Chapter 2
Steady Optimisation Time, ST

2.1 Steady Optimisation Time, ST

Figure 2.1 presents a model system of three interconnected reservoirs supplying water to independent consumers. The end conditions in the reservoir state trajectories are steady. This means that after an optimisation period W, the reservoir states should match previously determined values. For the vector of predicted inflows to the reservoirs, the optimising task formulated with index (2.1) comes down to:

- defining the control vector (outflows from reservoirs),

$$\hat{\boldsymbol{u}}(t-\boldsymbol{h}(t))_{(+)}, \quad \forall t \in [t_0, W] \tag{2.1}$$

  which will minimally differ from the vector consisting of partial water demands per individual system reservoir,

$$\mathbf{B}(t)\cdot\mathbf{Y}(t)\cdot\mathbf{S}\cdot\boldsymbol{1}, \forall t \in [t_0, W] \tag{2.2}$$

- achieving the target specified above at minimum transfer cost among reservoirs,
- obtaining at the end of optimisation horizon W the vector for the filling levels for the reservoirs $\boldsymbol{x}(W)$ satisfying the required values,

The reservoirs supply water to four consumers (WTPs) at the same time; therefore, to describe the system any further it is necessary to introduce a function for reservoir involvement in carrying out water demand functions $Y_j(t),\ j = 1, \ldots, 4\ \forall t \in [t_0, W]$ dividing (for each instant in time) the function $Y_j(t),\ j = 1, \ldots, 4$ among the reservoirs in the system:

$$Y_j(t) - \sum_{i=1}^{4} b_{i,j}(t)\cdot Y_j = 0, \quad j = 1, \ldots, 4 \tag{2.3}$$

W. Z. Chmielowski, *Management of Complex Multi-reservoir Water Distribution Systems Using Advanced Control Theoretic Tools and Techniques*, SpringerBriefs in Computational Intelligence, DOI: 10.1007/978-3-319-00239-2_2, © The Author(s) 2013

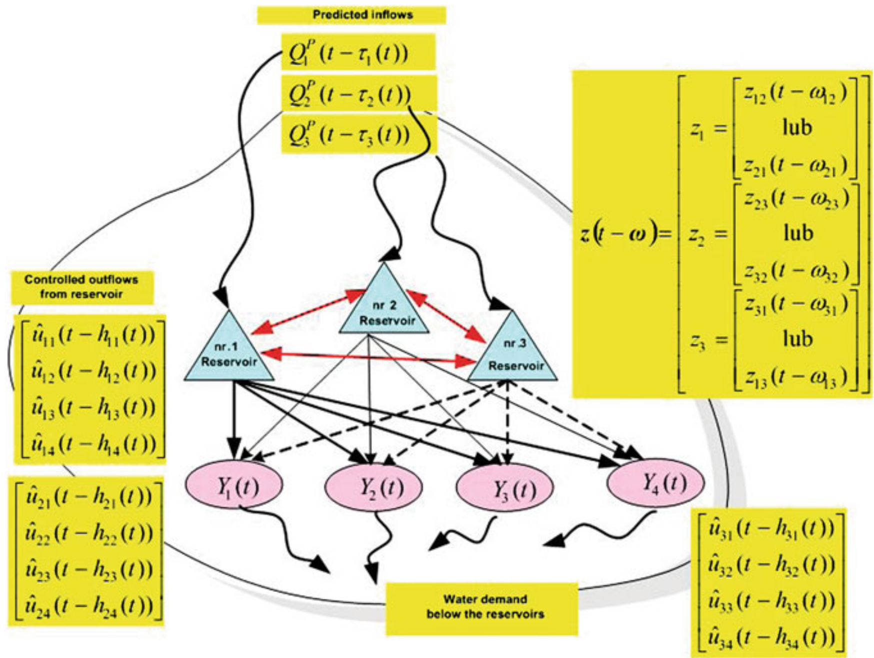

Fig. 2.1 The complex water-management system

In this problem of reservoir system functioning (Fig. 2.1), there are a number of delays in individual system input and output variables. The abovementioned delays, which substantially complicate the formal record of system function, come down to the following dependencies:

- vector of delays related to water flow through the reservoirs

$$\tau(t)^{\mathrm{T}} = [\tau_1(t)\ \ \tau_2(t)\ \ \tau_3(t)] \tag{2.4}$$

The water flow rate measurement point $\left[\mathrm{m}^3/\mathrm{s}\right]$ is usually located in the area where a river flows into a reservoir. Flow rates measured at the reservoir inlet will appear in the vicinity of a dam after a given time, that is, with a delay dependent on the reservoir's dimensions. The delay is also a function of time, because it may change with hydrological and climatic conditions within the reservoir area (higher, lower, variable rate of water flow through reservoir).
Therefore, the vector of inflows into the system reservoirs taking into account the abovementioned delays should be defined as follows:

$$\boldsymbol{Q}^P(t-\tau(t)) = \begin{bmatrix} Q_1^P(t-\tau_1(t)) \\ Q_2^P(t-\tau_2(t)) \\ Q_3^P(t-\tau_3(t)) \end{bmatrix} \tag{2.5}$$

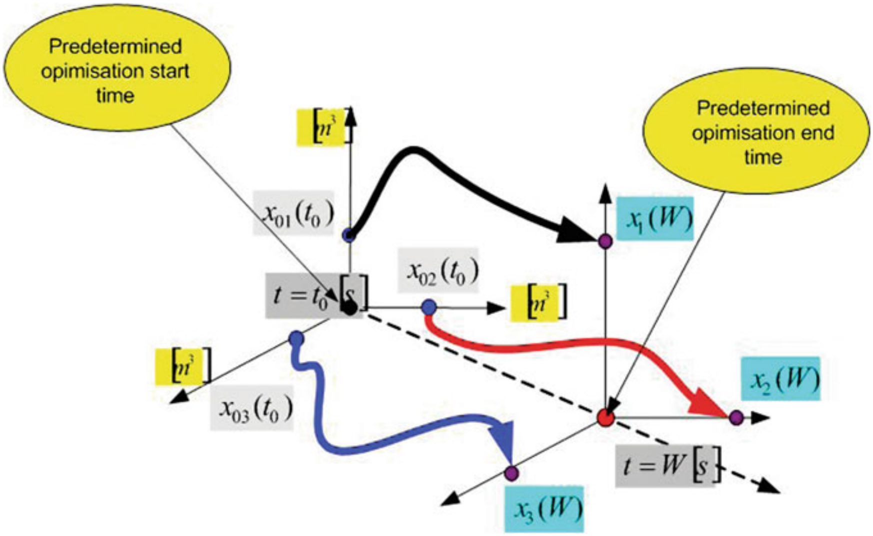

Fig. 2.2 Optimisation start and end time

- Another are transport-related delays in water distribution from the reservoir system to the WTPs (2.6). These delays result from the distance between individual reservoirs and water consumers. The vector of controlled outflows from reservoirs taking into account the abovementioned delays is defined by the following formula (2.7):

$$\boldsymbol{h}(t) = \begin{bmatrix} h_{11}(t) & h_{21}(t) & h_{31}(t) \\ h_{12}(t) & h_{22}(t) & h_{32}(t) \\ h_{13}(t) & h_{23}(t) & h_{33}(t) \\ h_{14}(t) & h_{24}(t) & h_{34}(t) \end{bmatrix} \tag{2.6}$$

$$\hat{\boldsymbol{u}}(t - \boldsymbol{h}(t))_{(+)} = \begin{bmatrix} \begin{bmatrix} u_{11}(t - h_{11}(t)) \\ u_{12}(t - h_{12}(t)) \\ u_{13}(t - h_{13}(t)) \\ u_{14}(t - h_{14}(t)) \end{bmatrix} \\ \\ \begin{bmatrix} u_{21}(t - h_{21}(t)) \\ u_{22}(t - h_{22}(t)) \\ u_{23}(t - h_{23}(t)) \\ u_{24}(t - h_{24}(t)) \end{bmatrix} \\ \\ \begin{bmatrix} u_{31}(t - h_{31}(t)) \\ u_{32}(t - h_{32}(t)) \\ u_{33}(t - h_{33}(t)) \\ u_{34}(t - h_{34}(t)) \end{bmatrix} \end{bmatrix} \tag{2.7}$$

- Also, there are transport-related delays in water transfers among the system reservoirs, which result from the location of reservoirs in the field. Additionally, delays in water transport between reservoirs are asymmetrical, which means that, for instance, a delay in water transport from reservoir no. 1 to reservoir no. 2 may differ from the delay in water transport in the opposite direction.

$$\boldsymbol{\omega}^{\mathrm{T}} = \left[\begin{bmatrix} \omega_{12} \\ \omega_{21} \end{bmatrix} \begin{bmatrix} \omega_{23} \\ \omega_{32} \end{bmatrix} \begin{bmatrix} \omega_{31} \\ \omega_{13} \end{bmatrix} \right] \tag{2.8}$$

The vector of controlled transfers among the reservoirs taking into account the delays in water transport may be defined using the following formula:

$$\hat{z}(t - \boldsymbol{\omega}) = \begin{bmatrix} \begin{bmatrix} z_1 = & \begin{matrix} z_{12}(t - \omega_{12}) \\ or \\ z_{21}(t - \omega_{21}) \end{matrix} \end{bmatrix} \\ \\ \begin{bmatrix} z_2 = & \begin{matrix} z_{23}(t - \omega_{23}) \\ or \\ z_{32}(t - \omega_{32}) \end{matrix} \end{bmatrix} \\ \\ \begin{bmatrix} z_3 = & \begin{matrix} z_{31}(t - \omega_{31}) \\ or \\ z_{13}(t - \omega_{13}) \end{matrix} \end{bmatrix} \end{bmatrix} \tag{2.9}$$

2.1.1 Quality Coefficient

Optimal reservoir state trajectories and trajectories of outflows from reservoirs (control room) will be determined through solving the dynamic optimising task specified below. Its requisite elements include:

$$F = 0,5 \int_{t_0}^{W} \left\{ \begin{matrix} [\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \boldsymbol{u}(t - \boldsymbol{h}(t))]_{(+)}^{\mathrm{T}} \\ \cdot \mathbf{A}_1 \cdot \\ [\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \boldsymbol{u}(t - \boldsymbol{h}(t))]_{(+)} \\ \cdot \mathbf{z}(t - \boldsymbol{\omega})^{\mathrm{T}} \cdot \mathbf{A}_2 \cdot \mathbf{z}(t - \boldsymbol{\omega}) \end{matrix} \right\} dt \tag{2.10}$$

Regarding Fig. 2.1, the following symbols are used in Eq. (2.10):

- Matrix $\mathbf{B}(\mathbf{t})$ is a diagonal block matrix with terms constituting diagonal matrixes. Their elements are functions of reservoir involvement $i = 1, \ldots, 3$ in carrying out the demand functions $Y_j(t)$, $j = 1, \ldots 4$, $[m^3/s]$.

$$\mathbf{B}(\mathbf{t}) = \begin{bmatrix} \mathbf{B}_1(t) & * & * \\ * & \mathbf{B}_2(t) & * \\ * & * & \mathbf{B}_3(t) \end{bmatrix} \tag{2.11}$$

where:

$$\mathbf{B}_1(t) = \begin{bmatrix} b_{11}(t) & 0 & 0 & 0 \\ 0 & b_{12}(t) & 0 & 0 \\ 0 & 0 & b_{13}(t) & 0 \\ 0 & 0 & 0 & b_{14}(t) \end{bmatrix} \tag{2.12}$$

$$\mathbf{B}_2(t) = \begin{bmatrix} b_{21}(t) & 0 & 0 & 0 \\ 0 & b_{22}(t) & 0 & 0 \\ 0 & 0 & b_{23}(t) & 0 \\ 0 & 0 & 0 & b_{24}(t) \end{bmatrix} \tag{2.13}$$

$$\mathbf{B_3}(t) = \begin{bmatrix} b_{31}(t) & 0 & 0 & 0 \\ 0 & b_{32}(t) & 0 & 0 \\ 0 & 0 & b_{33}(t) & 0 \\ 0 & 0 & 0 & b_{34}(t) \end{bmatrix} \tag{2.14}$$

$$* = \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix} \tag{2.15}$$

Matrix $\mathbf{Y}(t)$ is a diagonal block matrix with terms constituting diagonal matrixes. Their elements are the demand functions effective in the system $Y_j(t)$, $j = 1, \ldots 4$, $[m^3/s]$ Fig. 2.2

$$\mathbf{Y}(t) = \begin{bmatrix} \circ(t) & * & * \\ * & \circ(t) & * \\ * & * & \circ(t) \end{bmatrix} \tag{2.16}$$

where:

$$\circ(t) = \begin{bmatrix} Y_1(t) & 0 & 0 & 0 \\ 0 & Y_2(t) & 0 & 0 \\ 0 & 0 & Y_3(t) & 0 \\ 0 & 0 & 0 & Y_4(t) \end{bmatrix} \tag{2.17}$$

- The control vector (of controlled outflows from reservoirs) (7) is a block vector. Its elements are vectors, and their elements are reservoir outflows $i = 1, \ldots, 3$ to conurbations $j = 1, \ldots, 4$.
- Then, matrix $\mathbf{A_1}$ is a positively definite block matrix with terms on a diagonal also being diagonal matrixes. Their elements are weight coefficients related to proper control vector elements

$$\mathbf{A_1} = \begin{bmatrix} \bullet_1 & * & * \\ * & \bullet_2 & * \\ * & * & \bullet_3 \end{bmatrix} \tag{2.18}$$

where:

$$\bullet_1 = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{14} \end{bmatrix} \tag{2.19}$$

$$\bullet_2 = \begin{bmatrix} a_{21} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & a_{24} \end{bmatrix} \tag{2.20}$$

$$\bullet_3 = \begin{bmatrix} a_{31} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{34} \end{bmatrix} \tag{2.21}$$

- Further, positively definite diagonal matrix $\mathbf{A_2}$,

$$\mathbf{A_2} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \tag{2.22}$$

with elements, which are weight coefficients related to a subintegral section of the quality coefficient (2,10), corresponding to the costs of water transfers among the reservoirs, e.g. a_{11} all applies to transfer cost $z_1(t)$, that is from reservoir 1 to reservoir 2, or the other way round, etc. Regarding the matrix, this way of presenting the problem is a simplification, because in a general case the water transfer cost e.g. from reservoir no. 1 to reservoir no. 2 will not necessarily equal the cost of water transfer from reservoir no. 2 to reservoir no. 1 (e.g. gravitational flow and pumping: $a_{11}^{zb1\rightarrow zb2} \neq a_{11}^{zb1\leftarrow zb2}$).
- Then, unit vector

$$\boldsymbol{1}^{\mathrm{T}} = \begin{bmatrix} \otimes \ \otimes \ \otimes \end{bmatrix}, \quad \otimes = \begin{bmatrix} 1 \ 1 \ 1 \ 1 \end{bmatrix} \tag{2.23}$$

- and vector of transfers among reservoirs (2.9)

2.1.2 State Equation of Reservoirs

The following have been assumed: balance equation of state for the system reservoirs, and start and end conditions in reservoir state trajectories

$$f : \dot{\boldsymbol{x}}(t) = \boldsymbol{Q}^P(t - \boldsymbol{\tau}(t)) - \mathbf{S_1}\boldsymbol{u}(t - \boldsymbol{h}(t)) + \mathbf{S_2}\boldsymbol{z}(t - \boldsymbol{\omega}) \tag{2.24}$$

$$\boldsymbol{x}_0(t_0)^{\mathrm{T}} = \left[x_{01}(t_0)\ x_{02}(t_0)\ x_{03}(t_0) \right] \tag{2.25}$$

$$\boldsymbol{x}^U(W)^{\mathrm{T}} = \left[x_1{}^U(W)\ x_2{}^U(W)\ x_3{}^U(W) \right] \tag{2.26}$$

The following symbols are used in state equation (2.24):

- vector of reservoir state derivatives

$$\dot{\boldsymbol{x}}^{\mathrm{T}}(t) = [dx_1(t)/dt\ \ dx_2(t)/dt\ \ dx_3(t)/dt] \tag{2.27}$$

- vector of predicted inflows to reservoirs (2.5) $\left[\mathrm{m}^3/\mathrm{s}\right]$
- vector of predetermined initial filling levels in reservoirs (2.25) $\left[\mathrm{m}^{\mathrm{s}}\right]$
- vector of predetermined final filling levels in reservoirs (2.26) $\left[\mathrm{m}^{\mathrm{s}}\right]$
- t_0 predetermined optimisation start time $[s]$
- W predetermined optimisation end time $[s]$
- diagonal structural matrix $\mathbf{S}_1$, needed to record the system structure and the system state equation

$$\mathbf{S}_1 = \begin{bmatrix} \otimes & [0] & [0] \\ [0] & \otimes & [0] \\ [0] & [0] & \otimes \end{bmatrix} \quad [0] = \left[0\ 0\ 0\ 0 \right] \tag{2.28}$$

- structural matrix needed to record the relations between reservoirs with reference to transfers among reservoirs in the system state equation

$$\mathbf{S}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \tag{2.29}$$

- Matrix $\mathbf{S}$ is a structural matrix derived from operation $\mathbf{S} = ((\mathbf{S}_1{}^{\mathrm{T}} \cdot \mathbf{S}_1) * \mathbf{I})$, (asterisk $*$, table multiplication).

2.1.3 Solution for the Optimising Task

Hamilton's function for the system of Eqs. (2.10) and (2.24) has the following form:

$$H = -f_0 + \boldsymbol{\psi}^{\mathrm{T}} \cdot \boldsymbol{f} \tag{2.30}$$

f_0 —sub-integral function of coefficient (2.10)
$\boldsymbol{f}$ —state equation for reservoirs (2.24)
$\boldsymbol{\psi}$ —conjugate variable, vector [3*1]

$$H = -0,5 \cdot \left\{ \begin{array}{l} [\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \boldsymbol{u}(t - \boldsymbol{h}(t))]_{(+)}^{\mathrm{T}} \\ \cdot \mathbf{A}_1 \cdot \\ [\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \boldsymbol{u}(t - \boldsymbol{h}(t))]_{(+)} \\ \cdot \mathbf{z}(t - \boldsymbol{\omega})^{\mathrm{T}} \cdot \mathbf{A}_2 \cdot \mathbf{z}(t - \boldsymbol{\omega}) \end{array} \right\} + \boldsymbol{\psi}(t)^{\mathrm{T}} \cdot \left[\boldsymbol{Q}^P(t - \boldsymbol{\tau}(t)) - \mathbf{S_1} \boldsymbol{u}(t - \boldsymbol{h}(t)) + \mathbf{S_2} \boldsymbol{z}(t - \boldsymbol{\omega}) \right] \tag{2.31}$$

The system of equations for Hamilton's function in form (2.31) is shown below:

1.

$$\left[(\nabla_u H)_{\hat{u},\hat{x},\hat{\psi}} \right]^{\mathrm{T}} = \boldsymbol{0} \quad \Rightarrow \hat{\boldsymbol{u}}(t - \boldsymbol{h}(t))_{(+)} = \mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \mathbf{A}_1^{-1} \cdot \mathbf{S}_1{}^{\mathrm{T}} \cdot \hat{\boldsymbol{\psi}}(t) \tag{2.32}$$

2.

$$\left[(\nabla_z H)_{\hat{u},\hat{z},\hat{x},\hat{\psi}} \right]^{\mathrm{T}} = \boldsymbol{0} \quad \Rightarrow \quad \hat{\mathbf{z}}(t - \boldsymbol{\omega}) = \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}} \cdot \hat{\boldsymbol{\psi}}(t) \tag{2.33}$$

3.

$$\left[\left(\nabla_\psi H \right)_{\hat{u},\hat{x}} \right]^{\mathrm{T}} = \dot{\boldsymbol{x}}(t) \quad \Rightarrow \boldsymbol{Q}^P(t - \boldsymbol{\tau}(t)) - \mathbf{S_1} \boldsymbol{u}(t - \boldsymbol{h}(t)) + \mathbf{S_2} \boldsymbol{z}(t - \boldsymbol{\omega}) \tag{2.34}$$

4.

$$\left[-(\nabla_x H)_{\hat{u},\hat{x},\hat{\psi}} \right]^{\mathrm{T}} = \dot{\hat{\boldsymbol{\psi}}}(t) \quad \Rightarrow \quad \dot{\hat{\boldsymbol{\psi}}}(t) = \boldsymbol{0}_{(3*1)} \tag{2.35}$$

Equation (2.35) proves that:

$$\hat{\boldsymbol{\psi}}(t) = \boldsymbol{C}_1 \tag{2.36}$$

Integration of Eq. (2.34) enables us to obtain an equation characterising the general form of a state trajectory vector:

$$\hat{\boldsymbol{x}}(t) = \int_{t_0}^{t} \left\{ \begin{array}{l} \boldsymbol{Q}^P(\xi - \boldsymbol{h}(\xi)) + \\ -\mathbf{S}_1 \cdot \hat{\boldsymbol{u}}(\xi - \boldsymbol{h}(\xi))_{(+)} \\ +\mathbf{S}_2 \cdot \hat{z}(\xi - \boldsymbol{\omega}) \end{array} \right\} d\xi + \boldsymbol{C}_2 \tag{2.37}$$

We substitute term (2.36) in Eq. (2.32), and then the control vector is defined by the following equation:

$$\hat{\boldsymbol{u}}(t - \boldsymbol{h}(t))_{(+)} = \mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} \cdot \boldsymbol{C}_1 \tag{2.38}$$

We substitute term (2.36) in Eq. (2.33), and then the vector of controlled transfers among reservoirs is defined by the following equation:

$$\hat{\mathbf{z}}(t-\boldsymbol{\omega}) = \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}} \cdot \boldsymbol{C}_1 \tag{2.39}$$

Then, we attempt to determine the state vector. We substitute (2.38) and (2.39) in (2.37), thus receiving

$$\hat{\boldsymbol{x}}(t) = \int_{t_0}^{t} \left\{ \begin{array}{l} \boldsymbol{Q}^P(\xi - \boldsymbol{h}(\xi)) \\ -\mathbf{S}_1 \cdot [\mathbf{B}(\xi) \cdot \mathbf{Y}(\xi) \cdot \mathbf{S} \cdot \boldsymbol{1}] \\ +\mathbf{S}_1 \cdot \left[\mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} \cdot \boldsymbol{C}_1\right] \\ +\mathbf{S}_2 \cdot \left[\mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}} \cdot \boldsymbol{C}_1\right] \end{array} \right\} d\xi + \boldsymbol{C}_2 \tag{2.40}$$

for $t = t_0 \Rightarrow \boldsymbol{C}_2 = \boldsymbol{x}(t_0)$.
After regrouping and integrating the component constant in time, and for $t = W$, the state trajectory vector assumes the following form:

$$\begin{aligned} \boldsymbol{x}(W) = & \int_{t_0}^{W} \left\{ \begin{array}{l} \boldsymbol{Q}^P(t - \boldsymbol{h}(t)) \\ -\mathbf{S}_1 \cdot [\mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1}] \end{array} \right\} dt \\ & + \left(\mathbf{S}_1 \cdot \mathbf{A}_1^{-1} \cdot \mathbf{S}_1{}^{\mathrm{T}} + \mathbf{S}_2 \cdot \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}}\right) \cdot \boldsymbol{C}_1 \cdot (W - t_0) + \boldsymbol{x}(t_0) \end{aligned} \tag{2.41}$$

We calculate the vector of constants $\boldsymbol{C}_1$ from Eq. 2.41)

$$\begin{aligned} \boldsymbol{C}_1 = & \left[\left(\mathbf{S}_1 \cdot \mathbf{A}_1^{-1} \cdot \mathbf{S}_1^{\mathrm{T}} + \mathbf{S}_2 \cdot \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}}\right) \cdot (W - t_0)\right]^{-1} \cdot \\ & \left\{ \begin{array}{l} \boldsymbol{x}(W) - \boldsymbol{x}(t_0) - \\ + \int_{t_0}^{W} \left[\boldsymbol{Q}^P(t - \boldsymbol{h}(t)) - \mathbf{S}_1 \cdot \mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1}\right] dt \end{array} \right\} \end{aligned} \tag{2.42}$$

Further solving of the problem does not represent any major difficulties—substitute (2.42) in (2.36), and then obtained result in (2.32) and (2.33), thus receiving optimal control vector $\hat{\boldsymbol{u}}(t - \boldsymbol{h}(t))_{(+)}, \forall t \in [t_0, W]$, and vector of transfers among reservoirs $\hat{\mathbf{z}}(t - \omega)$. Transport-related delays in transfers among reservoirs essentially affect satisfying end condition of reservoir state trajectories (2.26). They force changes in transfer vector value so as to fulfil condition (2.26). Satisfying prerequisite condition (2.26) involves a need to introduce modified water distribution among the reservoirs. This change is a consequence of the scheme shown below. Formula (2.43) determines adequately increased/reduced transfer among the reservoirs during the distribution period, taking into account the delays.

$$\mathbf{z}(t - \omega) = \begin{bmatrix} z_1(t - \omega_{12 \vee 21}) : \otimes_1 \\ z_2(t - \omega_{23 \vee 32}) : \otimes_2 \\ z_3(t - \omega_{31 \vee 13}) : \otimes_3 \end{bmatrix}$$

$$\otimes_1 = \left\{ \begin{array}{l} |z| = |z_{12}| = |z_{21}| \\ (\mathrm{sgn}(z_1) = 1 \ \wedge \ \omega_{12} > 0) \ \Rightarrow \\ z_1(t - \omega_{12}) = |z_{12}| + \frac{\int_0^{\omega_{12}} |z_{12}| dt}{(W - \omega_{12})} ; \\ (\mathrm{sgn}(z_1) = -1 \ \wedge \ \omega_{21} > 0) \ \Rightarrow \\ z_1(t - \omega_{21}) = |z_{12}| + \frac{\int_0^{\omega_{21}} |z_{12}| dt}{(W - \omega_{21})} \end{array} \right\} \tag{2.43}$$

$$\otimes_2 = \left\{ \begin{array}{l} |z| = |z_{23}| = |z_{32}| \\ (\mathrm{sgn}(z_2) = 1 \ \wedge \ \omega_{23} > 0) \ \Rightarrow \\ z_2(t - \omega_{23}) = |z_{23}| + \frac{\int_0^{\omega_{23}} |z_{23}| dt}{(W - \omega_{23})} \\ (\mathrm{sgn}(z_2) = -1 \ \wedge \ \omega_{32} > 0) \ \Rightarrow \\ z_2(t - \omega_{32}) = |z_{32}| + \frac{\int_0^{\omega_{32}} |z_{23}| dt}{(W - \omega_{32})} \end{array} \right\} \tag{2.44}$$

$$\otimes_3 = \left\{ \begin{array}{l} |z| = |z_{31}| = |z_{13}| \\ (\mathrm{sgn}(z_3) = 1 \ \wedge \ \omega_{31} > 0) \ \Rightarrow \\ z_3(t - \omega_{31}) = |z_{31}| + \frac{\int_0^{\omega_{31}} |z_{31}| dt}{(W - \omega_{31})} \\ (\mathrm{sgn}(z_3) = -1 \ \wedge \ \omega_{13} > 0) \ \Rightarrow \\ z_3(t - \omega_{13}) = |z_{13}| + \frac{\int_0^{\omega_{13}} |z_{13}| dt}{(W - \omega_{13})} \end{array} \right\}$$

Formulas (2.43, 2.44) take into account the possibility of water transfer both ways (e.g. from reservoir no. 1 to reservoir no. 2 and vice versa) and asymmetrical transport-related delay (np. $\omega_{12} \neq \omega_{21}$). This principle is applicable to all relations among the reservoirs. As regards this water system (Fig. 2.1), notation of transfer vector modifications $\hat{\mathbf{z}}(t - \boldsymbol{\omega})$ formula (2.43, 2.44), is relatively simple. Its complexity degree increases very quickly with the system dimensionality, and is particularly dependent on the number and direction of transfers among reservoirs and the delay involved.

Having completed the abovementioned modifications, we substitute (2.38) and (2.39) in (2.41) to obtain the vector of optimal state trajectories satisfying condition (2.30). We receive the minimum quality coefficient value by substituting (2.38) and (2.30) in (2.10).

2.1.4 Computer Simulations

Hipothetical scenario of events. Example 1 In order to test whether the solution obtained is optimal and correct, in the first example we will make the following simple numerical assumptions, specifying that there will be no delays in inflow and further distribution of water:

- initial/start states in reservoirs

$$\boldsymbol{x}_0(t_0)^{\mathrm{T}} = \begin{bmatrix} 10 \; 10 \; 10 \end{bmatrix} \begin{bmatrix} m^3 \end{bmatrix}$$

- final/end states in reservoirs $\boldsymbol{x}^U(W)^{\mathrm{T}} = \begin{bmatrix} 30 \; 30 \; 30 \end{bmatrix} \begin{bmatrix} m^3 \end{bmatrix}$
- inflows to reservoirs

$$\boldsymbol{Q}^P(t - \boldsymbol{\tau}(t)) = \begin{bmatrix} (t - (\tau_1 = 0) + 1) \\ (t - (\tau_2 = 0) + 1) \\ (t - (\tau_3 = 0) + 1) \end{bmatrix} \begin{bmatrix} m^3/s \end{bmatrix}$$

- water demand below reservoirs

$$\mathbf{Y}(t) = \begin{bmatrix} [*] & 0^* & 0^* \\ 0^* & [*] & 0^* \\ 0^* & 0^* & [*] \end{bmatrix} \begin{bmatrix} m^3/s \end{bmatrix}$$

$$[*] = \begin{bmatrix} 4(t) & 0 & 0 & 0 \\ 0 & 4(t) & 0 & 0 \\ 0 & 0 & 4(t) & 0 \\ 0 & 0 & 0 & 4(t) \end{bmatrix}$$

$$0^* = \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix}$$

- the functions of reservoir involvement in carrying out the demand functions

$$\mathbf{B}(t) = \begin{bmatrix} [*_1] & 0^* & 0^* \\ 0^* & [*_1] & 0^* \\ 0^* & 0^* & [*_1] \end{bmatrix}$$

$$[*_1] = \begin{bmatrix} 0.33 & 0 & 0 & 0 \\ 0 & 0.33 & 0 & 0 \\ 0 & 0 & 0.33 & 0 \\ 0 & 0 & 0 & 0.33 \end{bmatrix} \cdot (t)$$

- matrices of weight coefficients

$$\mathbf{A}_1 = \begin{bmatrix} [*] & 0^* & 0^* \\ 0^* & [*] & 0^* \\ 0^* & 0^* & [*] \end{bmatrix}, \quad [*] = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}$$

$$\mathbf{A_2} = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{bmatrix}$$

- structural matrix

$$\mathbf{S}_1 = \begin{bmatrix} [*] & 0^{\bullet} & 0^{\bullet} \\ 0^{\bullet} & [*] & 0^{\bullet} \\ 0^{\bullet} & 0^{\bullet} & [*] \end{bmatrix}$$

$$[*] = \begin{bmatrix} 1\ 1\ 1\ 1 \end{bmatrix}, \quad 0^{\bullet} = \begin{bmatrix} 0\ 0\ 0\ 0 \end{bmatrix}$$

- structural matrix

$$\mathbf{S_2} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

- start time $t_0 = 0[s]$, same for all reservoirs,
- end time $W = 10[s]$, same for all reservoirs,

Solution

- acc. to (2.42) $\boldsymbol{C}_1 = [0, 32 \quad 0, 32 \quad 0, 32]$
- acc. to (2.38) $\hat{\boldsymbol{u}}(t - \boldsymbol{h}(t))_{(+)} = \mathbf{B}(t) \cdot \mathbf{Y}(t) \cdot \mathbf{S} \cdot \boldsymbol{1} - \mathbf{A}_1^{-1} \cdot \mathbf{S}_1{}^{\mathrm{T}} \cdot \boldsymbol{C}_1$

$$\hat{\boldsymbol{u}}(t - \boldsymbol{h}(t))_{(+)} = \begin{bmatrix} \begin{bmatrix} 1.32 \\ 1.32 \\ 1.32 \\ 1.32 \end{bmatrix} \\ \begin{bmatrix} 1.32 \\ 1.32 \\ 1.32 \\ 1.32 \end{bmatrix} \\ \begin{bmatrix} 1.32 \\ 1.32 \\ 1.32 \\ 1.32 \end{bmatrix} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} 0, 32 \\ 0, 32 \\ 0, 32 \\ 0, 32 \end{bmatrix} \\ \begin{bmatrix} 0, 32 \\ 0, 32 \\ 0, 32 \\ 0, 32 \end{bmatrix} \\ \begin{bmatrix} 0, 32 \\ 0, 32 \\ 0, 32 \\ 0, 32 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{bmatrix} \\ \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{bmatrix} \\ \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{bmatrix} \end{bmatrix}$$

- acc. to (2.39) $\hat{z}(t - \boldsymbol{\omega}) = \mathbf{A}_2^{-1} \cdot \mathbf{S}_2^{\mathrm{T}} \cdot \boldsymbol{C}_1$

$$\hat{\mathbf{z}}(t - \boldsymbol{\omega}) = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0, 32 \\ 0, 32 \\ 0, 32 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0, 0 \\ 0.0 \end{bmatrix}$$

It is worth observing that in the case of identical input data for the system of reservoirs and identical parameters related to WTPs, there are no transfers among reservoirs. The end reservoir filling level (for $W = 10\,[s]$) should be determined from Eq. 2.37), taking into account the values obtained for the vector of controlled outflows from reservoirs and the vector of transfers among reservoirs.

$$\hat{\boldsymbol{x}}(W) = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$$

The quality coefficient value in the discussed time interval $W = 10\,[s]$ for the assumed input data is: $F_{\min} = 6.144$, while the percentage execution of the water demand function for a given WTP is, respectively:

$$\mathbf{Y}(W) = \begin{bmatrix} 75\,\% & 0 & 0 & 0 \\ 0 & 75\,\% & 0 & 0 \\ 0 & 0 & 75\,\% & 0 \\ 0 & 0 & 0 & 75\,\% \end{bmatrix}$$

A graphical illustration of the example is shown in the diagrams (Fig. 2.4) obtained as a result of the system operation simulation carried out using an original application constructed in the Matlab/Simulink environment (Fig. 2.3). Each attempt to change the optimal control values obtained from the system operation simulation leads to an increase in the value of the assumed quality coefficient (2.10), or to a violation of the conditions imposed on reservoir end states (2.26). This proves the correctness of the solution, and ensures that it is optimum with regard to the coefficient and further assumptions applied.

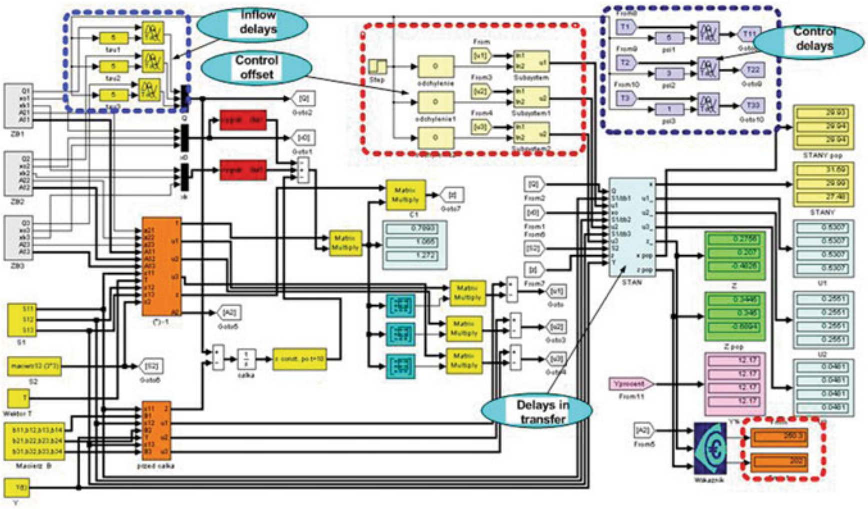

Fig. 2.3 Diagram of an analogue/digital simulation (Matlab/Simulink)

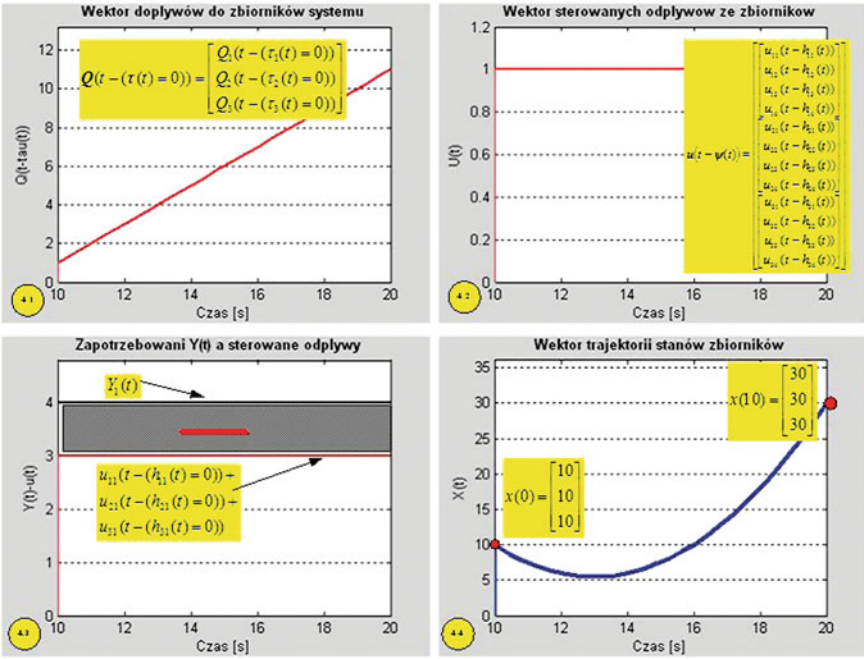

Fig. 2.4 Simulation results for a hypothetical scenario of events

Since the input data for the system operation simulation are identical for all reservoirs and consumers (Water Treatment Plants—WTP), the diagrams obtained are also identical for each of the reservoirs.

- on the left (4.1), the diagram shows the reservoir inflow vector,
- diagram (4.2) shows the trajectories of controlled outflows,
- diagram (4.3) illustrates the performance of the WTP demand functions, and
- diagram (4.4) shows reservoir state trajectories for the optimisation period

Hipothetical scenario of events. Example 2 The next stage in checking the optimality and correctness of the solution obtained involves introducing:

- delays in the vector of water inflows to reservoirs

$$\boldsymbol{Q}^P(t-\boldsymbol{\tau}(t)) = \begin{bmatrix} (t-(\tau_1 = 1s) + 1) \\ (t-(\tau_2 = 3s) + 1) \\ (t-(\tau_3 = 5s) + 1) \end{bmatrix} \left[\mathrm{m}^3/\mathrm{s}\right]$$

- transport-related delays in the controlled water outflows from reservoirs to consumers (WTP plants):

$$
\hat{\boldsymbol{u}}(t-\boldsymbol{h}(t))_{(+)} = \begin{bmatrix} \begin{bmatrix} u_{11}(t-h_{11}(t)=5s) \\ u_{12}(t-h_{12}(t)=5s) \\ u_{13}(t-h_{13}(t)=5s) \\ u_{14}(t-h_{14}(t)=5s) \end{bmatrix} \\ \begin{bmatrix} u_{11}(t-h_{11}(t)=3s) \\ u_{12}(t-h_{12}(t)=3s) \\ u_{13}(t-h_{13}(t)=3s) \\ u_{14}(t-h_{14}(t)=3s) \end{bmatrix} \\ \begin{bmatrix} u_{11}(t-h_{11}(t)=1s) \\ u_{12}(t-h_{12}(t)=1s) \\ u_{13}(t-h_{13}(t)=1s) \\ u_{14}(t-h_{14}(t)=1s) \end{bmatrix} \end{bmatrix}
$$

- other parameters remain unchanged

A graphical illustration of the example is shown in Fig. 2.5.

- Diagram (5.1) shows inflows to reservoirs taking into account delays during water flow through reservoirs.
- Diagram (5.2) contains the trajectories of controlled outflows as seen by the WTP water consumer. From this perspective, we see the impact of transport-related delays in water distribution from reservoirs to consumers.
- Diagram (5.3) clearly indicates that, for example, in different time intervals the demand function is carried out by 1, 2 or 3 reservoirs.
- Diagram (5.4) shows the significant trajectories of reservoir states, thus confirming that condition (26) is satisfied.
- Diagram (5.5) shows the trajectories of transfers among reservoirs.
- Diagram (5.6) demonstrates the directions of transfers among reservoirs (assumed and actual).

The quality coefficient value in the time interval discussed for assumed delays and input data is $F_{\min} = 84, 56$, while the percentage execution of water demand function for a WTP is, respectively:

$$
\mathbf{Y}(W) = \begin{bmatrix} 12.08\,\% & 0 & 0 & 0 \\ 0 & 12.08\,\% & 0 & 0 \\ 0 & 0 & 12.08\,\% & 0 \\ 0 & 0 & 0 & 12.08\,\% \end{bmatrix}
$$

Reduction of the quality coefficient value and percentage execution of the water demand function in WTP plants is dictated by the assumed transport-related delay in water delivery from the reservoirs to the WTPs. Water is distributed from reservoirs immediately as a result of receiving the optimising task solution, while, due to transport-related delays, the water is delivered to its consumer after a given time (with a delay) which, considering the quality coefficient, affects its value. This is so because, in different time intervals during the optimisation horizon, the WTP plants

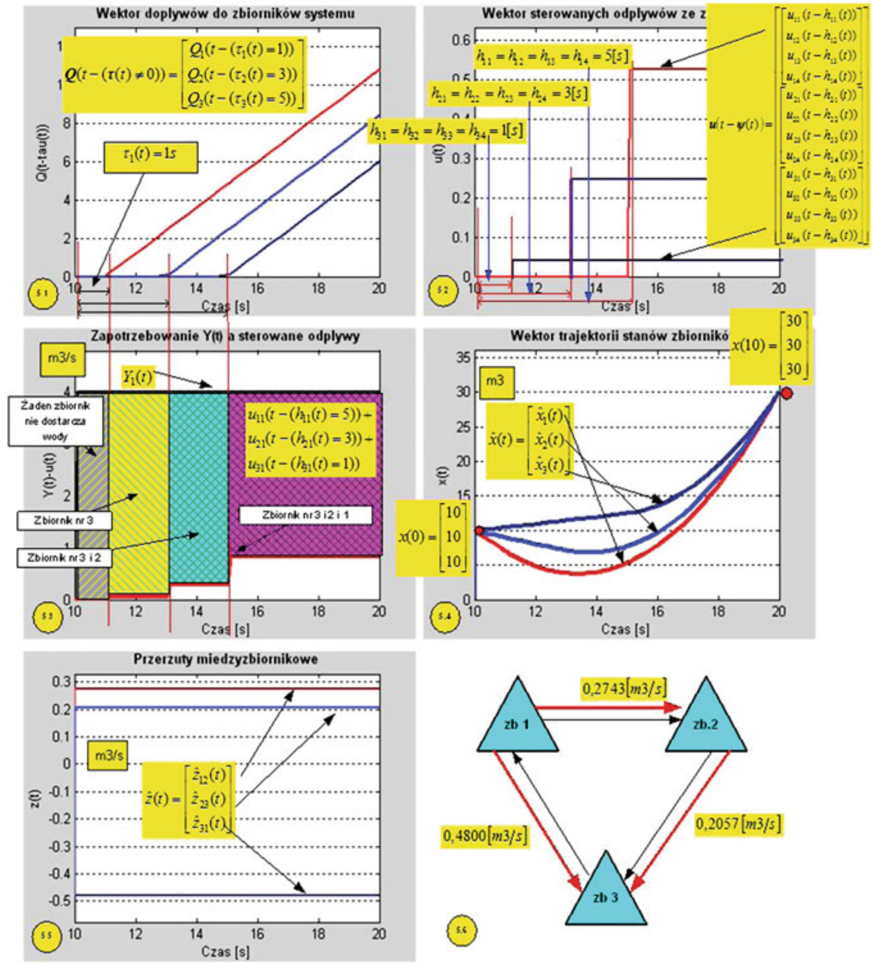

Fig. 2.5 Simulation results for a hypothetical scenario of events

either do not receive water at all or receive water from one, two or three reservoirs. In contrast, transport-related delays between reservoirs and water consumers (the WTP plants) have no impact on the end states in reservoirs and, as a result of this, condition (2.26) is always satisfied.

A completely different situation is observed when looking at the occurrence of delays during transfers among reservoirs. Delays in water transport among reservoirs affect the values of end states in reservoirs and, if they occur, condition (2.26) is not immediately satisfied. It is thus necessary to correct the values of water transfers among reservoirs in time, which does not affect the change in values of previously calculated controlled outflows from reservoirs to the WTP plants.

- examples of transport-related delays in water transfers among the system reservoirs, resulting from reservoir locations in the field;

$$\boldsymbol{\omega}^{\mathrm{T}} = \left[\begin{bmatrix} \omega_{12} = 2s \\ \omega_{21} = 4s \end{bmatrix} \begin{bmatrix} \omega_{23} = 4s \\ \omega_{32} = 6s \end{bmatrix} \begin{bmatrix} \omega_{31} = 2.5s \\ \omega_{13} = 3s \end{bmatrix} \right]$$

other parameters remain unchanged.

A graphical illustration of the example is shown in Figs. 2.6 and 2.7. Diagrams 6.1, 6.2 and 6.3 are the same as Fig. 2.5. Diagram 6.4 shows the trajectories of transfers among reservoirs, in which delays in water transport among reservoirs have been taken into account. Diagram 6.5 presents the trajectories of reservoir states showing

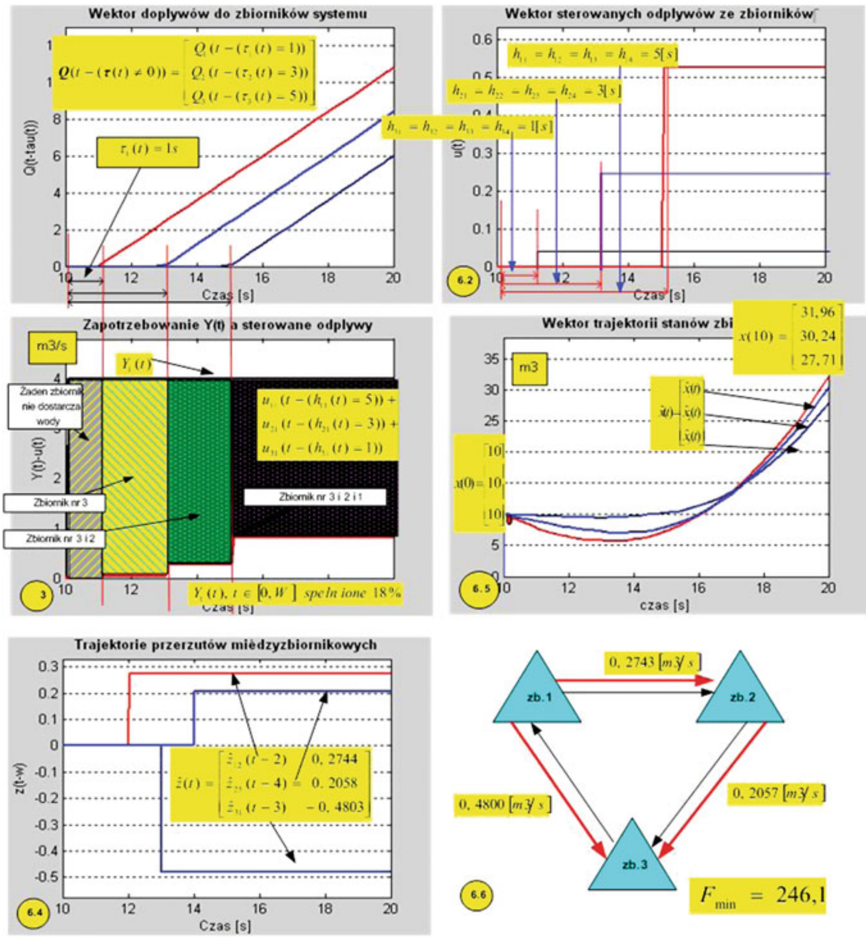

Fig. 2.6 Simulation results for a hypothetical scenario of events

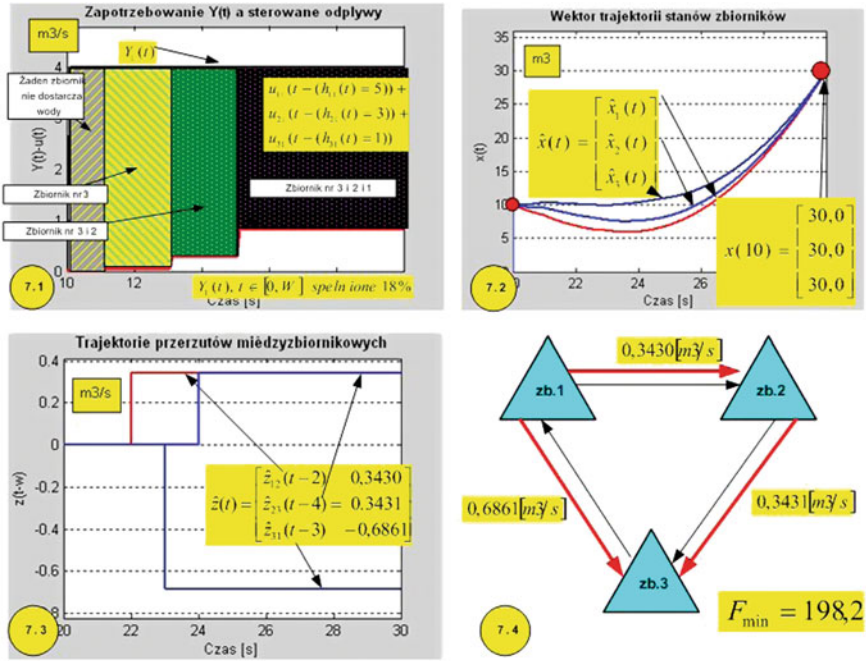

Fig. 2.7 Simulation results for a hypothetical scenario of events

those values which do not satisfy condition (2.26) concerning the end values of state trajectories at the instant of optimisation end. !!!!!. The quality coefficient value reaches $F_{\min} = 246, 1$. The further procedure satisfying condition (2.26) is based on formulas (2.43) and (2.44) and comes down to determining the corrections to the values of transfers among reservoirs and their duration so as to ensure that consequently, at the instant of optimisation end, condition (2.26) for trajectories of reservoir system states is satisfied. The graphical illustration is shown in Fig. 2.7.

2.1.5 Summary

The following conclusions may be derived as a result of many further simulations, carried out for systems characterised by various structure of connections with reference both to reservoirs and conurbations, and to transfers among reservoirs and different sets of delays in inflows, outflows and transfers:

1. The possibility of including and applying water transfers among reservoirs significantly affects the operation of a system of combined reservoirs, mainly with respect to leaving the reservoir end states at the required levels (condition (2.26)), and that these states will be reached with a minimum value for the quality coeffi-

cient (2.10). The possibility to take into account delays related to water transfers among reservoirs considerably raises the substantive aspect of the solution.

2. Cooperation of a system of reservoirs in a configuration without transfers among reservoirs comes down to the operation of reservoirs which have one shared purpose—satisfy the water demands imposed on the system. None of the reservoirs *sees* other reservoirs in the system while satisfying the system's water demands assigned to that particular reservoir. In some cases this cooperation may lead to a situation in which, within a system of reservoirs working together, some of the reservoirs will remain with very low end states/levels with unfavourable, low predicted inflow and after optimisation time *W*. This unfavourable effect may be mitigated as a result of including transfers among reservoirs. According to the optimising task conditions, these transfers will be selected (for value and transfer direction) so as to ensure the required end states for the system reservoirs at a given vector of predicted inflows to the reservoir system.

Management of Complex Multi-reservoir Water Distribution
Systems using Advanced Control Theoretic Tools and
Techniques
Chmielowski, W.
2013, VI, 85 p. 22 illus., Softcover
ISBN: 978-3-319-00238-5