Chapter 2
Karl Stellmacher 1938, the Cauchy problem for the Einstein equations

Original paper: Karl Stellmacher, "Zum Anfangswertproblem der Gravitationsgleichungen", *Mathematische Annalen* **115**, 136 – 152 (1938).

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GOLDEN OLDIE EDITORIAL

Editorial note to: Karl Stellmacher, On the initial value problem of the equations of gravitation

Helmut Friedrich

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Keywords Solutions of Einstein's equations · Initial value problem for Einstein's equations in vacuum and with a dust source · Golden Oldie

The general relativistic notion of 'domain of dependence' is important in the (global) analysis of gravitational fields. It is technically convenient because it is a geometrical concept defined, irrespective of any field equations, purely in terms of the metric. Its fundamental importance, however, comes from the fact that it gives precise meaning to the statement that the evolution process of the Einstein equations respects the notion of causality defined by the solution metrics of these equations.

This is taken for granted nowadays but it was not obvious in the early days of General Relativity. It is the merit of Stellmacher's article that it clarifies this issue for the first time. Previous work by A. Einstein, D. Hilbert, G. Darmois, C. Lanczos and others, which revealed important properties of the field equations, was based on the study of linearizations, formal expansions and Cauchy–Kowalevskaja type arguments. These techniques did not allow one to draw any conclusion of the type given in Stellmacher's article. The new ingredient he uses is an argument based on 'energy integrals' which was proposed by K. O. Friedrichs and H. Lewy (cf. [1] for earlier work in this direction). It is discussed in section 2 (we note that the inequality following equations (3) is slightly misleading and should be replaced by the statement that the mean value theorem is used to obtain these equations).

The republication of the original paper can be found in this issue following the editorial note and online via doi:10.1007/s10714-010-0960-1.

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It appears to be the first time that this powerful argument has been used in the context of general relativity. It has largely been generalized by now and ‘energy estimates’ represent an indispensable tool in the existence theory.

Stellmacher analyses the Einstein–Maxwell equations in harmonic and Lorentz gauge respectively, arguing that these gauge conditions themselves respect the causality relations defined by the solution metric. In this case the proof that the domains of (geometric) uniqueness for the field equations coincide with the domains of dependence defined by the metric is short, clear, and clean. He also analyses the case of pressure free matter (dust) which is more complicated and his results are in fact not so clear and complete because they only refer to regions where neither caustics nor matter-vacuum interfaces occur. This should not be held against him, the situation is complicated and poses problems even nowadays (cf. the discussion in [2]). He finally also remarks that he did not succeed in the case of more general matter fields. Again, this is not surprising.

While Stellmacher’s results and methods are standard now they were certainly new at the time and marked in a sense a kind of ‘phase transition’ in the analysis of the field equations. Had the times been better (K. O. Friedrichs who had suggested the work had left the country for political reasons half a year before the article was submitted) it might have led to a much earlier clarification of the local Cauchy problem.

Karl Ludwig Stellmacher—a brief biography

By Hubert Goenner, based on Ref. [3].

Karl Ludwig Stellmacher (1910–2001) studied mathematics, physics and chemistry at the University of Göttingen from 1927–1933 in order to become a teacher at a Gymnasium (high school/first two college years). His final exam did not recommend him for a PhD. Nevertheless, Richard Courant seems to have given him a problem in hyperbolic differential equations. After having lost Courant as an advisor, in 1935 the mathematician Gustav Herglotz (1881–1953) took him on and proposed propagation of gravity as a subject. Stellmacher finished his dissertation in 1937; it is the paper reproduced here. Surprisingly, he credited Kurt Friedrichs as responsible for the theme who, in the same year left his position in Braunschweig/Germany as a consequence of his political views and his jewish fiancée (whom he later married).

After his dissertation, Stellmacher worked as an assistant to Max Schuler (1882–1972), director and successor of the famous pioneer of aerodynamics, Ludwig Prandtl (1875–1853), in the Kaiser-Wilhelm Institute for Fluid Dynamics, on problems concerning gyroscopes. During the 2nd world war, he had to be a soldier from 1939 to 1944. After the war, his results on the gyroscope found no interest so that he went back to pure mathematics.

In 1948 he became both an assistant and a Privatdozent (lecturer) in the Mathematics Institute of the University of Göttingen. After having received the title of professor (but not the pay) in 1955, he left Göttingen immediately to become a research associate in the Institute for Fluid Dynamics and Applied Mathematics of the University of

Maryland in College Park. Within a year, he became full professor and stayed there until retirement in 1977.

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GOLDEN OLDIE

Republication of:
On the initial value problem of the equations of gravitation

Karl Stellmacher

Published online: 26 May 2010

An editorial note to this paper and a biography can be found in this issue preceding this Golden Oldie and online via doi:10.1007/s10714-010-0959-7.

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On the initial value problem of the equations of gravitation

By

Karl Stellmacher at Göttingen*):

For the solutions of the field equations of general relativity it is shown in the following that they possess the property of depending in a world point P only on that part of the remaining world, which lies inside the past of the temporal divide defined by the point P.

If, hence, the state variables are changed outside of the temporal divide, these variables will keep their values in P, respectively they can be returned to their original values by a transformation.

1.

In general relativity one defines: a world point P is "earlier"[1]) than a world point Q, if P and Q can be connected by an everywhere timelike line and if Q is associated with a larger value of the x^0-coordinate than P.

"Simultaneous" are called two points, which can be connected by an everywhere spacelike line.

By these definitions a certain causal connectedness is imprinted on the world, as one will have to demand that for simultaneity no cause–effect relationship can exist.

The value of a state variable in a world point P can therefore depend only on those world points, which can be connected to P by everywhere timelike lines; i.e., only on those points, which lie in the interior of the temporal divide constructed in P, namely in that part which is directed into the past. Hence, all field actions are allowed to propagate at most at the speed of light.

However, the cause–effect relationship is determined independently of these definitions by certain partial differential equations for the physical state variables.

Therefore, the question arises whether the cause–effect relationship which is fixed by the field equations is in agreement with the cause–effect

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*) This work is part of a dissertation accepted by the Faculty of Mathematics and the Natural Sciences of the University of Göttingen. It was suggested to me by Herr Professor Friedrichs. For this favour, as well as for his supporting interest, I feel obliged to thank him dearly.

[1]) See Hilbert, Math. Annalen **92**, p. 11ff. — Ges. Abh. Bd. III, p. 268ff.

relationship, which is imposed on the world by the introduction of an indefinite metric.

In the case of special relativity it is known of the Maxwell–Lorentzian equations that they yield the correct cause–effect relationship for the electromagnetic field strength.

That is, if the Cauchyian initial value problem is posed, i.e., if the field strength is specified at a certain time $x^0 = 0$, then its value in a later world point P depends only on that part of the initial values, which is cut out of the initial surface by the characteristic cone of this equation system to be constructed in P; the characteristic cone coincides with the temporal divide. Here, the demanded cause–effect relationship arises as a very special property of a solution to the Maxwell–Lorentzian equations.

Also in the case of general relativity we need to investigate if the field equations, and the gravitational equations in particular, possess the respective property to depend only on a part of the initial values.

This issue has already been raised a number of times. Initially it was Einstein, who settled the problem for weakly gravitating fields in first approximation[2]).

Next, it was shown by Vessiot[3]) that the characteristic cone of the equations of gravitation coincides with the temporal divide, i.e., that all discontinuities of derivatives of second or higher order of the field components propagate at the speed of light, wherein these singularities cannot be removed by a transformation, and wherein the respective derivatives of lower order are continuous in the point of concern.

A further step towards settling this problem was taken by de Donder[4]), who carried over the coordinate system used by Einstein for weak fields to the case of arbitrary gravitational fields; in such a coordinate system the equations of gravitation assume a form in which the propagation of gravitation at the speed of light and the correct causal structure of the world already become quite plausible. (This coordinate system will be the foundation of one of our two proofs. For further details, see Chapter 3.)

(In not so immediate connection to the present question are the famous

[2]) Ber. d. Berl. Akad. Wiss. 1916, p. 688, and 1918, p. 154; see thereto the comments and concerns of Eddington, Relativitätstheorie in math. Behandlung (Berlin 1925), § 57.

[3]) Compt. rend. **166** (1918); see on this also Levi–Civita, Atti d. Linc. **11** (1930), p. 1.

[4]) La gravifique Einsteinienne (Paris 1921), p. 40–41.

Hilbertian investigations on the causality problem[5]). Based on the theorem by Cauchy–Kowalewsky it is shown there, that the values of the field components and their first derivatives, to be analytically given on a space-like initial manifold, *uniquely* induce, up to transformations, an analytical solution in a neighbourhood of the initial surface.

The question which is of interest to us is barely touched upon hereby, as no statement is made on the domain of dependence of the solution, and, by means of the method of Cauchy–Kowalewsky, such a statement is impossible in principle. Also, uniqueness is not guaranteed inasmuch as the case is conceivable that a second non-analytical system of solutions may exist.)

In contrast to this we will conduct the proof that field actions propagate at most at the speed of light, thereby considering two cases; 1. arbitrary electrical and gravitating fields in empty space, 2. pure gravitational fields, which are superimposed on continuously distributed matter.

2.

The pursuit of our proof for the validity of the postulate of causality succeeds on the basis of applying a method devised by Friedrichs and Lewy[6]).

First we prove as a lemma that the uniqueness theorem by Friedrichs and Lewy can be carried over without difficulty to the case of an arbitrary, no longer necessarily linear hyperbolic differential equation. This allows us to repeat, simultaneously, the course of thoughts of the Friedrichs–Lewyian proof. It shall be shown that the value of the solution of a given differential equation of totally hyperbolic character depends only on that part of the initial values, which is cut out from the initial surface by the characteristic conoid to be constructed in P.

Let an equation be given of the type[7]):

$$L\,[u] \equiv a^{ik}\,u_{ik} + f = 0 \qquad \left(u_{ik} = \frac{\partial^2 u}{\partial x^i \partial x^k}\right). \tag{1}$$

[5]) See the work cited in footnote 1.

[6]) Math. Annalen **98**, p. 192ff.

[7]) One overlooks easily, that the present simple generalisation can also be carried over to the case that an arbitrary differential equation of the most general form $F(u) = 0$ is given. For then the considerations can be applied, under respective assumptions of differentiability, to the differentiated equation $\frac{\partial F}{\partial x^0} = \frac{\partial F}{\partial u_{ik}} u_{ik0} + W = 0$. See on this the work by Schauder, Fundam. Mathem. **34** (1935), p. 213.

In a domain G, which contains a part of the initial surface A, let a^{ik} and f be continuously differentiable functions of the four independent variables x^i, as well as of u and its first derivatives, the u_i. Equation (1) shall be totally hyperbolic for a certain given solution u, i.e. the quadratic form associated with a^{ik} shall possess the index $(- - - +)$; furthermore, the a^{ik} shall transform contravariantly, and the coordinate system shall be chosen such that the conditions of spacelikeness are satisfied:

$$(2) \qquad a^{00} > 0, \qquad \begin{vmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{vmatrix} < 0\,.$$

Then the matrix (a^{ik}) $(i, k = 1, 2, 3)$ is evidently non-degenerate negative definite; since the discriminant of the form is negative by assumption, and so only the indices of inertia $(+ + -)$ or $(- - -)$ are possible. However, since by assumption only *one* positive sign can occur, the claim thus follows.

Within the domain G there shall be given a well determined second solution $\breve{u}$[8]) of the differential equation (1), which coincides on a spacelike[9]) initial manifold A with respect to its value as well as the values of the first derivatives with the solution u. Let $\breve{u}$ in G, besides its derivatives of first and second order, be bounded from above; likewise, of course, u:

$$|u|,\ |u_i|,\ |u_{ik}|, \quad |\breve{u}|,\ |\breve{u}_i|,\ |\breve{u}_{ik}| < M \qquad i, k = 0, 1, 2, 3.$$

M is a fixed well determined number. Thus, it also holds:

$$L\,[\breve{u}] = \breve{a}^{ik}\,\breve{u}_{ik} + \breve{f} = 0,$$

where $\breve{a}^{ik}$ and $\breve{f}$ mean the functions a^{ik}, f, upon substitution therein of u and its derivatives by $\breve{u}$ and its derivatives. Under these premises there holds the

[8]) That we consider *one particular* second solution, therein lies the new aspect compared to Friedrichs and Lewy.

[9]) Let A be the surface $f = \text{const}$. Then f is called spacelike, if

$$a^{ik}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^k} > 0.$$

T h e o r e m. *Within a still to be determined neighbourhood of A both solutions are identical.*

To begin with the proof we form $L(u) - L(\breve{u})$:

$$a^{ik}\,(u_{ik} - \breve{u}_{ik}) + \breve{u}_{ik}\,(a^{ik} - \breve{a}^{ik}) + (f - \breve{f}) = 0\,.$$

If I now set:

$$\begin{aligned} u - \breve{u} &= v, \\ u_i - \breve{u}_i &= v_i, \\ u_{ik} - \breve{u}_{ik} &= v_{ik}, \end{aligned}$$

then I obtain an in v linear and homogeneous differential equation with vanishing initial values on A:[10]

$$(3) \qquad a^{ik}\,v_{ik} + b^\rho\,v_\rho + c\,v = 0 \qquad \left\{ \begin{aligned} b^\rho &= \frac{\partial a^{ik}(\overline{u})}{\partial u_\rho}\,\breve{u}_{ik} + \frac{\partial f(\overline{u})}{\partial u_\rho} \\ c &= \frac{\partial a^{ik}(\overline{u})}{\partial u}\,\breve{u}_{ik} + \frac{\partial f(\overline{u})}{\partial u} \end{aligned} \right\}$$

$$\breve{u} \le \overline{u} \le u\,.$$

From now on we can transfer the considerations by Friedrichs and Lewy (in particular l.c. Chapter 4) without further ado.

One constructs in a point P within G the characteristic cone B, of which we make the simplifying assumption that, together with the initial surface A, it defines a simply connected domain G'; this domain shall be completely contained within G. B in turn shall cut out from A a domain A' which is also simply connected. Furthermore, without loss of generality we assume that A be the surface $x^0 = 0$ (see Fig. 1).

We then imagine the interior of the cone to be filled by a foliation of surfaces $x^0 = \text{const}$, which we assume to be spacelike in the interior of G'. By integration of equation (3), multiplied by v_0, over a domain G'', which is bounded by A', $x^0 = c > 0$ and B', one obtains the equation

$$(4) \qquad \iiint\limits_{G''} \left(a^{ik}\,v_{ik} + b^\rho\,v_\rho + c\,v\right) v_0\,d\tau = 0.$$

[10] In the last $\partial f/\partial u$ in the equation below the argument of f was corrected from u to $\overline{u}$ by the translator.

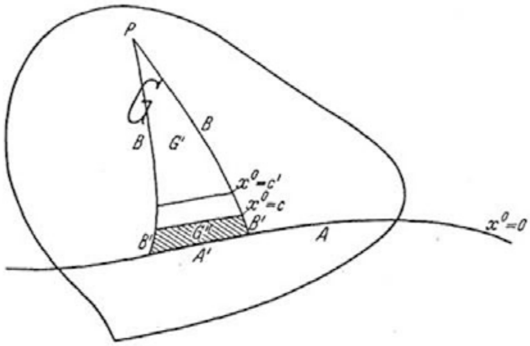

Fig. 1.

The integrand may be represented by a divergence and an additional quadratic form of the first derivatives of v and of v itself:

$$\iiint\limits_{G''} \left[\left(a^{ik}\, v_i\, v_0\right)_k - \tfrac{1}{2}\left(v_i\, v_k\, a^{ik}\right)_0\right] d\tau = \iiint\limits_{G''} Q(v_i, v)\, d\tau.$$

The left-hand side is converted into a surface integral by the Gaußian integral theorem:

$$\iint\limits_{O} a^{ik}\left(v_i\, v_0\, \xi_k - \tfrac{1}{2}\, v_i\, v_k\, \xi_0\right) d\omega = \iiint\limits_{G''} Q(v_i, v)\, d\tau.$$

The ξ_i denote the direction cosines of the inward-pointing normals. The part of the surface integral extending over A' vanishes; the remaining part subdivides into a surface integral over a strip M' of the mantle of the conoid M and a surface integral over the domain C of the surface $x^0 = c$. We recast the integrand of the surface integral and obtain:

$$-\frac{1}{2}\iint\limits_{C+M'} \frac{1}{\xi_0}\left[a^{ik}\left(v_i\, \xi_0 - v_0\, \xi_i\right)\left(v_k\, \xi_0 - v_0\, \xi_k\right) - a^{ik}\, \xi_i\, \xi_k\, v_0^2\right] d\omega$$
$$= \iiint\limits_{G''} Q(v_i, v)\, d\tau.$$

Under the surface integral on the left-hand side we have evidently, as on C $\xi_i = \delta_i^0$ and therefore $a^{ik}\, \xi_i\, \xi_k = a^{00}$ holds, as integrand of the part over

C a non-degenerate negative definite form of all first derivatives of v. The integrand of the part extending over M' certainly does not become negative, as on M $a^{ik}\xi_i\xi_k = 0$ holds. By suppressing this latter part, we can evidently estimate:

$$\iint\limits_{C} \sum v_i^2 \, d\omega \leq D \iiint\limits_{G''} \left(\sum v_i^2 + v^2\right) d\tau \leq E \iiint\limits_{G''} \sum v_i^2 \, d\tau.$$

E and D are constants which do not depend on the choice of the parameter c[11]). By integration of the inequality with respect to x^0 from $x^0 = 0$ up to $x^0 = c$ one obtains eventually:

$$\iiint\limits_{G''} \sum v_i^2 \, d\tau \leq cE \iiint\limits_{G''} \sum v_i^2 \, d\tau.$$

Now by choosing $c < 1/E$ it follows evidently that $v = 0$ within G''.

If, with the method of proof just given, I now proceed piecewise from the surface $x^0 = c$ up to the surface $x^0 = c'$, then from c' up to c'' and so on, I can eventually reach the tip P of the characteristic cone, as one can give an a priori estimate for E which holds evenly for every domain strip between $x^0 = c^{(n)}$ and $x^0 = c^{(n+1)}$, irrespective of how these two positive constants $c^{(n)}$ and $c^{(n+1)}$ may be chosen (as long as $c < c(P)$).

Accordingly, we have proven the following theorem: every within and on the boundary of G'' twice continuously differentiable function $\breve{u}$, which there, besides its derivatives of first and second order, is bounded, and which in the subdomain A' of the initial surface A (including its boundary), besides its first derivatives, agrees with u, is identical to u within G'.

Therefore, it follows, too: $u(P)$ does not change when arbitrary changes of the initial values on A are introduced outside of A'.

3.

In order to be able to apply this method to the equations of gravitation, we use a coordinate system in which the equations of gravitation appear considerably simplified with respect to those parts which contain the second derivatives of the g_{ik} with respect to the coordinates.

[11]) For further details see the original work.

In this regard we are considering a generalisation of the coordinate system used by Einstein when integrating the equations for non-stationary weak metric fields.

(The possibility of introducing such a coordinate system was first discovered by de Donder[12]); a little later Lanczos[13]) discovered it independently and gave the equations to be satisfied a particularly elegant and concise form (see on this also the remark by Darmois[14])).

According to Lanczos, in a coordinate systems in which the four relations

(1)
$$\frac{1}{\sqrt{-g}}\,\frac{\partial\sqrt{-g}\,g^{ri}}{\partial x^r}=0 \qquad (i=0,1,2,3)$$

are satisfied, the components of the contracted Riemannian tensor simplify to the following form $\left(\text{we write } \Box \text{ for } g^{\lambda\mu}\frac{\partial^2}{\partial x^\lambda\,\partial x^\mu}\right)$:

(2)
$$R_{ik}=\tfrac{1}{2}\Box\, g_{ik}+\Gamma_{rp,i}\Gamma_{qs,k}g^{rs}g^{pq}-\frac{\partial g_{pi}}{\partial x^r}\frac{\partial g_{qk}}{\partial x^s}g^{pq}g^{rs}.$$

We show that in the neighbourhood of an arbitrary spacelike hypersurface ($x^0=0$) we can always find a new proper[15]) coordinate system such that the coordinate surfaces $x^i=$ const are connected with the metric field in an invariant way, and that there the equations (1) are satisfied.

We thus look for a non-singular transformation

(3)
$$\overline{x}^i=\varphi^{(i)}(x^0,x^1,x^2,x^3) \qquad (i=0,1,2,3)$$

such that in the new coordinate system it is true that:[16]

(1a)
$$\frac{1}{\sqrt{-\overline{g}}}\,\frac{\partial\sqrt{-\overline{g}}\,\overline{g}^{ri}}{\partial\overline{x}^r}=0 \qquad (i=0,1,2,3).$$

For this purpose we consider the "generalised potential equation":

(4)
$$\frac{1}{\sqrt{-g}}\,\frac{\partial\sqrt{-g}\,g^{rs}\,\dfrac{\partial\varphi}{\partial x^s}}{\partial x^r}=0.$$

[12]) La gravifique Einsteinienne (Paris 1921), p. 40/41.
[13]) Phys. Zeitschr. **23** (1921), p. 537ff.
[14]) Mém. des Sc. Math. **25** (1926), p. 14–19.
[15]) See Hilbert, l. c.
[16]A typo in the formula below was corrected by the translator: the bar over x^r was missing.

It is invariant, if φ constitutes a scalar function. Then we determine four scalar functions $\varphi^{(i)}$ such that each one of them satisfies equation (4), and that in addition the initial conditions

$$\left\{\begin{aligned} &(\varphi^{(i)})_{x^0=0} = x^i, \\ &\left(\frac{\partial \varphi^{(i)}}{\partial x^0}\right)_{x^0=0} = \delta^i_0, \end{aligned}\right. \tag{5}$$

hold, from which follows immediately:

$$\frac{\partial \varphi^{(i)}}{\partial x^k} = \delta^i_k.$$

The solution of this initial value problem can be obtained following Hadamard[17]).

On the four equations obtained in this way,

$$\frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\, g^{rs} \dfrac{\partial \varphi^{(i)}}{\partial x^r}\right)}{\partial x^s} = 0 \qquad (i = 0, 1, 2, 3),$$

we apply the transformation (3); then there evidently results:[18]

$$\frac{1}{\sqrt{-\overline{g}}} \frac{\partial(\sqrt{-\overline{g}}\, \overline{g}^{rs}\, \delta^i_r)}{\partial \overline{x}^s} = 0 \qquad (i = 0, 1, 2, 3),$$

i.e., the equations (1a).

When the old coordinate system was a proper one, in particular on the initial surface, then, by reasons of continuity, this also holds in a certain neighbourhood of the initial surface for the new one.

A de Donderian coordinate system is determined uniquely by the property that, along a certain spacelike hypersurface, it coincides with an arbitrarily given second one up to the first derivatives of the transformation functions. Then the components of an arbitrary tensor agree on this initial

[17]) Le problème de Cauchy, App. 1. – There the existence proof is given only for equations with analytical coefficients. However, as Hadamard remarks himself, there are no essential difficulties involved in overcoming this premise.

[18]A typo in the formula below was corrected by the translator: the bar over g^{rs} was missing.

surface in both coordinate systems. The new coordinate surfaces are connected in an invariant way with the metric, since the equation (4) possesses an invariant form.

4.

Now the means have been made available to settle the problem. We imagine a world piece G, free of matter, in which, hence, the equations hold:

(1)
$$R_{ik} = kS_{ik} \qquad (i, k = 0, 1, 2, 3).$$

R_{ik} denotes the contracted Riemannian tensor, S_{ik} the electromagnetic energy tensor:

$$-S_{ik} = F_{i\alpha}F^{\alpha}_{\cdot\;k} - \tfrac{1}{4}g_{ik}F_{\alpha\beta}F^{\alpha\beta} \qquad (i, k = 0, 1, 2, 3)$$

with $S = 0$. The skew symmetrical tensor F_{ik} of the electric field strength can be represented by the vector potential Φ_i.

(2)
$$F_{ik} = \frac{\partial}{\partial x^i}\Phi_k - \frac{\partial}{\partial x^k}\Phi_i \qquad (i, k = 0, 1, 2, 3),$$

and there hold the Maxwell–Lorentzian field equations:

(3)
$$\frac{\partial\sqrt{-g}\,F^{i\alpha}}{\partial x^\alpha} = 0 \qquad (i = 0, 1, 2, 3).$$

It is well known that by (2) and (3) the vector Φ_i is determined only up to an arbitrary additive gradient, so that in general one imposes the additional constraint

(4)
$$\frac{\partial\sqrt{-g}\;\Phi^\alpha}{\partial x^\alpha} = 0.$$

From (2), (3) and (4) one obtains by simple algebra[19]):

(5)
$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^l}\left(g^{kl}\sqrt{-g}\,\frac{\partial\Phi_i}{\partial x^k}\right) + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}\left[\Phi_l\,\frac{\partial}{\partial x^i}\sqrt{-g}\,g^{kl}\right] + g_{ih}F_{j\cdot}^{\cdot k}\,\frac{\partial g^{hj}}{\partial x^k} = 0 \qquad (i = 0, 1, 2, 3).$$

[19]) v. Laue, Phys. Zeitschr. **21** (1920), p. 659ff.; see also Rel. Theor. Bd. II, 1st ed., p. 148.

Within G we now construct in analogy with Chapter 2 (Fig. 1) the characteristic conoid B (that is the temporal divide), which, together with the initial surface $x^0 = 0$, bounds the simply connected domain G'. Now suppose that in G' there exist two systems of solutions of equations (1), (5)

$$g_{ik},\ \Phi_i \quad \text{and} \quad \breve{g}_{ik},\ \breve{\Phi}_i \qquad (i, k = 0, 1, 2, 3),$$

which we assume both to be twice continuously and bounded differentiable within G' and on its boundary surfaces. Both systems of solutions may agree on the domain A' cut out by the temporal divide B from the surface $x^0 = 0$, including their first derivatives.

Under these premises we prove the T h e o r e m: *Within G' both systems of solutions are transferable into each other by a coordinate transformation*, hence, are physically equivalent.

For the proof we introduce in the world with the metric g_{ik} a de Donderian coordinate system x_{I}. In this world there thus holds:

$$\frac{\partial \sqrt{-g}\, g^{ir}}{\partial x_{\mathrm{I}}^r} = 0 \qquad (i = 0, 1, 2, 3).$$

Respecting the initial conditions (5) of Chapter 3, this coordinate system shall, besides the x_{I}^0-axes emerging from the initial surface, agree with that proper one on the initial surface $x^0 = 0$ in which the initial values are given.

Analogously we choose in the second world with the metric $\breve{g}_{ik}$ a de Donderian coordinate system x_{II}, which satisfies the conditions:

$$\frac{\partial \sqrt{-\breve{g}}\, \breve{g}^{ir}}{\partial x_{\mathrm{II}}^r} = 0 \qquad (i = 0, 1, 2, 3).$$

Again, on $x^0 = 0$ the coordinates x_{II} shall agree with the coordinates in which the initial values are given, likewise thereupon the x^0-axes, too.

If we then map the world II by the equations

$$x_{\mathrm{I}}^i = x_{\mathrm{II}}^i \qquad (i = 0, 1, 2, 3)$$

onto the world I, then in this mapping all invariant equations which existed in the world II remain valid. On the initial surface there then holds:[20]

$$(g_{ik})_{x^0=0} = (\breve{g}_{ik})_{x^0=0} \tag{6}$$

[20]Two typos in the formula below were corrected by the translator: (i) The sign ˘ over g on the right-hand side was missing in the original; (ii) Equation-number (6) did not exist anywhere; most probably it should be here and therefore it was added.

(see p. 9).[21] But also the first derivatives of the g_{ik} still agree on the initial surface, as the second derivatives of the transformation functions $\varphi_{\mathrm{I}}^{(i)}$ and $\varphi_{\mathrm{II}}^{(i)}$ agree on the initial surface, which can be easily confirmed with the help of (4), Chapter 3.

In this way we then have gained in the world I, according to Ch. 3, equation (2), the two equations:

$$(7)\qquad \begin{aligned} \Box\, g_{ik} + A_{ik} &= kS_{ik} \qquad \left(\Box = g^{\lambda\mu}\,\frac{\partial^2}{\partial x^\lambda\,\partial x^\mu}\right), \\ \breve{\Box}\, \breve{g}_{ik} + \breve{A}_{ik} &= k\breve{S}_{ik} \qquad \left(\breve{\Box} = \breve{g}^{\lambda\mu}\,\frac{\partial^2}{\partial x^\lambda\,\partial x^\mu}\right). \end{aligned}$$

The A_{ik} are expressions that only contain the g_{ik} and their first derivatives. This also holds for the $\breve{A}_{ik}$[22]).

In addition we have the equations (5) for the electromagnetic potentials. As in these the second derivatives of the g_{ik} occur only in the combination $\dfrac{1}{\sqrt{-g}}\,\dfrac{\partial\sqrt{-g}\,g^{ie}}{\partial x^e}$, respectively in derivatives of these quantities, we can write for these equations:

$$(8)\qquad \begin{aligned} \Box\, \Phi_i + B_i &= 0, \\ \breve{\Box}\, \breve{\Phi}_i + \breve{B}_i &= 0, \end{aligned}$$

where the B_i only contain first derivatives of the g_{ik} and the Φ_i. Also, the S_{ik} in the equations (7) contain at most first derivatives of the Φ_i. Similarly for the quantities flagged with a hook.

When we now set

$$g_{ik} - \breve{g}_{ik} = l_{ik} \quad \text{and} \quad \Phi_i - \breve{\Phi}_i = f_i,$$

then we obtain from forming differences of (7), (8):

$$\begin{aligned} L_{ik} &\equiv \Box\, l_{ik} - D_{ik} = 0, \\ L_i &\equiv \Box\, f_i - D_i = 0. \end{aligned}$$

[21] The page numbers referred to are at the bottom of each page [editor].

[22]) Although in Ch. 2 we did not need for the second solution $\breve{u}$ the assumption that for it the coefficients a^{ik} satisfy the conditions of total hyperbolicity, we here need to also assume of the $\breve{g}_{ik}$ that its matrix possesses the index of inertia $(- - - +)$ (in the proof of the next section this is not necessary), as otherwise in the world II the transformation of de Donder need not be feasible.

In analogy to equation (3), p. 5, the D are linear functions of the l_{ik} and their first derivatives, and of the f_i and their first derivatives. As coefficients of these linear forms D serve certain polynomial forms of the g_{ik}, $\breve{g}_{ik}$, Φ_i, $\breve{\Phi}_i$ and their first and second derivatives. The particular appearence of these forms is of no importance for the proof. On the equation system thus obtained, we can apply the Friedrichs–Lewyian considerations. Evidently, the g^{ik} satisfy all preconditions that we set in Chapter 2 for the a^{ik} with regard to the index of inertia and, in particular, also with regard to conditions (2), Section 2, which are equivalent to the Hilbertian demand for a proper coordinate system[23]).

According to the conclusion of Chapter 2, the demanded proof is thus completed. Instead of the integral (4), p. 5, one only has to form:

$$\iiint\limits_{G''} \left(\frac{\partial l_{ik}}{\partial x^0} \, L_{ik} \right) d\tau = 0 \quad \text{resp.} \quad \iiint\limits_{G''} \left(\frac{\partial f_i}{\partial x^0} \, L_i \right) d\tau = 0$$

and then proceed just as in Chapter 2. We thus obtain the domain of dependence of the solution in a point P by envisaging the temporal divide in this point. The domain of dependence is then cut out from the initial manifold by the temporal divide.

In the language of relativity theory we can also express this theorem by:

In parts of space which are free of matter, "simultaneously" positioned world points cannot influence one another.

5.

The proof just given cannot be extended to the case where, besides the state variables g_{ik} and Φ_i, there is also continuously distributed matter present[24]).

By a special trick, however, one can treat the case of incoherent matter in the absence of any electrical fields.

We then have the equation system:

$$\mathfrak{R}_{ik} = k(\mathfrak{T}_{ik} - \tfrac{1}{2} g_{ik} \mathfrak{T}) \tag{1}$$

with

$$\mathfrak{T}_{ik} = \mathfrak{m} u_i u_k .$$

[23]) l. c.

[24]) Attempts in this direction fail due to the form of the divergence equation of the matter, the treatment of which, in our sense, proves altogether unpleasant.

As usual, $\mathfrak{m}$ denotes the mass *density* of the matter; u^i the tangent vector of length 1 to the world lines of the matter. For determining these quantities we have the equations:

$$\frac{du^i}{ds} + \Gamma^i_{\alpha\beta} u^\alpha u^\beta = 0, \qquad g_{ik} u^i u^k = 1 \tag{2}$$

and

$$\frac{\partial (u^i \mathfrak{m})}{\partial x^i} = 0. \tag{3}$$

In G'[25]) let there be given two systems of solutions (twice continuously and bounded differentiable)

$$g_{ik},\ u^i,\ \mathfrak{m} \quad \text{and} \quad \breve{g}_{ik},\ \breve{u}^i,\ \breve{\mathfrak{m}}$$

of these equations, which agree in the domain A' of the initial surface $x^0 = 0$; in particular, the g_{ik} shall agree up to their first derivatives, while the functions u^i and $\mathfrak{m}$ must agree only themselves with $\breve{u}^i$ and $\breve{\mathfrak{m}}$ on $x^0 = 0$.

Again, we then have the T h e o r e m: under the stated assumptions both systems of solutions are transferable into one another by transformation.

The settling of the problem, i.e. the proof of the theorem, succeeds due to the introduction of a "rest coordinate system" (see Hilbert, l. c.); that is, a coordinate system in which the contravariant tangent vector u^i possesses the components $(1\ 0\ 0\ 0)$. In contrast to the de Donder coordinate system the rest coordinate system is fixed by differential equations of *first* order for the coordinates. This is as in the new coordinate system there shall hold:

$$\overline{u}^i = \frac{\partial \overline{x}^i}{\partial x^\alpha} u^\alpha = \delta^i_0 \qquad (i = 0, 1, 2, 3)\cdot \tag{2a}$$

Those world positions in which there is no matter present shall be viewed as filled in a continuously differentiable way by a unit vector field that also satisfies the differential equations of the geodesic lines.

In our rest coordinate system the continuity equation of the matter assumes the form:

$$\frac{\partial \mathfrak{m}}{\partial x^0} = 0. \tag{3a}$$

[25]) Cf. Ch. 2, Fig. 1.

Moreover, it follows from the equations of motion (2) by respecting the value for the vector u^i:

$$\Gamma^i_{00} = 0 \qquad (i = 0, 1, 2, 3),$$

therefore, too:

$$\Gamma_{00,i} = 0 \qquad (i = 0, 1, 2, 3). \tag{4}$$

Moreover, because of $u^i\, u_i = 1$ (unit vector!) it holds:

$$g_{00} = 1.$$

Therefore, we obtain from the equations (4)

$$\frac{\partial g_{0i}}{\partial x^0} = 0 \qquad (i = 0, 1, 2, 3). \tag{5}$$

In each of the two envisaged worlds we introduce a rest coordinate system x_{I}, respectively x_{II} (according to the procedure given on p. 11), which agree on $x^0 = 0$ with the original coordinate system. Due to the agreement of the initial values of the u^i, on the initial surface there then also agree, because of (2a), the derivatives

$$\frac{\partial x^i_{\mathrm{I}}}{\partial x^k} \quad \text{resp.} \quad \frac{\partial x^i_{\mathrm{II}}}{\partial x^k}$$

with the unit tensor δ^i_k. For the same reason also the second derivatives of the x_{I}, respectively x_{II} agree on the initial surface.

If we map again the world II by the equations $x_{\mathrm{I}} = x_{\mathrm{II}}$ onto the world I, i.e. if we let the world lines of the matter of the two worlds sit on top of one another, then we obtain agreeing initial values of the g_{ik} and $\mathfrak{m}$ with the $\breve{g}_{ik}$, respectively $\breve{\mathfrak{m}}$; moreover, there follows in the interior of G', due to (3a),

$$\breve{\mathfrak{m}} = \mathfrak{m} \tag{6}$$

and from (5)

$$g_{0i} = \breve{g}_{0i} \qquad (i = 0, 1, 2, 3). \tag{7}$$

Respecting the simplifications that our coordinate system yields, we obtain for the two solutions, instead of (1), the systems of equations:

$$\begin{aligned} R_{ik} &= k \frac{\mathfrak{m}}{\sqrt{-g}} \left(g_{0i}\, g_{0k} - \tfrac{1}{2}\, g_{ik}\right) \\ \breve{R}_{ik} &= k \frac{\breve{\mathfrak{m}}}{\sqrt{-\breve{g}}} \left(\breve{g}_{0i}\, \breve{g}_{0k} - \tfrac{1}{2}\, \breve{g}_{ik}\right). \end{aligned}$$

Employing (6) and (7), we form the differences:

(1a) $$R_{ik} - \breve{R}_{ik} =$$

$$k \cdot \mathfrak{m} \left[\left(\frac{1}{\sqrt{-g}} - \frac{1}{\sqrt{-\breve{g}}} \right) (g_{0i}\, g_{0k} - \tfrac{1}{2}\, g_{ik}) + \frac{1}{\sqrt{-\breve{g}}} (-\tfrac{1}{2})(g_{ik} - \breve{g}_{ik}) \right].$$

We set again $g_{ik} - \breve{g}_{ik} = l_{ik}$ and can express the right-hand side of (1a) as a homogeneous linear function of the *six* quantities l_{ik} $(i, k = 1, 2, 3)$. We will also express the left-hand side as a linear form of the l_{ik} and their first and second derivatives. In line with Section 2, we will be most interested in that part which contains the second derivatives of the l_{ik}.

Moreover, for the following it will be important that in the equation (1a) with index 0 0, the left-hand side allows for such a representation of the first derivatives of the l_{ik} in which only derivatives with respect to x^0 occur. Because of the equations (4), it evidently holds:

$$R_{00} = -\frac{\partial \Gamma^r_{0r}}{\partial x^0} - \Gamma^r_{0s}\Gamma^s_{0r},$$
$$\breve{R}_{00} = -\frac{\partial \breve{\Gamma}^r_{0r}}{\partial x^0} - \breve{\Gamma}^r_{0s}\breve{\Gamma}^s_{0r},$$

and from this by subtraction[26]

$$R_{00} - \breve{R}_{00} = -\tfrac{1}{2}\, g^{rs}\, \frac{\partial^2 (g_{rs} - \breve{g}_{rs})}{(\partial x^0)^2} + \Gamma^r_{0s}\, (\Gamma^s_{0r} - \breve{\Gamma}^s_{0r})$$
$$+ \breve{\Gamma}^s_{0r}\, (\Gamma^r_{0s} - \breve{\Gamma}^r_{0s}) + l_{\rho\mu}\, A^{\rho\mu}.$$

The $A^{\rho\mu}$ are polynomials of the g_{ik}, g^{ik} and the $\Gamma_{ik,l}$, as well as of the $\breve{g}_{ik}$, $\breve{g}^{ik}$ and the $\breve{\Gamma}_{ik,l}$, and of their first derivatives.

By equation (7) $l_{0i} = 0$ holds; it then follows:

$$R_{00} - \breve{R}_{00} = -\frac{1}{2}\, g^{rs}\, \frac{\partial^2 l_{rs}}{(\partial x^0)^2} + \frac{1}{2}\, \frac{\partial l_{rt}}{\partial x^0}\, (\Gamma^t_{0s} g^{sr} + \breve{\Gamma}^t_{0s} \breve{g}^{sr}) + l_{\rho\mu}\, \overline{A}^{\rho\mu}.$$

By integration with respect to x^0 from the initial surface up to x^0, and a suitable subsequent recasting by means of integration by parts, we obtain

[26] A typo in the equation below was corrected by the translator: the indices "r" and "s" in the last Γ of the first line were interchanged.

eventually:

$$(8)\quad \int_0^{x^0} (R_{00} - \breve{R}_{00})\, dx^0 = -\frac{1}{2}\frac{\partial l_{rs}}{\partial x^0}\cdot g^{rs} + l_{\rho\mu}\, B^{\rho\mu} + \int_0^{x^0} l_{\rho\mu}\, C^{\rho\mu}\, dx^0,$$

wherein the coefficients $C^{\rho\mu}$ are again formed by polynomial algebraic operations, the $B^{\rho\mu}$ in addition also by integration with respect to x^0, from the g_{ik}, $\breve{g}_{ik}$ and their first and second derivatives.

Before we write down the remaining components, we remark that the contracted Riemannian tensor can be written with respect to its second derivatives as follows:[27]

$$2\, R_{ik} = -\, g^{rs}\frac{\partial^2}{\partial x^r\, \partial x^s} g_{ik} + \frac{\partial}{\partial x^i}\beta_k + \frac{\partial}{\partial x^k}\beta_i + L_{ik}\ ;$$

herein we set

$$(9)\qquad \beta_i = g^{rs}\left(\frac{\partial g_{ri}}{\partial x^s} - \frac{1}{2}\frac{\partial g_{rs}}{\partial x^i}\right),$$

while the L_{ik} only contain first derivatives of the g_{ik}. By subtraction it follows from this:

$$(9a)\qquad 2(R_{ik} - \breve{R}_{ik}) = -\, g^{rs}\frac{\partial^2 l_{ik}}{\partial x^r\, \partial x^s} + \frac{\partial \gamma_k}{\partial x^i} + \frac{\partial \gamma_i}{\partial x^k} + M_{ik}$$

with

$$(9b)\qquad \gamma_i = g^{rs}\left(\frac{\partial l_{ir}}{\partial x^s} - \frac{1}{2}\frac{\partial l_{rs}}{\partial x^i}\right).$$

Then, respecting equations (8), (9a), (9b), equations (1a) can be written:[28]

$$(10)\quad \begin{aligned} &\text{a)}\ \ -2\,\gamma_0 = l_{\rho\mu}\, B^{\rho\mu} + \textstyle\int_0^{x^0} l_{\rho\mu}\, \overline{C}^{\rho\mu}\, dx^0, \\ &\text{b)}\ \ -\frac{\partial \gamma_0}{\partial x^i} - \frac{\partial \gamma_i}{\partial x^0} = \Lambda_i \qquad\qquad (i = 1, 2, 3), \\ &\text{c)}\ \ \Box\, l_{ik} - \frac{\partial \gamma_i}{\partial x^k} - \frac{\partial \gamma_k}{\partial x^i} = \Lambda_{ik} \qquad (i, k = 1, 2, 3). \end{aligned}$$

[27] A typo in the equation below was corrected by the translator: a missing index "k" in the third term on the right-hand side was added.

[28] Several identical typos were corrected by the translator: in the original text, the upper limit in the integral was x_0 in the first equation below, in the first equation after (11), in the text above (15), in eqs. (13) – (16) and in the next one after (16).

The Λ are again linear forms of the l_{ik} and their first derivatives.

By eliminating γ_0 between equations (10a) and (10b), we obtain the following three equations:

$$-\frac{\partial \gamma_i}{\partial x^0} = \Lambda_i - \int_0^{x^0} \frac{\partial}{\partial x^i}\, l_{rs}\, \overline{C}^{rs}\, dx^0 - \frac{\partial}{\partial x^i}\, l_{rs}\, B^{rs},$$

or

$$\frac{\partial \gamma_i}{\partial x^0} = \Lambda_i - \int_0^{x^0} V_i\, dx^0, \tag{11}$$

where Λ_i and V_i are again linear forms of the l_{ik} and their first derivatives.

Now we multiply, respectively, the equations (10c) by $\frac{\partial l_{ik}}{\partial x^0}$ and (11) by $2\,\frac{\partial l_{ik}}{\partial x^k}$, add these up, and obtain:

$$\Box\,(l_{ik}) \cdot \frac{\partial l_{ik}}{\partial x^0} - \frac{\partial l_{ik}}{\partial x^0}\left(\frac{\partial \gamma_i}{\partial x^k} + \frac{\partial \gamma_k}{\partial x^i}\right) + 2\,\frac{\partial l_{ik}}{\partial x^k}\,\frac{\partial \gamma_i}{\partial x^0} = L + \int_0^{x^0} N\, dx^0.$$

The left-hand side is a divergence up to quadratic forms of the $\frac{\partial l_{ik}}{\partial x^r}$; this is because the following holds (see Friedrichs and Lewy)

$$2\,\frac{\partial^2 l_{ik}}{\partial x^r\,\partial x^s}\,\frac{\partial l_{ik}}{\partial x^0} = \left(\frac{\partial l_{ik}}{\partial x^r}\,\frac{\partial l_{ik}}{\partial x^0}\right)_s - \left(\frac{\partial l_{ik}}{\partial x^r}\,\frac{\partial l_{ik}}{\partial x^s}\right)_0 + \left(\frac{\partial l_{ik}}{\partial x^s}\,\frac{\partial l_{ik}}{\partial x^0}\right)_r.$$

We thus obtain:

$$\left(g^{rs}\,\frac{\partial l_{ik}}{\partial x^r}\,\frac{\partial l_{ik}}{\partial x^0}\right)_s - \frac{1}{2}\left(g^{rs}\,\frac{\partial l_{ik}}{\partial x^r}\,\frac{\partial l_{ik}}{\partial x^s}\right)_0 - \frac{\partial\left(\frac{\partial l_{ik}}{\partial x^0}\gamma_i\right)}{\partial x^k} - \frac{\partial\left(\frac{\partial l_{ik}}{\partial x^0}\gamma_k\right)}{\partial x^i} + 2\,\frac{\partial\left(\frac{\partial l_{ik}}{\partial x^k}\cdot\gamma_i\right)}{\partial x^0} = \overline{L} + \int_0^{x^0} \overline{N}\, dx^0. \tag{12}$$

Furthermore, we also add to (12) the equations (11) correspondingly multiplied by $2\,P\,\gamma_i$:

$$+\;P\,\frac{\partial(\gamma_i)^2}{\partial x^0} = 2\,P\,\Lambda_i\,\gamma_i + 2\,\gamma_i\,P\int_0^{x^0} N\, dx^0. \tag{13}$$

(P is a constant, the choice of which we still keep open.) We integrate the sum of the left-hand sides of (12) and (13) over the domain G'' (see p. 6, Fig. 1). Then there results upon application of the Gaußian integral theorem under the surface integral a quadratic form of the $4 \cdot 6$ derivatives of first order of the l_{ik} and the three γ_i, namely

$$g^{rs}\left(\frac{\partial l_{ik}}{\partial x^r}\frac{\partial l_{ik}}{\partial x^0}\xi_s - \frac{1}{2}\frac{\partial l_{ik}}{\partial x^r}\frac{\partial l_{ik}}{\partial x^s}\xi_0\right) - \frac{\partial l_{ik}}{\partial x^0}\gamma_i\,\xi_k - \frac{\partial l_{ik}}{\partial x^0}\gamma_k\,\xi_i$$

$$+2\,\frac{\partial l_{ik}}{\partial x^k}\gamma_i\,\xi_0 + 2\,P\,(\gamma_i)^2\,\xi_0.$$

The part deriving form the $\Box$-expressions is positive definite on C (cf. Ch. 2). Hence, this quadratic form can be made positive definite on C by a sufficiently large choice of the constant P. By substituting for the γ_i again their values (9b),[29] we gain an equation:

(14)
$$\iint\limits_{M'+C} Q\left(\frac{\partial l_{ik}}{\partial x^r}\right)d\omega$$
$$= \iiint\limits_{G''}\left[\overline{Q}\left(\frac{\partial l_{ik}}{\partial x^r}\,l_{st}\right) + E^{irmn}\,\frac{\partial l_{mn}}{\partial x^r}\int_0^{x^0} V_i\,dx^0\right]d\tau.$$

Q and $\overline{Q}$ are quadratic forms of the 24 first derivatives of the l_{ik}, $\overline{Q}$ also of the l_{ik}, Q is on C non-degenerate positive definite, on M' possibly degenerate positive definite. Besides the quadratic form $\overline{Q}$, we also have under the volume integral on the right-hand side an expression which may be viewed as a quadratic form of the l_{ik} and their first derivatives, as well as of the three quantities $\int_0^{x^0} V^i\,dx^0$. Accordingly, we gain from (14) the inequality:

(15)
$$\iint\limits_{C}\left(\frac{\partial l_{ik}}{\partial x^0}\right)^2 d\omega \le D\iiint\limits_{G''}\left[\left(\frac{\partial l_{ik}}{\partial x^r}\right)^2 + (l_{ik})^2 + \left(\int_0^{x^0} V_i\,dx^0\right)^2\right]d\tau.$$

With the aid of the Schwarzian inequality one easily gains:

(16)
$$\left(\int_0^{x^0} V_i\,dx^0\right)^2 \le E\int_0^{x^0}(V_i)^2\,dx^0.$$

[29]This reference to the equation-number was corrected by the translator, in the original text it was (9a).

Furthermore, it evidently holds:

$$\int_0^c dx^0 \int_0^{x^0} (V_i)^2\, dx^0 \leq W \int_0^c (V_i)^2\, dx^0.$$

D, E and W are constants, which may be chosen independently of the parameter c. Then we obtain from (15):[30]

$$\iint\limits_{C} \left(\frac{\partial l_{ik}}{\partial x^r}\right)^2 d\omega \leq F \iiint\limits_{G''} \left[\left(\frac{\partial l_{ik}}{\partial x^r}\right)^2 + (l_{ik})^2\right] d\tau, \tag{17}$$

again F can be chosen independently of c (cf. p. 7). From equation (17) then follows, according to what was said in Chapter 2, the uniqueness within G'. Thus, the theorem is proven.

Therefore, also for the case of a world with arbitrarily distributed incoherent matter the correct causal structure is guaranteed.

6.

So far, I have not succeeded in the settling of the general case, in which the matter is in addition charged. However, on the basis of results from Chapter 4, some statements can be made on causality also in this case, given the charged matter is distributed discontinuously on several very thin world tubes. With the aid of the final result of Chapter 4 one can then easily derive the theorem:

All *actions* that a matter particle *generates* in a world point P lie within the future-directed part of the temporal divide to be constructed in P.

To show this, one only has to use in the course of the proof of Chapter 4 as the boundary of the domain G' instead of the temporal divide B' a hypersurface which is obtained as follows: one erects in P the future-directed temporal divide B and forms the intersection of it with an arbitrary spacelike surface R. Then the envelope of all those temporal divides, which are to be envisaged in points of the intersection manifold B, R, yields the desired new boundary surface of the domain G'.

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[30] A typo in (17) was corrected by the translator: the volume element on the right-hand side was $d\omega$ in the original text.

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