2 Metric and Topological Spaces I

A key to *rigorous* multivariable calculus is a basic understanding of point set topology in the framework of metric spaces. Covering these basic concepts is the purpose of this chapter. We will see that studying these concepts in detail will really pay off in the chapters below. While studying metric spaces, we will discover certain concepts which are independent of metric, and seem to beg for a more general context. This is why, in the process, we will introduce *topological spaces* as well.

1 Basics

1.1

Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and $+\infty$. A *metric space* is a set X endowed with a *metric* (or *distance function*, briefly *distance*) $d : X \times X \to \mathbb{R}_+$ such that

(M1) $d(x, y) = 0 \quad \text{if and only if} \quad x = y$,
(M2) $d(x, y) = d(y, x)$, and
(M3) $d(x, y) + d(y, z) \leq d(x, z)$.

Condition (M3) is called the *triangle inequality*; the reader will easily guess why. The elements of a metric space are usually referred to as *points*.

Very often one considers distance functions which take on finite values only, but allowing infinite distances comes in handy sometimes.

1.1.1 Examples

(a) The set $\mathbb{R}$ of real numbers with the distance function $d(x, y) = |x - y|$.
(b) The set (plane) $\mathbb{C}$ of complex numbers, again with the distance $|x - y|$; note, however, that here the fact that it satisfies the triangle inequality is much less trivial than in the previous case (see Theorem 1.3 of Chapter 1).
(c) The Euclidean space $\mathbb{R}_m = \{(x_1, \ldots, x_m) \mid x_j \in \mathbb{R}\}$

I. Kriz and A. Pultr, *Introduction to Mathematical Analysis*,
DOI 10.1007/978-3-0348-0636-7_2, © Springer Basel 2013

$$d((x_1, \dots, x_m), (y_1, \dots, y_m)) = \sqrt{\sum (x_j - y_j)^2}.$$

Comment: In linear algebra, there are good reasons for distinguishing row and column vectors, and equally good reasons why the ordinary Eucliean space $\mathbb{R}^n$ should consist of *column* vectors. This is the reason why we used the subscript $\mathbb{R}_n$ above for row vectors, which are easier to write down (compare with A.7.3). From the point of view of metric and topological spaces, however, the distinction between row and column vectors has no meaning. Because of that, in this chapter, we will use the symbols $\mathbb{R}^n$ and $\mathbb{R}_n$ interchangably, not distinguishing between row and column vectors.

(d) $C(\langle a, b\rangle)$, the set of all continuous real functions on the interval $\langle a, b\rangle$, with

$$d(f, g) = \max_x |f(x) - g(x)|.$$

(e) The set $F(X)$ of all bounded real functions on a set X with

$$d(f, g) = \sup_x |f(x) - g(x)|.$$

(f) The unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

where for two points $P, Q \in S^1$, $d(P, Q)$ is the lesser of the two angles between the lines $\{tP|t \in \mathbb{R}\}$ and $\{tQ|t \in \mathbb{R}\}$.

(g) Any set S with the metric given by $d(x, y) = 0$ if $x = y \in S$ and $d(x, y) = 1$ if $x \neq y \in S$. This is known as the *discrete* space.

1.2 Norms

The metrics in Examples 1.1.1 (a)–(e) in fact all come from a more special situation, which plays an especially important role. A *norm* on a vector space V (over real or complex numbers) is a mapping $\| - \| : V \to \mathbb{R}$ such that

(1) $\|x\| \geq 0$, and $\|x\| = 0$ only if $x = \mathbf{o}$,

(2) $\|x + y\| \leq \|x\| + \|y\|$, and

(3) $\|\alpha x\| = |\alpha| \cdot \|x\|$.

1.2.1

A normed vector space is a (real or complex) vector space V provided with a norm. (The term *normed linear space* is also common.) Since we have

$$\|\mathbf{x} - \mathbf{z}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{y} - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|,$$

the function $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is a metric on V, called the *metric associated with the norm*. In this sense, we can always view a normed linear space as a metric space.

1.2.2 Examples

1. Any of the following formulas yields a norm in $\mathbb{R}^n$.
   (a) $\|\mathbf{x}\| = \max x_j$,
   (b) $\|\mathbf{x}\| = \sum |x_j|$,
   (c) $\|\mathbf{x}\| = \sqrt{\sum x_j^2}$.
   Notice that (c) gives the metric space in Example 1.1.1 (c).
2. In the space of bounded real functions on a set X we can consider the norm

$$\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}.$$

The associated metric gives rise to Example 1.1.1 (e) above.

1.2.3 A particularly important example

Example 1.2.2 (c) is in fact, a special case of the following construction: On a (real or complex) vector space with an inner product (see 4.2 of Appendix A), we have a norm

$$\|\mathbf{x}\| = \sqrt{\mathbf{xx}}.$$

Indeed: (1) of 1.2 is obvious. Further, by the Cauchy-Schwarz inequality (see 4.4 of Appendix A),

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y})(\mathbf{x} + \mathbf{y}) &= \mathbf{xx} + \mathbf{xy} + \mathbf{yx} + \mathbf{yy} \\ &= |\mathbf{xx} + \mathbf{xy} + \mathbf{yx} + \mathbf{yy}| \leq \|\mathbf{x}\|^2 + |\mathbf{xy}| + |\mathbf{yx}| + \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.\end{aligned}$$

Finally, $\|\alpha\mathbf{x}\| = \sqrt{(\alpha\mathbf{x})(\alpha\mathbf{x})} = \sqrt{\alpha\overline{\alpha}(\mathbf{xx})} = |\alpha| \cdot \|\mathbf{x}\|$. □

1.3 Convergence

A sequence $x_1, x_2, \ldots$ of points of metric space *converges* to a point x whenever for every $\varepsilon > 0$, there exists an n_0 such that for all $n \geq n_0$, we have $d(x_n, x) < \varepsilon$. This is expressed by writing

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad \lim_n x_n = x \quad \text{or just} \quad \lim x_n = x.$$

We then speak of a *convergent* sequence. Note that obviously
(*) *any subsequence* $(x_{k_n})_n$ *of a convergent sequence converges to the same limit.*

1.3.1 Examples

(a) The usual convergence in $\mathbb{R}$ or $\mathbb{C}$.
(b) Consider the examples in 1.1.1 (d) and (e). Realize that the convergence of a sequence of functions $f_1, f_2, \dots$ in these spaces is what one usually calls uniform convergence of functions.

1.4

Two metrics d_1, d_2 on the same set X are said to be *equivalent* if there exist positive real numbers α, β such that for every $x, y \in X$,

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

Note that we have an obvious

1.4.1 Observation. *If d_1 and d_2 are equivalent then $(x_n)_n$ converges in (X, d_1) if and only if it converges in (X, d_2).*

1.5

Let (X, d) and (Y, d') be metric spaces. A map $f : X \to Y$ is said to be *continuous* if

for every $x \in X$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that, for every y in X,

$$d(x, y) < \delta \quad \Rightarrow \quad d'(f(x), f(y)) < \varepsilon. \tag{ct}$$

Later on we will need a stronger concept: a mapping $f : X \to Y$ is said to be *uniformly continuous* if

for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, y in X

$$d(x, y) < \delta \quad \Rightarrow \quad d'(f(x), f(y)) < \varepsilon. \tag{uct}$$

Note the subtle difference between the two concepts. In the former the δ can depend on x, while in the latter it depends on the ε only. For example,

$$f = (x \mapsto x^2) : \mathbb{R} \to \mathbb{R}$$

is continuous but not uniformly continuous.

It is easy to prove

1.5.1 Proposition. *A composition $g \circ f$ of continuous (resp. uniformly continuous) maps f and g is continuous (resp. uniformly continuous).*

1.5.2

Here is another easy but important

Observation. *Let d, d_1 be equivalent metrics on X and let d', d'_1 be equivalent metrics on Y. Then a map $f : X \to Y$ is continuous (resp. uniformly continuous) with respect to d, d' if and only if it is continuous (resp. uniformly continuous) with respect to d_1, d'_1.*

1.6 Proposition. *A map $f : (X, d) \to (Y, d')$ is continuous if and only if for every convergent sequence $(x_n)_n$ in (X, d), the sequence $(f(x_n))_n$ is convergent and*

$$f(\lim x_n) = \lim f(x_n).$$

(Compare with Proposition 3.2 of Chapter 1.)

Proof. $\Rightarrow$: Let $\lim x_n = x$. Consider the $\delta > 0$ from (ct) taken for the x and an $\varepsilon > 0$. There is an n_0 such that $n \geq n_0$ implies $d(x_n, x) < \delta$. Then for $n \geq n_0$, $d'(f(x_n)), f(x)) < \varepsilon$.

$\Leftarrow$: Suppose f is not continuous. Then there is an $x \in X$ and an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists an $x(\delta)$ such that $d(x(\delta), x) < \delta$ while $d'(f(x(\delta)) \geq \varepsilon_0$. Now set $x_n = x(\frac{1}{n})$; obviously $\lim x_n = x$ and $(f(x_n))_n$ does not converge to $f(x)$. ☐

2 Subspaces and products

2.1

Let (X, d) be a metric space and let $X' \subseteq X$ be an arbitrary subset. Obviously (X', d') where d' is d restricted to $X' \times X'$ is a metric space again.

Examples.

(a) Intervals in the real line.

(b) More generally, the typical subspaces of the Euclidean space $\mathbb{R}^m$ one usually works with: n-dimensional intervals (by which we mean cartesian products of n-tuples of intervals), polyhedra, balls, spheres, etc.

(c) The space $C(\langle a, b\rangle)$ from 1.1.1.(d) is a subspace of the $F(\langle a, b\rangle)$ from 1.1.1.(e).

Convention. Unless otherwise stated we will think of subsets of spaces automatically as subspaces.

2.1.1 Observations. *1. Let (X', d') be a subspace of (X, d). Then the embedding map $j = (x \mapsto x) : X' \to X$ is uniformly continuous. Consequently, a restriction $f|X' : (X', d') \to (Y, \overline{d})$ of a continuous (resp. uniformly continuous) $f : (X, d) \to (Y, \overline{d})$ is continuous (resp. uniformly continuous).*
2. Let $f : (X, d) \to (Y, \overline{d})$ be a continuous (resp. uniformly continuous) map and let $Y' \subseteq Y$ be a subspace such that $f[X] \subseteq Y'$. Then $f' = (x \mapsto f(x)) : (X, d) \to (Y, \overline{d}')$ is continuous (resp. uniformly continuous).

Proof. 1. For $\varepsilon > 0$ take $\delta = \varepsilon$. For the consequence recall 1.5.1 and the fact that $f|X' = fj$.
2. For x and $\varepsilon > 0$ use the same δ as for f. □

2.2

Let (X_i, d_i), $i = 1, \dots, m$, be metric spaces. On the cartesian product $\prod_{i=1}^{m} X_i = X_1 \times \cdots \times X_m$ consider the following distances:

$$\rho((x_1, \dots, x_m), (y_1, \dots, y_m)) = \sqrt{\sum_{i=1}^{m} d_i(x_i, y_i)^2},$$

$$\sigma((x_1, \dots, x_m), (y_1, \dots, y_m)) = \sum_{i=1}^{m} d_i(x_i, y_i), \text{ and}$$

$$d((x_1, \dots, x_m), (y_1, \dots, y_m)) = \max d_i(x_i, y_i).$$

(σ and d satisfy (M1), (M2) and (M3) obviously. The triangle inequality of ρ needs some simple reasoning – one can use, for instance, Theorem 4.4 from Appendix A. In fact, we will rarely use this metric in the context of the topology of multivariable functions. However, note its geometrical significance: it yields the standard Pythagorean metric in the space $\mathbb{R}^m$ viewed as $\mathbb{R} \times \cdots \times \mathbb{R}$.)

2.2.1 Proposition. *The distance functions ρ, σ and d are equivalent metrics.*

Proof.

$$\rho((x_i)_i, (y_i)_i) \leq \sqrt{\sum_{i=1}^{m} \max_j d_j(x_j, y_j)^2} = \sqrt{n} \cdot d((x_i)_i, (y_i)_i).$$

Obviously $d((x_i)_i, (y_i)_i) \leq \rho((x_i)_i, (y_i)_i), \sigma((x_i)_i, (y_i)_i)$ and finally $\sigma((x_i)_i, (y_i)_i) \leq \sum_{i=1}^{m} \max_j d_j(x_j, y_j) = n \cdot d((x_i)_i, (y_i)_i)$.

2.2.2
The space $\prod X_i$ endowed with any of the metrics ρ, σ, d (typically, by d) will be referred to as the product of the spaces (X_i, d_i), $i = 1, \dots, m$.

Theorem. *1. The projections* $p_j = ((X_1, \dots, x_m) \mapsto x_j) : \prod_i (X_i, d_i) \to (X_j, d_j)$ *are uniformly continuous.*

2. A sequence

$$(x_1^1 \dots, x_m^1), (x_1^2 \dots, x_m^2), (x_1^3 \dots, x_m^3), \dots \qquad (*)$$

converges in $\prod(X_i, d_i)$ *if and only if each of the sequences*

$$x_j^1, x_j^2, x_j^3 \dots \qquad (**)$$

converges in the respective (X_j, d_j).

3. Let $f_j : (Y, d) \to (X_j, d_j)$ *be continuous (resp. uniformly continuous). Then the mapping*

$$f = (y \mapsto (f_1(y), \dots, f_m(y))) : (Y, d') \to \prod(X_i, d_i)$$

(the unique mapping such that $p_j f = f_j$ *for all* j*) is continuous (resp.uniformly continuous).*

Proof. 1. We have $d((x_i)_i, (y_i)_i) \geq d_j(x_j, y_j)$. Thus, it suffices to take $\delta = \varepsilon$.
2. If $(*)$ converges then each $(**)$ converges by 1 and 1.6. For $\varepsilon > 0$ choose n_j such that for $k \geq n_j$, $d_j(x_j^k, x_j) < \varepsilon$, and consider $n_0 = \max_j n_j$. Then for $k \geq n_0$, $d_j(x_j^k, x_j) < \varepsilon$ for all j, and hence $\max d_j(x_j^k, x_j) < \varepsilon$.
3. immediately follows from 2 and 1.6.

3 Some topological concepts

3.1 Neighborhoods

First, define the ε-ball with center x as

$$\Omega(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}.$$

A subset $U \subseteq X$ is a *neighborhood* of a point $x \in X$ if there exists an $\varepsilon > 0$ such that

$$\Omega(x, \varepsilon) \subseteq U.$$

Remark: While the concept of an ε-ball depends on the concrete metric, the concept of neighborhood does not change if we replace a metric by an equivalent one. In fact, we can change the metric even much more radically – see Exercise (5) below.

3.1.1 Observations. *1. If U is a neighborhood of x and $U \subseteq V$ then V is a neighborhood of x.*
2. If U_1, U_2 are neighborhoods of x then so is $U_1 \cap U_2$.

(1: for V use the same $\Omega(x, \varepsilon)$. 2: if $\Omega(x, \varepsilon_i) \subseteq U_i$ then $\Omega(x, \min(\varepsilon_1, \varepsilon_2)) \subseteq U_1 \cap U_2$.)

3.2 Open and closed sets

A subset $U \subseteq (X, d)$ is *open* if it is a neighborhood of each of its points.

A subset $A \subseteq (X, d)$ is *closed* if for every sequence $(x_n)_n$, $x_n \in A$ convergent in (X, d), the limit $\lim x_n$ is in A.

3.2.1 Proposition. *1. X and $\emptyset$ are open. If U and V are open then $U \cap V$ is open, and if U_i, $i \in J$, are open (J arbitrary) then $\bigcup_{i \in J} U_i$ is open.*
2. U is open if and only if $X \smallsetminus U$ is closed.
3. X and $\emptyset$ are closed. If A and B are closed then $A \cup B$ is closed, and if A_i, $i \in J$, are closed then $\bigcup_{i \in J} A_i$ is closed.

Proof. 1 is straightforward (use 3.1.1).

2: Let U be open, $A = X \smallsetminus U$. The limit x of a sequence $(x_n)_n$ that is all in A cannot be in U since there is an $\varepsilon > 0$ such that $\Omega(x, \varepsilon) \subseteq U$, and the x_n's with sufficiently large n have to be in such $\Omega(x, \varepsilon)$.

On the other hand, if U is not open, then there is an $x \in U$ such that for every n, $\Omega(x, \frac{1}{n}) \nsubseteq U$. Therefore, we can choose points $x_n \in \Omega(x, \frac{1}{n}) \cap A$ with $x = \lim x_n \in U = X \smallsetminus A$.

3 follows from 1.3 and the formulas relating intersections and unions with complements. $\square$

3.3 Closure

Let A be a general subset of a metric space $X = (X, d)$. For a point $x \in X$, define the distance of x from A by

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

Note that if $x \in A$ then $d(x, A) = 0$ but $d(x, A)$ can be 0 even if $x \notin A$.

The *closure* of a set A in (X, d) is the set

$$\overline{A} = \{x \mid d(x, A) = 0\}.$$

This definition seems to depend heavily on the distance function. But we have

3.3.1 Proposition. *1. The set $\overline{A}$ is closed, and it is the smallest closed set containing A. In other words,*

$$\overline{A} = \bigcap\{B \text{ closed} \mid A \subseteq B\}.$$

2. A point $x \in X$ is in $\overline{A}$ if and only if for each of its neighborhoods U, $U \cap A \neq \emptyset$ (in other words, if and only if for each open $U \ni x$, $U \cap A \neq \emptyset$).

Proof. $1: U = X \smallsetminus \overline{A}$ is open, since if $x \notin \overline{A}$ there is an $\varepsilon > 0$ such that $\Omega(x, 2\varepsilon) \cap A = \emptyset$ and hence by the triangle inequality $\Omega(x, \varepsilon) \cap \overline{A} = \emptyset$.

Let B be closed and $B \supseteq A$. Let $x \in \overline{A}$. For each n choose an $x_n \in A$ (and hence in B) such that $d(x, x_n) < \frac{1}{n}$. Then $x = \lim x_n$ is in B. The correctness of the formula follows from 3.2.1.

2 is obvious: in yet other words we are speaking about the balls $\Omega(x, \varepsilon)$ intersecting A. □

3.3.2 Proposition. *1. $\overline{\emptyset} = \emptyset$, $A \subseteq \overline{A}$, and $A \subseteq B \;\Rightarrow\; \overline{A} \subseteq \overline{B}$,*
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$, and
3. $\overline{\overline{A}} = \overline{A}$.

Proof. 1 is trivial.

2: By 1, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Now let $x \in \overline{A \cup B}$; x is or is not in $\overline{A}$. In the latter case, all sufficiently close elements from $A \cup B$ have to be in B and hence $x \in \overline{B}$.

3: By 3.3.1 1, $\overline{A}$ is closed and since it contains $B = \overline{A}$, it also contains $\overline{B} = \overline{\overline{A}}$. □

We also define the *interior* $\mathrm{Int}(A) = X \sim \overline{X \smallsetminus A}$. The interior of A is also denoted by $A°$. It immediately follows from Proposition 3.3.1 that the interior is the union of all open sets contained in A. The *boundary* of A is defined as $\partial A = \overline{A} \smallsetminus \mathrm{Int}(A)$.

3.4

Continuity can be expressed in terms of the concepts introduced in this section. We have

Theorem. *The following statements on a mapping $f : (X, d) \to (Y, d')$ are equivalent.*

(1) *f is continuous.*

(2) *For every $\in X$ and every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f[U] \subseteq V$.*

(3) *For every U open in (Y, d') the preimage $f^{-1}[U]$ is open in (X, d).*

(4) *For every A closed in (Y, d') the preimage $f^{-1}[A]$ is closed in (X, d).*

(5) *For every subset $A \subseteq X$,*

$$f[\overline{A}] \subseteq \overline{f[A]}.$$

(6) *For every subset $B \subseteq Y$,*

$$f^{-1}[\overline{B}] \supseteq \overline{f^{-1}[B]}.$$

Proof. (1)⇒(2) : Let V be a neighborhood of $f(x)$ with $\Omega(f(x), \varepsilon) \subseteq V$. Choose a $\delta > 0$ as in (ct) for x and ε. Then $f[\Omega(x, \delta)] \subseteq \Omega(f(x), \varepsilon)$, and $\Omega(x, \delta)$ is a neighborhood of x.

(2)⇒(3) : If $U \subseteq Y$ is open and $x \in f^{-1}[U]$ then $f(x) \in U$ and U is a neighborhood. Hence there is a neighborhood V of x such that $f[V] \subseteq U$ and we have $x \in V \subseteq f^{-1}[U]$, making $f^{-1}[U]$ a neighborhood of x.

(3)⇔(4) by 3.2.1 2, since $f^{-1}[-]$ preserves complements.

(4)⇒(5) : We have

$$A \subseteq f^{-1}[[f[A]] \subseteq f^{-1}[\overline{f[A]}].$$

Since $f^{-1}[\overline{f[A]}]$ is closed, we have by 3.3.1 $\overline{A} \subseteq f^{-1}[\overline{f[A]}]$ and the statement follows.

(5)⇒(6) : We have, by (5), $f[\overline{f^{-1}[B]}] \subseteq \overline{f[f^{-1}[B]]} \subseteq \overline{B}$ and hence $\overline{f^{-1}[B]} \subseteq f^{-1}[\overline{B}]$.

(6)⇒(1) : If $f(y) \in \Omega(f(x), \varepsilon)$ then $f(y) \notin \overline{Y \smallsetminus \Omega(f(x), \varepsilon)}$ and hence $y \notin f^{-1}[\overline{B}]$ where $B = Y \smallsetminus \Omega(f(x), \varepsilon)$. Hence $y \notin \overline{f^{-1}[B]}$ and there is a $\delta > 0$ such that $\Omega(y, \delta) \cap f^{-1}[B] = \emptyset$. Thus if $d(x, y) > \delta$ then $f(y) \notin B$, that is, $f(y) \in \Omega(f(x), \varepsilon)$. □

3.5

A continuous mapping $f : (X, d) \to (Y, d')$ is called a *homeomorphism* if there is a continuous mapping $g : (Y, d') \to (X, d)$ such that

$$fg = \mathrm{id}_Y \quad \text{and} \quad gf = \mathrm{id}_X.$$

If there exists a homeomorphism $f : (X,d) \to (Y,d')$ we say that the spaces (X,d) and (Y,d') are *homeomorphic*.

Note that if d and d' are equivalent metrics then the identity map $\mathrm{id}_X : (X,d) \to (X,d')$ is a homeomorphism. But $\mathrm{id}_X : (X,d) \to (X,d')$ can be a homeomorphism even when d and d' are far from being equivalent (consider, e.g., the interval $\langle 0, \pi)$ with the standard metric d and with $d'(x,y) = |\tan x - \tan y|$).

A property of a space or a concept related to spaces is said to be *topological* if it is preserved under all homeomorphisms. For example, by Theorem 3.4, for a set to be a neighborhood of a point, or to be open resp, closed, or the closure, are topological concepts. By 1.6, convergence is a topological concept.

Continuity is a topological concept, but uniform continuity is not.

This suggests the possibility of formulating a notion of a space based only on topological properties. We will explore this in the next section.

4 First remarks on topology

Very often, a choice of metric is not really important. We may be interested just in continuity, and a concrete choice of metric may be somehow off the point. For example, note that the "natural" Pythagorean metric would have been a real burden in dealing with the product. Sometimes it even happens that one has a natural notion of continuity, or convergence, without having a metric defined first. It may even happen that there is no reasonable way to define a metric.

This leads to a more general notion of a space, called a *topological space*. The idea is to describe the structure of interest simply in distinguishing whether a subset $U \subseteq X$ containing x "surrounds" (is a neighborhood of) x, or declaring some subsets open resp. closed, or specifying an operator of closure. We will present here three variants of the definition, which turn out to be equivalent.

4.1

We will start with the neighborhood approach, which was historically the first one (introduced by Hausdorff in 1914). It is convenient to denote by $\mathfrak{P}(X)$ the *power set of* X, which means the set of all subsets of X (including the empty set and X). With every $x \in X$, one associates a set $\mathcal{U}(x) \subseteq \mathfrak{P}(X)$, called the system of the *neighborhoods of* x, satisfying the following axioms:

(1) For each $U \in \mathcal{U}(x)$, $x \in U$,
(2) If $U \in \mathcal{U}(x)$ and $U \subseteq V \subseteq X$ then $V \in \mathcal{U}(x)$,
(3) If $U, V \in \mathcal{U}(x)$ then $U \cap V \in \mathcal{U}(x)$, and
(4) For every $U \in \mathcal{U}(x)$ and every $y \in V$ there is a $V \in \mathcal{U}(x)$ such that $U \in \mathcal{U}(y)$.

One then *defines* a (possibly empty) subset U of X to be *open* if U is a neighborhood of each of its points. One defines a subset A of X to be *closed* if the complement

$X \smallsetminus A$ of A is open. The closure of a subset S of X is defined by the formula $\overline{S} = \{x \mid \forall U \in \mathcal{U}(x),\ U \cap S \neq \emptyset\}$.

4.2

Nowadays probably the most common approach to the structure of topology is to define *open* sets first as a set of subsets of X satisfying certain axioms. It may be perhaps less intuitive, but it turns out to be much simpler technically.

In this approach, a *topology* on a set X is a subset $\tau \subseteq \mathfrak{P}(X)$ satisfying
(1) $\emptyset, X \in \tau$,
(2) $U, V \in \tau \quad\Rightarrow\quad U \cap V \in \tau$,
(3) $U_i \in \tau,\ i \in J \quad\Rightarrow\quad \bigcup U_i \in \tau$.
In other words, we may simply say that a topology is a subset of the set $\mathfrak{P}(X)$ of all subsets of X which is closed under all unions and all *finite* intersections. (To include (1), we allow the union of an empty set of subsets of X, which is said to be $\emptyset$, and the intersection of an empty set of subsets of X, which is said to be X.)

One then defines a *closed* set as a complement of an open set; U is a *neighborhood* of x if there is an open V such that $x \in V \subseteq U$, and the *closure* is defined by the formula

$$\overline{A} = \bigcap\{B \mid A \subseteq B,\ B \text{ closed}\}.$$

A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Remark: It is possible to start equivalently with *closed* sets first and then define open sets as their complements; the axioms of closed sets are obtained by expressing the axioms for open sets in terms of their complements (see Exercise (9)).

4.3

Or, one can start with a closure operator $u : \mathfrak{P}(X) \to \mathfrak{P}(X)$ satisfying
(1) $u(\emptyset) = \emptyset$ and $A \subseteq u(A)$,
(2) $u(A \cup B) = u(A) \cup u(B)$ and
(3) $u(u(A)) = u(A)$.
A is declared *closed* if $u(A) = A$, the open sets are complements of the closed ones, and U is a neighborhood of x if $x \notin u(X \smallsetminus U)$.

4.4

In fact one usually thinks of a topological space as a set endowed with all the above mentioned notions simultaneously, and the only question is which of them one considers primitive concepts and which are defined afterwards. The resulting structure is the same. (See the Exercises.)

4.5

A topology is not always obtained from a metric (if it is we speak of a metrizable space). Here are two rather easy examples.

(a) Take an infinite set X and declare $U \subseteq X$ to be open if either it is void or if $X \smallsetminus U$ is finite.

(b) Take a partially ordered set $(X, \leq)$ and declare U to be open if $U = \{x \mid \exists y \in U,\ x \leq y\}$. (Note: this topology is metrizable for certain special choices of partial orderings, but certainly not in general.)

Non-metrizable spaces of importance are of course seldom defined as easily as this. But it should be noted that many non-metrizable spaces are of interest today.

4.6

A mapping $f : X \to Y$ between topological spaces is *continuous* if for every $x \in X$ and every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f[U] \subseteq V$ (cf. (2) in Theorem 3.4). If we replace in 3.4 the metric definition of continuity (1) with the definition we just made, we have the following more general result:

Theorem. *Let X, Y be topological spaces. Then the following statements on a mapping $f : X \to Y$ are equivalent.*

(1) *f is continuous.*

(2) *For every U open in Y the preimage $f^{-1}[U]$ is open in X.*

(3) *For every A open in Y the preimage $f^{-1}[A]$ is closed in X.*

(4) *For every subset $A \subseteq X$,*

$$f[\overline{A}] \subseteq \overline{f[A]}.$$

(5) *For every subset $B \subseteq Y$,*

$$f^{-1}[\overline{B}] \supseteq \overline{f^{-1}[B]}.$$

Proof. Most of the implications can be proved by the same reasoning as in 3.4. The only one needing a simple adjustment is

(5)⇒(1): Let (5) hold and let V be a neighborhood of $f(x)$. Thus, $f(x) \notin \overline{Y \smallsetminus V}$, that is, $x \notin f^{-1}[\overline{Y \smallsetminus V}]$. Hence, $U = X \smallsetminus f^{-1}[Y \smallsetminus V] = f^{-1}[V]$ is a neighborhood of x, and $f[U] = ff^{-1}[V] \subseteq V$. □

4.7

The system of open sets τ constituting a topology is often determined by a so-called *basis*, which means a subset $\mathcal{B} \subseteq \tau$ such that

$$B_1, B_2 \in \mathcal{B} \quad \Rightarrow \quad B_1 \cap B_2 \in \mathcal{B} \quad \text{and}$$
$$\text{for every } \ U \in \tau, \quad U = \bigcup \{B \mid B \in \mathcal{B},\ B \subseteq U\}.$$

(For example, the set of all open intervals, or the set of all open intervals with rational endpoints are bases of the standard topology of the real line $\mathbb{R}$).

One may wish to define a topological space where some particular subsets are open, thus specifying a subset $\mathcal{S} \subseteq \mathfrak{P}(X)$ of such sets without any a priori properties. One easily sees that the smallest topology containing $\mathcal{S}$ is the set of all unions of finite intersections of elements of $\mathcal{S}$. Then one speaks of $\mathcal{S}$ as of a *subbasis* of the topology obtained.

The preimages of (finite) intersections are (finite) intersections, and preimages of unions are unions of preimages. Consequently we obtain from 4.6 an important

Observation. *A mapping $f : (X, \tau) \to (Y, \theta)$ is continuous if and only if there is a subbasis $\mathcal{S}$ of θ such that each $f^{-1}[S]$ with $S \in \mathcal{S}$ is open.*

(Thus e.g. to make sure a real function $f : X \to \mathbb{R}$ is continuous it suffices to check that all the $f^{-1}[(-\infty, a)]$ and $f^{-1}[(a. + \infty)]$ are open.)

4.8

Let (X, τ) be a topological space and let $Y \subseteq X$ be a subset. We define the *subspace* of (X, τ) carried (or induced) by Y as

$$(Y, \tau|Y) \quad \text{where} \quad \tau|Y = \{U \cap Y \mid U \in \tau\}.$$

Since for the embedding map $j : Y \to X$, $j^{-1}[U] = U$, the map j is continuous; furthermore, if $f : (Z, \theta) \to (X, \tau)$ is a continuous map such that $f[Z] \subseteq Y$ then the map $(z \mapsto f(z)) : (Z, \theta) \to (Y, \tau|Y)$ is continuous as well.

Note that this is in accordance with the concept of subspace in the metric case: the metric subspace (cf. 2.1) has the topology just described, obtained from the topology of the larger metric space.

4.8.1 Convention

Unless otherwise stated, the subsets of a topological space will be understood to be endowed with the induced topology, and we will subject the terminology to this convention. Thus we will speak of "connected subsets" or "compact subsets" etc (see below) or on the other hand of an 'open subspace" or "closed subspace", etc.

5 Connected spaces

One of the simplest notions defined for topological spaces is connectedness.

5.1

A topological space X is said to be *connected* if for any two open sets $U, V \subseteq X$ which satisfy $U \cap V = \emptyset$ and $U \cup V = X$, we have $U = \emptyset$ (and hence $V = X$), or $V = \emptyset$ (and hence $U = X$). It is also common, for a subset $S \subseteq X$, to say that S is connected if S is a connected topological space with respect to the induced topology. Note that this is equivalent to saying that for open sets $U, V \subseteq X$ such that $U \cap V \cap S = \emptyset$ and $U \cup V \supseteq S$, we have $U \supseteq S$ or $V \supseteq S$. The following observations are immediate.

5.1.1 Proposition. *Let X be a connected space and $f : X \to Y$ a continuous map which is onto. Then Y is connected.*

Proof. Suppose $U, Y \subseteq Y$ are open, $U \cap V = \emptyset$, $U \cup V = Y$. Then $f^{-1}[U] \cap f^{-1}[V] = \emptyset$, $f^{-1}[U] \cup f^{-1}[V] = X$, so $f^{-1}[U] = \emptyset$ or $f^{-1}[V] = \emptyset$, which implies $U = \emptyset$ or $V = \emptyset$ since f is onto. □

5.1.2 Proposition. *Let $S_i \subseteq X$, $i \in I$, and let each S_i be connected. Suppose further for every $i, j \in I$, there exist $i_0, \dots, i_k \in I$, $i_0 = i$, $i_k = j$ such that $S_{i_t} \cap S_{i_{t+1}} \neq \emptyset$. Then*

$$S = \bigcup_{i \in I} S_i$$

is connected.

Proof. Suppose U, V are open in X, $U \cup V \supseteq S$, $U \cap V \cap S = \emptyset$. Suppose further U is non-empty. Then there exists an $i \in I$ such that $U \cap S_i \neq \emptyset$, and hence $U \supseteq S_i$ since S_i is connected. Now select any $j \in I$ and let $i_0, \dots, i_k$ be as in the statement of the Proposition. By induction on t, we see that $U \cap S_{i_t} \neq \emptyset$, and hence $U \supseteq S_{i_t}$ since S_{i_t} is connected. Thus, $U \supseteq S_j$. Since $j \in I$ was arbitrary, $U \supseteq S$. □

5.1.3 Corollary. *A product $X \times Y$ of two connected metric spaces X, Y is connected.*

Proof. Choose a point $x \in X$ and consider the sets $S_0 = \{x\} \times Y$, $S_y = X \times \{y\}$ for $y \in Y$. Then S_i, $i \in Y \amalg \{0\}$, satisfy the assumptions of Proposition 5.1.2. □

5.1.4 Proposition. *The closure of a connected subset S of a topological space is connected.*

Proof. If $U, V \subseteq \overline{S}$ satisfy $U \cap V = \emptyset$, $U \cup V = \overline{S}$ and U, V are non-empty open in $\overline{S}$, then $U \cap S$, $V \cap S$ are non-empty and open in S, their union is S and their intersection is non-empty, contradicting the assumption that S is connected. □

5.2 Connectedness of the real numbers

The fact that the set $\mathbb{R}$ of all real numbers is connected is "intuitively obvious", but must be proved with care. Let us start with a preliminary result.

5.2.1 Lemma. *Every open set $U \subseteq \mathbb{R}$ is a union of countably (or finitely) many disjoint open intervals.*

Proof. We know that U is a union of countably many open intervals U_i, $i = 1, 2, \dots$ since open intervals (q_1, q_2), $q_1, q_2 \in \mathbb{Q}$, form a basis of the topology of $\mathbb{R}$. Note also that if V, W are open intervals and $V \cap W \neq \emptyset$, then $V \cup W$ is an open interval, and that an increasing union of open intervals is an open interval. Now consider an equivalence class on $\{1, 2, \dots\}$ where $i \sim j$ if and only if there exist $i_0, \dots, i_k$ such that $i_0 = i$, $i_k = j$ and $U_{i_t} \cap U_{i_{t+1}} \neq \emptyset$. Then the sets

$$\bigcup_{i \in C} U_i$$

where C are equivalence classes with respect to $\sim$ are disjoint open intervals whose union is U. □

5.2.2 Theorem. *The connected subsets of $\mathbb{R}$ are precisely (open, closed, half-open, bounded, unbounded, etc.) intervals.*

Proof. Let us first prove that intervals are connected. Let J be an interval. Suppose U, V are open in $\mathbb{R}$, $U \cap V \supseteq J$, $U \cap V \cap J = \emptyset$. Suppose U is non-empty. By Lemma 5.2.1, U is a disjoint union of countably many open intervals U_i, $i \in I \neq \emptyset$. Without loss of generality, none of the sets U_i is disjoint with J. Choose $i \in I$, and suppose $U_i = (a, b)$ does not contain J. Then $(a, b) \cup J$ is an interval containing but not equal to (a, b), so $a \in J$ or $b \in J$. Let, without loss of generality, $b \in J$. Then $b \notin V$, $b \notin U_j$, $j \neq i$, since V, U_j, $j \neq i$ are open and disjoint with U_i. Thus, $b \in J \smallsetminus (U \cup V)$, which is a contradiction.

On the other hand, suppose that $S \subseteq \mathbb{R}$ is connected but isn't an interval. Then there exist points $x < z < y$, $x, y \in S$, $z \notin S$. But then $S \subseteq (-\infty, z) \cup (z, \infty)$, which contradicts the assumption that S is connected. □

5.2.3 Corollary. *The Euclidean space $\mathbb{R}^n$ is connected.*

Proof. This follows from Theorem 5.2.2 and Corollary 5.1.3. □

5.3 Path-connected spaces

A topological space X is called *path-connected* if for any two points $x, y \in X$, there exists a continuous map $\phi : \langle 0, 1\rangle \to X$ such that $\phi(0) = x$, $\phi(1) = y$. By Theorem 5.2.2, Proposition 5.1.1 and Proposition 5.1.2, a path-connected space is connected. See Exercise (14) for an example of a closed subset of $\mathbb{R}^2$ which is connected but not path-connected.

5.3.1 Proposition. *Let $U \subseteq \mathbb{R}^n$ be a connected open set (with the induced topology). Then U is path-connected.*

Proof. If U is empty, it is clearly path-connected. Suppose U is non-empty. Choose a point $x \in U$. Let $V \subseteq U$ be the set of all points $y \in U$ for which there exists a continuous map $\phi : \langle 0, 1\rangle \to U$ such that $\phi(0) = x$, $\phi(1) = y$. We claim that V is open in U: this is the same as being open in $\mathbb{R}^n$. If ϕ is as above, $\Omega(y, \varepsilon) \subseteq U$, and $z \in \Omega(y, \varepsilon)$, extend ϕ to a map $\langle 0, 2\rangle \to U$ by putting $\phi(1+t) = tz + (1-t)y$ for $t \in \langle 0, 1\rangle$. Clearly ϕ is continuous, and defining $\psi : \langle 0, 1\rangle \to U$ by $\psi(t) = \phi(2t)$ shows $z \in V$.

We also claim, however, that V is closed in U: Let $y_n \to y$, $y_n \in V$, $y \in U$. Since U is open, there exists an $\varepsilon > 0$, $\Omega(y, \varepsilon) \subseteq U$. Then there exists an n such that $y_n \in \Omega(y, \varepsilon)$. Then we proceed the same way as above: Let $\phi : \langle 0, \rangle \to U$, $\phi(0) = x$, $\phi(1) = y_n$. Extend ϕ to a map $\langle 0, 2\rangle \to U$ by putting $\phi(1 + t) = ty + (1-t)y_n$ for $t \in \langle 0, 1\rangle$. Putting again $\psi(t) = \phi(2t)$ shows that $y \in V$.

Since $V \neq \emptyset$ (since $x \in V$), and since V is open and closed in U, we must have $V = U$, since U is connected. $\square$

5.4 Connected components

Let X be a topological space. Let $\sim$ be a relation on X where $x \sim y$ if and only if there exists a connected subset $S \subset X$ such that $x, y \in S$. Then $\sim$ is an equivalence relation (transitivity follows from Proposition 5.1.2). The equivalence classes of $\sim$ are called the *connected components* of X. Also by Proposition 5.1.2, connected components are connected subsets of X.

An immediate consequence of Proposition 5.1.4 is the following:

5.4.1 Lemma. *Connected components of X are closed subsets of X.* $\square$

Connected components may not be open: consider $\mathbb{Q}$ (with the topology induced from $\mathbb{R}$). Then the connected components are single points. We have, however,

5.4.2 Lemma. *Let $U \subseteq \mathbb{R}^n$ be an open set. Then the connected components of U are open in U (hence in $\mathbb{R}^n$).*

Proof. Let $x \in U$. Then there exists $\varepsilon > 0$ such that $\Omega(x,\varepsilon) \subseteq U$, but $\Omega(x,\varepsilon)$ is homeomorphic to $\mathbb{R}^n$ and hence connected by Corollary 5.2.3, so $\Omega(x,\varepsilon)$ is contained in the connected component of x. Since this is true for every point x, the connected components are open. □

5.5 A result on bounded closed intervals

The proof of the following result will seem, in nature, related to the proof of the fact that the real numbers are connected. While this is true, it turns out to be mainly due to special properties of the real numbers. The result itself is a reformulation of *compactness*, a notion which we will discuss in the next section. An understanding of this connection for general metric spaces, however, will have to be postponed until Chapter 9 below.

By an *open interval* (resp. bounded closed interval) in $\mathbb{R}^n$ we mean a set of the form $\prod_{k=1}^{n}(a_k, b_k)$ (resp. of the form $\prod_{k=1}^{n}\langle a_k, b_k\rangle$, $-\infty < a_k, b_k < \infty$).

Theorem. *For every bounded closed interval K in $\mathbb{R}^n$ and every set of open intervals S such that $K \subseteq \bigcup_{I\in S} I$, there exists a finite subset $F \subseteq S$ such that $J \subseteq \bigcup_{I\in F} I$.*

Proof. Let us first consider the case $n = 1$. Let $\langle a, b\rangle$ be contained in a union of a set S open intervals. Let $t \in \langle a, b\rangle$ be the supremum of the set M of all $s \in \langle a, b\rangle$ such that $\langle a, s\rangle$ is contained in a union of some finite subset of S. We want to prove that $t = b$. Assume, then, that $t < b$. Then there exists a $J \in S$ such that $t \in J$. On the other hand, by the definition of supremum, there exist $s_i \in M$ such that $s_i \nearrow t$. Then, for some i, $s_i \in J$. But we also know that there exists a finite subset $F \subseteq S$ whose union contains $\langle a, s_i\rangle$. Then the union of the finite subset $F \cup \{J\}$ contains $\langle a, x\rangle$ for every $x \in J$, contradicting $t = \sup M$.

Now let us consider general n. Assume, by induction, that the statement holds with n replaced by $n-1$. Let $K = \langle a_1, b_1\rangle \times \cdots \times \langle a_n, b_n\rangle$. Then for every point $x \in \langle a_1, b_1\rangle$, there exists, by the induction hypothesis, a finite subset $F_x \subset S$ such that $\{x\} \times \langle a_2, b_2\rangle \times \cdots \times \langle a_n, b_n\rangle \subseteq F_x$. Let I_x be the intersection of all the (1-dimensional) intervals I_1 where $I_1 \times \cdots \times I_n \in F_x$. Then $\langle a_1, b_1\rangle$ is contained in the union of the open intervals I_x, $x \in \langle a_1, b_1\rangle$, and hence there are finitely many points $x_1, \ldots, x_k \in \langle a_1, b_1\rangle$ such that $\langle a_1, b_1\rangle \subseteq \bigcup_{i=1}^{k} I_{x_i}$. Then K is contained in the union of the open intervals in $F_{x_1} \cup \cdots \cup F_{x_k}$. □

Corollary. *For every bounded closed interval K in $\mathbb{R}^n$ and every set of open sets Q such that $K \subseteq \bigcup_{I \in Q} I$, there exists a finite subset $F \subseteq Q$ such that $J \subseteq \bigcup_{I \in F} I$.*

(Apply the theorem to the set S of all open intervals which are contained in one of the open sets in Q.)

6 Compact metric spaces

6.1

A metric space X is said to be *compact* if each sequence $(x_n)_n$ in X contains a convergent subsequence. Thus, in particular, a bounded closed interval $\langle a, b\rangle$ in $\mathbb{R}$ is compact (recall Theorem 2.3 of Chapter 1).

6.2 Proposition. *1. A subspace of a compact space is compact if and only if it is closed.*

2. If $f : X \to Y$ is continuous then the image $f[A]$ of any compact $A \subseteq X$ is compact.

Proof. 1. Let A be a closed subspace of a compact X. Let $(x_n)_n$ be a sequence of points of A. There is a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \dots$ converging in X. Since A is closed, the limit is in A.
Now let A not be closed. Then there is a sequence $(x_n)_n$ of elements of A convergent in X, with the limit x in $X \setminus A$; since each subsequence converges to x, there is none converging to a point in A.

2. Let $(y_n)_n$ be a sequence in $f[A]$. Choose $x_i \in A$ such that $y_i = f(x_i)$. Since A is compact we have a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \dots$ converging to an $x \in A$. Then by 1.5, $y_{k_1}, y_{k_2}, y_{k_3}, \dots$ converges to $f(x)$. □

6.2.1

Note that from the second part of the proof of the first statement we obtain an immediate

Observation. *A compact subspace of any metric space X is closed in X.*

Remark. Thus we have a slightly surprising consequence: if X is compact, Y is a general metric space and if $f : X \to Y$ is a continuous mapping then, besides *preimages* of closed sets being closed, also the *images* of closed sets are closed. We will learn more about this phenomenon in Chapter 9 below. For now, let us record the following

6.2.2 Corollary. *Let $f : X \to Y$ be a continuous bijective (i.e. one to one and onto) map of metric spaces where X is compact. Then f is a homeomorphism.*

6.3 Proposition. *Let X be a compact metric space. Then for each continuous real function f on X there exist $x_1, x_2 \in X$ such that*

$$f(x_1) = \min\{f(x) \mid x \in X\} \quad \textit{and} \quad f(x_2) = \max\{f(x) \mid x \in X\}.$$

(Compare with 3.4 of Chapter 1.)

Proof. A compact subspace A of $\mathbb{R}$ has a minimal and a maximal point, namely $\inf A$ and $\sup A$ that are obviously limits of sequences in A. Apply to $A = f[X]$, compact by 6.2. □

6.4 Proposition. *(Finite) products of compact spaces are compact.*

Proof. We will begin with the product $X \times Y$ of two compact metric spaces - the extension to a general finite product follows by induction.

Let

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots \tag{*}$$

be a sequence of points of $X \times Y$. In X, choose a convergent subsequence $(x_{k_n})_n$ of $(x_n)_n$. Now take the sequence $(y_{k_n})_n$ in Y and choose a convergent subsequence $(y_{k_{r_n}})_n$. Then by 2.2.2.2 (and (1.2.1)),

$$(x_{k_{r_1}} y_{k_{r_n}}), (x_{k_{r_2}}, y_{k_{r_2}}), (x_{k_{r_3}}, y_{k_{r_3}}), \dots$$

is a convergent subsequence of (*). □

A metric space (X, d) is *bounded* if there exists a number K such that for all $x, y \in X$, $d(x, y) < K$. From the triangle inequality we immediately see that this is equivalent to any of the following statements:

there is a K such that for every x, $\quad X \subseteq \Omega(x, K)$,

for every x there is a K such that $\quad X \subseteq \Omega(x, K)$.

6.5 Theorem. *A subspace of the Euclidean space $\mathbb{R}^m$ is compact if and only if it is bounded and closed.*

Proof. I. From Theorem 2.3 of Chapter 1, we already know that a bounded closed interval is compact.

II. Now let X be a bounded closed subspace of $\mathbb{R}^m$. Since it is bounded there are intervals $\langle a_i, b_i\rangle$, $i = 1, \dots, m$, such that

$$X \subseteq J = \langle a_1, b_1\rangle \times \cdots \times \langle a_m, b_m\rangle.$$

By 6.4 and I, J is compact. The subspace X is closed in $\mathbb{R}^m$, hence in J, and hence it is compact by 6.2.

III. Let X not be closed in $\mathbb{R}^m$. Then it is not compact, by 6.2.1.

IV. Let X not be bounded. Choose arbitrarily x_1 and then x_n such that $d(x_1, x_n) > n$. A convergent sequence is always bounded (all but finitely many of its elements are in the ε-ball of the limit). Thus, $(x_n)_n$ cannot have a convergent subsequence as it has no bounded one. □

6.6

We have already observed that uniform continuity is a much stronger property than continuity (even the real function $x \mapsto x^2$ is not uniformly continuous). But the situation is different for compact spaces. We have

Theorem. *Let X, Y be metric spaces and let X be compact. Then a mapping $f : X \to Y$ is uniformly continuous if and only if it is continuous.*

(Compare with Theorem 3.5.1 of Chapter 1.)

Proof. Let f be continuous but not uniformly continuous. Negating the definition,

there is an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are $x(\delta), y(\delta)$ such that

$$d(x(\delta), y(\delta)) < \delta \quad \text{while} \quad d'(f(x(\delta)), f(y(\delta))) \geq \varepsilon_0.$$

Consider $x_n = x(\frac{1}{n})$ and $y_n = y(\frac{1}{n})$. Choose a convergent subsequence $(x_{k_n})_n$ of $(x_n)_n$ and a convergent subsequence $(y_{k_{r_n}})_n$ of $(y_{k_n})_n$, set $\widetilde{x}_n = x_{k_{r_n}}$ and $\widetilde{y}_n = y_{k_{r_n}}$, and finally $x = \lim \widetilde{x}_n$ and $y = \lim \widetilde{y}_n$. As $d(\widetilde{x}_n, \widetilde{y}_n) < \frac{1}{n}$, $x = y$. This is a contradiction since by continuity $f(x) = \lim f(\widetilde{x}_n)$ and $f(y) = \lim f(\widetilde{y}_n)$ and $d(f(\widetilde{x}_n), f(\widetilde{y}_n))$ is always at least ε_0. □

7 Completeness

7.1

A sequence $(x_n)_n$ in a metric space (X, d) is said to be *Cauchy* if

$$\forall \varepsilon > 0\ \exists n_0 \text{ such that } \forall m, n \geq n_0,\ \ d(x_m, x_n) < \varepsilon.$$

7.2 Proposition. *1. Every convergent sequence is Cauchy.*

2. Let a Cauchy sequence $(x_n)_n$ contain a convergent subsequence; then the whole sequence $(x_n)_n$ converges.

3. Every Cauchy sequence is bounded.

Proof. 1. Let $\lim x_n = x$. For $\varepsilon > 0$ choose an n_0 such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq n_0$. Then for $m, n \geq n_0$,

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

2. Let $(x_n)_n$ be Cauchy and let $(x_{k_n})_n$ be a subsequence converging to a point x. Choose an n_1 such that for $m, n \geq n_1$, $d(x_m, x_n) < \frac{\varepsilon}{2}$, and an n_2 such that for $n \geq n_2$, $d(x_{k_n}, x) < \frac{\varepsilon}{2}$. Set $n_0 = \max(n_1, n_2)$. Since $k_n \geq n$ we have, for $n \geq n_0$,

$$d(x_n, x) \leq d(x_n, x_{k_n}) + d(x_{k_n}, x) < \varepsilon.$$

3. Choose n_0 such that for $m, n \geq n_0$, $d(x_m, x_n) < 1$. Then for any n,

$$d(x, x_{n_0}) < 1 + \max_{k \leq n_0} d(x_{n_0}, x_k).$$ □

7.3

A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

7.3.1 Proposition. *A subspace A of a complete space X is complete if and only if it is closed.*

Proof. Let A be closed. If a sequence is Cauchy in A, it is Cauchy in X and hence convergent. Since A is closed, the limit of the sequence has to be in A.

If A is not closed there is a sequence $(x_n)_n$ with $x_n \in A$, convergent in X to an $x \in X \smallsetminus A$. Then $(x_n)_n$ is Cauchy in X and hence in A as well; but all of its subsequences converge to x and hence do not converge in A. □

7.4 Proposition. *A compact metric space is complete.*

Proof. Let $(x_n)_n$ be a Cauchy sequence in a compact metric space X. Then it has a convergent subsequence, and by 6.2 2, it converges. □

7.5 Theorem. *The Euclidean space $\mathbb{R}^m$ (in particular, the real line $\mathbb{R}$) is complete. Consequently, a subspace of $\mathbb{R}^m$ is complete if and only if it is closed.*

Proof. Let $(x_n)_n$ be a Cauchy sequence in $\mathbb{R}^m$. By 6.2 it is bounded and hence

$$\{x_n \mid n = 1, 2, \dots\} \subseteq J = \langle a_1, b_1 \rangle \times \cdots \times \langle a_m, b_m \rangle$$

for sufficiently large intervals $\langle a_j, b_j \rangle$. By 6.4 $(x_n)_n$ converges in J and hence it converges in $\mathbb{R}^m$. □

Remark. The special case of the real line is the well-known *Bolzano-Cauchy Theorem* (Theorem 2.4 of Chapter 1).

7.6

The following is the well-known *Banach Fixed Point Theorem*. At first sight it may seem that its use will be rather limited: the assumption is very strong. But the reader will be perhaps surprised by the generality of one of the applications in 3.3 of Chapter 6.

Theorem. *Let (X, d) be a complete metric space. Let $f : X \to X$ be a mapping such that there is a $q < 1$ with*

$$d(f(x), f(y)) \le q \cdot d(x, y) \qquad (*)$$

for all $x, y \in X$. Then there is precisely one $x \in X$ such that $f(x) = x$.

Proof. Choose any $x_1 \in X$ and then, inductively,

$$x_{n+1} = f(x_n).$$

Set $C = d(x_1, x_2)$. By the assumption we have

$$d(x_2, x_3) \le Cq,\ d(x_3, x_4) \le Cq^2,\ \dots\ ,\ d(x_n, x_{n+1}) \le Cq^{n-1}.$$

Thus, by triangle inequality, for $m \ge n + 1$,

$$d(x_n, x_m) = C(q^{n-1} + q^n + \cdots + q^{m-2}) \le Cq^{n-1}(1 + q + q^2 + \cdots) = \frac{C}{1-q} \cdot q^{n-1}.$$

Hence, $(x_n)_n$ ia a Cauchy sequence and we have a limit $x = \lim x_n$. Now a mapping f satisfying (*) is clearly continuous and hence we have

$$f(x) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = x.$$

Finally, if $f(x) = x$ and $f(y) = y$ then

$$d(x, y) = d(f(x), f(y)) \leq q \cdot d(x, y) \text{ with } q < 1$$

which is possible only if $d(x, y) = 0$. □

7.7 An Example: Spaces of continuous functions

Let $X = (X, d)$ be a metric space. Denote by

$$C(X)$$

the space of all *bounded* continuous real functions $f : X \to \mathbb{R}$, endowed with the metric

$$d(f, g) = \sup_{x \in X} |f(x) - f(x)|.$$

(The function d thus defined really is a metric. Obviously $d(f, g) = 0$ implies $f = g$ and $d(f, g) = d(g, f)$. Suppose $d(f, g) + d(g, h) < d(f, g)$; then there is an $x \in X$ such that $d(f, g) + d(g, h) < |f(x) - h(x)|$, but then in particular $|f(x) - g(x)| + |g(x) - h(x)| < |f(x) - h(x)|$, a contradiction.)

Remark. Of course, by 2.4.2, if X is compact then $C(X)$ is the space of *all* continuous functions on X.

7.7.1 Observation. *The convergence in $C(X)$ is exactly the uniform convergence defined in 8.1.*

(We have $d(f, g) < \varepsilon$ if and only if for all $x \in X$, $|f(x) - g(x)| < \varepsilon$.)

7.7.2 Proposition. *The space $C(X)$ with the metric defined above is complete.*

Proof. Let $(f_n)_n$ be a Cauchy sequence in $C(X)$. Then, since $|f_n(x) - f_m(x)| \leq d(f_n, f_m)$ for each $x \in X$, every $(f_n(x))_n$ is a Cauchy sequence in $\mathbb{R}$, and hence a convergent one. Set

$$f(x) = \lim_n f_n(x).$$

Claim. *The sequence* $(f_n)_n$ *converges to* f *uniformly.*
Proof of the Claim. Consider an $\varepsilon > 0$. There exists an n_0 such that for $m, n \geq n_0$,

$$\forall x, \ |f_n(x) - f_m(x)| < \tfrac{\varepsilon}{2}$$

and hence $\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - \lim_{m\to\infty} f_m(x)| = |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$. Thus, for $n \geq n_0$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. □

Proof of the Proposition continued. By the Claim and 8.2, f is continuous. Now there exists an n_0 such that for all $n, m \geq n_0$, $d(f_n, f_m) = \sup_x |f_n(x) - f_m(x)| < 1$ and hence, taking the limit, we obtain $|f_n(x) - f(x)| \leq 1$ for all x. Thus, if $|f_{n_0}(x)| \leq K$ we have $|f(x)| \leq K$ for all x.

Now we know that f is bounded and continuous, hence $f \in C(X)$, and by 7.7.1 and the Claim again, $(f_n)_n$ converges to f in $C(X)$. □

7.7.3

Let $a, b \in \mathbb{R} \cup \{-\infty. + \infty\}$. Put

$$C(X; a, b) = \{f \in C(X) \mid \forall x, \ a \leq f(x) \leq b\}.$$

Proposition. *The subspace* $C(X; a, b)$ *is closed in* $C(X)$. *Consequently, it is complete.*

Proof. Recall 8.1.1. Since uniform convergence implies pointwise convergence, if $a \leq f_n(x) \leq b$ and f_n converge to f then $a \leq f(x) \leq b$ and $f \in C(X; a, b)$.

The consequence follows from 7.3.1. □

8 Uniform convergence of sequences of functions. Application: Tietze's Theorems

On various occasions we have seen that general facts the reader knew about real functions of one real variable held generally, and the proofs did not really need anything but replacing $|x - y|$ by the distance $d(x, y)$. For example, this was the case when studying the relationship between continuity with convergence, or when proving that continuous maps of compact spaces are automatically uniformly continuous; or the fact about maxima and minima of real functions on a compact space (where in fact the general proof was in a way simpler, or more transparent, due to the observation that the image of a compact space is compact).

In this section we will introduce yet another case of such a mechanical extension, namely the behavior of uniformly convergent sequences of mappings, resp. uniformly convergent series of real functions. As an application we will present rather important Tietze Theorems on extension of continuous maps.

8.1

Let (X, d), (Y, d') be metric spaces. A sequence of mappings

$$f_1, f_2, f_3, \dots \quad : X \to Y$$

is said to *converge uniformly* to f if

for every $\varepsilon > 0$ there is an n_0 such that for all $n \geq n_0$ and for all $x \in X$,

$$d'(f_n(x), f(x)) < \varepsilon.$$

This is usually indicated

$$f_n \rightrightarrows f.$$

8.1.1 Remarks

1. Note that if $f_n \rightrightarrows f$ then

$$\lim f_n(x) = f(x) \text{ for all } x. \tag{*}$$

   The statement (*) alone, (called *pointwise convergence*), is much weaker, and would not suffice as an assumption in 8.2 below.
2. Also note that in the above definition, one uses the metric structure in (Y, d') only. See 8.2.1 below.

8.2 Proposition. *Let $f_n \rightrightarrows f$ for mappings $(X, d) \to (Y, d')$. Let all the functions f_n be continuous. Then f is continuous.*

Proof. For $\varepsilon > 0$ choose n such that $d'(f_n(x), f(x)) < \frac{\varepsilon}{3}$ for all x. Since f_n is continuous there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d'(f_n(x) f(x)) < \frac{\varepsilon}{3}$. Now we have the implication

$$d(x,y) < \delta \quad \Rightarrow \quad d'(f(x), f(y))$$
$$\leq d'(f(x), f_n(x)) + d'(f_n(x), f_n(y)) + d'(f_n(y), f(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

8.2.1

Note that an analogous proposition also holds for a topological space (X, τ) instead of a metric one. In the proof replace the requirement of δ by a neighborhood U of x such that $f_n[U] \subseteq \Omega(f_n(x), \frac{\varepsilon}{3})$ and use for $y \in U$ the triangle inequality as before.

8.3 Corollary. *Let $f_n : (X, d) \to \mathbb{R}$ be continuous functions, let $\sum a_n$ be a convergent series of real numbers, and let for every n and every x, $|f_n(x)| \leq a_n$. Then $g_n(x) = \sum_{k=1}^{n} f_k(x)$ uniformly converge to $\sum_{k=1}^{\infty} f_k(x)$ and hence $g = (x \mapsto \sum_{k=1}^{\infty} f_k(x))$ is a continuous function.*

8.4 Lemma. *Let A, B be disjoint closed subsets of a metric space (X, d) and let α, β be real numbers. Then there is a continuous function*

$$\varphi = \Phi(A, B; \alpha, \beta) : X \to \mathbb{R}$$

such that

$$\varphi[A] \subseteq \{\alpha\}, \quad \varphi[B] \subseteq \{\beta\} \quad \textit{and} \quad \min\{\alpha, \beta\} \leq \varphi(x) \leq \max\{\alpha, \beta\}. \qquad (\Phi)$$

Proof. Set

$$\varphi(x) = \alpha + (\beta - \alpha)\frac{d(x, A)}{d(x, A) + d(x, B)}.$$

This definition is correct: $d(x, A) + d(x, B) = 0$ yields $d(x, A) = d(x, B) = 0$ and by closedness $x \in A$ and $x \in B$; but A and B are disjoint.

Furthermore, $\psi(x) = d(x, C)$ is continuous (by triangle inequality, $d(y, C) \leq d(x, C) + d(x, y)$ and hence $|d(x, C) - d(y, C)| \leq d(x, y)$) so that φ, obtained by arithmetic operations from continuous functions, is continuous as well.

The properties listed in (Φ) are obvious. □

8.5 Theorem. *(Tietze) Let A be a closed subspace of a metric space X and let J be a compact interval in $\mathbb{R}$. Then each continuous mapping $f : A \to J$ can be extended to a continuous $g : X \to J$ (that is, there is a continuous g such that $g|A = f$).*

Proof. For a degenerate interval $\langle a, a\rangle$ the statement is trivial and all the other compact intervals are homeomorphic; if the statement holds for J_1 and if $h : J \to J_1$ is a homeomorphism we can extend for $f : A \to J$ the hf to a $\overline{g} : X \to J_1$ and then take $g = h^{-1}\overline{g}$. Thus we can choose the J arbitrarily. For our purposes, $J = \langle -1, 1\rangle$ will be particularly convenient.

Set $A_1 = f^{-1}[\langle -1, -\frac{1}{3}\rangle]$ and $B_1 = f^{-1}[\langle \frac{1}{3}, 1\rangle]$ and consider

$$\varphi_1 = \Phi(A_1, B_1; -\frac{1}{3}, \frac{1}{3}).$$

We obviously have

$$\forall x \in A, \quad |f(x) - \varphi_1(x)| \le \frac{2}{3}.$$

Set $f_1 = f - \varphi_1$.

Suppose we already have continuous

$$f = f_1, f_2, \dots, f_n : A \to \langle -1, 1\rangle \quad \text{and} \quad \varphi_1, \varphi_2, \dots \varphi_n : X \to \langle -1, 1\rangle$$

such that for all $k = 1, \dots, n$,

$$|\varphi_k(x)| \le \frac{1}{3^k}, \quad f_k(x) = f_{k-1}(x) - \varphi_k(x) \quad \text{and} \quad |f_k(x)| \le \frac{2}{3^k}. \qquad (*)$$

Then set

$$A_{n+1} = f^{-1}[\langle -\frac{1}{3^n}, -\frac{1}{3^{n+1}}\rangle], \quad B_{n+1} = f^{-1}[\langle \frac{1}{3^{n+1}}, \frac{1}{3^n}\rangle],$$
$$\varphi_{n+1} = \Phi(A_{n+1}, B_{n+1}; -\frac{1}{3^{n+1}}, \frac{1}{3^{n+1}}) \quad \text{and} \quad f_{n+1} = f_n - \varphi_{n+1}.$$

Thus we obtain sequences of continuous functions $\varphi_1, \varphi_3, \dots, \varphi_k, \dots$ and $f = f_0, f_1, \dots, f_k, \dots$ satisfying (*) for all k. By 7.3, we have a continuous function $g = (x \mapsto \sum_{k=1}^{\infty} \varphi_k(x)) : X \to \mathbb{R}$ and since $|g(x)| \le \sum_{k=1}^{\infty} \frac{2}{3^k} = 1$, we can view it as a continuous function

$$g : X \to \langle -1, 1\rangle.$$

Now let $x \in A$. We have

$$f(x) = \varphi_1(x)+f_1(x) = \varphi_1(x)+\varphi_2(x)+f_2(x) = \cdots = \varphi_1(x)+\cdots+\varphi_n(x)+f_n(x)$$

and since $\lim_n f_n(x) = 0$ we conclude that $f(x) = g(x)$. □

8.5.1 Theorem. *(Tietze's Real Line Theorem) Let A be a closed subspace of a metric space X. Then each continuous mapping $f : A \to \mathbb{R}$ can be extended to a continuous $g : X \to \mathbb{R}$.*

Proof. We can replace $\mathbb{R}$ by any space homeomorphic with $\mathbb{R}$ (recall the first paragraph of the previous proof). We will take the open interval $(-1, 1)$ instead and extend a map $f : A \to (-1, 1)$.

By 8.5, f can be extended to a $\overline{g} : X \to \langle -1, 1\rangle$. Such $\overline{g}$ can, however reach the values -1 or 1 and hence is not an extension as desired. To remedy the situation, consider $B = \overline{g}^{-1}[\{-1, 1\}]$ which is a closed set disjoint with A, consider the $\varphi = \Phi(A, B, 0, 1)$ from 8.4, and define

$$g(x) = \overline{g}(x) \cdot \varphi(x).$$

Now we have $f(x) = \overline{g}(x) = g(x)$ for $x \in A$, and $|g(x)| < 1$ for all $x \in X$: if $\overline{g}(x) = 1$ or -1 then $\varphi(x) = 0$.

8.5.2

A subspace R of a space Y is said to be a *retract* of Y if there exists a continuous $r : Y \to R$ such that $r(x) = x$ for all $x \in R$.

A metric space Y is *injective* if for every metric space X and closed $A \subseteq X$, each continuous $f : A \to Y$ can be extended to a continuous $g : X \to Y$. (Thus, we have learned above that $\mathbb{R}$ and any compact interval are injective spaces.)

Theorem. *Every retract of a Euclidean space is injective.*

Proof. First we will prove that a Euclidean space itself is injective. Consider it as the product

$$\mathbb{R}^m = \mathbb{R} \times \cdots \times \mathbb{R} \quad m \text{ times}$$

with the projections $p_j((x_1, \ldots, x_m)) = x_j$. Let $f : A \to \mathbb{R}^m$ be a continuous mapping. Then we have by 8.5.1 continuous $g_j : X \to \mathbb{R}$ such that $g_j|A = p_j f$. By 2.2.2 we have the continuous $g = (x \mapsto (g_1(x), \ldots, g_m(x))) : X \to \mathbb{R}^m$ and for $x \in A$ we obtain $g(x) = (p_1 f(x), \ldots, p_m f(x)) = f(x)$.

Now let Y be a retract of $\mathbb{R}^m$ with a retraction $r : \mathbb{R}^m \to Y$ and an inclusion map $j : Y \to \mathbb{R}^m$ (thus, $rj = \mathrm{id}$). Now if $f : A \to Y$ (or, rather, $jf : A \to \mathbb{R}^m$) is extended to $\overline{g} : X \to \mathbb{R}^m$, the desired extension g is $r\overline{g}$. ☐

9 Exercises

(1) Prove 1.4.1.
(2) Prove Proposition 1.5.1.
(3) Prove Observation 1.5.2.
(4) Prove that $f : (X, d) \to (Y, d')$ is continuous if and only if for each convergent sequence $(x_n)_n$ in (X, d) the sequence $(f(x_n))$ is convergent (not specifying the limits.).
(5) (a) Consider the set of real numbers $\mathbb{R}$. Prove that the function

$$d'(x, y) = |x^3 - y^3|$$

is a metric which is not equivalent to the metric d given in example 1.1.1 (a).

(b) Prove that nevertheless, neighborhoods with respect to d are the same as neighborhoods with respect to d'.

(6) Each $\Omega(x, \varepsilon)$ is open (use the triangle inequality).
(7) Let Y be a subspace of (X, d). U is open (closed) in Y if and only if there exists an open (closed) V in X such that $U = V \cap Y$. The closure of A in y is $\overline{A} \cap Y$ where $\overline{A}$ is the closure in X (discuss this from the various aspects of closure as presented in 3.3.
(8) Find an example when uniform continuity is not preserved under homeomorphism.
(9) Write down a definition of topology based on closed subsets of X.
(10) Check that the closures as defined in 4.1 and 4.2 satisfy the requirements of 4.3).
(11) Starting with open sets, define neighborhoods, and from them define closure as indicated above. Prove that you get the same as the closure defined from open sets directly.
(12) Start with open sets, define neighborhoods, and then open sets as in 4.1. Prove that the open sets thus defined are precisely the same sets as the original ones (note the role of the somewhat clumsy requirement (4) in 4.1).
(13) Preserving connectedness is not the same as continuity. Give an example of a map $f : X \to Y$ such that for every connected $S \subseteq X$, $f[S]$ is connected (with the induced topology from Y), but f is not continuous. [Hint: Take $X = \mathbb{Q}$, the rational numbers.]
(14) Let $X \subset \mathbb{R}^2$ be the union of the set of all points $(0, y)$, $y \in \langle -1, 1 \rangle$ and the set of all points $(x, \sin(1/x))$, $x > 0$, with the induced topology.

(a) Prove that $X \subset \mathbb{R}^2$ is a closed subset.

(b) Prove that X is connected but not path-connected.

(15) Let $U \subseteq \mathbb{R}^n$ be a connected open set, and let $x, y \in U$. Prove that there exist $x_0, \dots, x_k \in U$, $x_0 = x$, $x_k = y$, such that the straight line segment connecting x_t, x_{t+1} is contained in U. [Hint: mimic the proof of Proposition 5.3.1.]

(16) Path-connected components are defined the same way as connected components in 5.4, with the word "connected" replaced by the word "path-connected". Are path-connected components necessarily closed? Prove or give a counterexample.

(17) Check that convergence in the metric spaces defined in 1.1.1 (d), (e) is precisely uniform convergence.

(18) Prove an analogue of Proposition 8.2 for uniform continuity instead of continuity.

(19) Let K be the set of all real numbers of the form $\sum_{k=1}^{\infty} a_k 3^{-k}$, where $a_k \in \{0, 2\}$. (This is called the *Cantor set*.) Prove that K is compact. Prove that K contains no compact interval with more than one point.

(20) Prove that a subspace of $\mathbb{R}^m$ is injective *if and only if* it is a retract.

Springer

http://www.springer.com/978-3-0348-0635-0

Introduction to Mathematical Analysis
Kriz, I.; Pultr, A.
2013, XX, 510 p. 1 illus. in color., Softcover
ISBN: 978-3-0348-0635-0
A product of Birkhäuser Basel