# Chapter 2
# Time-Varying Linear Processes

## 2.1 The State Equations for Linear Processes Varying in Time

The usual analytical modeling of linear processes, variant in time, can also be expressed using ordinary differential equations (ode) of the form (1.1), associated with algebraic equations of the form (1.2), with the observation that the elements of the four matrices can be in a complete form, or in an incomplete one—continuous functions of time, respectively: $\mathbf{A} = \mathbf{A}(t)$, $\mathbf{B} = \mathbf{B}(t)$, $\mathbf{C} = \mathbf{C}(t)$ and $\mathbf{D} = \mathbf{D}(t)$. If the known input vector $\mathbf{u}(t)$ presents a continuous evolution with respect to the time ($t$), the solutions of the ordinary differential equations (ode), in the vector form (2.1), respect the continuity conditions in the Cauchy sense. If (2.1) limits to four successive derivatives with respect to time, we have

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2.1}$$

$$\ddot{\mathbf{x}} = \dot{\mathbf{A}}\mathbf{x} + \mathbf{A}\dot{\mathbf{x}} + \dot{\mathbf{B}}\mathbf{u} + \mathbf{B}\dot{\mathbf{u}} \tag{2.2}$$

$$\dddot{\mathbf{x}} = \ddot{\mathbf{A}}\mathbf{x} + 2\dot{\mathbf{A}}\dot{\mathbf{x}} + \mathbf{A}\ddot{\mathbf{x}} + \ddot{\mathbf{B}}\mathbf{u} + 2\dot{\mathbf{B}}\dot{\mathbf{u}} + \mathbf{B}\ddot{\mathbf{u}} \tag{2.3}$$

$$\ddddot{\mathbf{x}} = \dddot{\mathbf{A}}\mathbf{x} + 3\ddot{\mathbf{A}}\dot{\mathbf{x}} + 3\dot{\mathbf{A}}\ddot{\mathbf{x}} + \mathbf{A}\dddot{\mathbf{x}} + \dddot{\mathbf{B}}\mathbf{u} + 3\ddot{\mathbf{B}}\dot{\mathbf{u}} + 3\dot{\mathbf{B}}\ddot{\mathbf{u}} + \mathbf{B}\dddot{\mathbf{u}} \tag{2.4}$$

In this case also, ($\dot{\mathbf{x}}$) in (2.2) is taken from (2.1), ($\ddot{\mathbf{x}}$) in (2.3) is taken from (2.2) and finally ($\dddot{\mathbf{x}}$) in (2.4) is taken from (2.3) and so on, which simplifies a lot the operation of progressive derivatives with respect to time of the state vector ($\mathbf{x}$).

## 2.2 The Complete Taylor Series

The numerical approximation of the vector ($\mathbf{x}_k$) corresponds to the general formula (1.7) of the decomposition in the Taylor series, with all the observations associated

T. Coloşi et al., *Numerical Simulation of Distributed Parameter Processes*,
DOI: 10.1007/978-3-319-00014-5_2, © Springer International Publishing Switzerland 2013

in Sect. 1.2. In the composition of the derivatives $(\overset{(m)}{\mathbf{x}_{\mathbf{k}-1}})$ from (1.7), it is considered $\overset{(m-1)}{\mathbf{A}}_{k-1} = \frac{d^{m-1}}{dt^{m-1}}[\mathbf{A}(t)]_{t_{k-1}}, \overset{(m-1)}{\mathbf{B}}_{k-1} = \frac{d^{m-1}}{dt^{m-1}}[\mathbf{B}(t)]_{t_{k-1}}$, which exist in (2.2), …, (2.4). The vector $(\mathbf{x}_k)$ approximated at the moment $t_k = k \cdot \Delta t$ corresponds to

$$\begin{aligned}\mathbf{x}_k \cong\ & \mathbf{x}_{k-1} + \frac{\Delta t}{1!}(\mathbf{Ax} + \mathbf{Bu})_{k-1} + \frac{\Delta t^2}{2!}\left(\dot{\mathbf{A}}\mathbf{x} + \mathbf{A}\dot{\mathbf{x}} + \dot{\mathbf{B}}\mathbf{u} + \mathbf{B}\dot{\mathbf{u}}\right)_{k-1} \\ & + \frac{\Delta t^3}{3!}\left(\ddot{\mathbf{A}}\mathbf{x} + 2\dot{\mathbf{A}}\dot{\mathbf{x}} + \mathbf{A}\ddot{\mathbf{x}} + \ddot{\mathbf{B}}\mathbf{u} + 2\dot{\mathbf{B}}\dot{\mathbf{u}} + \mathbf{B}\ddot{\mathbf{u}}\right)_{k-1} \\ & + \frac{\Delta t^4}{4!}\left(\dddot{\mathbf{A}}\mathbf{x} + 3\ddot{\mathbf{A}}\dot{\mathbf{x}} + 3\dot{\mathbf{A}}\ddot{\mathbf{x}} + \mathbf{A}\dddot{\mathbf{x}} + \dddot{\mathbf{B}}\mathbf{u} + 3\ddot{\mathbf{B}}\dot{\mathbf{u}} + 3\dot{\mathbf{B}}\ddot{\mathbf{u}} + \mathbf{B}\dddot{\mathbf{u}}\right)_{k-1} + \cdots\end{aligned} \tag{2.5}$$

a result that limits at $\omega = 4$. The notation $()_{k-1}$ underlines the fact that the entire expression inside the parenthesis is considered at the sequence $(k-1)$ and the moment $t_{k-1} = (k-1) \cdot \Delta t$.

Furthermore, the observations made after the formula (1.8) remain valid.

## 2.3 The Taylor Series with Odd Derivatives

The numerical approximation of the vector $(\mathbf{x}_k)$ is finally based on the Eq. (1.10), with the same advantages. By replacing in (1.10) the derivatives (2.1) and (2.3), also considered at the sequence $(k-1)$, we obtain

$$\mathbf{x}_k \cong \mathbf{x}_{k-2} + 2\left[\frac{\Delta t}{1!}(\mathbf{Ax} + \mathbf{Bu})_{k-1} + \frac{\Delta t^3}{3!}\left(\ddot{\mathbf{A}}\mathbf{x} + 2\dot{\mathbf{A}}\dot{\mathbf{x}} + \mathbf{A}\ddot{\mathbf{x}} + \ddot{\mathbf{B}}\mathbf{u} + 2\dot{\mathbf{B}}\dot{\mathbf{u}} + \mathbf{B}\ddot{\mathbf{u}}\right)_{k-1} + \cdots\right] + \cdots \tag{2.6}$$

a result that represents the vector $(\mathbf{x}_k)$, approximated by the method of Taylor series with odd derivatives.

## 2.4 The Taylor Series with Even Derivatives

For the numerical approximation of the vector $(\mathbf{x}_k)$, it is based on the Eq. (1.13), having the same advantages. By replacing in (1.13) the derivatives (2.2) and (2.4), also considered at the sequence $(k-1)$, we obtain

$$\begin{aligned}\mathbf{x}_k \cong\ & 2\mathbf{x}_{k-1} - \mathbf{x}_{k-2} + 2\left[\frac{\Delta t^2}{2!}\left(\dot{\mathbf{A}}\mathbf{x} + \mathbf{A}\dot{\mathbf{x}} + \dot{\mathbf{B}}\mathbf{u} + \mathbf{B}\dot{\mathbf{u}}\right)_{k-1} + \frac{\Delta t^4}{4!} \cdot \right. \\ & \left. \left(\dddot{\mathbf{A}}\mathbf{x} + 3\ddot{\mathbf{A}}\dot{\mathbf{x}} + 3\dot{\mathbf{A}}\ddot{\mathbf{x}} + \mathbf{A}\dddot{\mathbf{x}} + \dddot{\mathbf{B}}\mathbf{u} + 3\ddot{\mathbf{B}}\dot{\mathbf{u}} + 3\dot{\mathbf{B}}\ddot{\mathbf{u}} + \mathbf{B}\dddot{\mathbf{u}}\right)_{k-1} + \cdots\right] + \cdots\end{aligned} \tag{2.7}$$

The same as in Sects. 1.2–1.4, the three above methods based on the Taylor series, that is, the complete method (2.5), with odd derivatives (2.6) and even derivatives (2.7), appeal to the input vector $(\overset{(m)}{\mathbf{u}}_{k-1})$ for $m = 0, 1, 2, 3, \ldots$ only at the regressive sequence $(k-1)$, even if $(\overset{(m)}{\mathbf{u}}_{k})$ is also known at the current sequence $(k)$.

The following three methods, called Taylor series—LIL, in complete form, with odd derivatives and even derivatives, will also operate with the input vector, at the regressive sequence $(\overset{(m)}{\mathbf{u}}_{k-1})$, as well as at the current sequence $(\overset{(m)}{\mathbf{u}}_{k})$, which will lead to some significant advantages.

## 2.5 The Complete Taylor Series-LIL

If the approximation (1.16) is introduced in (2.5), for $m = 1$, after calculus, we have the same local-iterative linearization form (1.17), where

$$\mathbf{g}_k = \sum_{m=1}^{\omega} \frac{\Delta t^m}{m!} \overset{(m-1)}{\mathbf{B}}_{k-1} \tag{2.8}$$

and

$$\begin{aligned}
\mathbf{h}_k = & \left[ \mathbf{x} + \frac{\Delta t}{1!}\mathbf{A}\mathbf{x} + \frac{\Delta t^2}{2!}(\dot{\mathbf{A}}\mathbf{x} + \mathbf{A}\dot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{u}}) + \frac{\Delta t^3}{3!}(\ddot{\mathbf{A}}\mathbf{x} + 2\dot{\mathbf{A}}\dot{\mathbf{x}} + \mathbf{A}\ddot{\mathbf{x}} + 2\dot{\mathbf{B}}\dot{\mathbf{u}} + \mathbf{B}\ddot{\mathbf{u}}) \right. \\
& \left. + \frac{\Delta t^4}{4!}(\dddot{\mathbf{A}}\mathbf{x} + 3\ddot{\mathbf{A}}\dot{\mathbf{x}} + 3\dot{\mathbf{A}}\ddot{\mathbf{x}} + \mathbf{A}\dddot{\mathbf{x}} + 3\ddot{\mathbf{B}}\dot{\mathbf{u}} + 3\dot{\mathbf{B}}\ddot{\mathbf{u}} + \mathbf{B}\dddot{\mathbf{u}}) \right]_{k-1} \\
& + \left[ \sum_{m=1}^{\omega} \frac{\Delta t^m}{m!} \overset{(m-1)}{\mathbf{B}}_{k-1} \right] \left[ \sum_{m=1}^{\omega} (-1)^m \frac{\Delta t^m}{m!} \overset{(m)}{\mathbf{u}}_k \right].
\end{aligned} \tag{2.9}$$

The observations made after the Eq. (1.19) remain valid, but for $(\mathbf{g}_k)$ and $(\mathbf{h}_k)$, we have the expressions from (2.8) and (2.9). The vector of the forced component $(\mathbf{g}_k \cdot \mathbf{u}_k)$ is less simple than the one from (1.17), and the vector of the free component $(\mathbf{h}_k)$ in (2.9), which has in its constitution $(\overset{(m)}{\mathbf{u}}_{k-1})$ and $(\overset{(m)}{\mathbf{u}}_{k})$, has a more laborious expression than the one in (1.19).

## 2.6 The Taylor Series-LIL with Odd Derivatives

If the approximation (1.16) is introduced in (2.6), for $m = 1$, after calculus, we have the same local-iterative linearization form from (1.17), where

$$\mathbf{g}_k = 2\left(\frac{\Delta t}{1!}\mathbf{B} + \frac{\Delta t^3}{3!}\ddot{\mathbf{B}} + \cdots\right)_{k=1} \tag{2.10}$$

and

$$\begin{aligned}\mathbf{h}_k = \mathbf{x}_{k-2} + 2\left[\frac{\Delta t}{1!}\mathbf{A}\mathbf{x} + \frac{\Delta t^3}{3!}\left(\ddot{\mathbf{A}}\mathbf{x} + 2\dot{\mathbf{A}}\dot{\mathbf{x}} + \mathbf{A}\ddot{\mathbf{x}} + 2\dot{\mathbf{B}}\dot{\mathbf{u}} + \mathbf{B}\ddot{\mathbf{u}}\right)\right]_{k-1} \\ + 2\left(\frac{\Delta t}{1!}\mathbf{B} + \frac{\Delta t^3}{3!}\ddot{\mathbf{B}} + \cdots\right)_{k=1}\left[\sum_{m=1}^{\omega}(-1)^m\frac{\Delta t^m}{m!}\overset{(m)}{\mathbf{u}}_k\right]\end{aligned} \tag{2.11}$$

The observations made after the Eq. (1.19) remain valid, but for ($\mathbf{g}_k$) and ($\mathbf{h}_k$), we have the expressions from (2.10) and (2.11).

## 2.7 The Taylor Series-LIL with Even Derivatives

If in (2.7) we replace ($\mathbf{u}_{k-1}$) with (1.16), then we will also obtain after the computations the form (1.17), where

$$\mathbf{g}_k = 2\left(\frac{\Delta t^2}{2!}\dot{\mathbf{B}} + \frac{\Delta t^4}{4!}\dddot{\mathbf{B}} + \cdots\right)_{k=1} \tag{2.12}$$

and

$$\begin{aligned}\mathbf{h}_k = 2\mathbf{x}_{k-1} - \mathbf{x}_{k-2} + 2\Bigg[\frac{\Delta t^2}{2!}\left(\dot{\mathbf{A}}\mathbf{x} + \mathbf{A}\dot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{u}}\right)_{k-1} \\ + \frac{\Delta t^4}{4!}\left(\dddot{\mathbf{A}}\mathbf{x} + 3\ddot{\mathbf{A}}\dot{\mathbf{x}} + 3\dot{\mathbf{A}}\ddot{\mathbf{x}} + \mathbf{A}\dddot{\mathbf{x}} + 3\ddot{\mathbf{B}}\dot{\mathbf{u}} + 3\dot{\mathbf{B}}\ddot{\mathbf{u}} + \mathbf{B}\dddot{\mathbf{u}}\right)_{k-1} + \cdots\Bigg] \\ + 2\left(\frac{\Delta t^2}{2!}\dot{\mathbf{B}} + \frac{\Delta t^4}{4!}\dddot{\mathbf{B}} + \cdots\right)_{k=1}\left[\sum_{m=1}^{\omega}(-1)\frac{\Delta t^m}{m!}\overset{(m)}{\mathbf{u}}_k\right]\end{aligned} \tag{2.13}$$

The observations presented after (1.17), …, (1.19) remain valid, but for ($\mathbf{g}_k$) and ($\mathbf{h}_k$), we have the expressions from (2.12) and (2.13).

For the entire chapter, we have obtained a general form that approximates the numerical model Taylor—LIL for time-varying linear processes, respectively

$$\mathbf{x}_k = \mathbf{g}_k\mathbf{u}_k + \mathbf{h}_k, \tag{2.14}$$

$$\mathbf{y}_k = (\mathbf{C}_k\mathbf{g}_k + \mathbf{D}_k)\mathbf{u}_k + \mathbf{C}_k\mathbf{h}_k. \tag{2.15}$$

Of course, this system particularizes for the three variants of the method of Taylor series—LIL, which are the complete one, with odd derivatives and even derivatives, as it has been presented in Sects. 2.5–2.7.

There is a formal similarity between (1.27) and (1.28) with (2.14) and (2.15), with the difference that the constant matrices ($\mathbf{g}$), ($\mathbf{C}$) and ($\mathbf{D}$) are now becoming matrices with linear functions of time, sequentially expressed by ($\mathbf{g}_k$), ($\mathbf{C}_k$) and ($\mathbf{D}_k$).

Even if (2.14) and (2.15) can have laborious forms, we can underline the unitary and systemized character of the model of Taylor series—LIL, which gets more precise as ($\omega$) gets higher.

## 2.8 The Non-iterative Variant of Calculus of the Taylor Series

For time-varying linear processes is based on (2.1), …, (2.4), where we operate with successive replacements of the vectors ($\dot{\mathbf{x}}$), ($\ddot{\mathbf{x}}$), ($\dddot{\mathbf{x}}$),etc., formally identical with the ones presented in Sect. 1.8. In this case also, the non-iterative variant becomes more laborious than the results obtained in (Sects. 2.1, …, 2.7).

The general observation for this chapter needs to be reminded that all matrices ($\mathbf{g}_k$) in (2.8), (2.10) and (2.12) are dependent on the current sequence ($k$), unlike in Chap. 1, where all matrices ($\mathbf{g}$) were constant.

As a conclusion, we can notice that the entire mathematical formalism from this chapter operates with multiple derivatives with respect to time, of the matrix ($\mathbf{A}$) and the vectors ($\mathbf{B}$) and ($\mathbf{u}$), to which we progressively add the more complicated expressions of the derivatives in (2.1), …, (2.4).

Even if the results (2.5), …, (2.15) become laborious, it is underlined the unitary and systemized character of the method of Taylor series or the method of Taylor series—LIL (each one in its three variants), for the numerical modeling and simulation of linear processes, variant in time.

http://www.springer.com/978-3-319-00013-8

Numerical Simulation of Distributed Parameter Processes
Colosi, T.; Abrudean, M.-I.; Unguresan, M.-L.; Muresan, V.
2013, XX, 343 p., Hardcover
ISBN: 978-3-319-00013-8