Chapter II

Sharp Inequalities

10 Introduction: Outline of methods and results

Let again $\mathbb{R}^n$ be euclidean n-space and let

$$0 < p < \infty\,, \quad 0 < q \leq \infty\,, \quad s > \sigma_p = n\left(\frac{1}{p} - 1\right)_+ . \tag{10.1}$$

Then both $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are not only subspaces of $S'(\mathbb{R}^n)$ (the collection of all tempered distributions in $\mathbb{R}^n$) but also subspaces of $L_1^{loc}(\mathbb{R}^n)$ (the collection of all complex-valued locally Lebesgue-integrable functions in $\mathbb{R}^n$, interpreted in the usual way as distributions). Let A^s_{pq} be either B^s_{pq} or F^s_{pq} and let $f \in A^s_{pq}(\mathbb{R}^n)$ with (10.1). Of interest is the singularity behaviour of f, usually expressed in terms of the distribution function μ_f, the non-increasing rearrangement f^* of f and its maximal function f^{**}, which are given by

$$\mu_f(\lambda) = |\{x \in \mathbb{R}^n \, : \, |f(x)| > \lambda\}|\,, \quad \lambda \geq 0\,, \tag{10.2}$$

$$f^*(t) = \inf\{\lambda \, : \, \mu_f(\lambda) \leq t\}\,, \quad t \geq 0\,, \tag{10.3}$$

and

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)\, d\tau\,, \quad t > 0\,, \tag{10.4}$$

respectively. Of interest are only those spaces $A^s_{pq}(\mathbb{R}^n)$ for which there exist functions $f \in A^s_{pq}(\mathbb{R}^n)$ such that $f^*(t)$ tends to infinity if $t > 0$ tends to zero. As indicated in Fig. 10.1 one has to distinguish between three cases. If $s > \frac{n}{p}$ then all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are continuously embedded in $L_\infty(\mathbb{R}^n)$

H. Triebel, *The Structure of Functions*,
DOI 10.1007/978-3-0348-0569-8_2, © Springer Basel 2001

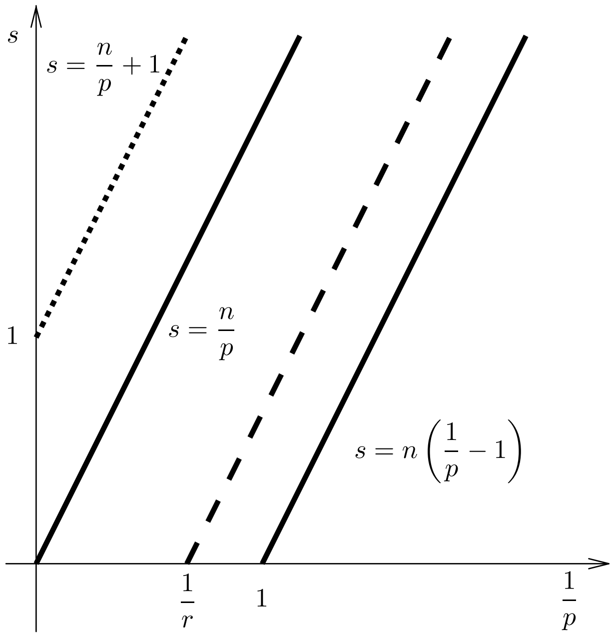

Fig. 10.1

and, hence, all functions $f^*(t)$ with $f \in A^s_{pq}(\mathbb{R}^n)$ are bounded. This is of no interest for us (at least as far as only the growth of functions is considered). The two remaining cases are called:

$$\textit{sub-critical} \quad \text{if} \quad 0 < p < \infty\,, \quad 0 < q \le \infty\,, \quad \sigma_p < s < \frac{n}{p}\,, \tag{10.5}$$

and

$$\textit{critical} \quad \text{if} \quad 0 < p < \infty\,, \quad 0 < q \le \infty\,, \quad s = \frac{n}{p}\,. \tag{10.6}$$

In all spaces belonging to the sub-critical case there are essentially unbounded functions f with $f^*(t) \to \infty$ if $t \downarrow 0$. In the critical case the situation is more delicate. It depends on the parameters p, q and whether A^s_{pq} is B^s_{pq} or F^s_{pq}. We give in Section 11 a detailed and definitive description of the relevant scenery surrounding the embedding of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in $L^{loc}_1(\mathbb{R}^n)$, $L_\infty(\mathbb{R}^n)$ and other classical target spaces. Let $A^s_{pq}(\mathbb{R}^n)$ be a space which is embedded in $L^{loc}_1(\mathbb{R}^n)$ but not in $L_\infty(\mathbb{R}^n)$. Then (in temporarily somewhat vague terms) the *growth envelope function* $\mathcal{E}_G A^s_{pq}$ is a function $t \mapsto \mathcal{E}_G A^s_{pq}(t)$ which is equivalent to

$$\mathcal{E}_G | A^s_{pq}(t) = \sup\left\{ f^*(t) \,:\, \|f \,| A^s_{pq}(\mathbb{R}^n)\| \le 1 \right\}\,, \quad 0 < t < \varepsilon\,, \tag{10.7}$$

where $0 < \varepsilon < 1$ is a given number. The notation $\mathcal{E}_G|A^s_{pq}$ indicates that this function is taken with respect to a given quasi-norm $\| \cdot |A^s_{pq}(\mathbb{R}^n)\|$. If

$$\| \cdot |A^s_{pq}(\mathbb{R}^n)\|_1 \sim \| \cdot |A^s_{pq}(\mathbb{R}^n)\|_2 \tag{10.8}$$

are two equivalent quasi-norms in a given space $A^s_{pq}(\mathbb{R}^n)$, then (in obvious notation)

$$\mathcal{E}_G|_1 A^s_{pq}(t) \sim \mathcal{E}_G|_2 A^s_{pq}(t)\,, \quad 0 < t \le \varepsilon\,, \tag{10.9}$$

are also equivalent. Since we never distinguish between equivalent quasi-norms of a given space $A^s_{pq}(\mathbb{R}^n)$ it is reasonable to extend this point of view to what we wish to call later on the *growth envelope function* $\mathcal{E}_G A^s_{pq}$. By definition one has the sharp inequality

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{\mathcal{E}_G|A^s_{pq}(t)} \le c\,\|f\,|A^s_{pq}(\mathbb{R}^n)\|\,, \quad f \in A^s_{pq}(\mathbb{R}^n)\,. \tag{10.10}$$

One may look at the left-hand side of (10.10) as the quasi-norm $L_\infty(I_\varepsilon, \mu)$ for a suitable Borel measure μ on the interval $I_\varepsilon = (0, \varepsilon]$. We wish to strengthen (10.10), replacing $L_\infty(I_\varepsilon, \mu)$ by $L_u(I_\varepsilon, \mu)$. Let $\mathcal{E}_G(t)$ be an unbounded positive, continuous, monotonically decreasing function on I_ε (this will apply in particular to all growth envelope functions with which we deal later on). Then the associated Borel measure $\mu = \mu_\Psi$ with respect to the distribution function $\Psi(t) = -\log \mathcal{E}_G(t)$ is the natural and distinguished choice for the above purpose. If $g(t)$ is a non-negative monotonically decreasing function on I_ε (with $g = f^*$ where $f \in A^s_{pq}(\mathbb{R}^n)$ as a typical example) then the corresponding quasi-norms are monotone,

$$\begin{aligned} \sup_{0<t<\varepsilon} \frac{g(t)}{\mathcal{E}_G(t)} \quad &\le \quad c_1 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_G(t)} \right)^{u_1} \mu_\Psi(dt) \right)^{\frac{1}{u_1}} \\ &\le \quad c_0 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_G(t)} \right)^{u_0} \mu_\Psi(dt) \right)^{\frac{1}{u_0}}\,, \end{aligned} \tag{10.11}$$

where $0 < u_0 < u_1 < \infty$. We refer to Proposition 12.2. Hence by (10.10) it makes sense to ask for the smallest number $u = u(A^s_{pq})$ with $0 < u \le \infty$ such that

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\mathcal{E}_G(t)} \right)^u \mu_\Psi(dt) \right)^{\frac{1}{u}} \le c\,\|f\,|A^s_{pq}(\mathbb{R}^n)\|\,, \quad f \in A^s_{pq}(\mathbb{R}^n)\,, \tag{10.12}$$

where

$$\mathcal{E}_G(t) = \mathcal{E}_G A^s_{pq}(t) \quad \text{and} \quad \Psi(t) = -\log \mathcal{E}_G A^s_{pq}(t)\,.$$

We denote provisionally the couple

$$\mathfrak{E}_G(A^s_{pq}) = \big(\mathcal{E}_G A^s_{pq}(t), u\big) \tag{10.13}$$

as the *growth envelope* of $A^s_{pq}(\mathbb{R}^n)$. We calculate the growth envelopes for all relevant spaces with the typical outcome

$$\mathfrak{E}_G(A^s_{pq}) = (t^{-\frac{1}{r}}, u) \quad \text{and} \quad \mathfrak{E}_G(A^s_{pq}) = (|\log t|^v, u) \tag{10.14}$$

in the sub-critical and critical case, respectively. Here r has the same meaning as in Fig. 10.1, whereas $0 < u \le \infty$ and $0 < v \le 1$ are suitable numbers depending on the parameters in B^s_{pq} and F^s_{pq}. In particular, the growth envelope exists in all cases. It is a natural and very precise description of the growth of functions belonging to $A^s_{pq}(\mathbb{R}^n)$. Inserting (10.14) in (10.12) one gets quasi-norms of the same type as in the Lorentz spaces $L_{ru}(I_\varepsilon)$ or in the special Lorentz-Zygmund spaces $L_{\infty,u}(\log L)_a(I_\varepsilon)$. However in the above outlined context neither spaces of this nor any other type are prescribed as target spaces in the course of setting up the required inequalities, and asking only afterwards for best parameters. (Of course all the spaces $L_{ru}(\log L)_a(I_\varepsilon)$ are reasonable refinements of classical target spaces like L_p, C, etc.) Here they emerge naturally and, hence, they are at the heart of the matter of the described singularity theory.

If $s > \frac{n}{p}$ then all spaces $A^s_{pq}(\mathbb{R}^n)$ are embedded in $L_\infty(\mathbb{R}^n)$ and a singularity theory in the above argument does not make much sense. However in the context of continuity there is one case which has attracted special attention in recent times and which corresponds to the line $s = 1 + \frac{n}{p}$ in Fig. 10.1. Hence we complement (10.5) and (10.6) by the case called:

$$\textit{super-critical} \quad \text{if} \quad 0 < p < \infty\,, \quad 0 < q \le \infty\,, \quad s = 1 + \frac{n}{p}\,. \tag{10.15}$$

First we recall that

$$\omega(f,t) = \sup_{x \in \mathbb{R}^n, |h| \le t} |f(x+h) - f(x)| \quad \text{and} \quad \widetilde{\omega}(f,t) = \frac{\omega(f,t)}{t} \tag{10.16}$$

with $t > 0$, are the usual modulus of continuity and the divided modulus of continuity, respectively. Then $Lip(\mathbb{R}^n)$ is the collection of all complex-valued functions in $\mathbb{R}^n$ such that

$$\| f \,| Lip(\mathbb{R}^n) \| = \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{0<t<1} \widetilde{\omega}(f,t) < \infty\,. \tag{10.17}$$

We have the remarkable fact (explained in greater detail in Section 11) that

$$A^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n) \quad \text{if, and only if,} \quad A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset Lip(\mathbb{R}^n)\,. \tag{10.18}$$

This makes (almost) clear that in the critical case (10.6) the growth envelope function $\mathcal{E}_G A^{\frac{n}{p}}_{pq}$ is unbounded if, and only if, the *continuity envelope function* $\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}$, defined as an equivalent function to

$$\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}(t) = \sup\left\{\widetilde{\omega}(f,t)\ :\ \|f\,|A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \le 1\right\}, \quad 0 < t < \varepsilon\,, \qquad (10.19)$$

for some $\varepsilon > 0$, is unbounded. Obviously, we rely here on the same notational agreement as in connection with (10.7), (10.8), (10.9). Furthermore, up to equivalences, the continuity envelope function is positive, continuous and monotonically decreasing in the interval $(0,\varepsilon]$. Then one has an immediate counterpart of (10.11), (10.12), which justifies the introduction of the *continuity envelope*

$$\mathfrak{E}_C(A^{1+\frac{n}{p}}_{pq}) = \left(\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}(t), u\right) \qquad (10.20)$$

in analogy to (10.13). We feel that the outcome is beautiful and perfect: One has in all cases of interest, this means all cases not covered by (10.18),

$$\mathfrak{E}_G(A^{\frac{n}{p}}_{pq}) = \mathfrak{E}_C(A^{1+\frac{n}{p}}_{pq})\,. \qquad (10.21)$$

We shall deal first with the critical case and afterwards lift not only the inequalities but also some extremal functions, responsible for the sharpness, by 1 to the super-critical case. In the one-dimensional case this is based on the simple but rather effective observation,

$$\widetilde{\omega}(f,t) \le c\,|f'|^{**}(t)\,, \quad 0 < t < 1\,, \qquad (10.22)$$

which provides at least an understanding of the method. (We use the notation introduced in (10.4)). There is a counterpart in $\mathbb{R}^n$, but it is more complicated. It may be found in 12.16.

The close connection between inequalities of Hardy type and rearrangement inequalities hidden in (10.12), (10.14), is based on the well-known observation,

$$\int_{\mathbb{R}^n} b^p(x)\,|f(x)|^p\,dx \le \int_0^\infty b^{*p}(t)\,f^{*p}(t)\,dt\,, \quad 0 < p < \infty\,, \qquad (10.23)$$

where $b(x)$ is a non-negative compactly supported weight function. This approach to Hardy inequalities has the advantage that both the singularity behaviour of the fixed weight function $b(x)$ and also of f, belonging to a given function space, are considered on a global scale. In particular, $b(x)$ may degenerate not only in points, hyper-planes, or smooth surfaces, but also on rather irregular sets. Let, as an example, Γ be a compact d-set in $\mathbb{R}^n$ according to (9.67) with $0 < d < n$, and let

$$D(x) = dist\,(x, \Gamma)\,, \quad x \in \mathbb{R}^n\,, \tag{10.24}$$

be the distance of $x \in \mathbb{R}^n$ to Γ. Then

$$b(x) = D(x)^a |\log D(x)|^b\,, \quad a < 0\,, \quad b \in \mathbb{R}\,, \quad 0 < D(x) < 1\,, \tag{10.25}$$

is a typical weight function in our context. One obtains, for example, in the critical case (10.6) for spaces $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ not covered by (10.18) (this means $1 < p < \infty$) the Hardy inequality

$$\int\limits_{D(x)<\varepsilon} \left| \frac{f(x)}{\log D(x)} \right|^p \frac{dx}{D^{n-d}(x)} \le c\, \| f \,| F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \|^p \tag{10.26}$$

where $0 < \varepsilon < 1$ and $0 < q \le \infty$. Recall that this applies in particular to the Sobolev spaces

$$H^{\frac{n}{p}}_p(\mathbb{R}^n) = F^{\frac{n}{p}}_{p,2}(\mathbb{R}^n)\,, \quad 1 < p < \infty\,. \tag{10.27}$$

If $d = 0$, then one may choose $\Gamma = \{0\}$ and gets

$$\int\limits_{|x|<\varepsilon} \left| \frac{f(x)}{\log |x|} \right|^p \frac{dx}{|x|^n} \le c\, \| f \,| F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \|^p\,, \quad 1 < p < \infty\,, \tag{10.28}$$

$0 < q \le \infty$, $0 < \varepsilon < 1$, where again (10.27) is an outstanding example. The indicated approach, which means the reduction of Hardy inequalities via (10.23) to rearrangement inequalities, is universal in our context. In particular it applies to all sub-critical cases and those critical cases which are not covered by (10.18). However the outcome is not always satisfactory. There seems to be a tricky interplay between weights, the geometry of Γ, and possible measures on Γ. We will discuss this point later on, although there are no final answers. For example, (10.26) looks better than it really is. On the other hand in case of $\Gamma = \{0\}$ one gets sharp assertions: The functions responsible for the sharpness of the rearrangement inequalities are also extremal functions for the related Hardy inequalities. Roughly speaking, these extremal functions convert the inequality (10.23) into an equivalence.

This chapter is organized as follows. In Section 11 we set the scene and collect (mostly without proofs) a few well-known classical embedding assertions in all three cases. Section 12 deals both with growth and continuity envelopes. Here we rely, at least partly, on recent work of D. D. Haroske, [Har01], where she introduced the notation of envelopes used above and studied growth and continuity envelopes systematically and, in particular, in the context of more general spaces. We restrict ourselves here to those properties which are more or less of direct use for later applications to the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. On this basis we study in Sections 13, 14 and 15 the critical, super-critical and sub-critical case, respectively. Hardy inequalities in the setting outlined above, will be considered in Section 16. Finally we collect in Section 17 some additional material and references.

11 Classical inequalities

11.1 Some notation

We use the notation introduced in the previous sections. In particular, let $\mathbb{R}^n$ be again euclidean n-space where $n \in \mathbb{N}$. The Schwartz space $S(\mathbb{R}^n)$, its dual $S'(\mathbb{R}^n)$, and the spaces $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$ have the same meaning as in 2.1, the latter quasi-normed by (2.1). Let $L_1^{loc}(\mathbb{R}^n)$ be the collection of all complex-valued locally Lebesgue-integrable functions in $\mathbb{R}^n$. Any $f \in L_1^{loc}(\mathbb{R}^n)$ is interpreted in the usual way as a regular distribution. Conversely, as usual, a distribution on $\mathbb{R}^n$ is called *regular* if, and only if, it can be identified (as a distribution) with a locally integrable function on $\mathbb{R}^n$. If $A(\mathbb{R}^n)$ is a collection of distributions on $\mathbb{R}^n$, then

$$A(\mathbb{R}^n) \subset L_1^{loc}(\mathbb{R}^n) \tag{11.1}$$

simply means that any element f of $A(\mathbb{R}^n)$ is a regular distribution $f \in L_1^{loc}(\mathbb{R}^n)$. Then, in particular, the distribution function $\mu_f(\lambda)$, the rearrangement $f^*(t)$ and its maximal function $f^{**}(t)$ in (10.2)–(10.4) make sense accepting that they might be infinite. If $A_1(\mathbb{R}^n)$ and $A_2(\mathbb{R}^n)$ are two quasi-normed spaces, continuously embedded in $S'(\mathbb{R}^n)$, then

$$A_1(\mathbb{R}^n) \subset A_2(\mathbb{R}^n) \tag{11.2}$$

always means that there is a constant $c > 0$ such that

$$\|f \,|A_2(\mathbb{R}^n)\| \le c\, \|f \,|A_1(\mathbb{R}^n)\| \quad \text{for all} \quad f \in A_1(\mathbb{R}^n)\,, \tag{11.3}$$

(*continuous embedding*). On the other hand we do not use the word embedding in connection with inequalities of type (10.12).

The spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ for the full scale of parameters

$$0 < p \le \infty\,, \quad 0 < q \le \infty\,, \quad s \in \mathbb{R}\,, \tag{11.4}$$

(with $p < \infty$ in the F-case) were introduced in Definitions 2.6 and 3.4 or in a more traditional (this means Fourier-analytical) way in connection with Theorems 2.9 and 3.6. A list of special cases, including Sobolev spaces, classical Sobolev spaces, Hölder-Zygmund spaces, and classical Besov spaces, may be found in 1.2. Although they are not special cases of the two scales $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ we need also the spaces $C(\mathbb{R}^n)$, $C^1(\mathbb{R}^n)$, and $Lip(\mathbb{R}^n)$. The latter space, including the modulus of continuity and the divided modulus of continuity were introduced in (10.17) and (10.16), respectively. Recall that $C(\mathbb{R}^n)$ is the space of all complex-valued, bounded, uniformly continuous functions in $\mathbb{R}^n$, normed by

$$\|f\,|C(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)|\,, \tag{11.5}$$

whereas

$$C^1(\mathbb{R}^n) = \left\{ f \in C(\mathbb{R}^n)\ :\ \frac{\partial f}{\partial x_j} \in C(\mathbb{R}^n) \text{ with } j = 1,\ldots,n \right\} \tag{11.6}$$

is the obviously normed related space of differentiable functions. Then $C^1(\mathbb{R}^n)$ is a closed subspace of $Lip(\mathbb{R}^n)$ and by the mean value theorem,

$$\|f\,|C^1(\mathbb{R}^n)\| \sim \|f\,|Lip(\mathbb{R}^n)\|\,, \quad f \in C^1(\mathbb{R}^n)\,, \tag{11.7}$$

(equivalent norms). First we clarify under what conditions $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ consist of regular distributions according to (11.1).

The word *classical* in the heading of this Section 11 has a double meaning. In Theorems 11.2 and 11.4 we collect (mainly without proofs) sharp embeddings in *classical* target spaces such as

$$L^{loc}_1(\mathbb{R}^n)\,, \quad L_r(\mathbb{R}^n)\,, \quad C(\mathbb{R}^n)\,, \quad C^1(\mathbb{R}^n)\,, \quad Lip(\mathbb{R}^n)\,,$$

whereas Theorem 11.7 describes those *classical* refined inequalities in limiting situations (from the middle of the 1960s up to around 1980) which are the roots of recent research and, in particular, of our further intentions in this chapter.

11.2 Theorem

(i) *Let*

$$0 < p < \infty\,, \quad 0 < q \le \infty\,, \quad s \in \mathbb{R}\,. \tag{11.8}$$

Then

$$F^s_{pq}(\mathbb{R}^n) \subset L^{loc}_1(\mathbb{R}^n) \tag{11.9}$$

if, and only if,

$$\text{either} \quad 0 < p < 1\,, \quad s \geq n\left(\frac{1}{p}-1\right), \quad 0 < q \leq \infty\,, \tag{11.10}$$

$$\text{or} \quad 1 \leq p < \infty\,, \quad s > 0\,, \quad 0 < q \leq \infty\,, \tag{11.11}$$

$$\text{or} \quad 1 \leq p < \infty\,, \quad s = 0\,, \quad 0 < q \leq 2\,. \tag{11.12}$$

(ii) *Let*

$$0 < p \leq \infty\,, \quad 0 < q \leq \infty\,, \quad s \in \mathbb{R}\,. \tag{11.13}$$

Then

$$B^s_{pq}(\mathbb{R}^n) \subset L^{loc}_1(\mathbb{R}^n) \tag{11.14}$$

if, and only if,

$$\text{either} \quad 0 < p \leq \infty\,, \quad s > n\left(\frac{1}{p}-1\right)_+, \quad 0 < q \leq \infty\,, \tag{11.15}$$

$$\text{or} \quad 0 < p \leq 1\,, \quad s = n\left(\frac{1}{p}-1\right), \quad 0 < q \leq 1\,, \tag{11.16}$$

$$\text{or} \quad 1 < p \leq \infty\,, \quad s = 0\,, \quad 0 < q \leq \min(p,2)\,. \tag{11.17}$$

11.3 Remark

This theorem coincides with Theorem 3.3.2 in [SiT95], where one finds also a proof. We refer also to [RuS96], pp. 32–35, where some of the key ideas of the proof are outlined. This theorem clarifies in a final way for which spaces $F^s_{pq}(\mathbb{R}^n)$ and $B^s_{pq}(\mathbb{R}^n)$ it makes sense to look at the distribution function $\mu_f(\lambda)$ in (10.2) and at the rearrangement $f^*(t)$ in (10.3) and to ask the questions sketched in the introduction in Section 10. We restrict ourselves in the sequel to spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with (10.1). In other words we exclude all borderline cases covered by Theorem 11.2 where either $p = \infty$ or $s = n(\frac{1}{p}-1)_+$. It would be of interest to have a closer look at these excluded spaces and also at a few other spaces not treated here, for example $F^s_{\infty q}(\mathbb{R}^n)$, including in particular $bmo(\mathbb{R}^n)$. Later on we return to these excluded spaces and add in 13.7 some comments and give a few references.

Next we wish to clarify the embedding of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in the sub-critical, critical, super-critical case, according to (10.5), (10.6), (10.15), respectively, in distinguished target spaces; this means $L_r(\mathbb{R}^n)$ in the sub-critical case (where r has the same meaning as in Fig. 10.1), $L_\infty(\mathbb{R}^n)$ and $C(\mathbb{R}^n)$ in the critical case, $Lip(\mathbb{R}^n)$ and $C^1(\mathbb{R}^n)$ in the super-critical case. These are the only cases of interest for us in the context outlined in Section 10. We do not repeat other assertions of the substantial embedding theory of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, including their special cases as Sobolev and Hölder-Zygmund spaces. The classical part of this theory may be found in [Triα], 2.8, and the almost classical part in [Triβ], 2.7.1. The final clarification of this type of embeddings goes back to [SiT95]. Descriptions of these results may be found in [RuS96], 2.2 and in [ET96], 2.3.3. As said, we restrict ourselves here to those special assertions which are directly related to the problems outlined in Section 10. Recall that the spaces $C(\mathbb{R}^n)$, $C^1(\mathbb{R}^n)$, and $Lip(\mathbb{R}^n)$ were introduced in 11.1 and (10.17).

11.4 Theorem

(i) (Sub-critical case) *Let*

$$1 < r < \infty\,, \quad s > 0\,, \quad s - \frac{n}{p} = -\frac{n}{r}\,, \quad \textit{and} \quad 0 < q \le \infty\,, \tag{11.18}$$

(*the dashed line in Fig. 10.1*). *Then*

$$B^s_{pq}(\mathbb{R}^n) \subset L_r(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < q \le r\,, \tag{11.19}$$

and

$$F^s_{pq}(\mathbb{R}^n) \subset L_r(\mathbb{R}^n) \quad \textit{for all} \quad 0 < q \le \infty\,. \tag{11.20}$$

(ii) (Critical case) *Let*

$$0 < p < \infty\,, \quad 0 < q \le \infty\,, \quad s = \frac{n}{p}\,. \tag{11.21}$$

Then

$$B^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < p < \infty\,, \quad 0 < q \le 1\,, \tag{11.22}$$

and

$$F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < p \le 1\,, \quad 0 < q \le \infty\,. \tag{11.23}$$

In (11.22) and (11.23) one can replace $C(\mathbb{R}^n)$ by $L_\infty(\mathbb{R}^n)$.

(iii) (Super-critical case) *Let*

$$0 < p < \infty\,, \quad 0 < q \le \infty\,, \quad s = 1 + \frac{n}{p}\,, \tag{11.24}$$

(*the dotted line in Fig. 10.1*). *Then*

$$B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset C^1(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0<p<\infty\,, \quad 0<q\le 1\,, \tag{11.25}$$

and

$$F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset C^1(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0<p\le 1\,, \quad 0<q\le \infty\,. \tag{11.26}$$

In (11.25) and (11.26) one can replace $C^1(\mathbb{R}^n)$ by $Lip(\mathbb{R}^n)$.

Proof (of part (iii)) Detailed proofs of parts (i) and (ii) may be found in [SiT95], Theorems 3.2.1 and 3.3.1, Remark 3.3.5; short descriptions are given in [RuS96], 2.2, and in [ET96], 2.3.3. Part (iii) is essentially the lifting of part (ii) by 1. But this is by no means obvious and must be justified. First we remark that

$$f \in F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{if, and only if,} \quad f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{and} \quad \frac{\partial f}{\partial x_j} \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \tag{11.27}$$

where $j=1,\dots,n$ and (equivalent quasi-norms)

$$\|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \sim \|f\,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| + \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j}\,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\right\|. \tag{11.28}$$

We refer to [Triβ], Theorem 2.3.8, pp. 58/59. Hence, the if-part of (11.26) follows from (11.23). Conversely, assume

$$F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset C^1(\mathbb{R}^n) \quad \text{for some} \quad 0<p<\infty\,, \quad 0<q\le\infty\,. \tag{11.29}$$

We construct special Fourier multipliers and introduce some cones in $\mathbb{R}^n$,

$$K_t = \{\xi=(\xi',\xi_n)\in\mathbb{R}^n \,:\, \xi_n>0\,,\ |\xi'|<t\xi_n\}\,, \quad t>0\,, \tag{11.30}$$

where obviously $\xi'=(\xi_1,\dots,\xi_{n-1})\in\mathbb{R}^{n-1}$. Let φ be a C^∞ function in $\mathbb{R}^n\backslash\{0\}$ with

$$\varphi(\xi)=\varphi\left(\frac{\xi}{|\xi|}\right), \quad \varphi(\xi)=1 \quad \text{if} \quad \xi\in K_t \quad \text{and} \quad supp\,\varphi\subset\overline{K_{2t}}\,. \tag{11.31}$$

Let $\psi(\xi)$ be a C^∞ function in $\mathbb{R}^n$ which is identically 1 if $|\xi|\ge 1$ and $0\notin supp\,\psi$. Then, by [Triβ], Theorem 2.3.7 on p. 57,

$$\varphi^j(\xi)=\varphi(\xi)\,\psi(\xi)\,\xi_j\xi_n^{-1}\,, \quad \text{where} \quad j=1,\dots,n\,, \tag{11.32}$$

are Fourier multipliers in all spaces $F^s_{pq}(\mathbb{R}^n)$ and $B^s_{pq}(\mathbb{R}^n)$. Let

$$f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad supp\, \widehat{f} \subset K_t \cap \{\xi \in \mathbb{R}^n \,:\, |\xi| > 1\}\,, \tag{11.33}$$

and

$$g(x) = \Big(\xi_n^{-1} \widehat{f}(\xi)\Big)^{\vee}(x) = \Big(\xi_n^{-1} \varphi^n(\xi)\, \widehat{f}(\xi)\Big)^{\vee}(x)\,, \quad x \in \mathbb{R}^n\,, \tag{11.34}$$

where we used the notation introduced in 2.8. Since $\varphi^j(\xi)$ are Fourier multipliers it follows by (11.28) that

$$g \in F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{and} \quad \frac{\partial g}{\partial x_n}(x) = i\, f(x)\,. \tag{11.35}$$

Hence if we assume (11.29) for some p and q, then it follows for functions f with (11.33) that $f \in C(\mathbb{R}^n)$ and

$$\|f\,|C(\mathbb{R}^n)\| \le c\,\|f\,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\|\,. \tag{11.36}$$

An arbitrary function $f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ can be decomposed in finitely many functions of type (11.33) and a harmless function $\Big((1-\psi)\widehat{f}\Big)^{\vee}$, where ψ has the above meaning. Then we have (11.36) for those p, q with (11.29). This is the converse we are looking for, and it proves (11.26). Finally we must show that one can replace $C^1(\mathbb{R}^n)$ in (11.26) by $Lip(\mathbb{R}^n)$. Since $Lip(\mathbb{R}^n)$ is the larger space we must disprove

$$F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset Lip(\mathbb{R}^n) \tag{11.37}$$

if $p > 1$ and $0 < q \le \infty$. By the monotonicity of the spaces $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ with respect to q we may assume $q < \infty$. Then $S(\mathbb{R}^n)$ is dense in $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ and we have (11.7) for $f \in S(\mathbb{R}^n)$. By completion it follows (11.37) with $C^1(\mathbb{R}^n)$ in place of $Lip(\mathbb{R}^n)$. But this contradicts (11.26) since $p > 1$. The proof for the B-spaces is the same.

11.5 Remark

Usually, $s - \frac{n}{p}$ is called the *differential dimension* of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. In particular, $-\frac{n}{r}$ is the differential dimension of $L_r(\mathbb{R}^n)$. This notion can obviously be extended to the above target spaces $C(\mathbb{R}^n)$, $L_\infty(\mathbb{R}^n)$ (differential dimension 0) and $C^1(\mathbb{R}^n)$, $Lip(\mathbb{R}^n)$ (differential dimension 1). Continuous embeddings between function spaces with the same differential dimension are often called *limiting embeddings*. The above theorem deals exclusively with embeddings of this type in the indicated specific situations:

sub-critical : differential dimension $-\frac{n}{r}$,

critical : differential dimension 0,

super-critical : differential dimension 1.

Furthermore, (10.18) is now an immediate consequence of the above theorem. As explained in Section 10, in connection with the growth envelope $\mathfrak{E}_G(A^{\frac{n}{p}}_{pq})$ in (10.13) for the critical case and the continuity envelope $\mathfrak{E}_C(A^{1+\frac{n}{p}}_{pq})$ in (10.20) for the super-critical case, we are interested only in those spaces which are not covered by (10.18), this means by the parts (ii) and (iii) of the above theorem in the spaces

$$B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\,, \quad B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 0<p<\infty\,, \quad 1<q\le\infty\,, \tag{11.38}$$

and

$$F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\,, \quad F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 1<p<\infty\,, \quad 0<q\le\infty\,. \tag{11.39}$$

If $q=2$ in (11.39) then we get by (1.9) the Sobolev spaces

$$H^{\frac{n}{p}}_{p}(\mathbb{R}^n)\,, \quad H^{1+\frac{n}{p}}_{p}(\mathbb{R}^n) \quad \text{with} \quad 1<p<\infty\,. \tag{11.40}$$

On the other hand, we have the famous Sobolev embedding

$$H^s_p(\mathbb{R}^n) \subset L_r(\mathbb{R}^n)\,, \quad 1<p<\infty\,, \quad -\frac{n}{r} = s-\frac{n}{p} < 0\,, \tag{11.41}$$

as a special case of (11.20). But it is just the failure to extend (11.41) from the sub-critical case $1<r<\infty$, to the critical case $r=\infty$, which triggered the search for adequate substitutes. Then we are back to the 1960s. At the same time refinements of sub-critical embeddings according to part (i) of Theorem 11.4 for the Sobolev spaces $H^s_p(\mathbb{R}^n)$ and the classical Besov spaces $B^s_{pq}(\mathbb{R}^n)$ in terms of Lorentz spaces L_{ru} have been studied (in the West inspired by interpolation theory). Around 1980 further refinements in the critical case and the first steps in the super-critical case for the Sobolev spaces in (11.40) were taken. We shall describe this nowadays historical part of these refined embeddings in Theorem 11.7 below, including in 11.8 the respective references. Mostly for this reason we discuss in 11.6 the relevant target spaces, whereas later on we prefer to formulate our results in terms of inequalities.

11.6 Lorentz-Zygmund spaces

For the reasons just explained we restrict ourselves to a brief description. The standard reference for Lorentz spaces and Zygmund spaces is [BeS88]. Their combination, the Lorentz-Zygmund spaces, were introduced in [BeR80]. Let $0 < \varepsilon < 1$ and let $I_\varepsilon = (0, \varepsilon]$. We use the rearrangement $f^*(t)$ of a complex-valued measurable function $f(t)$ on I_ε as introduced in (10.2), (10.3), temporarily with I_ε in place of $\mathbb{R}^n$.

(i) *Lorentz spaces* Let $0 < r < \infty$ and $0 < u \leq \infty$. Then $L_{ru}(I_\varepsilon)$ is the set of all measurable complex-valued functions f on I_ε such that

$$\int_0^\varepsilon \left(t^{\frac{1}{r}} f^*(t) \right)^u \frac{dt}{t} < \infty \quad \text{if} \quad 0 < u < \infty\,, \tag{11.42}$$

and

$$\sup_{t \in I_\varepsilon} t^{\frac{1}{r}} f^*(t) < \infty \quad \text{if} \quad u = \infty\,. \tag{11.43}$$

Of course, $L_{rr}(I_\varepsilon) = L_r(I_\varepsilon)$ with $0 < r < \infty$, are the usual Lebesgue spaces.

(ii) *Zygmund spaces* Let $0 < r < \infty$ and $a \in \mathbb{R}$. Then $L_r(\log L)_a(I_\varepsilon)$ is the set of all measurable complex-valued functions f on I_ε such that

$$\int_0^\varepsilon |f(t)|^r \, \log^{ar}(2 + |f(t)|)\, dt < \infty\,. \tag{11.44}$$

Let $a < 0$. Then $L_\infty(\log L)_a(I_\varepsilon)$ is the set of all measurable complex-valued functions f on I_ε such that there is a number $\lambda > 0$ with

$$\int_0^\varepsilon \exp\left\{ (\lambda |f(t)|)^{-\frac{1}{a}} \right\} dt < \infty\,. \tag{11.45}$$

When $r < \infty$, then this notation resembles that in [BeS88], pp. 252–253. The alternative notation $L_{\exp,-a}(I_\varepsilon)$ for $L_\infty(\log L)_a(I_\varepsilon)$ is closer to that employed in [BeS88]. The somewhat curious looking expression (11.45) may be justified by the following unified alternative representation, where $f \in L_r(\log L)_a(I_\varepsilon)$ if, and only if,

$$\left(\int_0^\varepsilon |\log t|^{ar} f^{*r}(t)\, dt \right)^{\frac{1}{r}} < \infty\,, \quad \text{when} \quad 0 < r < \infty\,, \quad a \in \mathbb{R}\,, \tag{11.46}$$

and

$$\sup_{t \in I_\varepsilon} |\log t|^a f^*(t) < \infty\,, \quad \text{when} \quad r = \infty\,, \quad a < 0\,. \tag{11.47}$$

We refer to [BeS88], p. 252, or [ET96], p. 66. In [ET96], 2.6.1, 2.6.2, one finds also another unifying representation, further properties and references. This way of looking at these spaces fits also in the scheme developed in the following Section 12. Both (11.46) and (11.47) are quasi-norms.

(iii) *Lorentz-Zygmund spaces* The combination of the above Lorentz spaces and Zygmund spaces results in the Lorentz-Zygmund spaces studied in detail in [BeR80]. Let

$$0 < r < \infty\,, \quad 0 < u \le \infty \quad \text{and} \quad a \in \mathbb{R}.$$

Then $L_{ru}(\log L)_a(I_\varepsilon)$ is the set of all measurable complex-valued functions f on I_ε such that

$$\left(\int_0^\varepsilon \left(t^{\frac{1}{r}} |\log t|^a\, f^*(t) \right)^u \frac{dt}{t} \right)^{\frac{1}{u}} < \infty \quad \text{if} \quad 0 < u < \infty\,, \tag{11.48}$$

and

$$\sup_{t \in I_\varepsilon} t^{\frac{1}{r}} |\log t|^a\, f^*(t) < \infty \quad \text{if} \quad u = \infty\,. \tag{11.49}$$

Again (11.48) and (11.49) are quasi-norms. Note that

$$L_{rr}(\log L)_a(I_\varepsilon) = L_r(\log L)_a(I_\varepsilon) \quad \text{where} \quad 0 < r < \infty\,, \quad a \in \mathbb{R}\,,$$

and

$$L_{ru}(\log L)_0(I_\varepsilon) = L_{ru}(I_\varepsilon) \quad \text{where} \quad 0 < r < \infty\,, \quad 0 < u \le \infty\,.$$

For details we refer to [BeS88], p. 253, and, in particular to [BeR80]. This reference covers also the interesting case $r = \infty$, hence $L_{\infty,u}(\log L)_a(I_\varepsilon)$. If $r = u = \infty$ then (11.49) coincides with (11.47), and one needs $a < 0$. If $r = \infty$ and $0 < u < \infty$, then one needs in (11.48) that $au < -1$, hence $a + \frac{1}{u} < 0$. Otherwise, if $r = \infty$ and $au \ge -1$, then there are no non-trivial functions f with (11.48). Some well-known embeddings between these spaces will come out later on in 12.4 in a natural way.

We are not so much interested in the Lorentz-Zygmund spaces for their own sake. We formulate later on our results in terms of inequalities using rearrangements. This will also be done in the next theorem where we collect the historical roots of the theory outlined in the introductory Section 10, although the original formulations looked sometimes quite different. This applies in particular when the spaces $L_r(\log L)_a(I_\varepsilon)$ in the original versions (11.44), (11.45) are involved. Not only the spaces themselves but also their equivalent characterizations via (11.46), (11.47) can be traced back to Hardy-Littlewood, Zygmund, Lorentz and Bennett. The corresponding references may be found in the Note Sections in [BeS88], pp. 288, 180–181, and also in [BeR80], in connection with Corollary 10.2 and Theorem 10.3. This resulted finally in the Lorentz-Zygmund spaces quasi-normed by (11.48), (11.49) in [BeR80]. In particular, all the equivalent (quasi-)norms mentioned above were known around 1980 (and often long before).

Recall that $H^s_p(\mathbb{R}^n)$ are the Sobolev spaces, $1 < p < \infty$, $s > 0$, according to (1.9) with the classical Sobolev spaces $W^s_p(\mathbb{R}^n)$ as special cases. The classical Besov spaces are normed by (1.14). The divided differences $\widetilde{\omega}(f, t)$ are given by (10.16). Let $0 < \varepsilon < 1$.

11.7 Theorem

(Classical refined inequalities in limiting situations)

(i) (Sub-critical case, Lorentz spaces, dashed line in Fig. 10.1)
Let $s > 0$,

$$1 < p < \infty\,, \quad -\frac{n}{r} = s - \frac{n}{p} < 0\,, \quad 1 \le q \le \infty\,. \tag{11.50}$$

Then there is a constant $c > 0$ *such that*

$$\left(\int_0^\varepsilon \left(t^{\frac{1}{r}} f^*(t) \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \le c\, \| f \,| H^s_p(\mathbb{R}^n) \| \quad \textit{for all} \quad f \in H^s_p(\mathbb{R}^n)\,, \tag{11.51}$$

and

$$\left(\int_0^\varepsilon \left(t^{\frac{1}{r}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le c\, \| f \,| B^s_{pq}(\mathbb{R}^n) \| \quad \textit{for all} \quad f \in B^s_{pq}(\mathbb{R}^n) \tag{11.52}$$

(with (11.43) if $q = \infty$*).*

(ii) (Sub-critical case, Zygmund spaces, dashed line in Fig. 10.1)
Let $s > 0$,

$$1 < p < \infty\,, \quad -\frac{n}{r} = s - \frac{n}{p} < 0\,, \quad r < q \le \infty\,, \quad -\infty < a < \frac{1}{q} - \frac{1}{r}\,. \tag{11.53}$$

Then there is a constant $c > 0$ *such that*

$$\left(\int_0^\varepsilon |\log t|^{ar} f^{*r}(t)\, dt \right)^{\frac{1}{r}} \le c\, \| f \,| B^s_{pq}(\mathbb{R}^n) \| \quad \textit{for all} \quad f \in B^s_{pq}(\mathbb{R}^n)\,. \tag{11.54}$$

(iii) (Critical case) *Let*

$$1 < p < \infty\,, \quad \frac{1}{p} + \frac{1}{p'} = 1\,. \tag{11.55}$$

Then there is a constant $c > 0$ *such that*

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{\frac{1}{p'}}} \le c\, \|f\,|H_p^{\frac{n}{p}}(\mathbb{R}^n)\| \quad \textit{for all} \quad f \in H_p^{\frac{n}{p}}(\mathbb{R}^n)\,, \tag{11.56}$$

and

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \le c\, \|f\,|H_p^{\frac{n}{p}}(\mathbb{R}^n)\| \quad \textit{for all} \quad f \in H_p^{\frac{n}{p}}(\mathbb{R}^n)\,. \tag{11.57}$$

(iv) (Super-critical case) *Let*

$$1 < p < \infty\,, \quad \frac{1}{p} + \frac{1}{p'} = 1\,, \quad 1 + \frac{n}{p} = k \in \mathbb{N}\,. \tag{11.58}$$

Then there is a constant $c > 0$ *such that*

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{p'}}} \le c\, \|f\,|W_p^k(\mathbb{R}^n)\| \quad \textit{for all} \quad f \in W_p^k(\mathbb{R}^n)\,. \tag{11.59}$$

11.8 Historical references and comments

We tried to collect in the above theorem those refined inequalities which we believe are the roots of the programme outlined in the introductory Section 10. "Refined" must be understood in comparison with Theorem 11.4 asking for a tuning of the admissible target spaces finer than used there. The spaces described in 11.6 may be considered as a reasonable choice for such an undertaking. A balanced or even comprehensive history of inequalities of this type seems to be rather complicated. Many mathematicians contributed to this flourishing field of research, and parallel developments in the East (in the Russian literature) have often passed unnoticed in the West, but also vice versa. In a sequence of points, denoted as 11.8(i) etc., we collect related papers, comment on a few topics, and try to clarify to what extent the above inequalities fit in our scheme. We shift more recent references to a later occasion (with exception of a few surveys which in turn describe the history) and restrict ourselves to those papers which stand for the early development of this theory (although this covers a period of some 20 years).

11.8(i) (Sub-critical case, Lorentz spaces) The inequalities (11.51), (11.52) came into being in the middle of the 1960s with the advent of the interpolation theory: there was no escape, as we outline in the next point. But there are more direct approaches, especially in connection with a wider scale of Besov spaces (generalized moduli of continuity), vector-valued Besov spaces, and the question about the sharpness of these inequalities. We refer to [Pee66], [Str67], [Her68], [Bru72], [Bru76], [Gol85], [Gol86], and the surveys [Kol89], [Kol98], [Liz86], which describe especially what has been done in the Russian literature.

11.8(ii) (Sub-critical case, interpolation) We use without further explanations the real interpolation $(A_0, A_1)_{\theta,q}$ of two (quasi-)Banach spaces A_0 and A_1, and $0 < \theta < 1$, $0 < q \leq \infty$. We refer to [Triα], [BeL76] or [BeS88], where one finds all that one needs. Recall that

$$(L_{r_0}, L_{r_1})_{\theta,p} = L_{rp}\,, \quad 1 < r_0 < r_1 < \infty\,, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}\,, \quad 0 < p \leq \infty\,, \tag{11.60}$$

on I_ε or on $\mathbb{R}^n$, where L_{rq} are the Lorentz spaces from 11.6(i). Lifting of (11.60) on $\mathbb{R}^n$ gives

$$\left(H^s_{p_0}, H^s_{p_1}\right)_{\theta,p} = H^s_p\,, \quad 1 < p_0 < p_1 < \infty\,, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,, \quad s \in \mathbb{R}\,. \tag{11.61}$$

Furthermore, again on $\mathbb{R}^n$,

$$\left(B^{s_0}_{p,1}, B^{s_1}_{p,1}\right)_{\theta,q} = B^s_{pq}\,, \quad 0 < s_0 < s_1 < \infty\,, \quad s = (1-\theta)s_0 + \theta s_1\,, \tag{11.62}$$

$0 < q \leq \infty$. Now (11.20) with $H^s_p = F^s_{p,2}$, and the interpolations (11.61), (11.60) give (11.51), whereas (11.19), and the interpolations (11.62), (11.60) result in (11.52).

11.8(iii) (Sub-critical case, Zygmund spaces) The inequality (11.54) is less satisfactory than the inequalities (11.51) and (11.52). We explain the reason in the next point. In addition, the above Zygmund spaces are not naturally linked with the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in the reasoning of Section 10 in sub-critical situations, in contrast to the Lorentz spaces. Although we could not find a direct formulation of (11.54) in the literature, assertions of this type are apparently included (in a somewhat hidden way) in a larger theory of embeddings of the form

$$B^{\omega(\cdot)}_{pq} \subset L_\Phi \quad 1 \leq p < \infty\,, \quad 0 < q \leq \infty\,, \tag{11.63}$$

where ω is a generalized modulus of continuity and L_Φ stands for an Orlicz space. This was studied in the 1980s in great detail in the Russian literature. In [Gol86], Theorem 5.4, one finds necessary and sufficient conditions for embeddings of type (11.63). A corresponding formulation may also be found in

[KaL87], Theorem 8.5, p. 27. This paper surveys what has been done by the Russian school of the theory of function spaces. With some modifications it is the English version of [Liz86] where the same result (11.63) may be found in D.1.8. At least in some cases $L_r(\log L)_a$ can be identified with an Orlicz space. We refer to [BeS88], p. 266, Example 8.3(e), with $\Phi(t) = t^r |\log t|^a$. Taking together all these remarks then some assertions of type (11.54) for classical Besov spaces are hidden in [Gol86], [Liz86], [KaL87].

11.8(iv) (Sub-critical case, Hölder inequalities) As said in the previous point, (11.54) with (11.53) does not fit optimally in our context. Furthermore, this estimate follows from (11.52) and Hölder's inequality: Let again $0 < \varepsilon < 1$,

$$1 < r < q \le \infty \,, \quad \frac{1}{r} = \frac{1}{q} + \frac{1}{u} \,, \tag{11.64}$$

and $h(t) > 0$ if $0 < t \le \varepsilon$. Then by Hölder's inequality,

$$\begin{aligned} \left(\int_0^\varepsilon h(t)^r \, f^{*r}(t) \, dt \right)^{\frac{1}{r}} &= \left(\int_0^\varepsilon \left(t^{\frac{1}{r}} \, h(t) \, f^*(t) \right)^r \, \frac{dt}{t} \right)^{\frac{1}{r}} \\ &\le \left(\int_0^\varepsilon \left(t^{\frac{1}{r}} \, f^*(t) \right)^q \, \frac{dt}{t} \right)^{\frac{1}{q}} \left(\int_0^\varepsilon h(t)^u \, \frac{dt}{t} \right)^{\frac{1}{u}} . \end{aligned} \tag{11.65}$$

The last factor with $h(t) = |\log t|^a$ converges if, and only if, $au < -1$. This proves the if-part of:

$$\left(\int_0^\varepsilon |\log t|^{ar} \, f^{*r}(t) \, dt \right)^{\frac{1}{r}} \le c \left(\int_0^\varepsilon \left(t^{\frac{1}{r}} \, f^*(t) \right)^q \, \frac{dt}{t} \right)^{\frac{1}{q}} \tag{11.66}$$

if, and only if, $a < \frac{1}{q} - \frac{1}{r}$. To disprove (11.66) if $a \ge \frac{1}{q} - \frac{1}{r}$ one may choose $a = \frac{1}{q} - \frac{1}{r}$ and, as a counter-example,

$$f(t) = t^{-\frac{1}{r}} \, |\log t|^{-\frac{1}{q}} \, \left(\log |\log t| \right)^{-\frac{1}{r}} \,, \quad 0 < t < \varepsilon \,, \tag{11.67}$$

assuming that $\varepsilon > 0$ is sufficiently small. But this makes clear that $|\log t|^a$ in (11.54) is a distinguished but not natural choice. There are better functions $h(t)$ such that the last factor in (11.65) converges. Maybe a systematic treatment in this direction would result in problems of type (11.63). Nevertheless we return to inequalities of type (11.54) later on in Corollary 15.4.

11.8(v) (Critical case) The inequality (11.56) has a rich history reflecting especially well parallel developments in East and West. First we recall that by the equivalence of (11.47) and (11.45) for some $\lambda > 0$, the left-hand side of (11.56) is finite if, and only if,

$$\int_0^\varepsilon \exp\left\{(\lambda|f(t)|)^{p'}\right\} dt < \infty \quad \text{for some} \quad \lambda > 0\,. \tag{11.68}$$

In this version, (11.56) is due to R. S. Strichartz, [Str72], including a sharpness assertion. Corresponding results for the classical Sobolev spaces

$$W_p^k(\mathbb{R}^n) = H_p^{\frac{n}{p}}(\mathbb{R}^n) \quad \text{with} \quad 1 < p < \infty\,, \quad \frac{n}{p} = k \in \mathbb{N}\,, \tag{11.69}$$

(especially if $k = 1$) in the version of (11.68) had been obtained before in [Tru67] and may also be found in [GiT77], Theorem 7.15 and (7.40) on p. 155. This paper by N. S. Trudinger made problems of this type widely known and influenced further development, in terms of spaces of type (11.68) and more general Orlicz spaces. For classical Sobolev spaces with the fixed norm (1.4), especially if $k = 1$, hence $W_n^1(\mathbb{R}^n)$, it makes sense to ask for the best constant λ in (11.68). This may be found in [Mos71]. As noted above there was a parallel and independent development in the East. We refer in particular to [Poh65] and the even earlier forerunner [Yud61]. Best constants for the embeddings of Sobolev spaces according to (11.69) in spaces of type (11.68), extending [Mos71] to all $k \in \mathbb{N}$, may be found in [Ada88]. This paper contains also a balanced history of this subject, including the Russian literature. The natural counterpart of (11.56) for classical Besov spaces $B_{pq}^{\frac{n}{p}}(\mathbb{R}^n)$, normed by (1.14), is given by

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{\frac{1}{q'}}} \le c\,\|f\,|B_{pq}^{\frac{n}{p}}(\mathbb{R}^n)\|\,, \quad 1 < p < \infty\,, \quad 1 < q \le \infty\,, \tag{11.70}$$

and $\frac{1}{q} + \frac{1}{q'} = 1$. Limiting embeddings of spaces $B_{pq}^{\frac{n}{p}}(\mathbb{R}^n)$ in Orlicz spaces were considered by J. Peetre in [Pee66]. If one takes his assertion in [Pee66], Theorem 9.1 on p. 303, and reformulates it in terms of later developments in the 1980s and 1990s, and which may be found in [ET96], 2.6.2, then one arrives at (11.70) or equivalently at (11.68) with q' in place of p'. As for (11.57) we first remark that

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{\frac{1}{p'}}} \le c \left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}}. \tag{11.71}$$

This is an assertion of type (10.11) and will be considered in detail in 12.4, especially (12.28). Hence (11.57) is sharper than (11.56). This improvement goes back to [Has79] and [BrW80] (as a consequence of Theorem 2 on p. 781), including sharpness assertion in [Has79]. We refer in this context also to [Zie89], 2.10.5, 2.10.6, pp. 100–103, where one finds improved and more detailed versions of the arguments in [BrW80].

11.8(vi) (Super-critical case) The first direct proof of (11.59) may be found in [BrW80], Corollary 5, p. 786. On the other hand, accepting (10.22), its n-dimensional counterpart, and that in (11.51), (11.52), (11.56), (11.57), f^* can be replaced by f^{**}, then all the inequalities in the super-critical case can be obtained by lifting from the critical case, in particular (11.59) by lifting of a special case of (11.56), and a stronger and more general version of (11.59) by lifting of (11.57), an inequality which has also been proved in [BrW80]. Maybe this connection passed unnoticed not only at that time but also in recent research dealing separately with the critical and super-critical case. This close interdependence is also well reflected by the parts (ii) and (iii) in Theorem 11.4, and by its short version (10.18).

12 Envelopes

12.1 Rearrangement and the growth of functions

In this section we introduce the concept of growth envelopes and continuity envelopes as outlined in the introductory Section 10. We restrict ourselves to those spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ which are of interest for us. On this basis we prove in the subsequent Sections 13–16 the main results of this chapter. The new notion of growth and continuity envelopes makes sense for a much wider range of function spaces and has been introduced and studied recently by D. D. Haroske in [Har01]. We take over a few results obtained there, including the relevant notation of envelopes and envelope functions.

First we recall again what is meant by rearrangement. For our purpose it is sufficient to assume that $f \in L_r(\mathbb{R}^n)$ with $1 \le r < \infty$. Then as in (10.2), (10.3), the distribution function μ_f and the non-increasing rearrangement f^* of f are given by

$$\mu_f(\lambda) = |\{x \in \mathbb{R}^n \,:\, |f(x)| > \lambda\}|\,, \quad \lambda \ge 0\,, \tag{12.1}$$

and

$$f^*(t) = \inf\{\lambda \,:\, \mu_f(\lambda) \le t\}\,, \quad t \ge 0\,. \tag{12.2}$$

We wish to measure the growth of functions f belonging to $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ either in the sub-critical situation according to Theorem 11.4(i) or in the critical situation for those parameters p, q not covered by Theorem 11.4(ii). Let $A^s_{pq}(\mathbb{R}^n)$ be such a space and let $f \in A^s_{pq}(\mathbb{R}^n)$. Then the behaviour of the rearrangement $f^*(t)$ if $t \downarrow 0$ indicates how singular the function f might be on the global scale, the whole of $\mathbb{R}^n$. Inspired by the spaces in 11.6 we try to measure the possible growth of f in terms of

$$\left(\int_0^\varepsilon \left(f^*(t)\, w(t)\right)^u \, \frac{dt}{t} \right)^{\frac{1}{u}} \tag{12.3}$$

for some fixed $0 < \varepsilon < 1$, $0 < u \le \infty$ (appropriately modified if $u = \infty$) and some positive continuous monotonically increasing weight functions $w(t)$ on $[0, \varepsilon)$ with $w(0) = 0$, $w(t) > 0$ if $0 < t < \varepsilon$, and

$$w(2^{-j-1}) \sim w(2^{-j}), \quad j \in \mathbb{N}, \quad j \ge J(\varepsilon). \tag{12.4}$$

Recall that we use the equivalence sign "$\sim$" as explained before, for example in (7.10). Then we have

$$\left(\int_0^\varepsilon \left(f^*(t)\, w(t)\right)^u \, \frac{dt}{t} \right)^{\frac{1}{u}} \sim \left(\sum_j \left(f^*(2^{-j})\, w(2^{-j})\right)^u \right)^{\frac{1}{u}}, \tag{12.5}$$

where the equivalence constants are independent of f. Hence (12.3) behaves like a sequence space ℓ_u, including the well-known monotonicity with respect to u.

The following reformulation of this type of singularity measurement might be helpful for a better understanding. We assume that the reader is familiar with the basic facts concerning rearrangement. They may be found in [BeS88]. In particular, rearrangement is measure-preserving,

$$|\{t > 0 \,:\, \tau_0 \ge f^*(t) \ge \tau_1\}| = |\{x \in \mathbb{R}^n \,:\, \tau_0 \ge |f(x)| \ge \tau_1\}|, \tag{12.6}$$

where $0 \le \tau_1 \le \tau_0 < \infty$. We refer also to [Tri$\alpha$], p. 132. Here "$\tau_0 \ge$" and/or "$\ge \tau_1$" can be replaced by "$\tau_0 >$" and/or "$> \tau_1$", respectively. Furthermore, the Lebesgue measure on $\mathbb{R}^n$ is divisible: If M is a Lebesgue-measurable set in $\mathbb{R}^n$ with $0 < |M| < \infty$ and if λ is a positive number with $0 < \lambda < |M|$, then there is a Lebesgue-measurable set M_λ with

$$M_\lambda \subset M \quad \text{and} \quad |M_\lambda| = \lambda. \tag{12.7}$$

Let again $f \in A^s_{pq}(\mathbb{R}^n)$ as above. By (12.6), (12.7) there is a set M with $|M| = 1$ and

$$M \subset \{x \in \mathbb{R}^n \,:\, |f(x)| \ge f^*(1)\}, \quad \mathbb{R}^n \backslash M \subset \{x \in \mathbb{R}^n \,:\, |f(x)| \le f^*(1)\}. \tag{12.8}$$

The set M can be decomposed by

$$M = \bigcup_{j=1}^{\infty} M_j\,, \quad M_j \cap M_k = \emptyset \text{ if } j \neq k\,, \quad |M_j| = 2^{-j} \text{ if } j \in \mathbb{N}\,, \tag{12.9}$$

such that

$$M_j \subset \{x \in \mathbb{R}^n \,:\, f^*(2^{-j+1}) \leq |f(x)| \leq f^*(2^{-j})\}\,, \quad j \in \mathbb{N}\,. \tag{12.10}$$

In other words, if (12.3), (12.4) or (12.5) is finite for all $f \in A^s_{pq}(\mathbb{R}^n)$, then we get some information on how rapidly $|f(x)|$ might grow on a sequence of sets M_j or M^j with $|M_j| = 2^{-j}$ or $|M^j| = 2^{-j+1}$ where $M^j = \cup_{l=j}^{\infty} M_l$.

For a closer look at (12.3) and (12.5) in connection with the above spaces $A^s_{pq}(\mathbb{R}^n)$ and inspired by the rearrangement formulations of Lorentz-Zygmund spaces according to 11.6, we adopt a slightly more general point of view. Let ψ be a real continuous monotonically increasing function on the interval $[0,\varepsilon]$, where $0 < \varepsilon < 1$, with

$$\psi(0) = 0 \quad \text{and} \quad \psi(t) > 0 \quad \text{if} \quad 0 < t \leq \varepsilon\,. \tag{12.11}$$

Let $\Psi(t) = \log \psi(t)$ and let μ_Ψ be the associated Borel measure. We refer to [Lan93], especially p. 285, for details of this notation. In particular, all the integrals below with respect to the distribution function $\Psi(t)$, but also with respect to the other distribution functions needed below, can be interpreted as Riemann-Stieltjes integrals (defined in the usual way via Riemann-Stieltjes sums). If, in addition, $\psi(t)$ is differentiable in $(0,\varepsilon)$ then

$$\mu_\Psi(dt) = \frac{\psi'(t)}{\psi(t)}\, dt \quad \text{in} \quad (0,\varepsilon). \tag{12.12}$$

Let $\psi_1(t)$ and $\psi_2(t)$ be two real continuous monotonically increasing functions on $[0,\varepsilon]$ with the counterpart of (12.11). Then, again, ψ_1 and ψ_2 are said to be equivalent, $\psi_1 \sim \psi_2$, if there are positive numbers c_1 and c_2 with

$$c_1\, \psi_1(t) \leq \psi_2(t) \leq c_2 \psi_1(t)\,, \quad 0 \leq t \leq \varepsilon\,. \tag{12.13}$$

12.2 Proposition

(i) *Let ψ be a real continuous monotonically increasing function on the interval $[0,\varepsilon]$ with (12.11). Let $0 < u_0 < u_1 < \infty$. There are two positive numbers*

c_0 and c_1 such that

$$\begin{aligned}\sup_{0<t\le\varepsilon} \psi(t)\,g(t) &\le c_1 \left(\int_0^\varepsilon (\psi(t)\,g(t))^{u_1}\,\mu_\Psi(dt)\right)^{\frac{1}{u_1}} \\ &\le c_0 \left(\int_0^\varepsilon (\psi(t)\,g(t))^{u_0}\,\mu_\Psi(dt)\right)^{\frac{1}{u_0}} \qquad (12.14)\end{aligned}$$

for all non-negative monotonically decreasing functions g on $(0,\varepsilon]$.

(ii) *Let ψ_1 and ψ_2 be two such functions which are equivalent according to (12.13), and let $\Psi_j(t) = \log\psi_j(t)$ and μ_{Ψ_j} be the corresponding distribution functions and measures ($j = 1,2$). Let $0 < u \le \infty$. Then*

$$\left(\int_0^\varepsilon (\psi_1(t)\,g(t))^u\,\mu_{\Psi_1}(dt)\right)^{\frac{1}{u}} \sim \left(\int_0^\varepsilon (\psi_2(t)\,g(t)))^u\,\mu_{\Psi_2}(dt)\right)^{\frac{1}{u}} \qquad (12.15)$$

(with the sup*-norm if $u = \infty$) for all non-negative monotonically decreasing functions g on $(0,\varepsilon]$, where the equivalence constants are independent of g.*

Proof *Step 1* We begin with some preparation. Let $0 < u < \infty$. We claim

$$\psi^u(t) \sim \int_0^t \psi^u(\tau)\,\mu_\Psi(d\tau)\,, \quad 0 < t \le \varepsilon\,, \qquad (12.16)$$

where the equivalence constants are independent of t and of all admitted ψ according to (i). Let $\psi(t)$ be fixed and let

$$2^{-l} < \psi(t) \le 2^{-l+1} \quad \text{for some} \quad l \in \mathbb{Z} \qquad (12.17)$$

(where, as in 2.1, $\mathbb{Z}$ is the collection of all integers). Let a_j be a decreasing sequence of positive numbers, tending to zero with $\psi(a_j) = 2^{-j}$. Then

$$a_l < t \le a_{l-1} \qquad (12.18)$$

(with the replacement of a_{l-1} by ε if $\psi(\varepsilon) < 2^{-l+1}$). Since $\Psi(a_j) - \Psi(a_{j+1}) = 1$ it follows that

$$\begin{aligned}\int_0^{a_l} \psi^u(\tau)\,\mu_\Psi(d\tau) &= \sum_{j=l}^\infty \int_{a_{j+1}}^{a_j} \psi^u(\tau)\,\mu_\Psi(d\tau) \\ &\sim \sum_{j=l}^\infty 2^{-ju} \sim 2^{-lu} \sim \psi^u(t)\,. \qquad (12.19)\end{aligned}$$

Similarly with a_{l-1} (respectively ε) in place of a_l on the left-hand side. This proves (12.16).

Step 2 We prove (i). Let $0 < u < \infty$. By (12.16) and the monotonicity of the non-negative function g it follows that

$$\begin{aligned} \psi(t)\, g(t) &\le c \left(\int_0^t g^u(\tau)\, \psi^u(\tau)\, \mu_\Psi(d\tau) \right)^{\frac{1}{u}} \\ &\le c \left(\int_0^\varepsilon g^u(\tau)\, \psi^u(\tau)\, \mu_\Psi(d\tau) \right)^{\frac{1}{u}}, \quad 0 < t \le \varepsilon\,. \end{aligned} \tag{12.20}$$

This is even sharper than the first estimate in (12.14). Let $0 < u_0 < u_1 < \infty$. Then

$$\int_0^\varepsilon (\psi(t)\, g(t))^{u_1}\, \mu_\Psi(dt) \le \int_0^\varepsilon (\psi(t)\, g(t))^{u_0}\, \mu_\Psi(dt) \cdot \left(\sup_{0<\tau\le\varepsilon} \psi(\tau)\, g(\tau) \right)^{u_1-u_0}. \tag{12.21}$$

Using (12.20) with u_0 in place of u we get the second inequality in (12.14).

Step 3 We prove (ii). This is obvious if $u = \infty$ (it is the left-hand side of (12.14)). Let $0 < u < \infty$. We begin with some preparation. Let $H(t)$ be a continuous monotonically increasing distribution function on $[0, \varepsilon]$ with $H(0) = 0$. Let μ_H be the associated measure. Let $G(t)$ be a bounded non-negative monotonically decreasing function on $[0, \varepsilon]$. Then the Riemann-Stieltjes sums

$$\sum_{j=0}^{N-1} G(b_j)(H(b_{j+1}) - H(b_j)) = \sum_{j=1}^{N-1} H(b_j)(G(b_{j-1}) - G(b_j)) + H(\varepsilon)G(b_{N-1}) \tag{12.22}$$

tend for suitable subdivisions

$$0 = b_0 < b_1 < \cdots < b_{N-1} < b_N = \varepsilon \tag{12.23}$$

of $[0, \varepsilon]$ to the Riemann-Stieltjes integral

$$\int_0^\varepsilon G(t)\, \mu_H(dt)\,. \tag{12.24}$$

If one has two such distribution functions $H_1(t)$ and $H_2(t)$ and related measures μ_{H_1} and μ_{H_2} which are equivalent, then by (12.22) and $G(t) \ge 0$ the corresponding Riemann-Stieltjes sums are also equivalent (with the same equivalence constants, independently of G) and this extends to the integrals (12.24).

With these facts established, one can prove the equivalence (12.15) as follows. We put $G(t) = g^u(t)$ (so far assumed to be bounded) and

$$\mu_H(dt) = \psi_1^u(t)\, \mu_{\Psi_1}(dt)\,.$$

The corresponding distribution function $H(t)$ is the right-hand side of (12.16) with ψ_1 , Ψ_1 in place of ψ , Ψ. By (12.16) this distribution function is equivalent to the distribution function $\psi_1^u(t)$. Hence the corresponding integrals of type (12.24) are also equivalent. Since $\psi_1 \sim \psi_2$ this assertion extends to ψ_2 and we obtain (12.15). Unbounded functions $g(t)$ can be approximated by bounded ones.

12.3 Discussion

The monotonicity (12.14) is the refined and more systematic version of what follows from (12.5). In the applications in the following sections we identify $\psi^{-1}(t)$ with

the growth envelope function $\mathcal{E}_G|A^s_{pq}(t)$ or

the continuity envelope function $\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}(t)$

from (10.7) or (10.19), respectively. They have essentially the required properties. However by our general point of view we do not distinguish between equivalent quasi-norms in a given space $A^s_{pq}(\mathbb{R}^n)$. With a few exceptions such as $L_p(\mathbb{R}^n)$ or, to a lesser extent, the classical Sobolev spaces $W^k_p(\mathbb{R}^n)$, there is no *primus inter pares* among the equivalent quasi-norms. The situation is much the same as in the final slogan in G. Orwell's novel, *Animal Farm*, [Orw51], p. 114, which reads, adapted to our situation, as follows,

All equivalent quasi-norms are equal
but some equivalent quasi-norms are more
equal than others.

In other words, any notation of relevance must be checked to see what happens when a quasi-norm is replaced by an equivalent one. This is the reason why we included part (ii) in the above proposition.

12.4 Examples

We discuss a few examples which, on the one hand, will be useful for our later considerations, and, on the other hand, shed new light on the Lorentz-Zygmund spaces mentioned in 11.6.

Example 1 Let $0 < r < \infty$ and $\psi(t) = t^{\frac{1}{r}}$. By (12.12) we have

$$\mu_\Psi(dt) = \frac{1}{r}\,\frac{dt}{t}\,, \quad 0 < t \le \varepsilon\,. \tag{12.25}$$

Let $0 < u_0 < u_1 < \infty$. Then (12.14) results in

$$\sup_{0<t\le\varepsilon} t^{\frac{1}{r}}\, g(t) \le c_1 \left(\int_0^\varepsilon \left(t^{\frac{1}{r}}\, g(t) \right)^{u_1} \frac{dt}{t} \right)^{\frac{1}{u_1}} \le c_0 \left(\int_0^\varepsilon \left(t^{\frac{1}{r}}\, g(t) \right)^{u_0} \frac{dt}{t} \right)^{\frac{1}{u_0}} \tag{12.26}$$

for all non-negative monotonically decreasing functions g on $(0,\varepsilon]$. With $g(t) = f^*(t)$ we have the Lorentz spaces $L_{ru}(I_\varepsilon)$ introduced in 11.6(i), and the well-known monotonicity with respect to u (for fixed r), [BeS88], p. 217.

Example 2 Let $b > 0$ and $\psi(t) = |\log t|^{-b}$ where $0 < t \le \varepsilon < 1$. By (12.12) we have

$$\mu_\Psi(dt) = b\,\frac{dt}{t|\log t|}\,, \quad 0 < t \le \varepsilon < 1\,. \tag{12.27}$$

Let $0 < u_0 < u_1 < \infty$. Then (12.14) results in

$$\begin{aligned} \sup_{0<t<\varepsilon} \frac{g(t)}{|\log t|^b} \quad &\le \quad c_1 \left(\int_0^\varepsilon \left(\frac{g(t)}{|\log t|^b} \right)^{u_1} \frac{dt}{t|\log t|} \right)^{\frac{1}{u_1}} \\ &\le \quad c_0 \left(\int_0^\varepsilon \left(\frac{g(t)}{|\log t|^b} \right)^{u_0} \frac{dt}{t|\log t|} \right)^{\frac{1}{u_0}} \end{aligned} \tag{12.28}$$

for all non-negative monotonically decreasing functions g on $(0,\varepsilon]$. Let $a = -b$ and $g(t) = f^*(t)$. Then the left-hand side of (12.28) coincides with (11.47). As mentioned there the corresponding spaces $L_\infty(\log L)_a(I_\varepsilon)$ can be equivalently described by (11.45). The right-hand side of (12.28), say, with $u = u_1$, fits in the scheme of the special Lorentz-Zygmund spaces $L_{\infty u}(\log L)_a(I_\varepsilon)$ in 11.6(iii) with $a = -b - \frac{1}{u}$. Then the requirement mentioned there, $au < -1$, coincides with $b > 0$, and looks more natural. Inequalities of type (12.28) in terms of $L_{\infty u}(\log L)_a(I_\varepsilon)$ may be found in [BeR80], Theorem 9.5, where the notation *diagonal* comes from the natural combination $a + \frac{1}{u} = -b$. As stated above, we are not so much interested in the spaces $L_{ru}(\log L)_a(I_\varepsilon)$ for their own sake. We formulate our assertions in terms of inequalities of the same type as in Theorem 11.7. In particular, (11.71) follows from (12.28) with $u_1 = p$ and $b = \frac{1}{p'}$.

Example 3 Let $0 < r < \infty$, $a \in \mathbb{R}$ and $\psi(t) = t^{\frac{1}{r}} |\log t|^a$. By (12.12),

$$\mu_\Psi(dt) \sim \frac{dt}{t}, \quad 0 < t \le \varepsilon, \tag{12.29}$$

where $\varepsilon > 0$ is chosen so small that $\psi(t)$ is monotone in the interval $[0, \varepsilon]$. Let $0 < u_0 < u_1 < \infty$. Then (12.14) results in

$$\begin{aligned} \sup_{0<t<\varepsilon} t^{\frac{1}{r}} |\log t|^a \, g(t) &\le c_1 \left(\int_0^\varepsilon \left(t^{\frac{1}{r}} |\log t|^a g(t) \right)^{u_1} \frac{dt}{t} \right)^{\frac{1}{u_1}} \\ &\le c_0 \left(\int_0^\varepsilon \left(t^{\frac{1}{r}} |\log t|^a g(t) \right)^{u_0} \frac{dt}{t} \right)^{\frac{1}{u_0}} \end{aligned} \tag{12.30}$$

for all non-negative monotonically decreasing functions g on $(0, \varepsilon]$.
With $g = f^*$ we have (11.48), (11.49), and hence the Lorentz-Zygmund spaces $L_{ru}(\log L)_a(I_\varepsilon)$ introduced there. The inequality (12.30) is known and may be found in [BeR80], Theorem 9.3.

12.5 Growth envelope functions

The concept of growth envelope functions $\mathcal{E}_G | A^s_{pq}$ (outlined so far in (10.7), (10.9) modulo equivalences) makes sense for all spaces $A^s_{pq}(\mathbb{R}^n)$ (where $A^s_{pq}(\mathbb{R}^n)$ always means either $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$) which are covered by Theorem 11.2. But we exclude borderline cases where $p = \infty$ or $s = n(\frac{1}{p} - 1)_+$. Hence we always assume

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > \sigma_p = n \left(\frac{1}{p} - 1 \right)_+ . \tag{12.31}$$

Furthermore the growth envelope function $\mathcal{E}_G | A^s_{pq}$ is designed to be a sharp instrument to measure on a global scale how singular (with respect to its growth) a function belonging to $A^s_{pq}(\mathbb{R}^n)$ can be. Hence it is reasonable to restrict the considerations to those spaces $A^s_{pq}(\mathbb{R}^n)$ with (12.31) which are, in addition, not embedded in $L_\infty(\mathbb{R}^n)$. To make clear which spaces are meant one must complement Theorem 11.4 by non-limiting embeddings. Since by Theorem 11.4(ii) one has in the critical case both embeddings and non-embeddings in $C(\mathbb{R}^n)$, one can combine this assertion with elementary embeddings for $A^s_{pq}(\mathbb{R}^n)$ with fixed p and variable s, q of type as in [Triβ], Proposition 2 on p. 47, to get a final answer. This results in all spaces in sub-critical situations (11.18) and in those spaces in critical situations $s = \frac{n}{p}$ which are not covered by (11.22) and (11.23). To avoid any misunderstanding we give a precise formulation which spaces we wish to exclude:

Under the assumption (12.31) the following three assertions (i), (ii), (iii) are equivalent to each other:

12.5(i) $A^s_{pq}(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)\,,$

12.5(ii) $A^s_{pq}(\mathbb{R}^n) \subset C(\mathbb{R}^n)\,,$

12.5(iii)

$$\textit{either} \qquad s > \frac{n}{p}\,,$$

$$\textit{or} \quad A^s_{pq}(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n) \quad \textit{with} \quad 0 < p < \infty,\ s = \frac{n}{p}\,,\ 0 < q \le 1\,,$$

$$\textit{or} \quad A^s_{pq}(\mathbb{R}^n) = F^s_{pq}(\mathbb{R}^n) \quad \textit{with} \quad 0 < p \le 1\,,\ s = \frac{n}{p}\,,\ 0 < q \le \infty\,.$$

A full proof of this assertion has been given in [SiT95], Theorem 3.3.1. A short description may be found in [RuS96], 2.2.4, p. 32 - 33.

Obviously, the concept of growth envelope functions makes sense for a much larger scale of function spaces than considered here. It has been studied recently in [Har01]. We do not go into detail, but we have a brief look at $L_r(\mathbb{R}^n)$ with $1 \le r < \infty$, obviously normed by (2.1) (we recall the Orwellian confession at the end of 12.3) and put

$$\mathcal{E}_G|L_r(t) = \sup\{f^*(t)\ :\ \|f\,|L_r(\mathbb{R}^n)\| \le 1\}\,, \quad 0 < t < \varepsilon\,. \tag{12.32}$$

Then

$$\mathcal{E}_G|L_r(t) \sim t^{-\frac{1}{r}}\,, \quad 0 < t < \varepsilon\,. \tag{12.33}$$

The estimate of $\mathcal{E}_G|L_r(t)$ from above by $t^{-\frac{1}{r}}$ follows from (12.26) with $u_1 = r$ and $g = f^*$. For the estimate from below one can choose the function $t^{-\frac{1}{r}}\,\chi_M(x)$, where $\chi_M(x)$ is the characteristic function of a set M with $|M| = t$.

As far as the growth envelope function $\mathcal{E}_G|A^s_{pq}$ for one of the spaces $A^s_{pq}(\mathbb{R}^n)$ of interest is concerned, we have first a closer look at $\mathcal{E}_G|A^s_{pq}$ with respect to a given quasi-norm $\|\cdot\,|A^s_{pq}(\mathbb{R}^n)\|$.

12.6 Proposition

(B) *Let either*

$$1 < r < \infty\,, \quad s > 0\,, \quad s - \frac{n}{p} = -\frac{n}{r}\,, \quad 0 < q \le \infty\,, \tag{12.34}$$

(sub-critical case, dashed line in Fig. 10.1) or

$$0 < p < \infty\,, \quad s = \frac{n}{p}\,, \quad 1 < q \le \infty\,, \tag{12.35}$$

(critical case) for the spaces $B^s_{pq}(\mathbb{R}^n)$.

(F) *Let either s, p, q be as in (12.34) or*

$$1 < p < \infty\,, \quad s = \frac{n}{p}\,, \quad 0 < q \le \infty\,, \tag{12.36}$$

(critical case) for the spaces $F^s_{pq}(\mathbb{R}^n)$.

Let $A^s_{pq}(\mathbb{R}^n)$ be either $B^s_{pq}(\mathbb{R}^n)$ with (B) or $F^s_{pq}(\mathbb{R}^n)$ with (F). Let, by definition, $\mathcal{E}_G|A^s_{pq}$,

$$\mathcal{E}_G|A^s_{pq}(t) = \sup\{f^*(t) \,:\, \|f\,|A^s_{pq}(\mathbb{R}^n)\| \le 1\}\,, \quad 0 < t < \varepsilon\,, \tag{12.37}$$

where ε is a given number. Then $\mathcal{E}_G|A^s_{pq}$ is a positive, monotonically decreasing, unbounded function on the interval $(0,\varepsilon]$ with

$$\mathcal{E}_G|A^s_{pq}(2^{-j}) \sim \mathcal{E}_G|A^s_{pq}(2^{-j+1})\,, \quad j = J, J+1, \dots\,, \tag{12.38}$$

(where the equivalence constants are independent of j). Furthermore, in the sub-critical case given by (12.34) we have

$$\mathcal{E}_G|A^s_{pq}(t) \le c\, t^{-\frac{1}{r}}\,, \quad 0 < t \le \varepsilon\,, \tag{12.39}$$

for some $c > 0$, and in the critical case given by (12.35) or (12.36),

$$\mathcal{E}_G|A^s_{pq}(t) \le c_\eta\, t^{-\eta}\,, \quad 0 < t \le \varepsilon\,, \tag{12.40}$$

for any $\eta > 0$ and a suitable constant $c_\eta > 0$.

Proof *Step 1* Obviously, $\mathcal{E}_G|A^s_{pq}(t)$ is monotonically decreasing (this means non-increasing) and positive for all $t > 0$. Assume that $\mathcal{E}_G|A^s_{pq}(t)$ is bounded. By (12.2) we have

$$\|f\,|L_\infty(\mathbb{R}^n)\| = f^*(0) \le \sup_{0<t<\varepsilon} \mathcal{E}_G|A^s_{pq}(t) \tag{12.41}$$

for all $f \in A^s_{pq}(\mathbb{R}^n)$ with $\|f\,|A^s_{pq}(\mathbb{R}^n)\| = 1$, and hence

$$\|f\,|L_\infty(\mathbb{R}^n)\| \le \left(\sup_{0<t<\varepsilon} \mathcal{E}_G|A^s_{pq}(t)\right) \|f\,|A^s_{pq}(\mathbb{R}^n)\|\,, \quad f \in A^s_{pq}(\mathbb{R}^n)\,. \tag{12.42}$$

However (B) and (F) collect just those cases with (12.31) which are not covered by 12.5(i)–12.5(iii). Hence $\mathcal{E}_G|A^s_{pq}(t)$ is unbounded if $t \downarrow 0$.

Step 2 Let s, p, q be given by (12.34). We prove (12.39). As remarked in 11.8(ii), the inequality (11.52) with $q = \infty$, hence

$$\sup_{0<t<\varepsilon} t^{\frac{1}{r}}\, f^*(t) \le c\,\|f\,|B^s_{p\infty}(\mathbb{R}^n)\| \le c'\,\|f\,|A^s_{pq}(\mathbb{R}^n)\|\,, \tag{12.43}$$

is very classical, and taken for granted here. The second inequality is an elementary embedding, [Triβ], Proposition 2 on p. 47. This proves (12.39) in all sub-critical cases. As for the critical cases (12.35) and (12.36) we note the elementary non-limiting embedding

$$A^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset L_r(\mathbb{R}^n) \quad \text{for any} \quad \max(p,1) < r < \infty\,. \tag{12.44}$$

Now (12.40) follows from (12.33) and, as a consequence of (12.44),

$$\mathcal{E}_G|A^{\frac{n}{p}}_{pq}(t) \leq c\,\mathcal{E}_G|L_r(t)\,. \tag{12.45}$$

Step 3 We prove (12.38). Let

$$f \in A^s_{pq}(\mathbb{R}^n) \quad \text{with} \quad \|f\,|A^s_{pq}(\mathbb{R}^n)\| \leq 1$$

and let $g(x) = f(2^{-\frac{1}{n}}x)$ where $x \in \mathbb{R}^n$. By (12.1) we have

$$\begin{aligned} \mu_g(\lambda) &= |\{x \in \mathbb{R}^n : |f(2^{-\frac{1}{n}}x)| > \lambda\}| \\ &= 2|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| = 2\mu_f(\lambda)\,, \quad \lambda > 0\,. \end{aligned} \tag{12.46}$$

Hence, by (12.2) (and by (12.39), (12.40)),

$$f^*(2^{-j}) = g^*(2^{-j+1})\,, \quad j = J+1, \ldots\,. \tag{12.47}$$

Furthermore with some $c > 0$ (independent of f)

$$c\,\|g\,|A^s_{pq}(\mathbb{R}^n)\| \leq \|f\,|A^s_{pq}(\mathbb{R}^n)\| \leq 1\,. \tag{12.48}$$

Now, by (12.37), and (12.47), (12.48), it follows that

$$\mathcal{E}_G|A^s_{pq}(2^{-j+1}) \geq c\,\mathcal{E}_G|A^s_{pq}(2^{-j})\,, \quad j = J+1, \ldots \quad , \tag{12.49}$$

with the same c as in (12.48). Since the converse inequality is obvious we obtain (12.38).

12.7 Equivalence classes of growth envelope functions

If one puts

$$w(t) = \frac{1}{\mathcal{E}_G|A^s_{pq}(t)}\,, \quad 0 < t \leq \varepsilon\,, \tag{12.50}$$

then (12.38) coincides with (12.4) and we have (12.5). This was one of our motivations. The refinement of this point of view at the end of 12.1, which resulted in Proposition 12.2 and in the discussion in 12.3, requires for the underlying monotonically increasing distribution function $w(t) = \psi(t)$ with (12.11) that it is in addition continuous. However one can circumvent the possibly somewhat delicate question as to whether or not $\mathcal{E}_G|A^s_{pq}(t)$ is continuous. First we remark that for two equivalent quasi-norms,

$$\| \cdot |A^s_{pq}(\mathbb{R}^n)\|_1 \sim \| \cdot |A^s_{pq}(\mathbb{R}^n)\|_2 \,, \tag{12.51}$$

of a given space $A^s_{pq}(\mathbb{R}^n)$ we have (in obvious notation)

$$\mathcal{E}_G|_1 A^s_{pq}(t) \sim \mathcal{E}_G|_2 A^s_{pq}(t) \,, \quad 0 < t \le \varepsilon \,, \tag{12.52}$$

as an immediate consequence of (12.37). Equivalence must always be understood according to (12.13) adapted to the above situation. This fits in our Orwellian point of view confessed at the end of 12.3.

The collection of all positive unbounded monotonically decreasing functions on the interval $(0, \varepsilon]$ can be subdivided into equivalence classes, where a class consists of all those admitted functions which are equivalent to one (and hence to all) functions in the given class.

By (12.52) all *growth envelope functions* for a space $A^s_{pq}(\mathbb{R}^n)$ covered by Proposition 12.6 belong to the same equivalence class, denoted by $[\mathcal{E}_G A^s_{pq}]$. This class contains also representatives which are continuous on $(0, \varepsilon]$ (in addition to the other required properties). For example, one can start with a fixed growth envelope function $\mathcal{E}_G|A^s_{pq}$ and define $\mathcal{E}_G A^s_{pq}$ (without the midline) as the polygonal line with

$$\mathcal{E}_G A^s_{pq}(2^{-j}) = \mathcal{E}_G|A^s_{pq}(2^{-j}) \,, \quad j = J, J+1, \ldots \,, \tag{12.53}$$

and linear in the intervals $2^{-j-1} \le t \le 2^{-j}$ (modification at ε). Then one can apply Proposition 12.2 with

$$\psi(t) = \mathcal{E}_G A^s_{pq}(t)^{-1} \,, \quad 0 < t \le \varepsilon \,.$$

One can even use (12.12).

12.8 Definition

Let $A^s_{pq}(\mathbb{R}^n)$ be either $B^s_{pq}(\mathbb{R}^n)$ with (B) or $F^s_{pq}(\mathbb{R}^n)$ with (F) according to Proposition 12.6. Let $[\mathcal{E}_G A^s_{pq}]$ be the equivalence class associated to $A^s_{pq}(\mathbb{R}^n)$ according to 12.7. Let

$$\mathcal{E}_G A^s_{pq} \in [\mathcal{E}_G A^s_{pq}] \quad \textit{be a continuous representative.} \tag{12.54}$$

Let

$$\psi(t) = \mathcal{E}_G A^s_{pq}(t)^{-1} \quad \textit{and} \quad \Psi(t) = \log \psi(t) = - \log \mathcal{E}_G A^s_{pq}(t)\,, \tag{12.55}$$

$0 < t \le \varepsilon$, *according to 12.1 and let* μ_Ψ *be the associated Borel measure on* $[0,\varepsilon]$. *Let* $0 < u \le \infty$. *Then the couple*

$$\mathfrak{E}_G A^s_{pq} = ([\mathcal{E}_G A^s_{pq}], u) \tag{12.56}$$

is called the growth envelope for $A^s_{pq}(\mathbb{R}^n)$ *when*

$$\left(\int_0^\varepsilon \left(\psi(t)\, f^*(t)\right)^v \, \mu_\Psi(dt) \right)^{\frac{1}{v}} \le c\, \| f \,| A^s_{pq}(\mathbb{R}^n) \| \tag{12.57}$$

(modified as on the left-hand side of (12.14) if $v = \infty$*) holds for some* $c = c_v > 0$ *and all* $f \in A^s_{pq}(\mathbb{R}^n)$ *if, and only if,* $u \le v \le \infty$.

12.9 Discussion and notational agreement

First we recall that under the restriction (12.31) (excluding borderline cases $p = \infty$ or $s = \sigma_p$) the conditions (B) and (F) cover all cases for which this concept is reasonable. Furthermore, the definition of the number u in (12.56) makes sense and is independent of the chosen representative $\mathcal{E}_G A^s_{pq}$. This follows from both parts of Proposition 12.2. However we must add a remark. By definition we have always

$$\sup_{0<t\le\varepsilon} \psi(t)\, f^*(t) = \sup_{0<t\le\varepsilon} \frac{f^*(t)}{\mathcal{E}_G A^s_{pq}(t)} \le c\, \| f \,| A^s_{pq}(\mathbb{R}^n) \| \tag{12.58}$$

for some $c > 0$ and all $f \in A^s_{pq}(\mathbb{R}^n)$. Hence by Proposition 12.2 it always makes sense to put

$$u = \inf\{ v \,:\, \text{(12.57) holds} \}\,. \tag{12.59}$$

But it is not clear from the very beginning whether (12.57) remains valid with u in place of v. However this will be always the case for all spaces considered here. This may justify *the incorporation here of this additional information* immediately *in the definition.* Furthermore we wish to simplify (12.56) by

$$\mathfrak{E}_G A^s_{pq} = (\mathcal{E}_G A^s_{pq}(t), u)\,, \tag{12.60}$$

where $\mathcal{E}_G A^s_{pq}$ is a continuous representative according to (12.54). The situation is similar to the usual simplification of writing $f \in L_p(\mathbb{R}^n)$ instead of $[f] \in$

$L_p(\mathbb{R}^n)$, where $[f]$ stands for the equivalence class of all measurable functions g which coincide with f almost everywhere. This is also justified by the typical examples in (10.14). Hence we prefer, for example,

$$\mathfrak{E}_G A^s_{pq} = (t^{-\frac{1}{r}}, u) \quad \text{compared with} \quad ([t^{-\frac{1}{r}}], u)\,, \tag{12.61}$$

or even more cumbersome versions avoiding the explicit appearance of the variable t. (Of course the use of $[\cdot]$ is much the same as above in $f \in L_p(\mathbb{R}^n)$ compared with $[f] \in L_p(\mathbb{R}^n)$.) Next we collect a few rather simple properties which make clear what type of sharp inequalities can be expected.

12.10 Proposition

Let $A^s_{pq}(\mathbb{R}^n)$ be either $B^s_{pq}(\mathbb{R}^n)$ with (B) or $F^s_{pq}(\mathbb{R}^n)$ with (F) according to Proposition 12.6. Let $0 < \varepsilon < 1$. Let $\mathcal{E}_G A^s_{pq}$ be a continuous growth envelope function as in (12.54), let, in notational modification of (12.55),

$$E(t) = -\log \mathcal{E}_G A^s_{pq}(t)\,, \quad 0 < t \le \varepsilon\,, \tag{12.62}$$

and let μ_E be the associated Borel measure on $(0,\varepsilon]$.

(i) *Let $\varkappa(t)$ be a positive function on $(0,\varepsilon]$. Then there is a number $c > 0$ such that*

$$\sup_{0<t\le\varepsilon} \frac{\varkappa(t)\, f^*(t)}{\mathcal{E}_G A^s_{pq}(t)} \le c\, \|f \,|A^s_{pq}(\mathbb{R}^n)\| \quad \textit{for all} \quad f \in A^s_{pq}(\mathbb{R}^n) \tag{12.63}$$

if, and only if, $\varkappa$ is bounded.

(ii) *Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0,\varepsilon]$ and let for some $0 < v < \infty$ and some $c > 0$*

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\mathcal{E}_G A^s_{pq}(t)} \right)^v \mu_E(dt) \right)^{\frac{1}{v}} \le c\, \|f \,|A^s_{pq}(\mathbb{R}^n)\| \tag{12.64}$$

for all $f \in A^s_{pq}(\mathbb{R}^n)$. Then for some $c' > 0$,

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\, f^*(t)}{\mathcal{E}_G A^s_{pq}(t)} \right)^v \mu_E(dt) \right)^{\frac{1}{v}} \le c'\, \|f \,|A^s_{pq}(\mathbb{R}^n)\| \tag{12.65}$$

for all $f \in A^s_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded.

Proof *Step 1* The proof of (i) is simple. On the one hand we have (12.58). On the other hand, if (12.63) holds for some $\varkappa$, then for any fixed t with $0 < t \le \varepsilon$,

$$\frac{\varkappa(t)\, f^*(t)}{\mathcal{E}_G A^s_{pq}(t)} \le c \quad \text{for all} \quad f \quad \text{with} \quad \|f\,|A^s_{pq}(\mathbb{R}^n)\| \le 1\,. \tag{12.66}$$

Now by (12.54) and (12.37) it follows $\varkappa(t) \le c'$ uniformly with respect to t.

Step 2 We prove (ii). The function $g(t) = \varkappa(t)\, f^*(t)$ is non-negative and monotonically decreasing on $(0, \varepsilon]$. Hence, by (12.14),

$$\frac{\varkappa(t)\, f^*(t)}{\mathcal{E}_G A^s_{pq}(t)} \le c\,\|f\,|A^s_{pq}(\mathbb{R}^n)\|\,, \quad 0 < t \le \varepsilon\,. \tag{12.67}$$

Then (ii) follows from (i).

12.11 Discussion

In part (ii) we assumed that $\varkappa$ is monotonically decreasing. This is natural in our context, where we ask for (12.65) under the assumption (12.64), and also in connection with the definition of the growth envelope in (12.60). On the other hand, if $\varkappa$ is non-negative on $(0, \varepsilon]$ and, maybe, wildly oscillating (or monotonically increasing), then at least formally the question (12.65) makes sense without assuming that (12.64) holds. To look at the discretised version of this problem we assume that the numbers a_l have the same meaning as in Step 1 of the proof of Proposition 12.2 with $\psi^{-1}(t) = \mathcal{E}_G A^s_{pq}(t)$. Then the discrete twin of the left-hand side of (12.65) is given by

$$\left(\sum_{j=J}^{\infty} 2^{-jv}\, f^*(a_j)^v \int\limits_{a_{j+1}}^{a_j} \varkappa(t)^v\, \mu_E(dt) \right)^{\frac{1}{v}} . \tag{12.68}$$

This suggests that not so much the pointwise behaviour of $\varkappa(t)$ but the behaviour of the indicated integral means is of interest. However we do not study problems of this type in the sequel.

12.12 Moduli of continuity

We outlined in Section 10 our methods and results. As explained there in connection with (10.22) we deal with the super-critical case by lifting the results obtained in the critical case. In rough terms, the role played by $f^*(t)$ in critical (and sub-critical) situations is taken over in super-critical cases by the divided modulus of continuity $\widetilde{\omega}(f,t)$. First we recall what we need in the sequel.

Let $f(x) \in C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ has been introduced in 11.1 as the set of all complex-valued, bounded, uniformly continuous functions in $\mathbb{R}^n$. Then

$$\omega(f,t) = \sup_{x\in\mathbb{R}^n, |h|\le t} |f(x+h) - f(x)|, \quad 0 \le t < \infty, \tag{12.69}$$

is called the modulus of continuity. Let $f \in C(\mathbb{R}^n)$ be fixed. Then $\omega(f,t)$ is a non-negative and monotonically increasing (this means non-decreasing) continuous function on $[0,\infty)$; in particular,

$$\omega(f,t) \to \omega(0) = 0 \quad \text{if} \quad t \downarrow 0. \tag{12.70}$$

Furthermore, $\omega(f,t)$ is almost concave in the following sense: Let $\overline{\omega}(f,t)$ be the least concave majorant of $\omega(f,t)$. Then

$$\frac{1}{2}\,\overline{\omega}(f,t) \le \omega(f,t) \le \overline{\omega}(f,t). \tag{12.71}$$

We refer to [DeL93], Ch. 2, §6, pp. 40–44, where one finds proofs of all these properties. Let

$$\widetilde{\omega}(f,t) = \frac{\omega(f,t)}{t}, \quad t > 0, \tag{12.72}$$

be the divided modulus of continuity. By (12.71) we have

$$\widetilde{\omega}(f,t) \sim \frac{\overline{\omega}(f,t)}{t}. \tag{12.73}$$

Since $\overline{\omega}(f,t)$ is concave and continuous on $[0,\infty)$ and $\overline{\omega}(f,0) = 0$ it follows that the right-hand side of (12.73) is monotonically decreasing on $(0,\infty)$. Hence $\widetilde{\omega}(f,t)$ is at least equivalent to a monotonically decreasing function. This is sufficient for our purpose. The concept of moduli of continuity has been widely used in the theory of function spaces. Our goal here is rather limited. We are interested exclusively in the super-critical case according to (10.15), and, even more restrictive, only in those spaces $B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ and $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ which are not continuously embedded in $Lip(\mathbb{R}^n)$. This means by Theorem 11.4(iii),

$$B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 0 < p < \infty, \quad 1 < q \le \infty, \tag{12.74}$$

and

$$F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 1 < p < \infty, \quad 0 < q \le \infty. \tag{12.75}$$

This is in good agreement with (10.18) on the one hand and (12.35), (12.36) on the other hand. We remark that

$$t \mapsto \sup\{\widetilde{\omega}(f,t) \,:\, \|f\,|A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \le 1\} \tag{12.76}$$

is a bounded function on the interval $(0,1)$ if, and only if, $A^s_{pq}(\mathbb{R}^n)$ is continuously embedded in $Lip(\mathbb{R}^n)$. Hence, (12.74) and (12.75) cover just those cases, where (12.76) is unbounded. Now we are very much in the same situation as in Proposition 12.6 with the following outcome.

12.13 Proposition

Let $A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ be either the space in (12.74) or the space in (12.75). Let, for some $\varepsilon > 0$, the continuity envelope function $\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}$, be defined by

$$\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}(t) = \sup\{\widetilde{\omega}(f,t) \,:\, \|f\,|A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \le 1\}\,, \quad 0 < t < \varepsilon\,. \tag{12.77}$$

Then $\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}$ is a positive, continuous, unbounded function on the interval $(0,\varepsilon]$ with

$$\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}(2^{-j}) \sim \mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}(2^{-j+1})\,, \quad j = J, J+1, \ldots\,, \tag{12.78}$$

(where the equivalence constants are independent of j). Furthermore , $\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}$ is equivalent to a monotonically decreasing function, and for any $\eta > 0$ there is a number $c_\eta > 0$ such that

$$\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}(t) \le c_\eta\, t^{-\eta}\,, \quad 0 < t \le \varepsilon\,. \tag{12.79}$$

Proof By the above remarks, $\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}$ is positive, unbounded, and equivalent to a monotonically decreasing function. By [DeL93], p. 41, we have

$$\omega(f,2t) \le 2\,\omega(f,t) \quad \text{and} \quad |\omega(f,t_1+t_2) - \omega(f,t_1)| \le \omega(f,t_2)\,. \tag{12.80}$$

This proves (12.78) and the continuity of $\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}$. Finally, for given η, $1 > \eta > 0$, we have the non-limiting embedding

$$\sup_{0<t<\varepsilon} \frac{\omega(f,t)}{t^{1-\eta}} \le c\,\|f\,|A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\|\,, \tag{12.81}$$

[Triβ], 2.7.1, p. 131, formula (12). This proves (12.79).

12.14 Definition

Let $A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ be either the space $B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ from (12.74) or the space $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ from (12.75). Let $0 < \varepsilon < 1$. Then $[\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}]$ is the equivalence class of all continuous monotonically decreasing functions on the interval $(0,\varepsilon]$ which are equivalent to one (and hence to all) continuity envelope function $\mathcal{E}_C|A^{1+\frac{n}{p}}_{pq}$ according to (12.77). Let

$$\mathcal{E}_C A^{1+\frac{n}{p}}_{pq} \in [\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}]\,,$$

$$\psi(t) = \mathcal{E}_C A^{1+\frac{n}{p}}_{pq}(t)^{-1} \quad \textit{and} \quad \Psi(t) = \log\psi(t) = -\log \mathcal{E}_C A^{1+\frac{n}{p}}_{pq}(t)\,, \tag{12.82}$$

$0 < t \le \varepsilon$, according to 12.1 and let μ_Ψ be the associated Borel measure on $[0,\varepsilon]$. Let $0 < u \le \infty$. Then the couple

$$\mathfrak{E}_C A^{1+\frac{n}{p}}_{pq} = ([\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}], u) \tag{12.83}$$

is called the continuity envelope for $A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ when

$$\left(\int_0^\varepsilon (\psi(t)\,\widetilde{\omega}(f,t))^v \; \mu_\Psi(dt) \right)^{\frac{1}{v}} \le c\,\|f\,|A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{12.84}$$

(modified as on the left-hand side of (12.14) if $v = \infty$) holds for some $c = c_v > 0$ and all $f \in A^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $u \le v \le \infty$.

12.15 Remark and notational agreement

This definition is the same as Definition 12.8, mutatis mutandis. In particular, all that had been said before 12.8 in 12.7, but also afterwards in 12.9, in Proposition 12.10, and in 12.11, has respective counterparts which will not be repeated here. But we mention that, much as in (12.60), we simplify (12.83) by

$$\mathfrak{E}_C A^{1+\frac{n}{p}}_{pq} = (\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}(t), u)\,, \tag{12.85}$$

where $\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}$ is a representative of $[\mathcal{E}_C A^{1+\frac{n}{p}}_{pq}]$. As stated above, we reduce later on the super-critical case to the critical case by lifting. If $n = 1$, then one has (10.22). In higher dimensions the situation is more complicated. In the next proposition we prove what we need later. Recall that

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right), \quad x \in \mathbb{R}^n\,, \tag{12.86}$$

and, hence,

$$|\nabla f(x)| = \left(\sum_{n=1}^{n} \left| \frac{\partial f}{\partial x_j}(x) \right|^2 \right)^{\frac{1}{2}} \sim \sum_{j=1}^{n} \left| \frac{\partial f}{\partial x_j}(x) \right| . \tag{12.87}$$

Furthermore we need the rearrangement $|\nabla f|^*(t)$ and its maximal function $|\nabla f|^{**}(t)$ according to (10.3) and (10.4) with $|\nabla f|$ in place of f. Let $\omega(f,t)$ and $\widetilde{\omega}(f,t)$ be the modulus of continuity and the divided modulus of continuity introduced in (12.69) and (12.72). Finally, $C^1(\mathbb{R}^n)$ has the same meaning as in (11.6).

12.16 Proposition

(i) *Let $0 < \varepsilon < 1$. There is a number $c > 0$ such that*

$$\widetilde{\omega}(f,t) \le c\, |\nabla f|^{**}(t^{2n-1}) + 3 \sup_{0<\tau\le t^2} \tau^{-\frac{1}{2}}\, \omega(f,\tau) \tag{12.88}$$

for all $0 < t < \varepsilon$ and all $f \in C^1(\mathbb{R}^n)$.

(ii) *Let $0 < p \le \infty$, $v > 0$, and $0 < \varepsilon < 1$. There is a number $c > 0$ such that*

$$\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|^v} \right)^p \frac{dt}{t|\log t|} \le c \int_0^\varepsilon \left(\frac{|\nabla f|^*(t)}{|\log t|^v} \right)^p \frac{dt}{t|\log t|} \quad \text{if} \quad p < \infty\,, \tag{12.89}$$

and

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^v} \le c \sup_{0<t<\varepsilon} \frac{|\nabla f|^*(t)}{|\log t|^v} \quad \text{if} \quad p = \infty\,, \tag{12.90}$$

for all $f \in C^1(\mathbb{R}^n)$.

Proof *Step 1* We prove (i). Let t with $0 < t < \varepsilon$ be fixed. Replacing $f(x)$ by $\varrho\, f(x)$ for some $\varrho > 0$ we may assume that the supremum in (12.88) equals 1, hence

$$|f(x+y) - f(x)| \le \tau^{\frac{1}{2}} \quad \text{for all } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^n \text{ with } |y| \le \tau\,, \tag{12.91}$$

where $\tau \le t^2$. Then (12.88) is equivalent to

$$t^{-1}\, |f(x+y) - f(x)| \le c\, |\nabla f|^{**}(t^{2n-1}) + 3 \tag{12.92}$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with $|y| \le t$. Of course it is sufficient to concentrate on those x and y for which the left-hand side of (12.92) is larger than 3. Without restriction of generality we may assume $x = 0$ and $y = y^1 = (y_1, 0, \ldots, 0)$. Hence,

$$A = |f(y^1) - f(0)| \ge 3t \quad \text{with} \quad y^1 = (y_1, 0, \ldots, 0)\,, \quad 0 < y_1 \le t\,. \tag{12.93}$$

Let $y^2 = (0, y_2, \dots, y_n) = (0, y')$ with $y' \in \mathbb{R}^{n-1}$ and $|y^2| = |y'| = \tau \le t^2$. With $y = y^1 + y^2$ we obtain by (12.93) and (12.91),

$$\begin{aligned} |f(y) - f(y^2)| &\ge |f(y^1) - f(0)| - |f(y) - f(y^1)| - |f(y^2) - f(0)| \\ &\ge A - 2t \ge \frac{A}{3}\,. \end{aligned} \tag{12.94}$$

Similarly one can estimate $|f(y) - f(y^2)|$ from above by $2A$. By construction y and y^2 differ only with respect to the first component. We fix $y' \in \mathbb{R}^{n-1}$ with $|y'| \le t^2$ and obtain

$$|f(y) - f(y^2)| = \left| \int_0^{y_1} \frac{\partial f}{\partial x_1}(\sigma, y')\, d\sigma \right| \le \int_0^t |\nabla f(\sigma, y')|\, d\sigma\,. \tag{12.95}$$

The left-hand side is equivalent to A. We integrate over $y' \in \mathbb{R}^{n-1}$ with $|y'| \le t^2$. Then we have for some $c > 0$,

$$t^{2n-2} A \le c \int_T |\nabla f(x)|\, dx\,, \tag{12.96}$$

where $T = [0, t] \times \{y' \,:\, |y'| \le t^2\}$ is a tube in $\mathbb{R}^n$ with the volume $|T| = t^{2n-1}$. By standard arguments for rearrangements we obtain (switching to arbitrary $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with $|y| \le t$ and the counterpart of (12.93))

$$\frac{|f(x+y) - f(x)|}{t} \le \frac{c}{t^{2n-1}} \int_0^{t^{2n-1}} |\nabla f|^*(\sigma)\, d\sigma \le c\, |\nabla f|^{**}(t^{2n-1})\,. \tag{12.97}$$

This proves (12.88).

Step 2 We prove (ii). Let $p < \infty$. By (i) we have

$$\begin{aligned} &\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|^v} \right)^p \frac{dt}{t|\log t|} \\ &\quad \le c \left(\sup_{0 < t \le \varepsilon} t^{\frac12}\, \widetilde{\omega}(f,t) \right)^p + c \int_0^\varepsilon \left(\frac{|\nabla f|^{**}(t^{2n-1})}{|\log t|^v} \right)^p \frac{dt}{t|\log t|} \\ &\quad \le c' \int_0^\varepsilon \left(t^{\frac12} \widetilde{\omega}(f,t) \right)^p \frac{dt}{t} + c' \int_0^\varepsilon \left(\frac{|\nabla f|^{**}(t)}{|\log t|^v} \right)^p \frac{dt}{t|\log t|}\,. \end{aligned} \tag{12.98}$$

Here we used that $\widetilde{\omega}(f,t)$ is equivalent to a monotonically decreasing function. Then application of (12.26) justifies the first term on the right-hand side of

(12.98). In connection with the second term we used the transformation $\tau = t^{2n-1}$. Let $g(t) = |\nabla f|^*(t)$. Then $|\nabla f|^{**}(t) = Mg(t)$ is the maximal function of $g(t)$ according to (10.4). We wish to prove that

$$\int_0^\varepsilon \left(\frac{Mg(t)}{|\log t|^v} \right)^p \frac{dt}{t|\log t|} \le c \int_0^\varepsilon \left(\frac{g(t)}{|\log t|^v} \right)^p \frac{dt}{t|\log t|}\,. \tag{12.99}$$

Let $\varepsilon \sim 2^{-J}$ and $t \sim 2^{-j}$ with $j \ge J$. Then

$$Mg(2^{-j}) \sim Mg(t) = \frac{1}{t} \int_0^t g(\tau)\, d\tau \sim \sum_{l=0}^\infty 2^{-l} g(2^{-l-j})\,. \tag{12.100}$$

Let $u = vp + 1$ and $q < p$. Then the left-hand side of (12.99) can be estimated from above by

$$\begin{aligned} c \sum_{j=J}^\infty \frac{Mg(2^{-j})^p}{j^u} \quad &\le \quad c' \sum_{j=J}^\infty \sum_{l=0}^\infty \frac{(j+l)^u}{j^u}\, 2^{-lq}\, \frac{g^p(2^{-j-l})}{(j+l)^u} \\ &\le \quad c'' \sum_{k=J}^\infty \frac{g^p(2^{-k})}{k^u} \sum_{l=0}^{k-J} \frac{k^u}{(k-l)^u}\, 2^{-lq}\,. \end{aligned} \tag{12.101}$$

Since $\frac{k}{k-l}$ can be estimated from above by $1 + \frac{l}{k-l} \le 1 + l$, it follows that the last factor in (12.101) can be estimated from above by a constant, which is independent of J. Then the right-hand side of (12.101) is equivalent to the right-hand side of (12.99). This proves (12.99). We return to (12.98) and remark in addition that for any $\eta > 0$ there is an ε_0, $0 < \varepsilon_0 < 1$, such that

$$t^{\frac{1}{2}} \le \eta\, |\log t|^{-u} \quad \text{if} \quad 0 < t \le \varepsilon_0\,, \tag{12.102}$$

where $u = vp+1$ has the above meaning. Inserting (12.99) with $g(t) = |\nabla f|^*(t)$ and (12.102) with a small η in the right-hand side of (12.98), then we have on the right-hand side the desired term from the right-hand side of (12.89) and in addition the same term as on the left-hand side with a factor, say, $\frac{1}{2}$. This proves (12.89) under the additional assumption $0 < \varepsilon \le \varepsilon_0$. We remove this restriction. Let $0 < \varepsilon < 1$ and let $0 < \varkappa < 1$. Since $\widetilde{\omega}(f,t)$ is equivalent to a monotonically decreasing function it follows that

$$\begin{aligned} \int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|^v} \right)^p \frac{dt}{t|\log t|} \quad &\le \quad c \int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,\varkappa t)}{|\log t|^v} \right)^p \frac{dt}{t|\log t|} \\ &\le \quad c' \int_0^{\varkappa\varepsilon} \left(\frac{\widetilde{\omega}(f,\tau)}{|\log \tau|^v} \right)^p \frac{d\tau}{\tau|\log \tau|}\,. \end{aligned} \tag{12.103}$$

This reduces the case $0 < \varepsilon < 1$ to $0 < \varepsilon \le \varepsilon_0$. Then we obtain (12.89). If $p = \infty$ then one can follow the above arguments with the necessary modifications and arrives at (12.90).

12.17 Remark

In the one-dimensional case, (12.88) with $n = 1$ reduces to

$$\widetilde{\omega}(f,t) \le c\,|f'|^{**}(t)\,, \quad 0 < t < \varepsilon\,, \quad f \in C^1(\mathbb{R})\,. \tag{12.104}$$

This follows from (12.95). It coincides with (10.22). The situation in $\mathbb{R}^n$ with $n \ge 2$ seems to be more complicated. Whether there is a direct counterpart of (12.104) with $|\nabla f|^{**}(t^n)$ is not so clear. On the other hand, the choice of $\tau^{-\frac{1}{2}}\,\omega(f,\tau)$ in the second term on the right-hand side of (12.88) is convenient and sufficient for us, but it can be modified. If one replaces $\tau^{-\frac{1}{2}}\,\omega(f,\tau)$ by $\varkappa(\tau)\tau^{-1}\omega(f,\tau)$ where $\varkappa(\tau)$ is a positive, say, monotonically increasing function with $\varkappa(\tau) \to 0$ if $\tau \to 0$, then one ends up with $\varrho(t)t^n$ in place of t^{2n-1} in the first term on the right-hand side of (12.88) with $\varrho(t) \to 0$ arbitrarily slowly if $t \to 0$. However if one wishes to apply modified versions of (12.88) to get (12.89) one needs a counterpart of (12.102) with $\varkappa(t)$ in place of $t^{\frac{1}{2}}$.

13 The critical case

13.1 Introduction

By the terminology of (10.6) the critical case covers the spaces

$$B^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{and} \quad F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 0 < p < \infty \quad \text{and} \quad 0 < q \le \infty\,. \tag{13.1}$$

This corresponds to the line of slope n in Fig. 10.1 starting from the origin. Generally in this Chapter II we are interested exclusively in spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ which are not only subspaces of $S'(\mathbb{R}^n)$ but also of $L_1^{loc}(\mathbb{R}^n)$ (and, hence, consist entirely of regular distributions). We refer to Section 10 where we outlined our intentions. Theorem 11.2 clarifies under what conditions $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are subspaces of $L_1^{loc}(\mathbb{R}^n)$. Recall that in all cases considered here (critical, super-critical, sub-critical) we always exclude borderline situations, which means in general

$$p = \infty \quad \text{and/or} \quad s = \sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{if} \quad 0 < p < \infty\,, \tag{13.2}$$

and especially according to Theorem 11.2,

$$B^0_{\infty,q}(\mathbb{R}^n) \quad \text{with} \quad 0 < q \le 2\,, \tag{13.3}$$

in the critical situation $s = \frac{n}{p}$. A further distinguished borderline space in connection with the critical situation not treated in this section is $bmo(\mathbb{R}^n) = F^0_{\infty,2}(\mathbb{R}^n)$. Here we add at least a brief remark at the end of this section in 13.7. Otherwise as a further restriction of (13.1) we are interested only in those spaces which are not continuously embedded in $L_\infty(\mathbb{R}^n)$ (or, which is the same, in $C(\mathbb{R}^n)$); this means by Theorem 11.4, and as has been detailed in 11.5, especially (11.38), (11.39), we deal only with the spaces

$$B^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 0 < p < \infty\,, \quad 1 < q \le \infty\,, \tag{13.4}$$

and

$$F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 1 < p < \infty\,, \quad 0 < q \le \infty\,. \tag{13.5}$$

This covers in particular the respective Sobolev spaces mentioned in (11.40). As outlined in the introductory Section 10 we wish to measure the singularity behaviour of functions belonging to the spaces in (13.4), (13.5) in terms of the growth envelope as introduced in Definition 12.8. Instead of $\mathfrak{E}_G A^s_{pq}$ in (12.56) we use the more handsome version (12.60). In the theorem below we calculate explicitly the growth envelopes for all spaces in (13.4) and (13.5). By Proposition 12.10 it is clear that one gets rather sharp assertions concerning the singularity behaviour of elements of these spaces in a very condensed form. Hence, it seems to be reasonable, after proving the theorem, to discuss what this means in detail. Finally we add references in 13.5 and, as said, a remark about $bmo(\mathbb{R}^n)$ in 13.7. Let $1 \le v \le \infty$. As usual v' is given by $\frac{1}{v} + \frac{1}{v'} = 1$.

13.2 Theorem

(i) *Let*

$$0 < p < \infty\,, \quad 1 < q \le \infty\,. \tag{13.6}$$

Then

$$\mathfrak{E}_G B^{\frac{n}{p}}_{pq} = (|\log t|^{\frac{1}{q'}}, q)\,. \tag{13.7}$$

(ii) *Let*

$$1 < p < \infty\,, \quad 0 < q \le \infty\,. \tag{13.8}$$

Then

$$\mathfrak{E}_G F^{\frac{n}{p}}_{pq} = (|\log t|^{\frac{1}{p'}}, p)\,. \tag{13.9}$$

Proof We break the rather long proof into 7 steps. Here is a guide. In Step 1 and Step 2 we prove those sharp inequalities which correspond to the right-hand sides of (13.7) and (13.9), respectively. In Step 3 we formulate what this means in terms of the growth envelope functions: They can be estimated from above by $|\log t|^{\frac{1}{q'}}$ and $|\log t|^{\frac{1}{p'}}$, respectively. To prove the sharpness we need extremal functions. They will be constructed in Steps 4 and 5. The outcome is of self-contained interest, also in connection with the super-critical case considered in Section 14, and will be formulated separately in Corollary 13.4. In Steps 6 and 7 we prove that $|\log t|^{\frac{1}{q'}}$ and $|\log t|^{\frac{1}{p'}}$ are envelope functions, and that q and p, respectively, are the correct exponents according to (13.7) and (13.9).

Step 1 Let p and q be given by (13.6), and let, as always, $0 < \varepsilon < 1$. We prove that there is a number $c > 0$ such that

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le c \, \| f \, | B^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \tag{13.10}$$

for all $f \in B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ with the interpretation

$$\sup_{0<t\le\varepsilon} \frac{f^*(t)}{|\log t|} \le c \, \| f \, | B^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n) \|$$

in case of $q = \infty$. Let $0 < p_1 < p_2 < \infty$. Then

$$B^{\frac{n}{p_1}}_{p_1 q}(\mathbb{R}^n) \subset B^{\frac{n}{p_2}}_{p_2 q}(\mathbb{R}^n) \,, \tag{13.11}$$

[Triβ], Theorem 2.7.1, p. 129. Hence it is sufficient to prove (13.10) for large values of p, in particular, we may assume

$$1 < p < \infty \,, \quad 1 < q \le \infty \,. \tag{13.12}$$

We rely on atomic decompositions for the spaces $B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$. Details (and also references to the original papers) may be found in [Triδ], Sections 13. (One could also use corresponding quarkonial decompositions according to Definition 2.6 and Theorem 2.9, but atoms are sufficient at the moment.) By [Triδ], Theorem 13.8, any $f \in B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ can be optimally decomposed in atoms $a_{jm}(x)$ and complex numbers b_{jm} such that

$$f(x) = \sum_{j=0}^{\infty} f_j(x) \quad \text{with} \quad f_j(x) = \sum_{m \in \mathbb{Z}^n} b_{jm} \, a_{jm}(x) \tag{13.13}$$

and

$$A = \left(\sum_{j=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |b_{jm}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \sim \|f \,|B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{13.14}$$

(obviously modified by $\sup_j$ if $q = \infty$). The equivalence constants are independent of f. Recall that the atoms $a_{jm}(x)$ have the following properties:

$$supp\, a_{jm} \subset \left\{ y \in \mathbb{R}^n \,:\, |y - 2^{-j}m| < d\, 2^{-j} \right\}, \tag{13.15}$$

$$|D^\gamma a_{jm}(x)| \le 2^{j|\gamma|} \quad \text{for all} \quad \gamma \in \mathbb{N}_0^n \quad \text{with} \quad |\gamma| \le \left[\frac{n}{p} \right] + 1, \tag{13.16}$$

for some $d > 0$ and all $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Let $\chi_{jl}(t)$ be the characteristic function of the interval $[C2^{-jn}(l-1), C2^{-jn}l)$ on $\mathbb{R}_+ = [0, \infty)$, where $C > 0$, $j \in \mathbb{N}_0$, and $l \in \mathbb{N}$. For fixed $j \in \mathbb{N}_0$ let b^*_{jl} with $l \in \mathbb{N}$ be the (decreasing) rearrangement of b_{jm} with $m \in \mathbb{Z}^n$. If $C > 0$ and $c > 0$ are chosen appropriately then

$$f^*_j(t) \le c \sum_{l=1}^{\infty} b^*_{jl}\, \chi_{jl}(t), \quad \text{where} \quad t > 0 \quad \text{and} \quad j \in \mathbb{N}_0. \tag{13.17}$$

Let $C2^{-jn}(l-1) < t \le C2^{-jn}l$. Then

$$f^{**}_j(t) = \frac{1}{t} \int_0^t f^*(\tau)\, d\tau \le \frac{c}{l} \sum_{k=1}^{l} b^*_{jk} = c\, b^{**}_{jl}. \tag{13.18}$$

Since $\{b^*_{jk}\}_{k=1}^{\infty}$ is monotonically decreasing and $1 < p < \infty$ we have

$$\sum_{l=1}^{\infty} b^{*p}_{jl} \le \sum_{l=1}^{\infty} b^{**p}_{jl} \le c \sum_{l=1}^{\infty} b^{*p}_{jl} = c \sum_{m \in \mathbb{Z}^n} |b_{jm}|^p = C^p_j. \tag{13.19}$$

The left-hand side is obvious since $b^*_{jl} \le b^{**}_{jl}$. The second estimate is the sequence version of the Hardy-Littlewood maximal inequality and can easily be reduced to the usual formulation of this maximal inequality. (A formulation and a proof of the latter may be found in [Ste70], p. 5.) Let

$$C2^{-(k+1)n} \le t < C2^{-kn} \quad \text{with} \quad k \in \mathbb{N}.$$

By the additivity property of f^{**} according to [BeS88], Theorem 3.4 on p. 55, and (13.13), (13.18) we obtain

$$f^{**}(t) \le \sum_{j=0}^{\infty} f^{**}_j(t) \le c \sum_{j=0}^{k} b^*_{j1} + c \sum_{j=k+1}^{\infty} b^{**}_{j,2^{(j-k)n}} \tag{13.20}$$

where we used $b^{**}_{j1} = b^{*}_{j1}$. If $1 < q < \infty$, then

$$\int\limits_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^q \frac{dt}{t} \le c_1 \sum_{k=1}^\infty \left(\frac{f^{**}(C2^{-kn})}{k} \right)^q$$

$$\le c_2 \sum_{k=1}^\infty \left(\frac{1}{k} \sum_{j=0}^k b^*_{j1} \right)^q + c_2 \sum_{k=1}^\infty \left(\frac{1}{k} \sum_{j=k+1}^\infty b^{**}_{j,2^{(j-k)n}} \right)^q = A_1^q + A_2^q \,. \quad (13.21)$$

Again we can apply the sequence version of the Hardy-Littlewood maximal inequality to the first sum A_1 and obtain

$$A_1 \le c_1 \left(\sum_{j=0}^\infty b^{*q}_{j1} \right)^{\frac{1}{q}} \le c_2 A \sim \| f \,| B^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \,, \qquad (13.22)$$

where we used (13.14). If $q = \infty$ then (13.21) must be replaced by

$$\sup_{0<t\le\varepsilon} \frac{f^*(t)}{|\log t|} \le c_2 \sup_k \frac{1}{k} \sum_{j=0}^k b^*_{j1} + c_2 \sup_k \frac{1}{k} \sum_{j=k+1}^\infty b^{**}_{j,2^{(j-k)n}} = A_1 + A_2 \,. \quad (13.23)$$

The term A_1 can be estimated from above by $\sup_j b^*_{j1}$, and hence by the right-hand side of (13.22). We estimate A_2. Since for fixed j the sequence b^{**}_{jl} is monotonically decreasing we have by (13.19),

$$\sum_{l=1}^\infty 2^{ln}\, b^{**p}_{j,2^{ln}} \le c\, C_j^p \,, \quad j \in \mathbb{N}_0 \,, \qquad (13.24)$$

and, hence,

$$b^{**}_{j,2^{ln}} \le c'\, 2^{-\frac{ln}{p}}\, C_j \,, \quad j \in \mathbb{N}_0 \,, \quad l \in \mathbb{N} \,. \qquad (13.25)$$

It follows that

$$\sum_{j=k+1}^\infty b^{**}_{j,2^{(j-k)n}} \le c \sum_{j=k+1}^\infty 2^{-\frac{(j-k)n}{p}} C_j \le c' \sup_l C_l \,. \qquad (13.26)$$

Now we get in both cases, $q = \infty$ by (13.23) and $1 < q < \infty$ by (13.21),

$$A_2 \le c \sup_l C_l \le c' A \sim \| f \,| B^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \,. \qquad (13.27)$$

Here we used again (13.14). Now (13.23) if $q = \infty$ and (13.21) if $1 < q < \infty$, and the estimates (13.22) and (13.27) prove (13.10).

Step2 Let p and q be given by (13.8). Let again $0 < \varepsilon < 1$. We prove that there is a number $c > 0$ such that

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \le c \, \| f \,| F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \tag{13.28}$$

for all $f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$. We reduce this case to (13.10) using the following consequence of an observation by Ju. V. Netrusov, [Net89a], Theorem 1.1 and Remark 4 on p. 191 (in the English translation): *For any $f \in F^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n)$ there is a function $g \in B^{\frac{n}{p}}_{pp}(\mathbb{R}^n)$ such that*

$$|f(x)| \le g(x) \text{ a.e. in } \mathbb{R}^n\,, \text{ and } \quad \| g \,| B^{\frac{n}{p}}_{pp}(\mathbb{R}^n) \| \le c \, \| f \,| F^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n) \| \,, \tag{13.29}$$

where c is independent of f and g. Since $1 < p < \infty$ we can apply (13.10) to g and $B^{\frac{n}{p}}_{pp}(\mathbb{R}^n)$. Together with (13.29) we obtain for $f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$,

$$\begin{aligned} \int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{dt}{t} &\le \int_0^\varepsilon \left(\frac{g^*(t)}{|\log t|} \right)^p \frac{dt}{t} \le c_1 \, \| g \,| B^{\frac{n}{p}}_{pp}(\mathbb{R}^n) \|^p \\ &\le c_2 \| f \,| F^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n) \|^p \le c_3 \, \| f \,| F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \|^p \,, \end{aligned} \tag{13.30}$$

where we used in addition the monotonicity of the F-spaces with respect to the q-index. This proves (13.28).

Step 3 Let p, q be given by (13.6) and let $b = \frac{1}{q'}$ in Example 2 in 12.4. Then we obtain by (12.28) and (13.10),

$$\begin{aligned} \sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{\frac{1}{q'}}} &\le c_1 \left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|^{\frac{1}{q'}}} \right)^q \frac{dt}{t|\log t|} \right)^{\frac{1}{q}} \\ &= c_1 \left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le c_2 \, \| f \,| B^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \,. \end{aligned} \tag{13.31}$$

If p, q are given by (13.8) then it follows in a similar way by (13.28) that

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{\frac{1}{p'}}} \le c \, \| f \,| F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \,. \tag{13.32}$$

Let $\mathcal{E}_G B^{\frac{n}{p}}_{pq}$ and $\mathcal{E}_G F^{\frac{n}{p}}_{pq}$ be the respective growth envelope functions according to Definition 12.8 and (12.37). Then it follows by (13.31) and (13.32) that

$$\mathcal{E}_G B^{\frac{n}{p}}_{pq}(t) \le |\log t|^{\frac{1}{q'}}, \quad 0 < p < \infty, \quad 1 < q \le \infty, \tag{13.33}$$

and

$$\mathcal{E}_G F^{\frac{n}{p}}_{pq}(t) \le |\log t|^{\frac{1}{p'}}, \quad 1 < p < \infty, \quad 0 < q \le \infty. \tag{13.34}$$

Step 4 To prove the converse of (13.33), (13.34) and to show that q, p are the correct numbers in (13.7), (13.9), respectively, we need some extremal functions. Let $\psi(x)$ be a non-trivial, non-negative, compactly supported C^∞ function in $\mathbb{R}^n$, for example,

$$\psi(x) = e^{-\frac{1}{1-|x|^2}} \quad \text{if} \quad |x| < 1 \quad \text{and} \quad \psi(x) = 0 \quad \text{if} \quad |x| \ge 1. \tag{13.35}$$

Let

$$1 < p < \infty, \quad 1 \le q \le \infty. \tag{13.36}$$

Let $b = \{b_j\}_{j=1}^\infty$ be a sequence of non-negative numbers with

$$b_1 \ge b_2 \ge \cdots \ge b_j \ge b_{j+1} \ge \cdots \quad \text{and} \quad \sum_{j=1}^\infty b_j^p < \infty, \tag{13.37}$$

and let

$$f(x) = \sum_{j=1}^\infty b_j \, \psi(2^{j-1}x). \tag{13.38}$$

We wish to prove

$$\sum_{j=1}^\infty b_j^p \sim \int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{dt}{t} \sim \| f \,| F^{\frac{n}{p}}_{pq} \mathbb{R}^n) \|^p, \tag{13.39}$$

where the equivalence constants are independent of b. We remark that (13.38) is an atomic or quarkonial decomposition in $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ according to [Triδ], Theorem 13.8, p. 75, or the above Definition 2.6, respectively. With the sequence space f_{pq}, given by (2.8), we obtain (in obvious notation)

$$\| f \,| F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \le c \, \| b \,| f_{pq} \| \sim \left(\sum_{j=1}^\infty b_j^p \right)^{\frac{1}{p}}. \tag{13.40}$$

The inequality in (13.40) is covered by the above references. The equivalence in (13.40) follows from the special structure of f in (13.38) and the modifications of f_{pq} indicated in 2.15 (which show that under the above circumstances q in f_{pq} is unimportant). Next we remark that $f(x)$ with, say, (13.35), is non-negative, rotationally invariant, and monotonically decreasing in radial directions. We have

$$f(x) \sim \sum_{j=1}^{k} b_j \quad \text{if} \quad |x| \sim 2^{-k} \quad \text{where} \quad k \in \mathbb{N}\,, \tag{13.41}$$

and, hence,

$$f^*(t) \sim \sum_{j=1}^{k} b_j \quad \text{if} \quad t \sim 2^{-kn} \quad \text{where} \quad k \in \mathbb{N}\,. \tag{13.42}$$

It follows that

$$\int_0^{\varepsilon} \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{dt}{t} \sim \sum_{k=K}^{\infty} \left(\frac{1}{k} \sum_{j=1}^{k} b_j \right)^p \sim \sum_{k=1}^{\infty} b_k^p\,, \tag{13.43}$$

where we used in the second equivalence again the sequence version of the Hardy-Littlewood maximal inequality as in connection with (13.19) and the monotonicity of the numbers b_j according to (13.37); (the number K is related to ε, but otherwise unimportant). Now (13.39) follows from (13.43), (13.28), (13.40). Similarly, but technically more simply, one obtains for $0 < p < \infty$,

$$\sum_{j=1}^{\infty} b_j^q \sim \int_0^{\varepsilon} \left(\frac{f^*(t)}{|\log t|} \right)^q \frac{dt}{t} \sim \|f \,| B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\|^q \quad \text{if} \quad 1 < q < \infty\,, \tag{13.44}$$

and

$$b_1 = \sup_j b_j \sim \sup_{0<t\le\varepsilon} \frac{f^*(t)}{|\log t|} \sim \|f \,| B^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n)\| \tag{13.45}$$

as follows: One has (13.40) with B in place of F and with q on the right-hand side, $1 < q \le \infty$. The first equivalences in (13.44), (13.45) follow as in (13.43), including $q = \infty$. Together with (13.10) one gets (13.44) and (13.45).

Step 5 The extremal functions $f(x)$ in (13.38) apply to all cases for the B-spaces, but, so far only to the F-spaces with (13.36). If $q < 1$ is small then the $\psi(2^{j-1}x)$ are no longer atoms or quarks in $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$. One needs moment conditions. We describe the respective repair. Let again ψ and $b = \{b_j\}_{j=1}^{\infty}$ be given by, say, (13.35) and (13.37) with $1 < p < \infty$. Let $x^0 \neq 0$. We modify (13.38) by

$$f(x) = \sum_{j=1}^{\infty} b_j\, \chi(2^{j-1}x) = \sum_{j=1}^{\infty} b_j \left(\psi(2^{j-1}x) - \psi(2^{j-1}x - x^0) \right) . \qquad (13.46)$$

Although not really necessary one may choose x^0 such that the supports of $\psi(2^{j-1}x - x^0)$ are pairwise disjoint. Furthermore the function $\chi(x)$ satisfies the first moment condition

$$\int_{\mathbb{R}^n} \chi(x)\, dx = 0 . \qquad (13.47)$$

Otherwise (13.40)–(13.43) remain unchanged and we get (13.39) for those q for which first moment conditions in the related atoms are sufficient. If higher moment conditions

$$\int_{\mathbb{R}^n} x^\beta\, \chi(x)\, dx = 0 \quad \text{for} \quad |\beta| \le L \qquad (13.48)$$

are needed, then the construction in (13.46) must be modified by

$$\chi(x) = \psi(x) - \psi_L(x - x^0) , \qquad (13.49)$$

where ψ_L is a suitable C^∞ function with a compact support, say, in the unit ball in $\mathbb{R}^n$, and $x^0 \neq 0$ chosen in such a way that the supports of $\psi_L(2^{j-1}x - x^0)$ are pairwise disjoint. An explicit construction of such a function may be found in [TrW96], pp. 665–666. We refer also to Corollary 13.4 below and its proof where we have for later purposes a second and more detailed look at constructions of this type. After this modification we get (13.39) now for all p, q with (13.8).

Step 6 We prove the converse of (13.33), (13.34). Let p, q, and ψ be given by (13.36) and (13.35), respectively, and let

$$f_J(x) = J^{-\frac{1}{p}} \sum_{j=1}^{J} \psi(2^{j-1}x) , \quad x \in \mathbb{R}^n , \quad J \in \mathbb{N} . \qquad (13.50)$$

Then by (13.42) and (13.39),

$$f_J^*(2^{-Jn}) \sim J^{\frac{1}{p'}} \quad \text{and} \quad \| f_J \,| F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \| \sim 1 \qquad (13.51)$$

uniformly in J. Hence by (12.54) and (12.37),

$$\mathcal{E}_G F^{\frac{n}{p}}_{pq}(2^{-Jn}) \ge f_J^*(2^{-Jn}) \sim J^{\frac{1}{p'}} , \quad J \in \mathbb{N} . \qquad (13.52)$$

This proves the converse of (13.34). If $q < 1$ then one has to replace f in (13.38) as indicated in Step 5. Similarly for $B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$. Hence $|\log t|^{\frac{1}{q'}}$ and $|\log t|^{\frac{1}{p'}}$ are the growth envelope functions for $B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ and $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$, respectively.

Step 7 We must prove that q and p are the correct numbers in (13.7) and (13.9), respectively. Since we know already (13.31) and its F-counterpart (13.28) we must prove that q, respectively p, cannot be improved. Assume that there is a number v with $v < q$ and

$$\int_0^{\varepsilon} \left(\frac{f^*(t)}{|\log t|^{\frac{1}{q'}}} \right)^v \frac{dt}{t|\log t|} \le c\, \|f\,|B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\|^v . \tag{13.53}$$

Let, according to (13.38),

$$f(x) = \sum_{j=2}^{\infty} b_j\, \psi(2^{j-1}x) \quad \text{with} \quad b_j = j^{-\frac{1}{q}}\, (\log j)^{-\frac{1}{v}} . \tag{13.54}$$

By (13.44) we have $f \in B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$. On the other hand, by (13.42), we can estimate the left-hand side of (13.53) from below for some $c > 0$ by

$$c \sum_{k=K}^{\infty} (k\, b_k)^v\, k^{-\frac{v}{q'}-1} = c \sum_{k=K}^{\infty} k^{-1} (\log k)^{-1} = \infty . \tag{13.55}$$

We get a contradiction. This proves (13.7). Similarly one obtains (13.9).

13.3 Inequalities

The above theorem covers all cases of interest (excluding borderline situations according to (13.2)). It describes in a rather condensed way very sharp inequalities. It seems be reasonable to make clear the outcome. We use Example 2 in 12.4, Definition 12.8 and Proposition 12.10. Let $0 < \varepsilon < 1$.

13.3(i) *The B-spaces* Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0, \varepsilon]$. Let $0 < u \le \infty$. Let p and q be given by (13.6). Then

$$\left(\int_0^{\varepsilon} \left(\frac{\varkappa(t)\, f^*(t)}{|\log t|^{\frac{1}{q'}}} \right)^u \frac{dt}{t|\log t|} \right)^{\frac{1}{u}} \le c\, \|f\,|B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{13.56}$$

for some $c > 0$ and all $f \in B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded and $q \le u \le \infty$ (with the modification (13.59) below if $u = \infty$). In particular, if $1 < q < \infty$, then

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{\frac{1}{q'}}} \le c_0 \left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le c_1 \| f \,|B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{13.57}$$

are the two end-point cases according to (12.28). If $q = \infty$ then one has

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|} \le c \, \| f \,|B^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n)\| \,. \tag{13.58}$$

Let $\varkappa(t)$ be an (arbitrary) positive function on $(0, \varepsilon]$ and again let $1 < q \le \infty$. Then

$$\sup_{0<t<\varepsilon} \frac{\varkappa(t)\, f^*(t)}{|\log t|^{\frac{1}{q'}}} \le c \, \| f \,|B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{13.59}$$

for some $c > 0$ and all $f \in B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded. However the difference between the assumptions for $\varkappa$ in (13.56) and in (13.59) is rather immaterial. We discussed this point in 12.11.

13.3(ii) *The F-spaces* Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0, \varepsilon]$. Let $0 < u \le \infty$. Let p and q be given by (13.8). Then

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\, f^*(t)}{|\log t|^{\frac{1}{p'}}} \right)^u \frac{dt}{t |\log t|} \right)^{\frac{1}{u}} \le c \, \| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{13.60}$$

for some $c > 0$ and all $f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded and $p \le u \le \infty$, with the modification

$$\sup_{0<t<\varepsilon} \frac{\varkappa(t)\, f^*(t)}{|\log t|^{\frac{1}{p'}}} \le c \, \| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \quad \text{if} \quad u = \infty \,. \tag{13.61}$$

In particular,

$$\sup_{0<t<\varepsilon} \frac{f^*(t)}{|\log t|^{\frac{1}{p'}}} \le c_0 \left(\int_0^\varepsilon \left(\frac{f^*(t)}{|\log t|} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \le c_1 \, \| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{13.62}$$

are the two end-point cases according to (12.28). As above, if $\varkappa$ is an arbitrary positive function, then we have (13.61) if, and only if, $\varkappa$ is bounded.

Let $a \in \mathbb{R}$. Then $a_+ = \max\,(0, a)$ and $[a]$ stands for the largest integer smaller than or equal to a.

13.4 Corollary

(i) *Let* $0 < \delta \le \frac{1}{4}$ *and let for* $y \in \mathbb{R}$,

$$h(y) = e^{-\frac{1}{\delta^2 - y^2}} \quad \textit{if}\, |y| < \delta \quad \textit{and} \quad h(y) = 0 \ \textit{if}\, |y| \ge \delta\,. \tag{13.63}$$

Let $L \in \mathbb{N}_0$ *and*

$$h_L(y) = h(y) - \sum_{l=0}^{L} \varrho_l\, h^{(l)}(y-1)\,. \tag{13.64}$$

There are numbers $\varrho_l \in \mathbb{R}$ *such that (moment conditions)*

$$\int_{\mathbb{R}} y^k\, h_L(y)\, dy = 0 \quad \textit{if} \quad k = 0, \ldots, L\,. \tag{13.65}$$

(ii) *Let* $L + 1 \in \mathbb{N}_0$ *and let* h_L *with (13.65) be complemented by* $h_{-1} = h$ *(then (13.65) is empty). Let*

$$f_b(x) = \sum_{j=1}^{\infty} b_j\, h_L(2^{j-1}x_1) \prod_{m=2}^{n} h(2^{j-1}\, x_m)\,, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n\,, \tag{13.66}$$

where $b = \{b_j\}_{j=1}^{\infty}$ *is a sequence of non-negative numbers with*

$$b_1 \ge b_2 \ge \cdots \ge b_j \ge b_{j+1} \ge \cdots \quad . \tag{13.67}$$

Let p, q *be given by (13.6) in the* B*-case, by (13.8) in the* F*-case, and*

$$L_B = -1\,, \quad L_F = \max\left(-1, \left[n\left(\frac{1}{\min(p,q)} - 1\right)_+ - \frac{n}{p}\right]\right)\,. \tag{13.68}$$

Let $L + 1 \in \mathbb{N}_0$ *with* $L \ge L_B$ *in the* B*-case and* $L \ge L_F$ *in the* F*-case. Let* $0 < \varepsilon < 1$. *If* $b \in \ell_q$, *then*

$$\left(\sum_{j=1}^{\infty} b_j^q\right)^{\frac{1}{q}} \sim \left(\int_0^{\varepsilon} \left(\frac{f_b^*(t)}{|\log t|}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \sim \|f\, |B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{13.69}$$

(usual modification if $q = \infty$*) and, if* $b \in \ell_p$, *then*

$$\left(\sum_{j=1}^{\infty} b_j^p\right)^{\frac{1}{p}} \sim \left(\int_0^{\varepsilon} \left(\frac{f_b^*(t)}{|\log t|}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \sim \|f\, |F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\|\,, \tag{13.70}$$

where the equivalence constants are independent of b.

Proof *Step 1* If one inserts (13.64) in (13.65) then one gets a triangular matrix for ϱ_l from which these coefficients can be uniquely calculated.

Step 2 By the product structure of the terms in (13.66) we have

$$\int_{\mathbb{R}^n} x^\beta\, f_b(x)\, dx = 0 \quad \text{for} \quad |\beta| \le L \tag{13.71}$$

(where (13.71) is empty if $L = -1$): Since the sequence b is bounded, all respective sums for $x^\beta f_b(x)$ converge at least in $L_1(\mathbb{R}^n)$. Recall that one needs moment conditions (13.71) for atoms in $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ up to order L, where

$$L \ge \max\left(-1, [\sigma_p - s]\right) \quad \text{and} \quad L \ge \max\left(-1, [\sigma_{pq} - s]\right), \tag{13.72}$$

respectively, with σ_p and σ_{pq} given by (2.20). We refer to [Triδ], Theorem 13.8 on p. 75. Here we have $s = \frac{n}{p}$, hence $L \ge -1$ for the B-spaces and $L \ge L_F$ for the F-spaces. This formalizes what we said in Step 5 of the proof of Theorem 13.2. Otherwise the proof of the corollary is covered by Steps 4 and 5 of this proof.

13.5 Further references and comments

We described in Theorem 11.7 and in (11.70) the classical inequalities related to the critical case considered in Theorem 13.2. Recall that

$$H_p^{\frac{n}{p}}(\mathbb{R}^n) = F^{\frac{n}{p}}_{p,2}(\mathbb{R}^n), \quad 1 < p < \infty, \tag{13.73}$$

are the Sobolev spaces with the classical Sobolev spaces $W^k_p(\mathbb{R}^n)$ in (11.69) as special cases. In 11.8(v) we tried to collect the historical references of (11.56), (11.57), and (11.70). Obviously all these cases are covered by Theorem 13.2 and by 13.3. In more recent times, inequalities of type (11.57) have again attracted some attention, mostly restricted to the case of classical Sobolev spaces according to (11.69), but in the context of general rearrangement-invariant (quasi-)norms. We refer in particular to [CwP98], [EKP00], and [Pic99]. The last paper surveys some aspects of embeddings of classical Sobolev spaces $W^k_p(\mathbb{R}^n)$, especially of $W^1_p(\mathbb{R}^n)$, in rearrangement-invariant spaces. Furthermore, there is a connection between inequalities of type (11.56), (11.57) and capacity estimates in function spaces. Details may be found in [EKP00] and [Pic99] with references to Maz'ya's results in this direction, especially in [Maz85], pp. 105, 109. As mentioned above, parallel or earlier developments in the East have often passed unnoticed in the West. In particular, Ju. V. Netrusov proved in [Net87b], Theorem 3 on p. 108, assertions, which are related to [CwP98] and [EKP00], in the framework of spaces of type F^s_{pq}, including optimality of range spaces. He generalizes earlier results in the Russian literature by Brudnyi, Kaljabin, and especially by Gold'man. A good description of and detailed references to this earlier work may be found in [Liz86], D.1.8 and D.1.9, pp. 398–404. Our own contributions started in [Tri93] and were repeated in a slightly improved form in [ET96], Theorem 2.7.1, p. 82, and Theorem 2.7.3, p. 93. The main new point is the construction of extremal functions f belonging both to $H^{\frac{n}{p}}_p(\mathbb{R}^n)$ and $B^{\frac{n}{p}}_{pp}(\mathbb{R}^n)$ with $1 < p < \infty$ and having the singularity behaviour

$$f(x) = |\log|x||^{\frac{1}{p'}} \, (\log|\log|x||)^{-\sigma} \quad \text{where} \quad \sigma > \frac{1}{p}\,, \tag{13.74}$$

near the origin. This is now essentially covered by the function f given by (13.38) with, say, $b = \{b_j\}_{j=2}^{\infty}$,

$$b_j = j^{-\frac{1}{p}} \, |\log j|^{-\sigma}\,; \quad j = 2, 3, \ldots\,. \tag{13.75}$$

Then $b \in \ell_p$, and hence we have on the one hand (13.39) especially for $H^{\frac{n}{p}}_p(\mathbb{R}^n)$, and (13.44), especially for $B^{\frac{n}{p}}_{pp}(\mathbb{R}^n)$. On the other hand, if $|x| \sim 2^{-k}$, then it follows by (13.41),

$$\begin{aligned} f(x) \quad &\sim \quad \sum_{j=2}^{k} b_j \sim \int_2^k y^{-\frac{1}{p}} \, (\log y)^{-\sigma} \, dy \sim k^{\frac{1}{p'}} \, (\log k)^{-\sigma} \\ &\sim \quad |\log|x||^{\frac{1}{p'}} \, (\log|\log|x||)^{-\sigma}\,. \end{aligned} \tag{13.76}$$

At the same time it is now clear that the functions in (13.38) improve the earlier developments in [Tri93] and [ET96]. The equivalence (13.39) in Step 4 of the above proof coincides essentially with [EdT99b], Theorem 2.1. This paper might be considered as a forerunner of Theorem 13.2, restricted to $H^{\frac{n}{p}}_p(\mathbb{R}^n)$ and $B^{\frac{n}{p}}_{pp}(\mathbb{R}^n)$. Even worse, we used there (11.57), going back to [Has79] and [BrW80], as a starting point and derived the corresponding inequality for $B^{\frac{n}{p}}_{pp}(\mathbb{R}^n)$, this means (13.56) with $u = p = q$ and $\varkappa = 1$, via non-linear interpolation from (11.57). Otherwise the sharpness in [EdT99b] is on the $\varkappa$-level as described in 13.3. All other parts of Theorem 13.2 and its proof are new and published here for the first time. Especially the concept of growth envelopes in the above context came out very recently in collaboration with D. D. Haroske, [Har01]. Finally we mention the extension of the related results in [Tri93] and in [ET96], Theorem 2.7.1, to spaces with dominating mixed derivatives in [KrS96], including optimality results via extremal functions.

13.6 Spaces on domains

Let Ω be a domain in $\mathbb{R}^n$. The spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ have been introduced in Definition 5.3 for all admitted s, p, q. The concept of the *growth envelope* and the *growth envelope function* according to Definition 12.8 and the notational agreement (12.60) can be carried over under the same natural restrictions as there to the respective spaces $A^s_{pq}(\Omega)$. We denote them by

$$\mathfrak{E}_{G,\Omega} A^s_{pq} = \left(\mathcal{E}_{G,\Omega} A^s_{pq}(t), u\right). \tag{13.77}$$

In the critical case, considered in this section, Theorem 13.2 can be extended to spaces on domains: If p, q are given by (13.6), then

$$\mathfrak{E}_{G,\Omega} B^{\frac{n}{p}}_{pq} = \mathfrak{E}_G B^{\frac{n}{p}}_{pq} = \left(|\log t|^{\frac{1}{q'}}, q\right) \tag{13.78}$$

and, if p, q are given by (13.8), then

$$\mathfrak{E}_{G,\Omega} F^{\frac{n}{p}}_{pq} = \mathfrak{E}_G F^{\frac{n}{p}}_{pq} = \left(|\log t|^{\frac{1}{p'}}, p\right). \tag{13.79}$$

To justify these assertions we remark first

$$\mathcal{E}_{G,\Omega} A^s_{pq}(t) \le \mathcal{E}_G A^s_{pq}(t), \quad 0 < t \le \varepsilon, \tag{13.80}$$

as a more or less immediate consequence of the definition of spaces on domains by restriction of corresponding spaces on $\mathbb{R}^n$. On the other hand, the construction of extremal functions in Steps 4 and 5 of the proof of Theorem 13.2 is strictly local. Hence the arguments in Steps 6 and 7 of this proof can be carried over from $\mathbb{R}^n$ to Ω. Then one obtains (13.78) and (13.79).

13.7 The space *bmo*

We always exclude borderline situations. In our context, described by Theorem 11.2, this means in general (13.2), and with respect to the critical case, (13.3). Furthermore, we excluded in all our considerations so far the spaces $F^s_{\infty q}(\mathbb{R}^n)$. If $1 < q < \infty$, these spaces were introduced in [Tri78], 2.5.1, p. 118, and may also be found in [Triβ], 2.3.4, p. 50. This has been modified and, in particular, extended to all q, $0 < q < \infty$, in [FrJ90], Section 5. In the critical situation we have $s = 0$. At least some of these spaces fit in the scheme (11.9),

$$F^0_{\infty q}(\mathbb{R}^n) \subset F^0_{\infty 2}(\mathbb{R}^n) = bmo(\mathbb{R}^n) \subset L^{loc}_1(\mathbb{R}^n) \quad \text{if} \quad 0 < q \le 2, \tag{13.81}$$

where $bmo(\mathbb{R}^n)$ is the inhomogeneous space consisting of those locally integrable functions with bounded mean oscillation for which

$$\|f\,|bmo(\mathbb{R}^n)\| = \sup_{|Q|\le 1} \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx + \sup_{|Q|>1} \frac{1}{|Q|}\int_Q |f(x)|\,dx < \infty. \tag{13.82}$$

Here Q stands for cubes in $\mathbb{R}^n$ and f_Q is the mean value of f with respect to Q. We refer for details and further information to [Triβ], 2.2.2, p. 37, and 2.5.8, p. 93. Let $\psi(x)$ be a C^∞ function with a compact support near the origin, for example ψ from (13.35). It is well known and can be checked easily that $\psi(x)\,|\log|x|\,|$ belongs to $bmo(\mathbb{R}^n)$. But this is a local matter and can be extended by (13.82) to

$$f(x) = \sum_{m\in\mathbb{Z}^n} \psi(x-m)\,|\log|x-m|\,| \in bmo(\mathbb{R}^n)\,. \tag{13.83}$$

This makes clear that there is no growth envelope function $\mathcal{E}_G bmo$ according to Definition 12.8 and (12.37), or in other words,

$$\mathcal{E}_G\, bmo(t) = \infty \quad \text{for all} \quad 0 < t < \infty\,. \tag{13.84}$$

However in sharp contrast to the situation described in 13.6 if $p < \infty$, the growth envelope and the growth envelope function are reasonable for the spaces $bmo(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ and where $bmo(\Omega)$ is again defined by restriction of $bmo(\mathbb{R}^n)$ on Ω. Let, for example, $\Omega = Q$ be a cube with $|Q| = \varepsilon < 1$. A detailed study of the spaces $bmo(Q)$ may be found in [BeS88], Chapter 5, Section 7. In particular by [BeS88], Corollary 7.11, on p. 383, we have

$$bmo(Q) \subset L_\infty(\log L)_{-1}(Q) \tag{13.85}$$

where we used that L_{exp} according to 11.6(ii) (again with reference to [BeS88]) coincides with the space on the right-hand side of (13.85). In particular,

$$\sup_{0<t\le\varepsilon} \frac{f^*(t)}{|\log t|} \le c\,\|f\,|bmo(Q)\|\,. \tag{13.86}$$

On the other hand, J. Marschall proved in [Mar95], Lemma 16 on p. 253 (with a forerunner in [Mar87b])

$$B^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n) \subset F^0_{\infty q}(\mathbb{R}^n)\,, \quad 0 < p < \infty\,, \quad 0 < q \le \infty\,, \tag{13.87}$$

in particular,

$$B^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n) \subset bmo(\mathbb{R}^n)\,, \quad 0 < p < \infty\,. \tag{13.88}$$

However by (13.7), 13.6 (and the notation introduced there) and (13.86) one gets

$$\mathfrak{E}_{G,Q} bmo = (|\log t|, \infty)\,. \tag{13.89}$$

In any case in borderline situations one has to distinguish carefully between global and local singularity behaviour.

14 The super-critical case

14.1 Introduction

By the terminology of (10.15) the super-critical case covers the spaces

$$B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{and} \quad F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with } 0<p<\infty \text{ and } 0<q\le\infty\,. \tag{14.1}$$

This corresponds to the dotted line in Fig. 10.1. Recall that we always exclude in this chapter borderline situations as described in (13.2). This means in the super-critical case that we do not deal with the spaces $B^1_{\infty q}(\mathbb{R}^n)$ and also not with the spaces $F^1_{\infty q}(\mathbb{R}^n)$ briefly mentioned in 13.7. As a further restriction of (14.1) we are interested only in those spaces which are not continuously embedded in $C^1(\mathbb{R}^n)$ (or, which is the same, in $Lip(\mathbb{R}^n)$); this means by Theorem 11.4, and has been detailed in 11.5, especially in (11.38), (11.39), we deal with the spaces

$$B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 0<p<\infty\,, \quad 1<q\le\infty\,, \tag{14.2}$$

and

$$F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 1<p<\infty\,, \quad 0<q\le\infty\,. \tag{14.3}$$

This covers in particular the Sobolev spaces mentioned in (11.40). As outlined in the introductory Section 10 we wish to measure the continuity of functions belonging to the spaces (14.2), (14.3) in terms of the continuity envelope as introduced in Definition 12.14. Instead of $\mathfrak{E}_C A^{1+\frac{n}{p}}_{pq}$ in (12.83) we use the more handsome version (12.85). In the theorem below we calculate explicitly the continuity envelope for all spaces in (14.2) and (14.3). Afterwards we describe what this means in detail. Finally we add a few references. Otherwise we try to keep the presentation of the super-critical case as close as possible in its formulations to the critical case considered in the previous section (this applies also to this introduction compared with 13.1). In rough terms, using Proposition 12.16 as a vehicle, we lift Theorem 13.2 from the critical to the super-critical situation. Let $1\le v\le\infty$. As usual, v' is given by $\frac{1}{v}+\frac{1}{v'}=1$.

14.2 Theorem

(i) *Let*

$$0<p<\infty\,, \quad 1<q\le\infty\,. \tag{14.4}$$

Then

$$\mathfrak{E}_C B^{1+\frac{n}{p}}_{pq} = (|\log t|^{\frac{1}{q'}}, q)\,. \tag{14.5}$$

(ii) *Let*

$$1 < p < \infty \,, \quad 0 < q \le \infty \,. \tag{14.6}$$

Then

$$\mathfrak{E}_C F^{1+\frac{n}{p}}_{pq} = (|\log t|^{\frac{1}{p'}}, p) \,. \tag{14.7}$$

Proof *Step 1* Recall that

$$\|f \,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \sim \|f \,|B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| + \sum_{j=1}^n \| \frac{\partial f}{\partial x_j} \,|B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.8}$$

and similarly for $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$, [Tri$\beta$], Theorem 2.3.8, pp. 58–59. Let p, q be given by (14.4). Using (12.87) we obtained by Theorem 13.2 and (13.57) with $0 < \varepsilon < 1$,

$$\left(\int_0^\varepsilon \left(\frac{|\nabla f|^*(t)}{|\log t|} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le c \, \|f \,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.9}$$

(obviously modified according to (13.58) if $q = \infty$). Similarly for $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if p, q are given by (14.6), based on (13.62). We apply Proposition 12.16 and obtain, if $q < \infty$, by completion

$$\left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le c \, \|f \,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \,, \quad f \in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \,, \tag{14.10}$$

(and again similarly in the F-case). If $q = \infty$ then we wish to have

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|} \le c \, \|f \,|B^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n)\| \,, \quad f \in B^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n) \,. \tag{14.11}$$

Let $f \in B^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n)$ and let φ be as in (2.33). Then we can apply (12.90) to

$$f_j = (\varphi(2^{-j}\cdot)\widehat{f})^\vee \,.$$

We obtain (14.11) with f_j in place of f, where the corresponding right-hand sides can be estimated uniformly with respect to j; hence

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f_j,t)}{|\log t|} \le c \, \|f \,|B^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n)\| \,, \quad f \in B^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n) \,. \tag{14.12}$$

By elementary embedding, $f_j(x)$ converges pointwise to $f(x)$. Then (14.11) follows from (14.12) and $j \to \infty$. Similarly for $F^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n)$. Since $\widetilde{\omega}(f,t)$ is equivalent to a monotonically decreasing function we are now in the same situation as in Step 3 of the proof of Theorem 13.2. We get

$$\mathcal{E}_C B^{1+\frac{n}{p}}_{pq}(t) \le |\log t|^{\frac{1}{q'}}\,, \quad 0<p<\infty\,, \quad 1<q\le\infty\,, \tag{14.13}$$

and

$$\mathcal{E}_C F^{1+\frac{n}{p}}_{pq}(t) \le |\log t|^{\frac{1}{p'}}\,, \quad 1<p<\infty\,, \quad 0<q\le\infty\,. \tag{14.14}$$

Step 2 To construct extremal functions we rely on Corollary 13.4 and put

$$h^L(y) = \int\limits_{-\infty}^{y} h_L(z)\,dz\,, \quad y\in\mathbb{R}\,, \tag{14.15}$$

where h_L has the same meaning as in part (i) of this corollary with $L \in \mathbb{N}_0$. Then h^L is a compactly supported C^∞ function. Integration by parts and (13.65) prove that

$$\int_{\mathbb{R}} y^k\, h^L(y)\,dy = \frac{1}{k+1}\int_{\mathbb{R}} (y^{k+1})' \int\limits_{-\infty}^{y} h_L(z)\,dz\,dy = 0 \quad \text{if} \quad k=0,\ldots,L-1\,, \tag{14.16}$$

(if $L=0$ then (14.16) is empty). We replace $f_b(x)$ in (13.66) by

$$f^b(x) = \sum_{j=1}^{\infty} b_j 2^{-j+1}\, h^L(2^{j-1}x_1) \prod_{m=2}^{n} h(2^{j-1}x_m)\,, \quad x=(x_1,\ldots,x_n)\in\mathbb{R}^n\,, \tag{14.17}$$

where $b=\{b_j\}_{j=1}^{\infty}$ is a sequence with $b_j \ge 0$ and (13.67). This can be interpreted as an atomic decomposition in $B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ and $F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$, where the necessary moment conditions according to (13.72), now with $s = 1+\frac{n}{p}$, may be assumed to be satisfied by the above construction. Let $0<\varepsilon<1$. We claim that we have in analogy to (13.69) and (13.70),

$$\left(\sum_{j=1}^{\infty} b_j^q\right)^{\frac{1}{q}} \sim \left(\int\limits_0^{\varepsilon} \left(\frac{\widetilde{\omega}(f^b,t)}{|\log t|}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \sim \|f^b\,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.18}$$

if $b \in \ell_q$ (usual modification when $q = \infty$) and

$$\left(\sum_{j=1}^{\infty} b_j^p \right)^{\frac{1}{p}} \sim \left(\int_0^{\varepsilon} \left(\frac{\widetilde{\omega}(f^b, t)}{|\log t|} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \sim \| f^b \,| F_{pq}^{1+\frac{n}{p}}(\mathbb{R}^n) \| \,. \tag{14.19}$$

Of course we always assume that (14.4) and (14.6) are satisfied. First we remark that

$$\| f^b \,| F_{pq}^{1+\frac{n}{p}}(\mathbb{R}^n) \| \le c \left(\sum_{j=1}^{\infty} b_j^p \right)^{\frac{1}{p}} \tag{14.20}$$

in analogy to (13.40). Secondly we claim

$$\widetilde{\omega}(f^b, t) \sim \sum_{j=1}^{k} b_j \quad \text{if} \quad t \sim 2^{-k} \quad \text{where} \quad k \in \mathbb{N} \,, \tag{14.21}$$

in analogy (but also in slight modification) of (13.42). Let $\eta > 0$ be small and $k \in \mathbb{N}$. Then one obtains by (14.17), and (14.15), (13.64),

$$f^b(0) - f^b(-\eta 2^{-k}, 0, \ldots, 0) \ge \sum_{j=1}^{k} b_j \, (h^L)'(z_{j,k}) \eta 2^{-k} \,, \tag{14.22}$$

with $-\frac{\delta}{2} < z_{j,k} < 0$, where δ has the same meaning as in (13.63). Since $(h^L)' = h_L$, all factors $h_L(z_{j,k}) \ge c > 0$ for some c which is independent of j and k. Hence the left-hand side of (14.21) can be estimated from below by its right-hand side. To prove the converse we note that the terms with $j \ge k$ in (14.17) are harmless. Together with the monotonicity (13.67) of the coefficients b_j, and the converse of (14.22) we get (14.21). But now we are very much in the same situation as in Step 4 of the proof of Theorem 13.2. The counterparts of (13.43) and (13.39) prove (14.19). Similarly one obtains (14.18). We are now in the same situation as in Steps 6 and 7 of the proof of Theorem 13.2. First we get equality in (14.13), (14.14) and that q and p are the correct numbers in (14.5) and (14.7), respectively.

14.3 Inequalities

The above Theorem 14.2 is the counterpart of Theorem 13.2. Even more, with Proposition 12.16 as a vehicle, 14.2 is a consequence of 13.2. It covers all cases of interest (excluding borderline situations as described in (13.2), which means here $p = \infty$). It describes in a rather condensed way very sharp inequalities. In analogy to 13.3 we discuss the outcome, where now the harvest is even richer, since we have not only inequalities in terms of moduli of continuity hidden in Theorem 14.2, but even sharper inequalities of type (14.9). As in 13.3 we formulate the corresponding assertions for the B-spaces in (i) and for the F-spaces in (ii), but in contrast to 13.3 a few explanations and justifications are needed. This will be done afterwards in (iii). We always assume that $0 < \varepsilon < 1$.

14.3(i) *The B-spaces* Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0, \varepsilon]$. Let $0 < u \leq \infty$. Let p and q be given by (14.4). Then: (i_1)

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\, \widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{q'}}} \right)^u \frac{dt}{t|\log t|} \right)^{\frac{1}{u}} \leq c\, \| f \,| B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.23}$$

for some $c > 0$ and all $f \in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded and $q \leq u \leq \infty$ (with the modification (14.28) below if $u = \infty$) and (i_2)

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\, |\nabla f|^*(t)}{|\log t|^{\frac{1}{q'}}} \right)^u \frac{dt}{t|\log t|} \right)^{\frac{1}{u}} \leq c\, \| f \,| B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.24}$$

for some $c > 0$ and all $f \in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded and $q \leq u \leq \infty$ (again with the indicated modification if $u = \infty$). In particular, if $1 < q < \infty$, then

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{q'}}} \leq c_0 \left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq c_1\, \| f \,| B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.25}$$

(and similarly with $|\nabla f|^*(t)$ in place of $\widetilde{\omega}(f,t)$) are the two end-point cases according to (12.28). The two types of inequalities (14.23) and (14.24) are connected by

$$\begin{aligned} \left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{q'}}} \right)^u \frac{dt}{t|\log t|} \right)^{\frac{1}{u}} \quad &\leq \quad c_0 \left(\int_0^\varepsilon \left(\frac{|\nabla f|^*(t)}{|\log t|^{\frac{1}{q'}}} \right)^u \frac{dt}{t|\log t|} \right)^{\frac{1}{u}} \\ &\leq \quad c_1\, \| f \,| B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \end{aligned} \tag{14.26}$$

for some $c_0 > 0$, $c_1 > 0$, and all $f \in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$, where again $q \leq u \leq \infty$ (with the modification (14.28) below if $u = \infty$). If $q = \infty$ then one has

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|} \leq c_0 \sup_{0<t<\varepsilon} \frac{|\nabla f|^*(t)}{|\log t|} \leq c\, \| f \,| B^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n)\| \,. \tag{14.27}$$

Let $\varkappa(t)$ be an (arbitrary) positive function on $(0,\varepsilon]$ and let again $1<q\le\infty$. Then

$$\sup_{0<t<\varepsilon} \frac{\varkappa(t)\,\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{q'}}} \le c\,\|f\,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.28}$$

for some $c>0$ and all $f\in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded (and the same assertion with $|\nabla f|^*(t)$ in place of $\widetilde{\omega}(f,t)$). But as discussed in 12.11 the above additional assumption that $\varkappa$ is monotone is rather immaterial.

14.3(ii) *The F-spaces* Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0,\varepsilon]$. Let $0<u\le\infty$. Let p and q be given by (14.6). Then: (ii$_1$)

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)\,\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{p'}}}\right)^u \frac{dt}{t|\log t|}\right)^{\frac{1}{u}} \le c\,\|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.29}$$

for some $c>0$ and all $f\in F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded and $p\le u\le\infty$, with the modification

$$\sup_{0<t<\varepsilon} \frac{\varkappa(t)\,\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{p'}}} \le c\,\|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \quad \text{if} \quad u=\infty\,; \tag{14.30}$$

and (ii$_2$)

$$\left(\int_0^\varepsilon \left(\frac{\varkappa(t)|\nabla f|^*(t)}{|\log t|^{\frac{1}{p'}}}\right)^u \frac{dt}{t|\log t|}\right)^{\frac{1}{u}} \le c\,\|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.31}$$

for some $c>0$ and all $f\in F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded and $p\le u\le\infty$ with a similar modification as in (14.30) if $u=\infty$. In particular,

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{p'}}} \le c_0 \left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \le c_1\,\|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \tag{14.32}$$

(and similarly with $|\nabla f|^*(t)$ in place of $\widetilde{\omega}(f,t)$) are the two end-point cases according to (12.28). The two types of inequalities (14.29) and (14.31) are connected by

$$\begin{aligned}\left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{p'}}}\right)^u \frac{dt}{t|\log t|}\right)^{\frac{1}{u}} &\le c_0 \left(\int_0^\varepsilon \left(\frac{|\nabla f|^*(t)}{|\log t|^{\frac{1}{p'}}}\right)^u \frac{dt}{t|\log t|}\right)^{\frac{1}{u}} \\ &\le c_1\,\|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| \end{aligned} \tag{14.33}$$

for some $c_0 > 0$, $c_1 > 0$, and all $f \in F_{pq}^{1+\frac{n}{p}}(\mathbb{R}^n)$, where again $p \le u \le \infty$ (with the modification as in (14.30) if $u = \infty$). As in connection with (14.28) one does not need for the sharpness assertion in (14.30) that $\varkappa$ is monotone.

14.3(iii) *Explanations* The above inequalities with respect to $\widetilde{\omega}(f,t)$ follow from Theorem 14.2, Example 2 in 12.4, Definition 12.14 and the modified Proposition 12.10 with $\mathcal{E}_C A_{pq}^{1+\frac{n}{p}}(t)$ and $\widetilde{\omega}(f,t)$ in place of $\mathcal{E}_G A^s_{pq}(t)$ and $f^*(t)$, respectively. Or in other words, they simply describe what is meant by a continuity envelope. Furthermore, (14.26), (14.27), and (14.33) are covered by Step 1 of the proof of Theorem 14.2. The only point which is not immediately clear by the above theorem and its proof is the boundedness of $\varkappa$ in (14.24) and (14.31). By (12.14) this question can be reduced to

$$\sup_{0<t<\varepsilon} \frac{\varkappa(t)\,|\nabla f|^*(t)}{|\log t|^{\frac{1}{q'}}} \le c\, \| f \,|B_{pq}^{1+\frac{n}{p}}(\mathbb{R}^n)\| \tag{14.34}$$

and its F-counterpart. Hence we assume that we have (14.34) with $1 < q \le \infty$ for some positive function $\varkappa$ on $(0,\varepsilon]$. We wish to show that $\varkappa$ must be bounded. Let f^b be given by (14.17) with $b_j = 1$ if $j = 1, \ldots, J$ and $b_j = 0$ if $j > J$. Then by an argument similar to that in (14.22) we have

$$|\nabla f|(x) \ge c\, J \quad \text{in a cube} \quad [-\eta 2^{-J}, 0]^n\,, \quad J \in \mathbb{N}\,, \tag{14.35}$$

with $2^{-K-1} \le \eta \le 2^{-K}$ where $c > 0$ and $K \in \mathbb{N}$ are independent of J. Then $|\nabla f|^*(\eta^n\, 2^{-Jn}) \ge c\, J$ and it follows by (14.34) and (14.18)

$$\frac{\varkappa(\eta^n\, 2^{-Jn})\, J}{J^{\frac{1}{q'}}} \le c\, J^{\frac{1}{q}}\,. \tag{14.36}$$

Hence $\varkappa$ is bounded. This proves the $\varkappa$-sharpness also in (14.24) and (14.31).

14.4 More handsome inequalities

In 14.3 we tried to unwrap what is hidden in Theorem 14.2 and, with a switch from f^* to $|\nabla f|^*$, in Theorem 13.2. This may also be taken as an excuse for the undue length of 14.3 (compared with the lengths of the respective theorems). Nevertheless the formulations remain somewhat involved. But in case of $u = \infty$, this means (14.28) and (14.30) with $\varkappa = 1$, one can convert these assertions into more handsome inequalities which come also near to what is done in the literature. Let $v > 0$. By the definitions of the moduli of continuity in (12.69) and (12.72) we have with $0 < \varepsilon < 1$,

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^v} = \sup_{0<t<\varepsilon} \frac{1}{t|\log t|^v} \sup_{x\in\mathbb{R}^n, |h|\le t} |f(x+h)-f(x)|$$
$$= \sup_{|x-y|\le\varepsilon} \frac{|f(x)-f(y)|}{|x-y|\,|\log|x-y|\,|^v} \,. \tag{14.37}$$

Then (14.28) and (14.30) with $\varkappa = 1$ can be reformulated as

$$|f(x)-f(y)| \le c\,|x-y|\,|\log|x-y|\,|^{\frac{1}{q'}}\, \|f\,|B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| , \quad x\in\mathbb{R}^n ,\ y\in\mathbb{R}^n , \tag{14.38}$$

and $|x-y| < \varepsilon$, for some $c>0$ and all $f\in B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$, where

$$0<p<\infty , \quad 1<q\le\infty , \quad \frac{1}{q}+\frac{1}{q'}=1 ,$$

and

$$|f(x)-f(y)| \le c\,|x-y|\,|\log|x-y|\,|^{\frac{1}{p'}}\, \|f\,|F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)\| , \quad x\in\mathbb{R}^n ,\ y\in\mathbb{R}^n , \tag{14.39}$$

and $|x-y| < \varepsilon$, for some $c>0$ and all $f\in F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n)$, where

$$1<p<\infty , \quad 0<q\le\infty , \quad \frac{1}{p}+\frac{1}{p'}=1;$$

with the special case

$$|f(x)-f(y)| \le c\,|x-y|\,|\log|x-y|\,|^{\frac{1}{p'}}\, \|f\,|H^{1+\frac{n}{p}}_{p}(\mathbb{R}^n)\| , \quad x\in\mathbb{R}^n ,\ y\in\mathbb{R}^n , \tag{14.40}$$

and $|x-y| < \varepsilon$, where $H^{1+\frac{n}{p}}_{p}(\mathbb{R}^n) = F^{1+\frac{n}{p}}_{p,2}(\mathbb{R}^n)$ are the Sobolev spaces, $1 < p < \infty$.

14.5 Borderline cases

This means in our context here $p=\infty$ and $s=1$, hence the Besov spaces $B^1_{\infty q}(\mathbb{R}^n)$, where $0<q\le\infty$, with the Zygmund class $\mathcal{C}^1(\mathbb{R}^n) = B^1_{\infty\infty}(\mathbb{R}^n)$ as a special case. We refer to 1.2(iv), (v), especially (1.11). The extension of (14.38) to $p=q=\infty$ is given by

$$|f(x)-f(y)| \le c\,|x-y|\,|\log|x-y||\ \left\|f\,|\mathcal{C}^1(\mathbb{R}^n)\right\| , \quad x\in\mathbb{R}^n ,\ y\in\mathbb{R}^n , \tag{14.41}$$

and $|x-y| < \varepsilon < 1$, for some $c > 0$, and all $f \in \mathcal{C}^1(\mathbb{R}^n)$. It is due to A. Zygmund, [Zyg45], and may also be found in [Zyg77], Chapter II, Theorem 3.4, p. 44. It was apparently A. Zygmund who coined the word *smooth functions* in this context in his paper [Zyg45]. In [Zyg77], Notes, p. 375, he mentioned that B. Riemann was the first who considered smooth functions. B. Riemann discussed in his Habilitationsschrift [Rie'54] the possibility to represent a continuous periodic function on the interval $[0, 2\pi]$ in terms of trigonometric series: First he surveyed what had been done so far. Afterwards he studied in Sections 7–13 the indicated problem in detail based on the systematic use of second differences. This is just what A. Zygmund called almost 100 years later in [Zyg45] *smooth functions.* The extension of (14.25) to $p = \infty$ (and $1 < q \le \infty$) is given by

$$\sup_{0<t<\varepsilon} \frac{\widetilde{\omega}(f,t)}{|\log t|^{\frac{1}{q'}}} \le c_0 \left(\int_0^\varepsilon \left(\frac{\widetilde{\omega}(f,t)}{|\log t|} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le c_1 \, \| f \, | B^1_{\infty q}(\mathbb{R}^n) \| \tag{14.42}$$

(with the obvious modification if $q = \infty$ which is essentially (14.41)). This has been proved very recently in [BoL00], Proposition 1. Furthermore, (14.25) with (14.4) and (14.32) with (14.6) follow from (14.42) and the embeddings

$$B^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset B^1_{\infty q}(\mathbb{R}^n) \quad \text{and} \quad F^{1+\frac{n}{p}}_{pq}(\mathbb{R}^n) \subset B^1_{\infty p}(\mathbb{R}^n)\,, \tag{14.43}$$

[Triβ], Theorem 2.7.1, p. 129. (In the second embedding we used the first one and [Triβ], (15) on p. 131.) Our own approach is characterized by lifting the critical case, considered in Section 13, to the super-critical one considered here. This results in the sharper inequalities (14.26), (14.27), (14.33), where always $p < \infty$. But it is unclear whether there is something of this type if $p = \infty$. As discussed in connection with (13.3), based on Theorem 11.2, the question itself makes sense at least for the spaces $B^1_{\infty q}(\mathbb{R}^n)$ with $1 < q \le 2$, also for the space $bmo(\Omega)$, lifted by 1, according to 13.7. By (14.42) and (14.43) the sharpness assertions available for the spaces with $p < \infty$ can be carried over to the spaces $B^1_{\infty q}(\mathbb{R}^n)$. We get the complement

$$\mathfrak{E}_C B^1_{\infty q} = (|\log t|^{\frac{1}{q'}}, q)\,, \quad 1 < q \le \infty\,, \tag{14.44}$$

of (14.5). In particular, the f^b, given by (14.17), are extremal functions also for $B^1_{\infty q}(\mathbb{R}^n)$ and we have (14.18) with $p = \infty$. If $p = \infty$ then

$$f(x) = h(x)\, x_1 \, \log |x|\,, \quad x \in \mathbb{R}^n\,, \tag{14.45}$$

with h given by (13.63), for example, is an extremal function in $\mathcal{C}^1(\mathbb{R}^n)$. This follows from

$$\frac{\partial f}{\partial x_1} \sim \log |x| \in bmo(\mathbb{R}^n) \subset \mathcal{C}^0(\mathbb{R}^n)\,, \quad |x| \le \frac{\delta}{2}\,, \tag{14.46}$$

the boundedness of the other first derivatives, and elementary calculations. The embedding mentioned is well known. We refer to [RuS96], p. 33. Other extremal functions may be found in 17.1.

14.6 Envelope functions and non-compactness

This remark applies equally to growth envelope functions and continuity envelope functions and to all cases (critical, super-critical, sub-critical). But it will be clear what is meant by looking at an example connected with the above considerations. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ (one might think of the unit ball). Then $B^s_{pq}(\Omega)$ has the usual meaning according to Definition 5.3. Let $\alpha \geq 0$. Let $Lip^{(1,-\alpha)}(\Omega)$ be, by definition, the Banach space of all (complex-valued) continuous functions in Ω, such that

$$\|f\,|Lip^{(1,-\alpha)}(\Omega)\| = \sup_{x\in\Omega} |f(x)| + \sup_{\substack{x,y\in\Omega \\ 0<|x-y|\leq\frac{1}{2}}} \frac{|f(x)-f(y)|}{|x-y|\,|\log|x-y|\,|^\alpha} < \infty\,. \tag{14.47}$$

We use here the notation introduced in [EdH99]. Then (14.38), restricted to Ω, is equivalent to the continuous embedding

$$B^{1+\frac{n}{p}}_{pq}(\Omega) \subset Lip^{(1,-\frac{1}{q'})}(\Omega)\,, \quad 0<p<\infty\,, \quad 1<q<\infty\,, \tag{14.48}$$

(where we excluded $q=\infty$). However this embedding is not compact. We prove this assertion by looking at the growth envelope function $|\log t|^{\frac{1}{q'}}$ for $B^{1+\frac{n}{p}}_{pq}(\Omega)$ (as for spaces on domains we refer also to 13.6). Since $q<\infty$ it follows that $C^\infty(\overline{\Omega})$, the restriction of $S(\mathbb{R}^n)$ on Ω, is dense in $B^{1+\frac{n}{p}}_{pq}(\Omega)$. Assume that the embedding (14.48) is compact. We fix a quasi-norm in $B^{1+\frac{n}{p}}_{pq}(\Omega)$. Then we find for any $\delta>0$ finitely many functions

$$f_j \in C^\infty(\overline{\Omega})\,, \quad \|f_j\,|B^{1+\frac{n}{p}}_{pq}(\Omega)\| \leq 1 \quad \text{with} \quad j=1,\ldots,M(\delta)\,, \tag{14.49}$$

such that for any f with $\|f\,|B^{1+\frac{n}{p}}_{pq}(\Omega)\| \leq 1$,

$$\inf_j \frac{\widetilde{\omega}(f-f_j,t)}{|\log t|^{\frac{1}{q'}}} \leq \delta\,, \quad \text{uniformly for} \quad 0<t<\frac{1}{2}\,. \tag{14.50}$$

Here $\{f_j\}$ is a δ-net. Furthermore we used (14.37) with $\varepsilon=\frac{1}{2}$. Since the functions f_j are smooth one obtains

$$\widetilde{\omega}(f,t) \leq C_\delta + \delta|\log t|^{\frac{1}{q'}} \quad \text{where} \quad 0<t<\frac{1}{2}\,, \tag{14.51}$$

and hence by (12.77),

$$\mathcal{E}_C B_{pq}^{1+\frac{n}{p}}(t) \le C_\delta + \delta\,|\log t|^{\frac{1}{q'}}\,, \quad 0 < t < \frac{1}{2}\,. \tag{14.52}$$

If $\delta > 0$ is small one gets a contradiction to

$$\mathcal{E}_C B_{pq}^{1+\frac{n}{p}}(t) \sim |\log t|^{\frac{1}{q'}}$$

when t is tending to zero. Hence the embedding (14.48) is not compact. But it was not so much our aim to prove this specific assertion. We wanted to make clear what happens if both source and target space have the *same* envelope functions (growth or continuity): The respective embeddings are not compact (at least in those cases where smooth functions are dense in the source space).

14.7 References

First we recall that (11.59) coincides with (14.40) if $1 + \frac{n}{p} = k \in \mathbb{N}$. This inequality in this version is due to [BrW80]. We refer also to our remarks in 11.8(vi). The extension (14.40) from the classical Sobolev spaces $W_p^k(\mathbb{R}^n)$ to $H_p^{1+\frac{n}{p}}(\mathbb{R}^n)$ may be found in [EdK95]. As mentioned in 14.5 the borderline case, in our notation used here, (14.41) goes back to [Zyg45] (and may be found with a new proof in [Zyg77], Chapter II, Theorem 3.4, p. 44). A Fourier-analytical proof of (14.38), at least in some cases, has been given in [Vis98]. The additional point of interest here is the use of spaces of type $Lip^{(1,-\alpha)}$ (on domains and in $\mathbb{R}^n$) according to (14.47) in connection with problems from physics. We refer in this context also to [Lio98], pp. 146, 152. The first full proof of (14.38) and (14.39), including sharpness assertions, was given in [EdH99]. In the context of this paper sharpness means that the exponents $\frac{1}{q'}$ and $\frac{1}{p'}$ in the log-terms in (14.38) and (14.39), respectively, cannot be replaced by a smaller exponent. The proofs are based on atomic decompositions. The borderline inequality (14.42) (without the middle term) has been derived in [KrS98] using extrapolation techniques. By (14.43), and as has also been mentioned in [KrS98] explicitly, this results in new proofs of (14.38), (14.39). As remarked in 14.5 the decisive improvement concerning the middle term in (14.42) is due to [BoL00]. Our own approach which resulted not only in Theorem 14.2, including the $\varkappa$-sharpness as described in 14.3, but also in the sharp assertions in 14.3 concerning $|\nabla f|^*(t)$, especially (14.26), (14.27), (14.33), is published here for the first time. Especially the concept of envelope functions and envelopes (here in connection with the super-critical case in the understanding of Theorem 14.2) came out in recent discussions with D. D. Haroske. A more systematic treatment will be given in [Har01]. In a different context lifting arguments have also been used in [EdK95] with a reference to [Adm75], Theorem

8.36, pp. 254–255. Somewhat different types of envelopes appear in [Net87b] in connection with optimal embeddings of F^s_{pq}-spaces in rearrangement-invariant spaces, preferably in sub-critical situations which will be treated in Section 15 below. Finally we add a remark in connection with 14.6. As mentioned, there is no hope that the continuous embeddings (14.48) are compact. However the situation is different if one replaces the target space in (14.48) by $Lip^{(1,-\alpha)}(\Omega)$ according to (14.47) with $\alpha > \frac{1}{q'}$. Then one has compact embeddings. The adequate notation to measure the degree of compactness are entropy numbers and approximation numbers. As for the general background we refer to [ET96], Chapter 1. But later on in connection with the spectral theory for fractal elliptic operators we repeat in 19.16 what is needed. A detailed study of entropy numbers and approximation numbers for problems treated in the present section (in the modification indicated above) has been given in [EdH99] and [EdH00]. This has been complemented in [Har00a]. The small survey [Har00b] summarizes these results.

15 The sub-critical case

15.1 Introduction

By the terminology of (10.5) the sub-critical case covers the spaces

$$B^s_{pq}(\mathbb{R}^n) \quad \text{and} \quad F^s_{pq}(\mathbb{R}^n) \quad \text{with } 0<p<\infty\,,\ 0<q\le\infty\,,\ \sigma_p<s<\frac{n}{p}\,. \tag{15.1}$$

Recall our standard abbreviations

$$\sigma_p = n\left(\frac{1}{p}-1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)}-1\right)_+ \tag{15.2}$$

where $0<p<\infty$, $0<q\le\infty$. We are interested in sharp limiting embeddings (or better related inequalities) corresponding to the foot-point of the dashed line in Fig. 10.1 and given by

$$1<r<\infty\,, \quad s>0\,, \quad s-\frac{n}{p}=-\frac{n}{r}\,, \quad \text{and} \quad 0<q\le\infty\,. \tag{15.3}$$

We characterized in Theorem 11.4(i) those spaces (15.1) which are embedded in $L_r(\mathbb{R}^n)$. In Theorem 11.7(i) and (ii) we collected the classical more refined inequalities in the sub-critical context and we described their rather rich history in the points 11.8(i)–(iv). Again we are interested only in spaces which consist entirely of regular distributions; this means that they are embedded in $L_1^{loc}(\mathbb{R}^n)$. A final description has been given in Theorem 11.2. Compared with

(15.1) we exclude as previously borderline cases, we mean here those spaces $F^s_{pq}(\mathbb{R}^n)$ and $B^s_{pq}(\mathbb{R}^n)$ with $0 < p < \infty$ and $s = \sigma_p$ which are covered by (11.9) and (11.14), respectively. Otherwise we are very much in the same general situation as in 13.1. Again, as outlined in the introductory Section 10 we wish to measure the singularity behaviour of functions belonging to the spaces (15.1) in terms of the growth envelope as introduced in Definition 12.8. Instead of $\mathfrak{E}_G A^s_{pq}$ in (12.56) we use the more handsome version (12.60). Similarly as in Section 13, in the theorem below we first calculate explicitly the growth envelopes for all spaces in (15.1). Afterwards we describe what this means in terms of inequalities. As explained in detail in 11.8(i), (ii) the Lorentz spaces and their (quasi-)norms come in naturally, whereas as described in 11.8(iii), (iv) the Zygmund spaces and their (quasi-)norms are distinguished but (from the above point of view) not so natural target spaces. Nevertheless we collect in a corollary below sharp assertions concerning related inequalities also in these cases. Finally we complement the references given so far.

15.2 Theorem

Let

$$0 < q \le \infty\,, \quad s > 0\,, \quad \textit{and} \quad s - \frac{n}{p} = -\frac{n}{r} \quad \textit{with} \quad 1 < r < \infty\,, \tag{15.4}$$

(the dashed line in Fig. 10.1). Then

$$\mathfrak{E}_G B^s_{pq} = (t^{-\frac{1}{r}}, q) \tag{15.5}$$

and

$$\mathfrak{E}_G F^s_{pq} = (t^{-\frac{1}{r}}, p)\,. \tag{15.6}$$

Proof *Step 1* Let $0 < \varepsilon < 1$. In 11.8(ii) we proved (11.52). The interpolation argument used there applies to all cases covered by (15.4), [Triβ], Theorem 2.4.2, p. 64. Hence, together with (12.26), we obtain always

$$\sup_{0<t<\varepsilon} t^{\frac{1}{r}}\, f^*(t) \le c_0 \left(\int_0^\varepsilon \left(t^{\frac{1}{r}}\, f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \le c_1 \, \| f \,| B^s_{pq}(\mathbb{R}^n) \| \,. \tag{15.7}$$

If $q = \infty$ then one has only the first and the last term. By (12.37) it follows that

$$\mathcal{E}_G B^s_{pq}(t) \le c\, t^{-\frac{1}{r}}\,, \quad 0 < t < \varepsilon\,. \tag{15.8}$$

As for the F-spaces we use Netrusov's observation described in (13.29). Similarly as in (13.30) it follows that

$$\sup_{0<t<\varepsilon} t^{\frac{1}{r}} f^*(t) \le c_0 \left(\int_0^\varepsilon \left(t^{\frac{1}{r}} f^*(t) \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \le c_1 \, \| f \, | F^s_{pq}(\mathbb{R}^n) \| \, . \tag{15.9}$$

Hence we have also

$$\mathcal{E}_G F^s_{pq}(t) \le c \, t^{-\frac{1}{r}} \, , \quad 0 < t < \varepsilon \, . \tag{15.10}$$

Step 2 Let ψ be given by (13.35). By [Triδ], Theorem 13.8, p. 75,

$$f_j(x) = 2^{j\frac{n}{r}} \, \psi(2^j x) \, , \quad j \in \mathbb{N} \, , \tag{15.11}$$

are atoms in all spaces $B^s_{pq}(\mathbb{R}^n)$ and at least in those spaces $F^s_{pq}(\mathbb{R}^n)$ where no moment conditions are needed, say $q \ge 1$ (ignoring constants, which may be chosen independent of j). Again by (12.37),

$$\mathcal{E}_G B^s_{pq}(d 2^{-jn}) \ge c \, f_j^*(d 2^{-jn}) \sim 2^{\frac{jn}{r}} \, , \quad j \in \mathbb{N} \, , \tag{15.12}$$

for some $d > 0$ and $c > 0$. Together with (15.8) we obtain

$$\mathcal{E}_G B^s_{pq}(t) = t^{-\frac{1}{r}} \, , \quad 0 < t < \varepsilon \, . \tag{15.13}$$

Similarly for the F-spaces as far as they are covered. If $q > 0$ is small then the moment conditions needed for the atoms can be incorporated in the same way as in Step 5 of the proof of Theorem 13.2. Then we have also

$$\mathcal{E}_G F^s_{pq}(t) = t^{-\frac{1}{r}} \, , \quad 0 < t < \varepsilon \, , \tag{15.14}$$

without any restriction for q.

Step 3 It remains to prove that q and p in (15.5) and (15.6), respectively, are the correct numbers. Since we have already (15.7) and (15.9) we must show that q and p, respectively, cannot be improved by smaller numbers. Let $v < q$ and let

$$\left(\int_0^\varepsilon \left(t^{\frac{1}{r}} f^*(t) \right)^v \frac{dt}{t} \right)^{\frac{1}{v}} \le c \, \| f \, | B^s_{pq}(\mathbb{R}^n) \| \tag{15.15}$$

for some $c > 0$ and all $f \in B^s_{pq}(\mathbb{R}^n)$. Let

$$f(x) = \sum_{j=1}^J 2^{j\frac{n}{r}} \, \psi(2^j x - x^0) \tag{15.16}$$

with $x^0 \in \mathbb{R}^n$. If $|x^0|$ is large then the supports of the atoms in (15.16) are disjoint. It follows for some $d > 0$,

$$f^*(d2^{-jn}) \sim 2^{j\frac{n}{r}} \quad \text{where} \quad j = 1, \ldots, J\,. \tag{15.17}$$

We insert (15.17) in (15.15). Since (15.16) is an atomic decomposition we get

$$\left(\sum_{j=1}^{J} 1\right)^{\frac{1}{v}} \leq c_0 \, \|f \,|B^s_{pq}(\mathbb{R}^n)\| \leq c_1 \left(\sum_{j=1}^{J} 1\right)^{\frac{1}{q}}, \tag{15.18}$$

where $c_0 > 0$ and $c_1 > 0$ are independent of J. But this is a contradiction. In case of the F-spaces we assume $v < p$ and that we have (15.15) with F^s_{pq} in place of B^s_{pq}. Let first q be large, say $q \geq 1$, such that no moment conditions in the atomic decomposition (15.16) are needed. We apply the considerations in connection with the proof of (13.40). Then we get (15.18) with p in place of q on the right-hand side. We have again a contradiction. Finally if moment conditions are needed, then one has to modify the above constructions as indicated in Step 5 of the proof of Theorem 13.2.

15.3 Inequalities

The above theorem covers all cases (15.3). It excludes borderline situations as described in 15.1. Parallel to 13.3 we explain also in the sub-critical case considered now, which is hidden in the above theorem. We use Example 1 in 12.4, Definition 12.8 and Proposition 12.10. Let $0 < \varepsilon < 1$.

15.3(i) *The B-spaces* Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0, \varepsilon]$. Let $0 < u \leq \infty$. Let p, q, s be given by (15.4). Then

$$\left(\int_0^{\varepsilon} \left(\varkappa(t)\, t^{\frac{1}{r}}\, f^*(t)\right)^u \frac{dt}{t}\right)^{\frac{1}{u}} \leq c \, \|f \,|B^s_{pq}(\mathbb{R}^n)\| \tag{15.19}$$

for some $c > 0$ and all $f \in B^s_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded and $q \leq u \leq \infty$, with the modification

$$\sup_{0<t<\varepsilon} \varkappa(t)\, t^{\frac{1}{r}}\, f^*(t) \leq c \, \|f \,|B^s_{pq}(\mathbb{R}^n)\| \tag{15.20}$$

if $u = \infty$. Furthermore, (15.7) deals with the two end-point cases according to (12.26). Let $\varkappa$ be an arbitrary positive function on $(0, \varepsilon]$. Then (15.20) holds if, and only if, $\varkappa$ is bounded.

15.3(ii) *The F-spaces* Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0,\varepsilon]$. Let $0 < u \le \infty$. Let p, q, s be given by (15.4). Then

$$\left(\int_0^\varepsilon \left(\varkappa(t)\, t^{\frac{1}{r}}\, f^*(t) \right)^u \frac{dt}{t} \right)^{\frac{1}{u}} \le c\, \| f \,| F^s_{pq}(\mathbb{R}^n) \| \tag{15.21}$$

for some $c > 0$ and all $f \in F^s_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded and $p \le u \le \infty$ (modified by (15.20) with F^s_{pq} in place of B^s_{pq} when $u = \infty$). Also the other assertions for the B-spaces after (15.20) have obvious counterparts, in particular the two end-point cases (15.9) according to (12.26).

The Lorentz spaces $L_{ru}(I_\varepsilon)$ were introduced in 11.6(i). The above theorem and the explanations just given can be reformulated in terms of natural and sharp embeddings of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with (15.4) into $L_{ru}(I_\varepsilon)$. We complement these assertions by looking at corresponding *optimal* embeddings into Zygmund spaces $L_r(\log L)_a(I_\varepsilon)$ according to 11.6(ii). By (11.46) the original definition (11.44) can be reformulated in terms of rearrangement. *Optimal* means here that for given r in (15.4) and in (11.46), (11.54), one asks for all numbers a for which we have the desired embedding, again formulated in terms of inequalities.

15.4 Corollary

Let p, q, s be given by (15.4) and let $0 < \varepsilon < 1$.

(i) *Then*

$$\left(\int_0^\varepsilon \left(|\log t|^a\, f^*(t) \right)^r dt \right)^{\frac{1}{r}} \le c\, \| f \,| F^s_{pq}(\mathbb{R}^n) \| \tag{15.22}$$

for some $c > 0$ and all $f \in F^s_{pq}(\mathbb{R}^n)$ if, and only if, $a \le 0$.

(ii) *Let, in addition, $0 < q \le r$. Then*

$$\left(\int_0^\varepsilon \left(|\log t|^a\, f^*(t) \right)^r dt \right)^{\frac{1}{r}} \le c\, \| f \,| B^s_{pq}(\mathbb{R}^n) \| \tag{15.23}$$

for some $c > 0$ and all $f \in B^s_{pq}(\mathbb{R}^n)$ if, and only if, $a \le 0$.

(iii) *Let, in addition, $r < q \le \infty$. Then (15.23) holds if, and only if, $a < \frac{1}{q} - \frac{1}{r}$.*

Proof *Step 1* If $a = 0$, then (15.22) and (15.23) with $q \le r$ follow from Theorem 11.4(i). Let $r < q$. Then (15.23) with $a < \frac{1}{q} - \frac{1}{r}$ is a consequence of (11.66) and (15.7). This covers all if-parts.

Step 2 It remains to prove the only-if-parts of the corollary. First we insert f_j, given by (15.11) in (15.22) and (15.23). By the equivalence in (15.12) we get

$$j^a \le c \quad \text{for all} \quad j \in \mathbb{N} \quad \text{and some} \quad c > 0\,. \tag{15.24}$$

Hence $a \le 0$. This completes the proof of (i) and (ii). As for (iii) we modify (15.16) by

$$f(x) = \sum_{j=2}^{\infty} b_j\, 2^{j\frac{n}{r}}\, \psi(2^j x - x^0) \quad \text{with} \quad b_j = j^{-\frac{1}{q}}\, (\log j)^{-\frac{1}{r}}\,. \tag{15.25}$$

This is again an atomic decomposition and we have

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\| \le c \left(\sum_{j=2}^{\infty} b_j^q \right)^{\frac{1}{q}} < \infty\,. \tag{15.26}$$

On the other hand, (15.17) must be modified by

$$f^*(d\,2^{-jn}) \sim j^{-\frac{1}{q}}\, (\log j)^{-\frac{1}{r}}\, 2^{j\frac{n}{r}}\,, \quad j = 2, 3, \ldots\,. \tag{15.27}$$

Inserted in the left-hand side of (15.23) with $a = \frac{1}{q} - \frac{1}{r}$ we obtain

$$\int_0^{\varepsilon} (|\log t|^a\, f^*(t))^r\, dt \sim \sum_{j=2}^{\infty} j^{-1}\, (\log j)^{-1} = \infty\,. \tag{15.28}$$

This proves the only-if-part of (iii).

15.5 Further references

Embeddings and related inequalities in sub-critical cases have a long and rich history. We tried to collect the relevant papers in 11.8(i), with respect to the Lorentz spaces L_{rq}, and in 11.8(iii), with respect to the Zygmund spaces $L_r(\log L)_a$. This will not be repeated here. In connection with the critical case we gave some additional references in 13.5, which apply at least partly also to the sub-critical case considered here. In particular, Ju. V. Netrusov anticipated in [Net87a], and also in [Net89a], in a somewhat different context, the concept of envelope functions and optimal embeddings in rearrangement-invariant spaces. More recent (and independent of each other and of Netrusov's work) treatments have been given in [CwP98] and in [EKP00] (restricted to Sobolev spaces, in contrast to Netrusov, who considered F^s_{pq} spaces). As for related capacity estimates we refer again to [Maz85] and to the recent paper [Sic99], where one finds also further references. This section is based on [Tri99d] and might be considered as an improved and extended version.

16 Hardy inequalities

16.1 Introduction

In this book we dealt so far several times with Hardy inequalities. But first we wish to mention that the whole story began with Hardy's note [Had28] and the famous Theorem 330 in [HLP52], p. 245 (in small print). As a consequence (ignoring constants) one gets the following assertion: *Let* $1 < p < \infty$ *and* $m \in \mathbb{N}$. *There is a number* $c > 0$ *such that*

$$\int_{\mathbb{R}} |t|^{-mp}\, |u(t)|^p \, dt \le c \int_{\mathbb{R}} \left| \frac{d^m u(t)}{dt^m} \right|^p \, dt \tag{16.1}$$

for all

$$u \in S(\mathbb{R}) \quad \textit{with} \quad \frac{d^j u}{dt^j}(0) = 0 \quad \textit{for} \quad j = 0, \ldots, m-1 \, .$$

In the years after, and especially in the last decades, hundreds of papers and dozens of books have appeared dealing with numerous variations of inequalities of this type. The reader may consult [OpK90] and the references given there. As far as this book is concerned we refer to 5.7–5.12, making clear how different natural inequalities for F-spaces and B-spaces might be. Of special interest in this section is the following consequence of the previous results. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. Let

$$\Gamma = \partial\Omega \, , \quad D(x) = \mathit{dist}\,(x, \Gamma) = \inf_{y \in \Gamma} |x - y| \, , \quad x \in \mathbb{R}^n \, , \tag{16.2}$$

be the distance to Γ and, for $\varepsilon > 0$,

$$\Gamma_\varepsilon = \{ x \in \mathbb{R}^n \, : \, D(x) < \varepsilon \} \tag{16.3}$$

be a neighbourhood of Γ. Let

$$0 < p < \infty \, , \quad 0 < q \le \infty \, , \quad n \left(\frac{1}{p} - 1 \right)_+ = \sigma_p < s < \frac{1}{p} \, . \tag{16.4}$$

There is a number $c > 0$ such that

$$\int_{\Gamma_\varepsilon} D^{-sp}(x)\, |f(x)|^p \, dx \le c \, \| f \, | F^s_{pq}(\mathbb{R}^n) \|^p \tag{16.5}$$

for all $f \in F^s_{pq}(\mathbb{R}^n)$. This follows from (5.104), (5.105). There one finds also the necessary explanations and further assertions of this type. This measures how singular a function f belonging to $F^s_{pq}(\mathbb{R}^n)$ near $\Gamma = \partial\Omega$ can be. Let Γ be an arbitrary, say, compact set on $\mathbb{R}^n$. Of interest is the behaviour of functions f belonging to a given space $F^s_{pq}(\mathbb{R}^n)$ or $B^s_{pq}(\mathbb{R}^n)$ near or at Γ. There are two different, but closely related aspects: traces on Γ and Hardy inequalities of type (16.5). If Γ is smooth (maybe $\Gamma = \partial\Omega$ as above) then the trace problem is more or less settled and treated in detail in most of the books mentioned in 1.1. Specific references and rather final formulations and proofs (excluding borderline cases) may be found in [Triβ], 2.7.2, 3.3.3, pp. 132, 200, and [Triγ], 4.4.2, 4.4.3, pp. 213–221. Sophisticated borderline cases have been treated recently in [Joh00] and [FJS00]. If Γ is an irregular, say, compact, set in $\mathbb{R}^n$ then the situation is different. We considered this problem in some detail in Section 9 and refer in particular to Theorems 9.3, 9.9, 9.21, and 9.33. There we quoted also the relevant literature. Special attention has been paid to d-sets. Of interest here is Proposition 9.13. One aim in the present section is to complement these trace assertions by a discussion about related Hardy inequalities. We outlined our intentions at the end of Section 10 and added also a warning concerning the outcome. As stated there we are interested with some preference in $\Gamma = \{0\}$, where we get sharp results.

But we look also at more general sets. In principle the method to get, for example (10.26) or (10.28), is quite simple. We use (10.23) as a vehicle to reduce Hardy inequalities to Theorems 13.2 and 15.2, and the related inequalities in 13.3 and 15.3, respectively. A few points should be mentioned.

First, in case of $\Gamma = \{0\}$ we deal both with F^s_{pq}-spaces and B^s_{pq}-spaces, although really satisfactory inequalities for the B-spaces look somewhat different. We refer to (5.77). This may justify that we later on concentrate on the F-spaces.

Secondly, if Γ is the boundary of a C^∞ domain or (part of) a hyper-plane and if one deals with the full spaces $F^s_{pq}(\mathbb{R}^n)$ or $B^s_{pq}(\mathbb{R}^n)$ (and not with appropriate subspaces) then there seems to be a clear distinction between those spaces having traces on Γ and those spaces with substantial Hardy inequalities. But if Γ is irregular the situation might be different. As examples, (16.5) may serve on the one hand and (10.26) on the other hand. In case of irregular compact sets Γ we have no final assertions, and the later parts of this section might be considered as a discussion of how to shed light on the possibly tricky interplay between Hardy inequalities, the geometry of irregular sets Γ and related measures. This justifies our restriction to examples, mostly $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$.

We complement (16.3) by

$$K_\varepsilon = \{x \in \mathbb{R}^n \,:\, |x| < \varepsilon\}\,. \tag{16.6}$$

16.2 Theorem

(Critical case)

Let $0 < \varepsilon < 1$ *and let* $\varkappa(t)$ *be a positive monotonically decreasing function on* $(0, \varepsilon]$.

(i) *Let*

$$1 < p < \infty \quad \textit{and} \quad 0 < q \leq \infty \,. \tag{16.7}$$

Then

$$\int_{K_\varepsilon} \left| \frac{\varkappa(|x|)\, f(x)}{\log |x|} \right|^p \frac{dx}{|x|^n} \leq c \, \left\| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \right\|^p \tag{16.8}$$

for some $c > 0$ *and all* $f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ *if, and only if,* $\varkappa$ *is bounded.*

(ii) *Let*

$$0 < p < \infty \quad \textit{and} \quad 1 < q < \infty \,. \tag{16.9}$$

Then

$$\int_{K_\varepsilon} \left| \frac{\varkappa(|x|)\, f(x)}{\log |x|} \right|^q \frac{dx}{|x|^n} \leq c \, \left\| f \,|B^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \right\|^q \tag{16.10}$$

for some $c > 0$ *and all* $f \in B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ *if, and only if,* $\varkappa$ *is bounded.*

Proof *Step 1* We prove (16.8) with $\varkappa = 1$. We may assume that the $\varepsilon > 0$ is so small that

$$a(x) = \Big|\, |x|^{\frac{n}{p}} \, \log |x| \,\Big|^{-1} \quad \text{is monotone in } K_\varepsilon$$

and, hence,

$$a^*(t) \sim t^{-\frac{1}{p}} |\log t|^{-1} \tag{16.11}$$

if $0 < t < c\varepsilon$ for some $c > 0$. Recall that $a^*(t)$ is the measure-preserving rearrangement of $a(x)$. Then (16.11) follows from the behaviour of $a(x)$ and $a^*(t)$ at $|x| \sim 2^{-\frac{j}{n}}$ and $t \sim 2^{-j}$, respectively, where $j \in \mathbb{N}$ is sufficiently large. We obtain

$$\begin{aligned} \int_{K_\varepsilon} \left| \frac{f(x)}{\log |x|} \right|^p \frac{dx}{|x|^n} &= \int_{K_\varepsilon} a(x)^p \, |f(x)|^p \, dx = \int_0^{c\varepsilon^n} (af)^{*p}(t) \, dt \\ &\leq \int_0^{c\varepsilon^n} a^{*p}(t) \, f^{*p}(t) \, dt \sim \int_0^{c\varepsilon^n} \left| \frac{f^*(t)}{\log t} \right|^p \frac{dt}{t} \leq c' \, \left\| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \right\|^p . \end{aligned} \tag{16.12}$$

The first inequality is a well-known property of rearrangement and may be found in [BeS88], p. 44. It goes back to [HLP52] (first edition 1934), Theorems 368 and 378. The last inequality comes from Theorem 13.2 or, more explicitly, from (13.62). Similarly one proves (16.10) with $\varkappa = 1$, where one has to use (13.57).

Step 2 We prove that $\varkappa$ in (16.8) must be bounded. Let $f(x)$ be a positive monotonically decreasing function in K_ε in radial directions. Since $\varkappa$ is also assumed to be monotone it follows in analogy to (13.62) by (12.28) and (16.8),

$$\begin{aligned} \sup_{0<t<c\varepsilon} \frac{\varkappa(t)\, f^*(t)}{|\log t|^{\frac{1}{p'}}} &\le c_1 \left(\int_0^{c\varepsilon} \left(\frac{\varkappa(t)\, f^*(t)}{|\log t|} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \\ &= c_1 \left(\int_{K_\varepsilon} \left| \frac{\varkappa(|x|)\, f(x)}{\log |x|} \right|^p \frac{dx}{|x|^n} \right)^{\frac{1}{p}} \le c_2 \left\| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \right\| . \end{aligned} \tag{16.13}$$

Let $q \ge 1$. We insert f_J with (13.50), (13.51). This proves that $\varkappa$ must be bounded. If $q > 0$ is small, then one has to modify f_J as indicated in Steps 5 and 6 of the proof of Theorem 13.2. But this does not influence the above argument. Similarly one proves that $\varkappa$ in (16.10) must be bounded.

16.3 Theorem

(Sub-critical case)

Let $\varepsilon > 0$ and let $\varkappa(t)$ be a positive monotonically decreasing function on $(0, \varepsilon]$. Let

$$s > 0 \quad \text{and} \quad s - \frac{n}{p} = -\frac{n}{r} \quad \text{with} \quad 1 < r < \infty \tag{16.14}$$

(the dashed line in Fig. 10.1).

(i) *Let $0 < q \le \infty$. Then*

$$\int_{K_\varepsilon} \left| \varkappa(|x|)\, |x|^{\frac{n}{r}}\, f(x) \right|^p \frac{dx}{|x|^n} \le c\, \|f \,|F^s_{pq}(\mathbb{R}^n)\|^p \tag{16.15}$$

for some $c > 0$ and all $f \in F^s_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded.

(ii) *Let $0 < q \le r$. Then*

$$\int_{K_\varepsilon} \left| \varkappa(|x|)\, |x|^{\frac{n}{r}}\, f(x) \right|^q \frac{dx}{|x|^n} \le c\, \|f \,|B^s_{pq}(\mathbb{R}^n)\|^q \tag{16.16}$$

for some $c > 0$ and all $f \in B^s_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded.

Proof Let $\alpha \geq 0$. Then

$$\left(|x|^{-\alpha n}\right)^* (t) \sim t^{-\alpha} \quad \text{where} \quad t > 0\,. \tag{16.17}$$

This can be applied to the left-hand sides of (16.15) and (16.16) with $\alpha = 1 - \frac{p}{r} > 0$ and $\alpha = 1 - \frac{q}{r} \geq 0$, respectively. Then (16.15) and (16.16) with $\varkappa = 1$ follow from the counterpart of (16.12) on the one hand, and (15.21) with $u = p$ and (15.19) with $u = q$, respectively, on the other hand. If one inserts $f_j(x)$ given by (15.11) (with the indicated modification for the F-spaces when $q > 0$ is small) in (16.15) and (16.16), then it follows that $\varkappa$ must be bounded.

16.4 Comments and references

First we look at the sub-critical case. Using (16.14), the inequality (16.15) can be reformulated as

$$\int_{K_\varepsilon} \frac{|f(x)|^p}{|x|^{sp}}\, dx \leq c\, \| f \,| F^s_{pq}(\mathbb{R}^n)\|^p\,, \tag{16.18}$$

where again $0 < q \leq \infty$. As mentioned in 1.2 if $q = 2$ and $1 < p < \infty$ then $F^s_{pq}(\mathbb{R}^n)$ are the Sobolev spaces $H^s_p(\mathbb{R}^n)$ with the classical Sobolev spaces $W^s_p(\mathbb{R}^n)$ as a subclass if, in addition, $s \in \mathbb{N}$. Then inequalities of type (16.18) are known although explicit formulations are rare in the literature (especially in higher dimensions). But everything is included in the extensively treated problem of embeddings of Sobolev spaces in weighted L_p spaces, or more generally in L_p spaces with respect to Radon measures in $\mathbb{R}^n$. We dealt in Section 9 with questions of this type in the different context of traces. But the references given there apply also to the above case, in particular [Maz85], [AdH96], [Ver99]. Switching to general spaces $F^s_{pq}(\mathbb{R}^n)$ and $B^s_{pq}(\mathbb{R}^n)$ the situation is different. The first explicit inequality of type (16.18) for the general spaces $F^s_{pq}(\mathbb{R}^n)$ may be found in [Triβ], 2.8.6, p. 155, which covers also the one-dimensional version of (16.5). Such inequalities also have anisotropic counterparts, at least for anisotropic spaces of type $B^s_{pp}(\mathbb{R}^n)$ and $H^s_p(\mathbb{R}^n)$. We refer to [ST87], 4.3, pp. 202–209, and the literature mentioned there. If $p \neq q$ then inequalities of type (16.16) are not optimally adapted to the spaces $B^s_{pq}(\mathbb{R}^n)$. More natural inequalities may be found in [Triα], p. 319, and more general ones in [Tri99b]. But they do not fit in our scheme here. The above Theorem 16.3 is a modification of [Tri99d]. There one finds also additional discussions concerning the interrelation of rearrangement and Hardy inequalities. In the critical case as considered in Theorem 16.2 there are only very few papers. Restricted to classical Sobolev spaces $W_2^{\frac{n}{2}}(\mathbb{R}^n)$ inequalities of type (16.8) with log-terms may be found in [EgK90], Lemma 8, p. 155, and in [Sol94]. Restricted to $H^s_p(\mathbb{R}^n)$ and $B^s_{pp}(\mathbb{R}^n)$, Theorem 16.2 has been proved in [EdT99b].

We reduced the inequalities in the Theorems 16.2 and 16.3 to 13.2, 13.3 and 15.2, 15.3, respectively. It is clear that all the other inequalities mentioned there in 13.3 and 15.3 produce also sharp Hardy inequalities: One has to modify (16.12). Another possibility is to replace $\Gamma = \{0\}$ by more general sets. In principle this does not cause much trouble. But it is unclear to what extent or for which Γ one gets sharp and natural inequalities. We formulate a few results and complement them by some discussions. It comes out that under some additional geometrical restrictions the outcome is far from being optimal. In other words, the main aim of the rest of this section is to shed light on these problems. This may also justify that we restrict our attention to the critical case and in particular to $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$. The first candidates beyond $\Gamma = \{0\}$ and, maybe, compact smooth surfaces are d-sets. Let $0 < d < n$. Then a compact set Γ in $\mathbb{R}^n$ is called a d-set if there are a Borel measure μ in $\mathbb{R}^n$ and two positive numbers c_1 and c_2 such that $supp\,\mu = \Gamma$ and

$$c_1 t^d \le \mu(B(\gamma,t)) \le c_2 t^d \quad \text{for all} \quad 0 < t < 1\,, \tag{16.19}$$

and $\gamma \in \Gamma$, where $B(\gamma, t)$ is a ball centred at $\gamma \in \Gamma$ and of radius t. Further details and references may be found in 9.12.

16.5 Proposition

Let $0 < d < n$ and let Γ be a compact d-set in $\mathbb{R}^n$. Let

$$D(x) = dist(x, \Gamma)\,, \quad x \in \mathbb{R}^n\,, \tag{16.20}$$

be the distance of $x \in \mathbb{R}^n$ to Γ. Let p, q be given by (16.7). Let $0 < \varepsilon < 1$ and let Γ_ε be an ε-neighbourhood of Γ as in (16.3). Then

$$\int_{\Gamma_\varepsilon} \left| \frac{f(x)}{\log D(x)} \right|^p \frac{dx}{D^{n-d}(x)} \le c \left\| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n) \right\|^p \tag{16.21}$$

for some $c > 0$ and all $f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$.

Proof Let

$$\Gamma_j = \left\{ x \in \mathbb{R}^n \,:\, 2^{-\frac{j+1}{n-d}} < D(x) \le 2^{-\frac{j}{n-d}} \right\}, \quad j \ge J\,. \tag{16.22}$$

Then $vol\,\Gamma_j \sim 2^{-j}$. With

$$a(x) = |\log D(x)|^{-p}\, D^{d-n}(x)$$

one gets

$$a^*(t) \sim t^{-1}\, |\log t|^{-p}\,, \quad 0 < t < \delta < 1\,. \tag{16.23}$$

Now we obtain (16.21) in the same way as in (16.12).

16.6 Discussion

Let Γ be a hyper-plane in $\mathbb{R}^n$, say,

$$\Gamma = \mathbb{R}^{n-1} = \{x = (x', x_n) \in \mathbb{R}^n \, , \; x_n = 0\} \, , \tag{16.24}$$

with $x' \in \mathbb{R}^{n-1}$ and $n \geq 2$. Let p, q be given by (16.7). For fixed $x' \in \mathbb{R}^{n-1}$ we use the one-dimensional version of (16.8) and obtain for $0 < \varepsilon < 1$,

$$\int\limits_{-\varepsilon}^{\varepsilon} \left| \frac{f(x', x_n)}{\log |x_n|} \right|^p \frac{dx}{|x_n|} \leq c \left\| f(x', \cdot) \, | F^{\frac{1}{p}}_{pq}(\mathbb{R}) \right\|^p , \quad x' \in \mathbb{R}^{n-1} \, . \tag{16.25}$$

If $1 < p < \infty$ and $1 \leq q \leq \infty$, then by Theorem 4.4, the spaces $F^{\frac{1}{p}}_{pq}(\mathbb{R}^n)$ have the Fubini property. Together with (16.25) one obtains

$$\int_{\mathbb{R}^{n-1}_\varepsilon} \left| \frac{f(x)}{\log |x_n|} \right|^p \frac{dx}{|x_n|} \leq c \left\| f \, | F^{\frac{1}{p}}_{pq}(\mathbb{R}^n) \right\|^p \tag{16.26}$$

for some $c > 0$ and all $f \in F^{\frac{1}{p}}_{pq}(\mathbb{R}^n)$, where $\mathbb{R}^{n-1}_\varepsilon$ is an ε-neighbourhood of $\mathbb{R}^{n-1}$ given by (16.24) according to (16.3). Since for fixed p with $1 < p < \infty$, the spaces $F^{\frac{1}{p}}_{pq}(\mathbb{R}^n)$ are monotone with respect to q, the inequality (16.26) holds for all p, q with (16.7). Even the $\varkappa$-sharpness of Theorem 16.2 extends from the one-dimensional case to the above situation:

Let $\varkappa(t)$ be a positive monotonically decreasing function on $(0, \varepsilon]$, where again $0 < \varepsilon < 1$, let p, q be given by (16.7). Then

$$\int_{\mathbb{R}^{n-1}_\varepsilon} \left| \frac{\varkappa(|x_n|) \, f(x)}{\log |x_n|} \right|^p \frac{dx}{|x_n|} \leq c \left\| f \, | F^{\frac{1}{p}}_{pq}(\mathbb{R}^n) \right\|^p \tag{16.27}$$

for some $c > 0$ and all $f \in F^{\frac{1}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded.

The if-part is covered by (16.26). We outline how the only-if-part can be proved by modification of previous arguments. Let

$$S_j = \left\{ x \in \mathbb{R}^n \, : \, |x'| < 1 \, , \; |x_n| < 2^{-j} \right\} , \quad j \in \mathbb{N} \, . \tag{16.28}$$

We modify (13.50) by

$$f_J(x) = J^{-\frac{1}{p}} \sum_{j=1}^{J} \sum_{l=1}^{2^{j(n-1)}} 2^{-j\frac{n-1}{p}} \left[2^{j\frac{n-1}{p}} \psi \left(2^{j-1}(x - x^{j,l}) \right) \right] , \tag{16.29}$$

where $[\cdots]$ are correctly normalized atoms or quarks in $F^{\frac{1}{p}}_{pq}(\mathbb{R}^n)$ (we refer to (2.16)) and where $x^{j,l}$ stands for suitable lattice-points. (We assume, say,

$q \geq 1$, such that no moment conditions are needed. The necessary additional modifications if $q > 0$ is small have been indicated in Step 5 of the proof of Theorem 13.2.) We have a counterpart of (13.51) with $n = 1$ in the first equivalence and with $F^{\frac{1}{p}}_{pq}(\mathbb{R}^n)$ in place of $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ in the second equivalence: We note that the arguments in connection with and after (13.40) with a reference to 2.15 apply also to (16.29). Then the desired $\varkappa$-sharpness follows as in Step 2 of the proof of Theorem 16.2. If one compares the sharp assertion (16.26) with (16.21) where now $d = n - 1$, then it is quite clear that in this special case, (16.21) does not say very much. Even worse: Since for any $\delta > 0$ the space $F^{\frac{1}{p}+\delta}_{pq}(-\varepsilon, \varepsilon)$ is continuously embedded in $C(-\varepsilon, \varepsilon)$ (in obvious notation and with a reference to, say, [Triβ], 2.7.1) one has an immediate and rather obvious counterpart of (16.26) with $F^{\frac{1}{p}+\delta}_{pq}(\mathbb{R}^n)$ in place of $F^{\frac{1}{p}}_{pq}(\mathbb{R}^n)$ and with an arbitrary positive integrable function $a(x_n)$ in place of $|x_n|^{-1}|\log|x_n||^{-p}$. Then, in this special case, (16.21) with $n \geq 2$ is obvious. On the other hand, the above arguments depend on the special structure of Γ in (16.24) and on the possibility to apply the Fubini Theorem 4.4. But this is not the case if Γ is a general d-set or an arbitrary fractal. In other words, the problem arises under which geometrical conditions for Γ the inequality (16.21) is substantial and sharp. Finally one can use (16.26) to complement our considerations in 5.23 and also of (16.5). We formulate the outcome.

16.7 Corollary

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let Γ, Γ_ε, and $D(x)$ be given by (16.2), (16.3). Let $0 < \varepsilon < 1$ and let $\varkappa(t)$ be a positive monotonically decreasing function on $(0, \varepsilon]$. Let p, q be given by (16.7). Then

$$\int_{\Gamma_\varepsilon} \left| \frac{\varkappa(D(x))\, f(x)}{\log D(x)} \right|^p \frac{dx}{D(x)} \leq c \left\| f \,|F^{\frac{1}{p}}_{pq}(\mathbb{R}^n) \right\|^p \tag{16.30}$$

for some $c > 0$ and all $f \in F^{\frac{1}{p}}_{pq}(\mathbb{R}^n)$ if, and only if, $\varkappa$ is bounded.

Proof This follows from (16.27) and standard localization arguments.

16.8 Remark

If p, q, s are given by (5.104), then we have the sharp Hardy inequality (5.105). If now p, q are restricted by (16.7) and $s = \frac{1}{p}$, then

$$\int_{\Omega} \left| \frac{f(x)}{1 + |\log D(x)|} \right|^p \frac{dx}{D(x)} \leq c \left\| f \,|F^{\frac{1}{p}}_{pq}(\Omega) \right\|^p. \tag{16.31}$$

This is an immediate consequence of (16.30).

16.9 Proposition

Let μ be a finite Radon measure in $\mathbb{R}^n$ and let $\Gamma = \operatorname{supp}\mu$ be compact. Let p, q be given by (16.7), $0 < \varepsilon < 1$, and

$$I_{p,\varepsilon}(x) = \int\limits_{B(x,\varepsilon)} \frac{\mu(d\gamma)}{|x-\gamma|^n |\log|x-y||^p} \le \infty, \quad x \in \mathbb{R}^n, \tag{16.32}$$

where $B(x,\varepsilon)$ is a ball centred at $x \in \mathbb{R}^n$ and of radius ε. Then

$$\int_{\mathbb{R}^n} I_{p,\varepsilon}(x)\, |f(x)|^p \, dx \le c \left\| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\right\|^p \tag{16.33}$$

for some $c > 0$ and all $f \in F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$.

Proof Let χ_ε be the characteristic function of K_ε given by (16.6). Let $\gamma \in \Gamma$. Then it follows by (16.8) that

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{\log|x-\gamma|} \right|^p \chi_\varepsilon(x-\gamma) \frac{dx}{|x-\gamma|^n} \le c \left\| f \,|F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)\right\|^p . \tag{16.34}$$

Integration with respect to μ and application of Fubini's theorem results in (16.33).

16.10 Remark

As mentioned at the end of 16.4 the Propositions 16.5 and 16.9 are far from final. This is also clear from the discussion in 16.6 and the more satisfactory assertions in 16.7 and 16.8. We mainly wanted to make clear that there might be a sophisticated interplay between the geometry of irregular fractal sets Γ and the singularity behaviour of functions belonging to spaces $F^s_{pq}(\mathbb{R}^n)$ and $B^s_{pq}(\mathbb{R}^n)$ near Γ. We restricted ourselves in the course of this discussion to the critical case extending Theorem 16.2. But of course one can deal in the same way with the sub-critical case as considered in Theorem 16.3.

17 Complements

17.1 Green's functions as envelope functions

Looking at (13.7) or (13.9) one may ask whether there are functions f belonging to $B^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ or $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$ such that $f^*(t)$ is equivalent to $|\log t|^{\frac{1}{q'}}$ or $|\log t|^{\frac{1}{p'}}$, respectively. If $q < \infty$ in (13.7), then it follows from (13.57) that this is impossible since in such a case the middle term diverges. Because always $p < \infty$, one has by (13.62) a corresponding argument for the spaces $F^{\frac{n}{p}}_{pq}(\mathbb{R}^n)$. Similarly for the sub-critical case according to Theorem 15.2 and (15.7). Corresponding questions can also be asked for the super-critical case considered in Theorem 14.2. If $q = \infty$ then the situation is different. We deal first with the critical case as covered by Theorem 13.2 and by 13.3. Let δ be the usual δ-distribution in $\mathbb{R}^n$ with the origin as the off-point. Then

$$\delta \in B^{n(\frac{1}{p}-1)}_{p\infty}(\mathbb{R}^n) \quad \text{where} \quad 0 < p \le \infty . \tag{17.1}$$

This is well known and also an easy consequence of (2.37). Let again $-\Delta$ be the Laplacian in $\mathbb{R}^n$. By well-known lifting properties of $-\Delta + id$ it follows that

$$G = (id - \Delta)^{-\frac{n}{2}} \delta \in B^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n) , \quad 0 < p \le \infty , \tag{17.2}$$

where G might be considered as the Green's function of the fractional power $(id - \Delta)^{\frac{n}{2}}$ of $id - \Delta$. We claim that $G(x)$ is a C^∞ function in $\mathbb{R}^n \backslash \{0\}$ which decays exponentially if $|x| \to \infty$ and

$$G(x) \sim |\log |x|| \quad \text{if} \quad |x| < \varepsilon \quad \text{and hence} \quad G^*(t) \sim |\log t| \tag{17.3}$$

if $0 < t < \varepsilon < 1$. Hence $G(x)$ is an extremal function for $B^{\frac{n}{p}}_{p\infty}(\mathbb{R}^n)$. By Definition 12.8 and (12.60), and in agreement with (13.7), (13.58) we have

$$\mathcal{E}_G B^{\frac{n}{p}}_{p\infty}(t) = G^*(t) \sim |\log t| , \quad 0 < t < \varepsilon < 1 . \tag{17.4}$$

We outline a proof. Let

$$g(x) = \int_0^\infty e^{-t-\frac{|x|^2}{4t}} \, \frac{dt}{t} , \quad x \in \mathbb{R}^n , \quad x \ne 0 . \tag{17.5}$$

By well-known properties of the Fourier transform of $e^{-\frac{|x|^2}{2}}$ and with respect to dilations $x \to cx$, $c \ne 0$, e.g., [Tri92], pp. 100/101, it follows that

$$\begin{aligned} (Fg)(\xi) &= \int_0^\infty e^{-t} \, F\left(e^{-\frac{|x|^2}{4t}}\right)(\xi) \, \frac{dt}{t} \\ &= \int_0^\infty (2t)^{\frac{n}{2}} \, e^{-t} \, e^{-t|\xi|^2} \, \frac{dt}{t} = c\,(1 + |\xi|^2)^{-\frac{n}{2}} \end{aligned} \tag{17.6}$$

for some $c > 0$. Here we used the Fubini theorem. This is possible since an integration over $\mathbb{R}^n$ in (17.5) results in a convergent integral over $\mathbb{R}^n \times [0, \infty)$. Application of the Fourier transform to G, introduced in (17.2), gives

$$G(x) = c \int_0^\infty e^{-t - \frac{|x|^2}{4t}} \, \frac{dt}{t} \,, \quad x \in \mathbb{R}^n \,, \quad x \neq 0 \,, \tag{17.7}$$

for some $c > 0$. We estimate $G(x)$. Let $|x| \geq 1$. We split the integral in (17.7) in

$$\int_0^{|x|} e^{-t - \frac{|x|^2}{4t}} \, \frac{dt}{t} \leq \int_0^{|x|} e^{- \frac{|x|^2}{4t}} \, \frac{dt}{t} = c \int_{\frac{|x|}{4}}^\infty e^{-\tau} \, \frac{d\tau}{\tau} \leq c' \, e^{-\frac{|x|}{4}} \tag{17.8}$$

and in

$$\int_{|x|}^\infty e^{-|x|} \, e^{-(\sqrt{t} - \frac{|x|}{2\sqrt{t}})^2} \, \frac{dt}{t} \leq \int_{|x|}^\infty e^{-|x|} \, e^{-tc} \, \frac{dt}{t} \leq c' \, e^{-c''|x|} \,. \tag{17.9}$$

This proves the exponential decay of $G(x)$ if $|x| \to \infty$. (Of course all constants in the above estimate are positive.) Let $|x| > 0$ be small. Then by (17.7) and $t = |x|^2 \, \tau$,

$$\begin{aligned} G(x) \quad &\sim \quad 1 + \int_0^1 e^{-t - \frac{|x|^2}{4t}} \, \frac{dt}{t} \sim 1 + \int_0^1 e^{- \frac{|x|^2}{4t}} \, \frac{dt}{t} \\ &\sim \quad 1 + \int_0^{|x|^{-2}} e^{-\frac{1}{4\tau}} \, \frac{d\tau}{\tau} \sim |\log |x|| \,. \end{aligned} \tag{17.10}$$

Hence by (17.2), the decay assertions, and (17.3) it follows that $G(x)$ materializes the envelope function in (13.7) with $q = \infty$. Furthermore by (17.2) and (13.87), the Green's function G belongs also to $F^0_{\infty q}(\mathbb{R}^n)$ for all $0 < q \leq \infty$, in particular

$$G \in bmo(\mathbb{R}^n) \,.$$

Finally we mention that (17.3) in case of $n = 2$ is essentially the well-known behaviour of the Green's function of the Laplacian in the plane.

The super-critical case can be reduced to the critical one as follows. Let g be given by (17.5). Then it comes out that

$$h(x) = x_1 \, g(x) \in B^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n) \,, \quad 0 < p \leq \infty \,, \tag{17.11}$$

and

$$|\nabla h|^*(t) \sim \widetilde{\omega}(h,t) \sim |\log t|\,, \quad 0 < t < \varepsilon\,, \tag{17.12}$$

where $\widetilde{\omega}(h,t)$ are the divided differences introduced in (12.72). In particular, h is an extremal function according to (14.27) and we have by Definition 12.14, (12.85) and (14.5) ,

$$\mathcal{E}_C B^{1+\frac{n}{p}}_{p\infty}(t) = \widetilde{\omega}(h,t) \sim |\log t|\,. \tag{17.13}$$

We outline a proof. By similar arguments as in (17.6) it follows that

$$h(x) = c\,\frac{\partial}{\partial x_1}\int_0^\infty e^{-t-\frac{|x|^2}{4t}}\,dt = c'\,\frac{\partial}{\partial x_1}\,(id-\Delta)^{-\frac{n}{2}-1}\,\delta \in B^{1+\frac{n}{p}}_{p\infty}(\mathbb{R}^n)\,. \tag{17.14}$$

Furthermore, as in (17.10) we obtain

$$\frac{\partial h}{\partial x_1} = g(x) + cx_1^2\int_0^\infty e^{-t-\frac{|x|^2}{4t}}\,\frac{dt}{t^2} \sim |\log|x|\,| + \frac{x_1^2}{|x|^2}\,. \tag{17.15}$$

This proves (17.12) (one needs only an estimate from below, since the estimate from above is covered by (14.27)).

Finally, formulas like (17.7) originate from heat kernels and their relations to Green's functions. We refer to [Dav89], 3.4, pp. 99–105, for details.

17.2 Further limiting embeddings

In all three cases, critical, super-critical, sub-critical, treated in Sections 13, 14, 15, respectively, we avoided borderline situations. This means in the critical and sub-critical case spaces with parameters as described in (13.2), (13.3) as far as they are covered by Theorem 11.2. In the super-critical case we excluded $p=\infty$ in the source spaces and concentrated on the target spaces exclusively on $s=1$, $p=\infty$ (the dotted line in Fig. 10.1). This might be justified by the history of the topic which we tried to collect in Theorem 11.7 and on which we commented in 11.8. It would be of interest to have a closer look at these omitted spaces. In the context of a more systematic study it might be even reasonable to modify the sub-division of the spaces covered by Theorem 11.2 in the above three distinguished cases as follows:

(i) to extend the *sub-critical* case as described in (10.5) to those spaces with $s=\sigma_p = n(\frac{1}{p}-1)_+$ covered by Theorem 11.2,

(ii) to extend the *critical* case as described in (10.6) to those spaces with $p = \infty$ covered by Theorem 11.2, and

(iii) to call all other spaces covered by Theorem 11.2 *super-critical*.

Any subdivision of the spaces covered by Theorem 11.2 depends on the admitted target spaces. This means in the sub-critical and critical case spaces with $s = 0$ according to Fig. 10.1 and in the super-critical case $s = 1$, $p = \infty$. A somewhat more general case of interest in connection with target spaces is given by

$$s = 1\,, \quad 1 < p \leq \infty\,, \tag{17.16}$$

(again in the understanding of Fig. 10.1). We give a brief description of the set-up in a slightly more general context. We use standard notation. Let $m \in \mathbb{N}$ and $1 < p \leq \infty$. Then

$$\omega_m(f,t)_p = \sup_{|h| \leq t} \|\Delta_h^m f \,| L_p(\mathbb{R}^n)\|\,, \quad 0 < t < \infty\,, \tag{17.17}$$

is the usual mth order modulus of continuity, [BeS88], 5.3, p. 332 or [DeL93], 2.7, p. 44. Here $\Delta_h^m f$ is given by (1.12). The classical Besov spaces

$$B^1_{pq}(\mathbb{R}^n) \quad \text{with} \quad 1 \leq p \leq \infty\,, \quad 1 \leq q \leq \infty\,,$$

described in 1.2(v), can be normed by

$$\|f \,| B^1_{pq}(\mathbb{R}^n)\| = \|f \,| L_p(\mathbb{R}^n)\| + \left(\int_0^\varepsilon \left(\frac{\omega_2(f,t)_p}{t} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \tag{17.18}$$

if $q < \infty$ and by

$$\|f \,| B^1_{p\infty}(\mathbb{R}^n)\| = \|f | L_p(\mathbb{R}^n)\| + \sup_{0<t<\varepsilon} \frac{\omega_2(f,t)_p}{t} \tag{17.19}$$

if $q = \infty$. Here $0 < \varepsilon < 1$, [Triβ], 2.5.12, p. 110. Now we incorporate a log-term in (17.18), (17.19). Let $b \in \mathbb{R}$. Then

$$B^{(1,-b)}_{pq}(\mathbb{R}^n) \quad \text{with} \quad 1 \leq p \leq \infty\,, \quad 1 \leq q \leq \infty\,,$$

is the collection of all $f \in L_p(\mathbb{R}^n)$ with

$$\|f \,| B^{(1,-b)}_{pq}(\mathbb{R}^n)\| = \|f \,| L_p(\mathbb{R}^n)\| + \left(\int_0^\varepsilon \left(\frac{\omega_2(f,t)_p}{t|\log t|^b} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \tag{17.20}$$

(obviously modified if $q = \infty$). These spaces can be characterized in Fourier-analytical terms. Let φ_k be the same functions as in (2.33)–(2.35). In generalization of (2.37),

$$\left(\sum_{j=0}^{\infty} 2^{jq} (1+j)^{-bq} \, \| (\varphi_j \widehat{f})^\vee \, | L_p(\mathbb{R}^n) \| \right)^{\frac{1}{q}} \tag{17.21}$$

(obviously modified if $q = \infty$) is an equivalent norm in $B^{(1,-b)}_{pq}(\mathbb{R}^n)$. These spaces, in their general version of $B^{(s,-b)}_{pq}(\mathbb{R}^n)$ with (2.36) and $b \in \mathbb{R}$ go back to H.-G. Leopold in 1998 and may be found in [Leo98] and [Leo00a]. The point of interest in our context of distinguished target spaces is to replace the second differences $\omega_2(f,t)_p$ in (17.20) by the first differences $\omega(f,t)_p = \omega_1(f,t)_p$ and to introduce in this way spaces $Lip^{(1,-\alpha)}_{pq}(\mathbb{R}^n)$ of Lipschitz type, consisting of all $f \in L_p(\mathbb{R}^n)$ such that

$$\| f \, | Lip^{(1,-\alpha)}_{pq}(\mathbb{R}^n) \| = \| f \, | L_p(\mathbb{R}^n) \| + \left(\int_0^{\varepsilon} \left(\frac{\omega(f,t)_p}{t |\log t|^{\alpha}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \tag{17.22}$$

is finite (with the usual modification if $q = \infty$). Here

$$1 \le p \le \infty \, , \quad 0 < q \le \infty \, , \quad \alpha > \frac{1}{q} \, ,$$

(with $\alpha \ge 0$ if $q = \infty$). The restriction on α is natural. This follows from the considerations in 12.12: If $\alpha \le \frac{1}{q}$ (with $\alpha < 0$ in case of $q = \infty$) then, with exception of $f = 0$, there are no functions f such that (17.22) is finite. These spaces were introduced in [Har00a], Definition 1. If $p = \infty$ in (17.22) then the inequalities in 14.3 for the super-critical case can be reformulated (at least locally) in terms of these target spaces. Hence it is reasonable to extend these considerations from $p = \infty$ to, say, $1 < p < \infty$. But there is a decisive difference between these two cases. Recall that the classical Sobolev space $W^1_p(\mathbb{R}^n)$ with $1 < p < \infty$ can be equivalently normed by

$$\| f \, | L_p(\mathbb{R}^n) \| + \sup_{0<t<\varepsilon} \frac{\omega(f,t)_p}{t} \, , \quad f \in W^1_p(\mathbb{R}^n) \, . \tag{17.23}$$

We refer to [Ste70], Proposition 3, p. 139, [Nik77], 4.8, p. 213 (first edition 1969) and [DeL93], p. 53. In other words, $W^1_p(\mathbb{R}^n)$ with $1 < p < \infty$ coincides with $Lip^{(1,0)}_{p\infty}(\mathbb{R}^n)$. Replacing the first differences in (17.23) by second or higher differences, one gets (17.19) and hence the larger spaces $B^1_{p\infty}(\mathbb{R}^n)$. We do not go into detail. A thorough investigation of all these spaces, especially their mutual embeddings, may be found in [Har00a] with [EdH99] and [EdH00] as forerunners. We refer also to the small survey [Har00b]. Finally we mention that limiting embeddings especially in the super-critical case for spaces with dominating mixed derivatives have been considered in [KrS98].

17.3 Logarithmic spaces

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. We assume that $|\Omega| = \varepsilon < 1$. Let $1 < p < \infty$ and $a \in \mathbb{R}$. Then the spaces $L_p(\log L)_a(\Omega)$ can be introduced much as in 11.6(ii) as the collection of all $f \in L_1(\Omega)$ such that

$$\int_\Omega |f(x)|^p \, \log^{ap}(2 + |f(x)|)\, dx < \infty \,. \tag{17.24}$$

As in (11.46), these spaces can also be characterized as the collection of all $f \in L_1(\Omega)$ such that

$$\left(\int_0^\varepsilon (|\log t|^a \, f^*(t))^p \, dt \right)^{\frac{1}{p}} < \infty \tag{17.25}$$

(equivalent quasi-norms). Details and references are given in 11.6(ii). Based on [EdT95] we proved in [ET96], 2.6.2, Theorem 1, pp. 69/70, another characterization of these spaces with the consequence that they have the same mapping properties with respect to pseudodifferential operators and fractional powers of elliptic operators as the space $L_p(\Omega)$ with $1 < p < \infty$, itself. In particular one can define logarithmic Sobolev spaces $H^s_p(\log H)_a(\Omega)$ by lifting of $L_p(\log L)_a(\Omega)$ in the following way. Let

$$A_m f = (id - \Delta)^m f \,, \quad m \in \mathbb{N} \,, \tag{17.26}$$

and let $A_{m,N} f = A_m f$ be the corresponding Neumann operator with the domain of definition

$$dom\, A_{m,N} = \left\{ f \in H^{2m}_p(\Omega) \,:\, \frac{\partial^{j+m} f}{\partial \nu^{j+m}} |\partial\Omega = 0 \text{ if } j = 0, \ldots, m-1 \right\} \,, \tag{17.27}$$

where ν is the outer normal with respect to $\partial\Omega$. Let

$$a \in \mathbb{R} \,, \quad 1 < p < \infty \,, \quad 0 < \tau \le \frac{1}{2} \quad \text{and} \quad s = 2m\tau \,.$$

Then one can define

$$H^s_p(\log H)_a(\Omega) = A^{-\tau}_{m,N} \, L_p(\log L)_a(\Omega) \,. \tag{17.28}$$

We refer for details and explanations to [ET96], 2.6.3, pp. 75–81. In particular, (17.28) imitates the lifting (1.8). If $s \in \mathbb{N}$, then one obtains, as should be the case,

$$H^s_p(\log H)_a(\Omega) = \{f \in L_1(\Omega) \,:\, D^\alpha f \in L_p(\log L)_a(\Omega)\,,\; |\alpha| \le s\}\;, \tag{17.29}$$

with the equivalent norms

$$\sum_{|\alpha|\le s} \|D^\alpha f\,|L_p(\log L)_a(\Omega)\|\;. \tag{17.30}$$

If $s \in \mathbb{N}$ and $s = \frac{n}{p}$, then one is in the critical case with logarithmically modified classical Sobolev spaces (the dotted line in Fig. 10.1). One may ask for counterparts of Theorem 13.2 and related inequalities in 13.3. Some results of Trudinger type as in (11.56) and 11.8(v) may be found in [FLS96]. Extensions to the fractional case, including Sobolev-Orlicz spaces, have been given in [EdK95]. The interest in these logarithmic Sobolev spaces comes also from the regularity properties of the Jacobian. References can be found in [FLS96]. This may justify having a closer look at these logarithmic spaces from the point of view of sharp inequalities as treated in this chapter.

17.4 Compact embeddings

We proved in 14.6 by geometrical reasoning that the sharp embeddings (14.48) cannot be compact. If one replaces $-\frac{1}{q'}$ in (14.48) by $-\alpha$ with $-\alpha < -\frac{1}{q'}$, then one gets a larger space and it turns out that the corresponding embedding is compact. The degree of compactness can be measured in terms of entropy numbers and approximation numbers. Definitive results in this direction have been obtained in [EdH99], [EdH00], [Har00b] and recently in [CoK00]. This covers also the more general spaces normed by (17.22), and the delicate interplay between these spaces and also in relation to the spaces introduced in (17.18), (17.19), (17.20). As for the latter spaces we refer also to [Leo98] and [Leo00a]. The general background may be found in [ET96] and, in connection with weighted and logarithmic spaces as discussed in 17.3, in [Har97], [Har98], and [Har00c].

Springer

http://www.springer.com/978-3-0348-0568-1

The Structure of Functions
Triebel, H.
2001, XII, 425 p., Softcover
ISBN: 978-3-0348-0568-1
A product of Birkhäuser Basel