact graph of a circle packing is defined to be the graph with set of vertices which correspond to the set of circles and an edge between two circles if and only if they are tangent.

Let $\mathcal{T}_n^S$ be the set of all triangulations of the sphere $\mathbb{S}_2$ with n faces with no loops or multiple edges. We recall the well known circle packing theorem (see Wikipedia, [HS95]):

[1]The real theorem actually deals directly with maps.

Theorem 6.5 (Circle Packing Theorem). *If T is a finite triangulation without loops or multiple edges then there exists a circle packing $P = \{P_c\}_{c \in C}$ in the sphere $\mathbb{S}_2$ such that the contact graph of P is T. In addition this packing is unique up to Möbius transformations.*

Recall that the group of Möbius transformations $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ can be identified with $\mathrm{PSL}_2(\mathbb{C})$ and that it acts transitively on triplets (x, y, z) of $\mathbb{S}_2$. The circle packing enables us to take a "nice" representation of a triangulation $T \in T_n$, nevertheless the non-uniqueness is somehow disturbing because to fix a representation we can, for example, fix the images of three vertices of a distinguished face of T. This specification breaks all the symmetry, because sizes of some circles are chosen arbitrarily. Here is how to proceed:

The action on $\mathbb{S}_2$ of an element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ can be continuously extended to $\mathbb{B}_3 := \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 \leq 1\}$: this is the Poincaré-Beardon extension. We will keep the notation γ for transformations $\mathbb{B}_3 \to \mathbb{B}_3$. The action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{B}_3$ is now transitive on points. The group of transformations that leave 0 fixed is precisely the group $\mathrm{SO}_2(\mathbb{R})$ of rotations of $\mathbb{R}^3$.

Theorem 6.6 (Douady-Earle). *Let μ be a measure on $\mathbb{S}_2$ such that $\#\mathrm{supp}(\mu) \geq 2$. Then we can associate to μ a "barycenter" denoted by $\mathrm{Bar}(\mu) \in \mathbb{B}_3$ such that for all $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ we have*

$$\mathrm{Bar}(\gamma^{-1}\mu) = \gamma(\mathrm{Bar}(\mu)).$$

We can now describe the renormalization of a circle packing. If P is a circle packing associated to a triangulation $T \in \mathcal{T}_n^S$, we can consider the atomic measure μ_P formed by the Dirac's at tangency point of the disks in P

$$\mu_P := \frac{1}{\#\text{tangency points}} \sum_{\substack{x \text{ is a tangency} \\ \text{point}}} \delta_x.$$

By transitivity there exists a conformal map $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ such that $\mathrm{Bar}(\gamma^{-1}\mu_P)=0$. The renormalized circle packing is by Definition $\gamma(P)$, this circle packing is unique up to rotation of $\mathrm{SO}_2(\mathbb{R})$, we will denote it by $\mathbf{P}_T$. This constitutes a canonical discrete conformal structure for the triangulation.

Here are some open problems regarding circle packing on the sphere:

Open problem 6.7. *If T_n is a random variable distributed uniformly over the set $\mathcal{T}_n^S$, then the variable $\mu_{\mathbf{P}_{T_n}}$ is a random probability measure over $\mathbb{S}_2$ seen up to rotations of $\mathrm{SO}_2(\mathbb{R})$. By classical arguments there exist weak limits μ_∞ of $\mu_{\mathbf{P}_{T_n}}$.*

1. *(Schramm) Determine coarse properties (invariant under $\mathrm{SO}_2(\mathbb{R})$) of μ_∞, e.g. what is the dimension of the support? Start by showing singularity.*

2. *Uniqueness (in law) of μ_∞? In particular can we describe μ_∞ in terms of the Gaussian Free Field? Is it* $\exp((8/3)^{1/2}GFF)$, *does KPZ hold? See [dup] for more details.*
3. *The random measure μ_∞ can come together with* d_∞ *a random distance on* $\mathbb{S}_2$. *Can you describe links between μ_∞ and* d_∞*? Does one characterize the other?*

6.3 Stochastic Hyperbolic Infinite Quadrangulation (SHIQ)

Recently Guth et al. [GPY11] studied pants decomposition of random surfaces chosen uniformly in the moduli space of hyperbolic metrics equipped with the Weil-Peterson volume form and a combinatorial analogue obtained by randomly gluing Euclidean triangles (with unit side length) together. They showed that such a random compact surfaces with no genus restriction have large pants decomposition, growing with the volume of the surface. This suggests that the injectivity radius around a typical point is growing to infinity. Gamburd and Makover [GM02] showed that as N grows the genus will converge to $N/4$ and using the Euler's characteristic the average degree will grow to infinity.

Take a uniform measure on triangulations with N triangles conditioned on the genus to be CN for some fixed $C < 1/4$, then we *conjecture* that as N grows to infinity the random surface will locally converge in the sense of [BS01b] (see Sect. 5 above) to a random triangulation of the hyperbolic plane with average degree $\frac{6}{1-4C}$. In particular we believe that the local injectivity radius around a typical vertex will go to infinity on such a surface as $N \to \infty$.

We would like to present here a natural quadrangulation that might describe such a local limit in the context of quadrangulations. A variant for triangulations might describe the limit with a specific supercritical random tree.

There exist nice and useful bijections between maps and labeled trees especially the so-called Schaeffer bijection. A variant of the UIPT (for quadrangulation) can be constructed from a labeled critical Galton-Watson tree conditioned to survive, see [CMM12] for details. Here we propose the study of a random quadrangulation constructed from a labeled super critical Galton-Watson trees.

Consider T_3 the full ternary tree given with a root vertex $\rho \in T_3$ and embedded in the plane $\mathbb{R}^2$. Assign independently to each edge e of the tree a random variable d_e uniformly distributed over $\{-1, 0, +1\}$. This procedure yields a labeling ℓ of the tree T_3 by setting the label of any vertex u as the sum of the d_e's along the geodesic line between ρ and u.

A *corner* c of the tree T_3 is an angular sector between two adjacent edges. There is a natural (partial) order on the corners of T_3 given by the clockwise contour of the tree T_3. We then extend the Schaeffer construction to the labeled tree (T_3, ℓ) as follows: For each corner c of T_3 associated to a vertex of label l, draw an edge between c and the first corner in the clockwise order whose associated vertex has label $l - 1$. Consider the quadrangulation obtained using only the edges added and not the original tree we started with. T_3 can be replaced by any tree.

It can be checked that all these edges can be drawn such that they are non-crossing and the resulting map is a infinite quadrangulation (with a root vertex ρ) that we call the Stochastic Hyperbolic Infinite Quadrangulation. It should be thought as a hyperbolic analogue of the UIPT/Q.

Here are several questions and observations regarding SHIQ:

- Does the SHIQ admits spatial Markovity? If it is indeed a local limit then yes.
- Starting with a super critical Galton Watson tree it easily follows that a.s. the quadrangulation has exponential volume growth. Estimate it. Are there limit theorem for ball size analogous to the branching process theory?
- Does the SHIQ has positive anchored expansion a.s. (see [Vir00] for the study of anchored expansion). This will imply positive speed and bounds on return probability.
- Using [BLS99] it is possible to show that simple random walk has positive speed.
- Is the sphere at infinity topologically S^1? Does SRW converges to a point on the sphere at infinity? Is the sphere at infinity the Martin boundary? See [Anc88] for details.
- Show that the Self Avoiding Walk is a.s. ballistic on the SHIQ? Adapt the theory of Poisson Voronoi percolation on the hyperbolic plain [BS01a] to the SHIQ. Study SHIQ coupled with spin systems such as Ising as for the UIPQ.

6.4 Sphere Packing of Graphs in Euclidean Space

One way to extend the notion of planar graphs in order to hopefully make initial steps in the context of three dimensional random geometry is to consider graphs sphere pack in $\mathbb{R}^3$. Some partial results extending ideas from planar circle packing to higher dimension were presented in [MTTV98, BS09] and [BC11]. The general theory of packing was recently developed by Pierre Pansu in [Pan]. See [BC11] for a collection of problems on the subject.

Maybe an extension of Schaeffer's bijection can used to create graphs sphere packed in $\mathbb{R}^3$. In Schaeffer's bijection the edges of a planar tree are labeled $-1, +1$ or 0. Walking around the tree as in depth first search and summing the labels, this defines a height function on the vertices, two values for each vertex. If an edge is added between any vertex and the closest vertex in the direction of the walk with a smaller height a quadrangulation is generated.

We hope that replacing the tree by a planar graph in a related recipe will create a packable graph. Let G be a planar graph, $f : G \to \mathbb{Z}$, with value differ by at most one between neighbors. Circle pack G in the Euclidean plain, for any vertex $v \in G$, add an edge from v to the vertex u which is among the closest to v in the Euclidean metric, with $f(u) < f(v)$, (were vertices are identified with the center of the corresponding circles).

Open problem 6.8. *Is the resulting graph a sphere packed in* $\mathbb{R}^3$*?*

Start with G the square grid. By [MTTV98] we know that packable graphs has separation function bounded by $n^{d-1/d}$. Can this be used to construct a counter example by maybe realizing large expanders in this way?

If the conjecture is true than a natural family of packable graph (perhaps) can be obtained by taking G to be a random quadrangulation and f the Gaussian free field on it. We don't know an example of a transient graph which does not contain a transient subgraph which is sphere packed in $\mathbb{R}^3$.

Theorem 6.9. *Assume G is a finite vertex transitive graph which is sphere packed in* $\mathbb{R}^d$*. The diameter of G is bigger than* $C_d|G|^{1/d}$*. For some universal constant depending only on d.*

Exercise 6.10. (Level 3) Prove this by combining the fact from [MTTV98] that packable graphs has separation function bounded by $n^{d-1/d}$ and Theorem 2.1.

For planar finite vertex transitive graphs this follows also from a known structure theorem [FI79].

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