

# Chapter II

## Diffusion Processes and their Transformations

This chapter is devoted to a brief exposition on the theory of diffusion processes in order to fix definitions and notations which will be necessary in the following chapters. Moreover, transformations of diffusion processes by means of multiplicative functionals, a renormalization of Kac's semi-groups, and Feller's one-dimensional diffusion processes will be explained.

### 2.1. Time-Homogeneous Diffusion Processes

Let us denote by  $A$  an elliptic differential operator

$$(2.1) \quad A = \frac{1}{2} \Delta + \mathbf{b}(x) \cdot \nabla.$$

In this monograph  $\Delta$  denotes the Laplace-Beltrami operator<sup>1</sup>

$$(2.2) \quad \Delta = \frac{1}{\sqrt{\sigma_2(x)}} \frac{\partial}{\partial x^i} (\sqrt{\sigma_2(x)} (\sigma^T \sigma(x))^{ij} \frac{\partial}{\partial x^j}),$$

unless otherwise stated, where  $\sigma^T \sigma(x)$  is a positive definite diffusion matrix,  $\sigma_2(x) = |\sigma^T \sigma(x)|_{ij}$ , and  $\mathbf{b}(x)$  denotes a drift coefficient. Moreover, let us assume the existence of a unique fundamental solution  $p_t(x, y)$ , for  $t > 0$  and  $x, y \in \mathbf{R}^d$ , of the diffusion equation

$$(2.3) \quad \frac{\partial p}{\partial t}(t, x) = A p(t, x).$$

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<sup>1</sup> This is for convenience to discuss duality in the following chapters, but it is not absolutely necessary

It is well known that there exists such a fundamental solution  $p_t(x, y)$  of the diffusion equation (2.3) if  $\partial^2(\sigma^T \sigma(x))^{ij} / \partial x^k \partial x^h$  and  $\partial b^i(x) / \partial x^k$  are locally uniformly Hölder continuous.<sup>2</sup>

In terms of the fundamental solution  $p_t(x, y)$  we define a transition probability  $P_t(x, B)$  through

$$\begin{aligned} P_t(x, B) &= \int p_t(x, y) 1_B(y) dy, \quad \text{for } t > 0, \\ &= \delta_x(B), \quad \text{for } t = 0, \end{aligned}$$

where  $\delta_x(B)$  denotes a point measure at  $x$ , and  $p_t(x, y)$  will be called the transition probability density. In this chapter we assume that a transition probability density  $p_t(x, y)$  is given. The transition probability satisfies the Chapman-Kolmogoroff equation

$$P_{s+t}(x, B) = \int P_s(x, dy) P_t(y, B), \quad \text{for } s, t \geq 0,$$

and the normality condition

$$P_t(x, \mathbf{R}^d) = 1.$$

For a given initial distribution  $\mu$  and a transition probability  $P_t(x, B)$ , following Kolmogoroff (1931, 33), we can construct a probability measure on the space  $\Omega = C([0, \infty), \mathbf{R}^d)$  of  $\mathbf{R}^d$ -valued continuous functions by means of finite dimensional distributions:

$$\begin{aligned} (2.4) \quad P_\mu[f(X_0, X_{t_1}, \dots, X_{t_{n-1}}, X_{t_n})] \\ = \int \mu(dx_0) P_{t_1}(x_0, dx_1) P_{t_2 - t_1}(x_1, dx_2) \cdots \\ \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n) f(x_0, \dots, x_n), \end{aligned}$$

in terms of transition probabilities  $P_t(x, dy)$ , where  $0 = t_0 < t_1 < \dots < t_n$ , and  $f$  is any bounded measurable function on the product space  $(\mathbf{R}^d)^{n+1}$ .<sup>3</sup>

<sup>2</sup> Cf. e.g., S. Ito (1957), Friedman (1964), Dynkin (1965)

<sup>3</sup> In general the diffusion coefficient may degenerate, in which case there is no transition density. This happens when space-time processes will be considered

For  $\omega \in \Omega$  we denote

$$(2.5) \quad X_t(\omega) = X(t, \omega) = \omega(t).$$

We call  $\{X_t, P_\mu\}$  a diffusion process with the initial distribution  $\mu$  and the transition probability density  $p_t(x, y)$ . The space  $\Omega = C([0, \infty), \mathbf{R}^d)$  is called the "path space" and  $\omega(t)$  a (sample) path of the diffusion process.

The probability measure  $P_\mu$  is defined first on the product space  $(\mathbf{R}^d)^{[0, \infty)}$  through the formula (2.4) and then Kolmogoroff's continuity theorem is applied, in order to restrict it on  $\Omega = C([0, \infty), \mathbf{R}^d)$ :

**Kolmogoroff's continuity Theorem.** *Let  $\{X_t, P\}$  be a stochastic process. If there are positive constants  $\alpha, \beta$ , and  $c$  such that*

$$P[|X_t - X_s|^\alpha] \leq c|t - s|^{1+\beta}, \text{ for any } s, t \in [a, b],$$

*then there exists a continuous modification of the process  $X_t, t \in [a, b]$ .*<sup>4</sup>

By  $P[f]$  we denote the expectation of  $f$  with respect to a measure  $P$  (we avoid the notation  $E[f]$  for the expectation), and by  $P[f|\mathcal{F}]$  the conditional expectation of  $f$  under a condition of a  $\sigma$ -field  $\mathcal{F}$ .

Since the fundamental solution satisfies

$$p_t(x, y) \leq \kappa t^{-d/2} e^{-\lambda|x-y|^2/t}$$

in any finite time interval with positive constants  $\kappa$  and  $\lambda$ ,<sup>5</sup> we have

$$\begin{aligned} P_\mu[|X_t - X_s|^4] &\leq \kappa \int x^4 (t-s)^{-d/2} e^{-\lambda|x|^2/(t-s)} dx \\ &= c|t-s|^2 \end{aligned}$$

with a positive constant  $c$ .

Therefore, the probability measure  $P_\mu$  can be defined on the space  $\Omega = C([0, \infty), \mathbf{R}^d)$  by Kolmogoroff's continuity theorem.

<sup>4</sup> Cf., e.g. Bauer (1981, 90, 91), Dynkin (1965), or any other standard textbooks on Markov and diffusion processes

<sup>5</sup> Cf., e.g. Friedman (1964), Dynkin (1965)

Formula (2.4) yields immediately the *Markov property* of the diffusion process  $\{X_t, P_\mu\}$  in the following practical form: with  $P_x = P_{\delta_x}$

$$(2.6) \quad P_x[Gf(X_{t+s})] = P_x[G P_{X_t}[f(X_s)]],$$

for a bounded measurable function  $f$  on  $\mathbf{R}^d$  and  $G = \prod_{k=1}^n f_k(X_{t_{k-1}})$ , where  $f_k$  are bounded measurable functions on  $\mathbf{R}^d$ ,  $t_{k-1} \leq t$ , and  $k \leq n$ .

With the help of the monotone class lemma it is easy to see that the Markov property (2.6) holds for any bounded  $\mathcal{F}_t$  measurable function  $G$  on  $\Omega$ , where  $\mathcal{F}_t$  denotes the standard  $\sigma$ -field generated by  $\{X_r: \forall r \leq t\}$ .

The Markov property (2.6) is often written in a more general form in terms of the conditional expectation

$$(2.6') \quad P_\mu[F \circ \theta_t | \mathcal{F}_t] = P_{X_t}[F], \quad P_\mu\text{- a.e.},$$

where  $F$  is any bounded measurable function on  $\Omega$ , and  $\theta_t$  is the shift operator defined by  $\theta_t \omega(s) = \omega(t+s)$  for  $\omega \in \Omega$ . This can be shown easily from (2.6) with the help of the monotone class lemma.

Formula (2.6) of the Markov property implies immediately the semi-group property

$$P_{s+t}f = P_s P_t f, \quad s, t \geq 0,$$

of the system of non-negative operators  $\{P_t: t \geq 0\}$  defined through

$$P_t f(x) = P_x[f(X_t)].$$

The semi-group  $\{P_t: t \geq 0\}$  is defined on the space of bounded measurable functions. In many cases it can be considered as a semi-group on the space of bounded continuous functions. The semi-group description of diffusion processes will be needed in Chapter 3, when duality of diffusion equations will be discussed in connection with time reversal of diffusion processes.

If we are given a normalized semi-group  $\{P_t: t \geq 0\}$  of non-negative linear operators on the space of, say, bounded continuous functions, we can get a transition probability  $P_t(x, B)$  with the help of the Riesz-Markov theorem (cf. e.g., Yosida (1965)). In terms of the transition probability a Markov process can be constructed. The method by means of semi-groups is

a well known powerful analytic tool and often adopted to construct Markov processes. However, this analytic semi-group method is really not well-suitable to handle singular diffusion processes which will be treated in later chapters, and hence we will not use it in this monograph to construct (singular) diffusion processes.

The diffusion process such as  $\{X_t, P_\mu\}$  explained above will be a basic (unperturbed) stochastic motion, which is our starting point but not our main interest. There exist always such diffusion processes under mild regularity conditions on diffusion and drift coefficients as explained (see also Section 2.4). Therefore, it is harmless to assume the existence of the basic diffusion process, relying on standard textbooks on diffusion equations and diffusion processes and we will do so. Then, we shall apply "perturbation" to the basic diffusion processes. Severely perturbed diffusion processes will be our main concern in this monograph. Since the perturbation which will be treated is necessarily singular, as will be seen, the existence of perturbed diffusion processes is no longer evident and cannot be found in standard textbooks. Chapters 5 and 6 will be devoted to the existence problem of such singular diffusion processes.

## 2.2. Time-Inhomogeneous Diffusion Processes <sup>6</sup>

If time dependent diffusion and drift coefficients  $(\sigma^T \sigma)^{ij}(t, x)$  and  $\mathbf{b}(t, x)$ , respectively, are given, we consider a time-inhomogeneous diffusion equation

$$(2.7) \quad \tilde{L}p(t, x) = 0, \quad (t, x) \in (a, b) \times \mathbf{R}^d,$$

where  $-\infty < a < b < \infty$ , and  $\tilde{L}$  is a time-dependent parabolic differential operator

$$(2.8) \quad \tilde{L} = \frac{\partial}{\partial t} + \frac{1}{2} (\sigma^T \sigma(t, x))^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \tilde{\mathbf{b}}(t, x)^i \frac{\partial}{\partial x^i},$$

or as a special case

$$(2.8') \quad L = \frac{\partial}{\partial t} + \frac{1}{2} \Delta + \mathbf{b}(t, x) \cdot \nabla,$$

where  $\Delta$  denotes the Laplace-Beltrami operator.

<sup>6</sup> Cf., e.g. Chapter 1 of Gihman-Skorohod (1975), Vol. II

If  $\partial(\sigma^T\sigma(t,x))^{ij}/\partial t$ ,  $\partial^2(\sigma^T\sigma(t,x))^{ij}/\partial x^k\partial x^h$ , and  $\partial\tilde{b}^i(t,x)/\partial x^k$  are locally uniformly Hölder continuous, then there exists a unique fundamental solution  $p(s,x;t,y)$  to the parabolic differential equation (2.7).<sup>7</sup> We consider a time-inhomogeneous diffusion process  $\{X_t, t \in [a, b], P_\mu\}$  characterized by the fundamental solution  $p(s,x;t,y)$  in terms of the finite dimensional distributions:

$$(2.9) \quad P_\mu[f(X_a, X_{t_1}, \dots, X_{t_{n-1}}, X_b)] \\ = \int \mu(dx_0)p(a, x_0; t_1, x_1)dx_1p(t_1, x_1; t_2, x_2)dx_2 \cdots \\ \cdots p(t_{n-1}, x_{n-1}; b, x_n)dx_n f(x_0, x_1, \dots, x_n),$$

where  $a < t_1 < \cdots < t_{n-1} < b$ ,  $f(x_0, x_1, \dots, x_n)$  is any bounded measurable function on the product space  $(\mathbf{R}^d)^{n+1}$ , and  $P_\mu$  is a probability measure on the path space  $\Omega = C([a, b], \mathbf{R}^d)$ . Since the fundamental solution satisfies

$$p(s, x; t, y) \leq \kappa(t - s)^{-d/2} e^{-\lambda|x - y|^2/(t-s)}$$

in any finite time interval with positive constants  $\kappa$  and  $\lambda$ ,<sup>8</sup> we have

$$P_\mu[|X_t - X_s|^4] \leq c|t - s|^2, \text{ for any } s, t \in [a, b],$$

with a positive constant  $c$ , and hence with the help of Kolmogoroff's continuity theorem, one can define  $P_\mu$  on the path space  $\Omega = C([a, b], \mathbf{R}^d)$  through formula (2.9).<sup>9</sup>

It is clear by the definition that a time-homogeneous diffusion process is a special case of time-inhomogeneous diffusion processes. However, when we treat a time-inhomogeneous diffusion process  $\{X_t, t \in [a, b], P_\mu\}$ , we employ a standard trick: Instead of the process  $X_t$  on  $\mathbf{R}^d$  we consider a *space-time diffusion process*  $(t, X_t)$  on an enlarged (space-time) state space  $[a, b] \times \mathbf{R}^d$ .

It is well known that space-time diffusion processes are *time-homogeneous* with transition probabilities

<sup>7</sup> Cf., e.g. S. Ito (1957), Friedman (1964)

<sup>8</sup> Cf., e.g. Friedman (1964), Dynkin (1965)

<sup>9</sup> We can define a diffusion process in terms of solutions of a stochastic differential equation. See the following sections

$$(2.10) \quad \tilde{P}_r((s, x), d(t, y)) = p(s, x; t, y) \delta_{s+r}(dt) dy,$$

where  $\delta_r(dt)$  denotes the point measure at  $r$ , and hence we can apply the theory of time-homogeneous processes to space-time processes.

Let us denote by  $\mathcal{F}_s^t$  the standard  $\sigma$ -field generated by the random variables  $\{X_r: s \leq \forall r \leq t\}$ . With the help of the monotone class lemma, we define  $P_{(s,x)}[F]$  for any bounded  $\mathcal{F}_s^b$ -measurable function  $F$  through (2.9), replacing  $\mu$  by  $\delta_x$ , and  $a$  by  $s$ , and requiring  $s < t_1 < \dots < t_{n-1} < b$ .

The *time inhomogeneous Markov property* :

$$(2.11) \quad P_\mu[F | \mathcal{F}_a^s] = P_{(s,X_s)}[F], \quad P_\mu\text{- a.e.},$$

follows from (2.9) with the help of the monotone class lemma, where  $F$  is any bounded (or non-negative)  $\mathcal{F}_s^b$ -measurable function on  $\Omega$ . Formula (2.11) can be regarded as a space-time version of the time-homogeneous Markov property (2.6) or (2.6'). We apply the Markov property (2.11) often in the following practical form

$$P_{(r,x)}[Gf(t, X_t)] = P_{(r,x)}[G P_{(s,X_s)}[f(t, X_t)]],$$

where  $a \leq r \leq s \leq t \leq b$ ,  $f$  is a bounded measurable function on  $[a, b] \times \mathbf{R}^d$ , and  $G$  a bounded  $\mathcal{F}_r^s$ -measurable function on  $\Omega$ .

The semi-group of a space-time process can be defined with

$$P_{t-s}f((s, x)) = P_{(s,x)}[f(t, X_t)], \quad \text{if } t - s \geq 0,$$

for any bounded measurable functions  $f$  on  $[a, b] \times \mathbf{R}^d$ . The semi-group property follows immediately from the Markov property (2.11).

## 2.3. Brownian Motions

The most typical and fundamental time-homogeneous diffusion processes are  $d$ -dimensional **Brownian motions**,<sup>10</sup> which we can define, applying (2.4), with the Brownian transition probability density

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<sup>10</sup> Cf. Einstein (1905, b)

$$(2.12) \quad p_t(x, y) = (2\pi t)^{-d/2} \exp\left(-\frac{\|x - y\|^2}{2t}\right),$$

which is the fundamental solution of the diffusion equation (2.3) with  $\sigma = \delta_{ij}$  and  $\mathbf{b} \equiv 0$ .

An explicit construction of one-dimensional Brownian motion is due to P. Lévy:<sup>11</sup> let  $\{g_0, g_{k, 2^{-n}} : \text{odd } k < 2^{-n}\}$  be a Gaussian family of random variables on a probability space  $\{W, \mathcal{F}, P\}$ , and by  $\{f_0, f_{k, 2^{-n}} : \text{odd } k < 2^{-n}\}$  denote the family of Schauder functions

$$f_{k, 2^{-n}}(t) = \int_0^t h_{k, 2^{-n}}(s) ds,$$

where the system of functions  $\{h_{k, 2^{-n}}(s)\}$  is defined by

$$\begin{aligned} h_{k, 2^{-n}}(s) &= 2^{(n-1)/2}, & (k-1)2^{-n} \leq s < k2^{-n} \\ &= -2^{(n-1)/2}, & k2^{-n} \leq s < (k+1)2^{-n} \\ &= 0, & \text{otherwise.} \end{aligned}$$

The family of Schauder functions gives a collection of little tents, though superposition of which with independent Gaussian random coefficients  $g$ 's, we define

$$(2.13) \quad B(t) = g_0 f_0(t) + \sum_{n=1}^{\infty} \sum_{\text{odd } k < 2^n} g_{k, 2^{-n}} f_{k, 2^{-n}}(t).$$

It is not difficult to show<sup>12</sup> that this converges uniformly in  $t \in [0, 1]$ , and has the distribution density

$$P[B(t) \in dx] = (2\pi t)^{-1/2} \exp\left(-\frac{|x|^2}{2t}\right) dx.$$

Therefore,  $B(t)$  is a one-dimensional Brownian motion starting from the origin. It is routine to extend the time parameter from  $[0, 1]$  to  $[0, \infty)$  and for arbitrary starting points.

<sup>11</sup> N. Wiener adopted another basis system and constructed Brownian motions (Wiener measure), cf. Itô-Nisio (1968)

<sup>12</sup> Cf. McKean (1969)



A  $d$ -dimensional Brownian motion can be obtained by

$$\mathbf{B}(t) = (B_1(t), \dots, B_d(t)),$$

where  $B_i(t)$ 's are independent copies of  $B(t)$ . This is a probabilists' favorite construction of Brownian motions, since the continuity of the process is evident by definition; formula (2.13) shows clearly that Brownian motions are sums of independent random increments, and finally the method does not depend too heavily on analysis.

There is another method of constructing a Brownian motion as a limit of Markov chains with an appropriate scaling. This method plays an important role in diffusion approximations, but we will not discuss it in this monograph.

*Remark.* For Brownian motions there are many excellent textbooks and monographs which have different characters. One can refer to *e.g.* Revuz-Yor (1991) and the references given there.

## 2.4. Stochastic Differential Equations

If diffusion and drift coefficients are smooth or merely Lipschitz continuous, we apply Itô's theory of stochastic differential equations.<sup>13</sup> When we handle the parabolic differential operator  $L$  with the Laplace-Beltrami operator  $\Delta$  given at (2.8'), we rewrite it as

$$(2.14) \quad L = \frac{\partial}{\partial t} + \frac{1}{2} (\sigma^T \sigma(x))^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \tilde{\mathbf{b}}(t, x)^i \frac{\partial}{\partial x^i},$$

as appears in Itô's formula, where

$$\tilde{\mathbf{b}}(t, x) = \mathbf{b}(t, x) + \mathbf{b}_{\sigma}(x),$$

$$\mathbf{b}_{\sigma}(x)^i = \frac{1}{2} \frac{1}{\sqrt{\sigma_2(x)}} \frac{\partial}{\partial x^i} (\sigma^T \sigma(x))^{ij} \sqrt{\sigma_2(x)}.$$

By  $\{B_t\}$  we denote a  $d$ -dimensional Brownian motion defined on a probability space  $\{W, \mathcal{F}, P\}$ . Then we consider the stochastic differential equation (SDE)

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<sup>13</sup> For stochastic differential equations cf., *e.g.*, McKean (1969), Ikeda-Watanabe (1981, 89), Chung-Williams (1983, 88)

$$(2.15) \quad X_t = X_a + \int_a^t \sigma(r, X_r) dB_r + \int_a^t \tilde{b}(r, X_r) dr.$$

Let us assume, for simplicity, that the initial value  $X_a$  is square integrable and independent of the Brownian motion  $B_r$ . If  $\sigma(t, x)$  and  $\tilde{b}(t, x)$  are Lipschitz continuous and satisfy a growth condition

$$\|\sigma(t, x)\|^2 + \|\tilde{b}(t, x)\|^2 \leq c_1(1 + \|x\|^2),$$

then there exists a pathwise unique solution  $X_t$  which is square integrable.

In terms of the solution we can define a diffusion process  $\{X_t, P_\mu\}$  corresponding to the parabolic differential operator  $L$  given in (2.14) or more generally  $\tilde{L}$  given in (2.8). Solutions of the SDE (2.15) can be obtained through the standard successive approximation as follows: To demonstrate the standard way we consider (2.15) simply in one-dimension and assume  $\sigma(t, x)$  and  $\tilde{b}(t, x)$  are bounded in the following.

By the assumption  $\sigma(t, x)$  and  $\tilde{b}(t, x)$  are Lipschitz continuous:

$$|\sigma(t, x) - \sigma(t, y)|, |\tilde{b}(t, x) - \tilde{b}(t, y)| \leq c_0|x - y|.$$

A solution can be obtained through successive approximation. We set

$$X_t^{(0)} = X_a,$$

$$X_t^{(n)} = X_a + \int_a^t \sigma(r, X_r^{(n-1)}) dB_r + \int_a^t \tilde{b}(r, X_r^{(n-1)}) dr, \text{ for } n \geq 1.$$

First of all it is clear that

$$P[|X_t^{(1)} - X_t^{(0)}|^2] \leq c, \text{ for } t \in [a, b].$$

For arbitrary  $n > 1$ , we have

$$(2.16) \quad P[|X_t^{(n)} - X_t^{(n-1)}|^2] \leq cK^{n-1} \frac{t^{n-1}}{(n-1)!},$$

which can be verified through induction as follows:

$$P[|X_t^{(n+1)} - X_t^{(n)}|^2] \leq 2P\left[\left|\int_a^t \{\sigma(s, X_s^{(n)}) - \sigma(s, X_s^{(n-1)})\} dB_s\right|^2\right] \\ + 2P\left[\left|\int_a^t \{\tilde{b}(s, X_s^{(n)}) - \tilde{b}(s, X_s^{(n-1)})\} ds\right|^2\right],$$

where we have applied an inequality  $(a + b)^2 \leq 2(a^2 + b^2)$ . The first integral on the right-hand side is dominated by

$$2(c_0)^2 P\left[\int_a^t |X_s^{(n)} - X_s^{(n-1)}|^2 ds\right],$$

because of the Lipschitz continuity of  $\sigma(t, x)$ , and the second one by

$$2P\left[\int_a^t |\tilde{b}(s, X_s^{(n)}) - \tilde{b}(s, X_s^{(n-1)})|^2 ds \int_a^t ds\right] \\ \leq 2(c_0)^2 (b - a) P\left[\int_a^t |X_s^{(n)} - X_s^{(n-1)}|^2 ds\right],$$

where we have applied Schwarz's inequality and the Lipschitz continuity of  $\tilde{b}(t, x)$ . Thus we have shown, with a constant  $K > 0$ ,

$$P[|X_t^{(n+1)} - X_t^{(n)}|^2] \leq KP\left[\int_a^t |X_s^{(n)} - X_s^{(n-1)}|^2 ds\right].$$

A substitution of (2.16) on the right hand side verifies (2.16) for any  $n$ .

Therefore, we have

$$P\left[\sup_{t \in [a, b]} |X_t^{(n)} - X_t^{(n-1)}|^2\right] \leq \text{const.} \frac{(K(b - a))^{n-1}}{(n - 1)!},$$

and hence

$$P\left[\sup_{t \in [a, b]} |X_t^{(n)} - X_t^{(n-1)}|^2 > \frac{1}{2^{n-1}}\right] \leq \text{const.} \frac{(K(b - a))^{n-1}}{(n - 1)!}.$$

With the help of the Borel-Cantelli lemma,  $X_t^{(n)}$  converges uniformly in  $t \in [a, b]$ , P-a.e. and also in  $L^2$ . Moreover,

$$X_t = \lim_{n \rightarrow \infty} X_t^{(n)}$$

satisfies the SDE (2.15).

The uniqueness of solutions follows immediately from

**Gronwall's Lemma.** *Let  $A(t)$  be a non-negative integrable function on  $[a, b]$  satisfying*

$$A(t) \leq \kappa \int_a^t A(s) ds + C(t), \quad \kappa > 0,$$

where  $C(t)$  is also integrable. Then

$$A(t) \leq \kappa \int_a^t e^{\kappa(t-s)} C(s) ds + C(t).$$

In particular if  $C(t)$  is non-negative and non-decreasing, then

$$A(t) \leq e^{\kappa(t-a)} C(t).$$

*Proof* is a good exercise (apply iteration).<sup>14</sup>

Now let  $X_t^1$  and  $X_t^2$  be solutions of equation (2.15) and set

$$Z_t = |X_t^1 - X_t^2|^2.$$

Then,  $P[Z_t] < \infty$  and

$$P[Z_t] \leq K \int_a^t P[Z_s] ds.$$

Therefore, by Gronwall's lemma  $P[Z_t] = 0$ , for  $t \in [a, b]$ , which proves the uniqueness of solutions.

Applying Itô's formula, we will show that the diffusion process  $\{X_t, P\}$  is determined by the parabolic differential operator  $\tilde{L}$ :

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<sup>14</sup> Cf. e.g., Gikhman-Skorokhod (1969), p. 393, or Revuz-Yor (1991), p. 499

Let us denote the  $i$ -th component of  $X_t$  by  $X_t^i$ . Then, the random variable  $Y_t = f(t, X_t^1, \dots, X_t^d)$ , for any bounded  $f \in C^2([a, b] \times \mathbf{R}^d)$ , satisfies

$$(2.17) \quad dY_t = \partial_t f dt + \sum_{i=1}^d (\partial_i f) \cdot dX_t^i + \frac{1}{2} \sum_{i,j=1}^d (\partial_i \partial_j f) \cdot dX_t^i dX_t^j,$$

which is called *Itô's formula*, where  $\partial_t = \partial/\partial t$ ,  $\partial_i = \partial/\partial x^i$ ,

$$dX_t^i = \sigma_j^i dB_t^j + \tilde{b}^i dt,$$

$$dB_t^i dB_t^j = \delta^{ij} dt, \quad \text{and} \quad dt dB_t^i = 0.$$

Therefore, we get

$$(2.18) \quad f(t, X_t) - f(s, X_s) - \int_s^t \tilde{L}f(r, X_r) dr = \sum_{i,j=1}^d \int_s^t (\sigma_j^i \partial_i f)(r, X_r) dB_r^j,$$

the right-hand side of which is a martingale as a sum of Itô's stochastic integrals with respect to Brownian motions, the expectation of which vanishes consequently. Let  $t = b$ ,  $s = a$ , and let  $f$  be any  $C^\infty$ -function with a compact support in  $(a, b) \times \mathbf{R}^d$ , and take the expectation of both sides of (2.18). Then, except for the third term on the left-hand side, all terms vanish, and hence

$$\int_a^b P[\tilde{L}f(t, X_t)] dt = 0.$$

Therefore, denoting by  $\mu_t(x)$ , the probability density of the diffusion process  $X_t$ , we have

$$(2.19) \quad \int_{(a,b) \times \mathbf{R}^d} \mu_t(x) \tilde{L}f(t, x) dt dx = 0,$$

namely,  $\mu_t(x)$  is a weak solution of

$$(2.20) \quad \tilde{L}^\circ \mu = 0,$$

where  $\tilde{L}^\circ$  is the formal adjoint of  $\tilde{L}$ . When  $L$  of (2.14), which is the same as (2.8') with the Laplace-Beltrami operator  $\Delta$ , is handled, replace  $\tilde{L}^\circ$  by

$$(2.21) \quad L^\circ g = -\frac{\partial g}{\partial t} + \frac{1}{2} \Delta g - \frac{1}{\sqrt{\sigma_2}} \nabla \{ \sqrt{\sigma_2} \mathbf{b}(t, x) g \}.$$

Let us denote by  $X$  the mapping from  $W$  to  $\Omega = C([a, b], \mathbf{R}^d)$

$$X : w \rightarrow X_t(w),$$

and define a probability measure  $P_\mu$  on  $\Omega = C([a, b], \mathbf{R}^d)$  by

$$P_\mu = P \circ X^{-1}.$$

In this way we get a diffusion process  $\{X_t, P_\mu\}$  defined on the space of continuous paths  $\Omega = C([a, b], \mathbf{R}^d)$ .

Maruyama (1954) proved that the probability measure  $P_\mu$  is absolutely continuous with respect to the probability measure of a  $d$ -dimensional Brownian motion defined on the space  $\Omega = C([a, b], \mathbf{R}^d)$ , if the diffusion coefficient is non-degenerate,<sup>15</sup> and hence the transition probability of the diffusion process  $\{X_t, P_\mu\}$  has a density function  $p(s, x; t, y)$ . This will be called Maruyama's absolute continuity theorem.

Assume the diffusion coefficient is non-degenerate. Then, the equation (2.19) or (2.20) implies that the transition probability density  $p(s, x; t, y)$  of the diffusion process  $X_t$  satisfies

$$(2.22) \quad \tilde{L}p = 0,$$

weakly, as we have wanted to show.

*Remark.* In later chapters, diffusion process with singular drift will be treated. Truncating and approximating drift coefficients, we can apply the SDE method even in such singular cases. However, since this approximation procedure makes things complicated, we will not do it. Instead we will construct singular diffusion process in other ways, cf. Chapters 5 and 6. In some cases the truncation method is dangerous and must be handled with great care; see examples in Section 7.9.

*Remark.* Based on formula (2.18) a method of the so-called martingale problem was developed by Stroock-Varadhan (1970), which allows diffusion

<sup>15</sup> This is known as Girsanov's theorem (1960), being not aware of Maruyama's paper. See the next section

and drift coefficients merely to be continuous. This martingale method combines SDE and semi-group methods and generalizes both. For this we refer to Stroock-Varadhan (1970).

*Remark.* For a detailed treatment of stochastic differential equations one can refer to e.g. McKean (1969), Liptser-Shiryayev (1977) and Ikeda-Watanabe (1981, 89).

## 2.5. Transformation by a Multiplicative Functional

In the theory of diffusion processes a transformation of probability measures in terms of multiplicative functionals is a strong tool in perturbing a given diffusion process into a new one.

Let  $\{(t, X_t), P_{(s,x)}, (s, x) \in [a, b] \times \mathbb{R}^d\}$  be a space-time (unperturbed) basic diffusion process.<sup>16</sup>

A functional  $M_s^t(\omega)$ ,  $a \leq s \leq t \leq b$ , is a *multiplicative functional*, if it is  $\mathcal{F}_s^t$ -measurable and satisfies the *multiplicativity*

$$(2.23) \quad M_r^t(\omega) = M_r^s(\omega)M_s^t(\omega), \quad \text{for } r \leq s \leq t.^{17}$$

For simplicity we assume  $M_s^t(\omega)$  is continuous in  $t$  for fixed  $s$ .

If a multiplicative functional satisfies in addition the *normality condition*

$$(2.24) \quad P_{(s,x)}[M_s^t] = 1,$$

then it will be called a normal multiplicative functional, or martingale multiplicative functional, since it turns out to be a martingale; in fact

$$\begin{aligned} P_{(s,x)}[M_s^t | \mathcal{F}_s^r] &= P_{(s,x)}[M_s^r M_r^t | \mathcal{F}_s^r] = M_s^r P_{(s,x)}[M_r^t | \mathcal{F}_s^r] \\ &= M_s^r P_{(r,X_r)}[M_r^t] \\ &= M_s^r, \quad P_{(s,x)} - a.e., \end{aligned}$$

<sup>16</sup> We consider a family of probability measures  $P_{(s,x)}$  with arbitrary starting points  $(s, x)$

<sup>17</sup> In general the equality may have an exceptional set of measure zero, cf Meyer (1962, 63), Blumenthal-Gettoor (1968), Sharpe (1988)

for  $s < r < t$ , where we have applied at the third and fourth equalities the Markov property (2.11) and normality condition (2.24), respectively.

Let us define a system of new probability measures  $Q_{(s,x)}$  by

$$(2.25) \quad Q_{(s,x)}[F] = P_{(s,x)}[M_s^b F],$$

for any bounded  $\mathcal{F}_s^b$ -measurable function  $F$ , namely,  $M_s^b$  is a density of the probability measure  $Q_{(s,x)}$  with respect to the (unperturbed) probability measure  $P_{(s,x)}$ . We will denote  $Q_{(s,x)} = M_s^b P_{(s,x)}$  (instead of the standard notation  $dQ_{(s,x)} = M_s^b dP_{(s,x)}$ ).

In this way we can obtain a new space-time diffusion process  $\{(t, X_t), Q_{(s,x)}, (s,x) \in [a, b] \times \mathbf{R}^d\}$ . This is the so-called *transformation of a space-time diffusion process*  $\{(t, X_t), P_{(s,x)}, (s,x) \in [a, b] \times \mathbf{R}^d\}$  by a *multiplicative functional*  $M_s^t$ . The Markov property of the transformed process is immediate: For any bounded  $\mathcal{F}_s^r$ -measurable  $G$  on  $\Omega$  and bounded measurable  $f$  on  $[a, b] \times \mathbf{R}^d$ ,

$$(2.26) \quad \begin{aligned} Q_{(s,x)}[G f(t, X_t)] &= P_{(s,x)}[M_s^b G f(t, X_t)] \\ &= P_{(s,x)}[M_s^r G P_{(s,x)}[M_r^b f(t, X_t) | \mathcal{F}_s^r]] \\ &= P_{(s,x)}[M_s^r G P_{(r, X_r)}[M_r^b f(t, X_t)]] \\ &= Q_{(s,x)}[G Q_{(r, X_r)}[f(t, X_t)]]. \end{aligned}$$

The transformed space-time process  $\{(t, X_t), Q_{(s,x)}, (s,x) \in [a, b] \times \mathbf{R}^d\}$  also inherits the strong Markov property of the unperturbed space-time diffusion process  $\{(t, X_t), P_{(s,x)}, (s,x) \in [a, b] \times \mathbf{R}^d\}$ , since one can apply the same manipulation as in (2.26) for random stopping (or optional) times.<sup>18</sup>

Thus we have shown

**Theorem 2.1.** *Let  $\{(t, X_t), P_{(s,x)}, (s,x) \in [a, b] \times \mathbf{R}^d\}$  be a space-time diffusion process. If a continuous multiplicative functional  $M_s^t(\omega)$  satisfies normality condition (2.24), then the system of transformed probability measures  $Q_{(s,x)} = M_s^b P_{(s,x)}$  defines a new (perturbed) space-time diffusion process  $\{(t, X_t), Q_{(s,x)}, (s,x) \in [a, b] \times \mathbf{R}^d\}$ .*

<sup>18</sup> Cf. Meyer (1962, 63), Blumenthal-Gettoor (1968), Sharpe (1988)



A typical example of multiplicative functionals is Kac's multiplicative functional

$$m_s^t = \exp\left(\int_s^t c(r, X_r) dr\right),$$

where  $c(r, x)$  is a bounded measurable function. The multiplicativity (2.23) of the functional is immediate. However, the Kac multiplicative functional does not satisfy normality condition (2.24), and hence Theorem 2.1 cannot be applied. To overcome the difficulty, we will renormalize Kac's functionals in Section 2.7. The renormalization of Kac's multiplicative functionals will play an important role in this monograph (see Chapters 5 and 6).

Another well-known multiplicative functional for a  $d$ -dimensional Brownian motion  $B_t$  is the *Maruyama-Girsanov density*

$$(2.27) \quad M_s^t = \exp\left(\int_s^t \mathbf{b}(r, B_r) \cdot dB_r - \frac{1}{2} \int_s^t \|\mathbf{b}(r, B_r)\|^2 dr\right),$$

(cf. Maruyama (1954), Girsanov (1960). See also Liptser-Shiryayev (1977), Ikeda-Watanabe (1981, 89)). If the vector function  $\mathbf{b}(t, x)$  is bounded measurable, then it is easy to see, applying Itô's formula, that the  $M_s^t$  in (2.27) is well-defined and satisfies normality condition (2.24).

The boundedness assumption on the drift coefficient  $\mathbf{b}(t, x)$  is not necessary, but for simplicity. A well-known sufficient condition for the normality is Novikov's condition<sup>19</sup>

$$(2.28) \quad P_{(s,x)}\left[\exp\left(\frac{1}{2} \int_s^b \|\mathbf{b}(r, B_r)\|^2 dr\right)\right] < \infty.$$

This condition allows the drift vector  $\mathbf{b}(t, x)$  to be singular to some extent.<sup>20</sup>

We will prove that the transformation in terms of the Maruyama-Girsanov density (2.27) induces the drift term with  $\mathbf{b}(t, x)$ . Therefore, it is often called "drift transformation" or the "Maruyama-Girsanov transformation". Let us consider the case of one-dimensional Brownian motion for simplicity, and assume normality condition (2.24).

<sup>19</sup> Cf., e.g. Liptser-Shiryayev (1977), Ikeda-Watanabe (1981, 89), Revuz-Yor (1991)

<sup>20</sup> For interesting examples see Stummer (1990)

For a proof we apply formulae of Itô's stochastic calculus:

$$(2.29) \quad d(X_t Y_t) = (dX_t)Y_t + X_t(dY_t) + (dX_t)(dY_t),$$

$$(2.30) \quad (dB_t)^2 = dt \quad \text{and} \quad (dt)(dB_t) = 0.$$

We apply formula (2.29) to

$$X_t = f(t, B_t) \quad \text{and} \quad Y_t = M_s^t = e^\alpha,$$

$$\alpha = \int_s^t b(r, B_r) dB_r - \frac{1}{2} \int_s^t b(r, B_r)^2 dr,$$

where  $f$  is any  $C^\infty$ -function of compact support in  $(s, b) \times \mathbf{R}$ .

Since

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dt, \quad \text{and} \quad dY_t = e^\alpha b(t, B_t) dB_t,$$

we get

$$(2.31) \quad d(X_t Y_t) = Lf(t, B_t) e^\alpha dt + \left\{ f(t, B_t) b(t, B_t) + \frac{\partial f}{\partial x}(t, B_t) \right\} e^\alpha dB_t,$$

because of Itô's formula (2.17) with (2.30), where

$$L = \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} + b(t, x) \frac{\partial}{\partial x}.$$

Therefore, taking the expectation of both sides of (2.31) and integrating over  $[s, b]$ , since  $f$  is of compact support in  $(s, b) \times \mathbf{R}$ , we get

$$\int_s^b P_{(s, x)} [Lf(t, B_t) M_s^t] dt = 0.$$

If we denote by  $\mu_t(x)$  the probability density of  $B_t$ , with respect to the transformed probability measure  $Q_{(s, x)} = M_a^b P_{(s, x)}$ , then

$$(2.32) \quad \int_{(s, b) \times \mathbf{R}} \mu_t(x) Lf(t, x) dt dx = 0,$$

namely, the *diffusion process under the transformed probability measure*  $Q_{(s,x)} = M_a^b P_{(s,x)}$  has an additional drift term  $b(t, x)$ .

To extend the above arguments to higher dimensions is more or less routine.<sup>21</sup>

## 2.6. Feynman-Kac Formula

Let  $\{(t, X_t), P_{(s,x)}, (s, x) \in [a, b] \times \mathbf{R}^d\}$  be the diffusion process determined by the parabolic differential operator  $\tilde{L}$  given in (2.8) and let  $c(t, x)$  be a measurable function. Then, the Feynman-Kac formula, which will be given in (2.35), represents the solution  $u(s, x)$  of the diffusion equation

$$(2.33) \quad \left\{ \frac{\partial}{\partial s} + \frac{1}{2} (\sigma^T \sigma(s, x))^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \tilde{b}(s, x)^i \frac{\partial}{\partial x^i} + c(s, x) \right\} u = 0,$$

with

$$u(t, x) = f(t, x),$$

in terms of the diffusion process. In (2.33),  $s \in (a, t)$  and  $t \in (a, b]$  is arbitrary but fixed. We assume in this section that  $c(s, x)$  and  $u(s, x)$  are bounded.

If the function  $c(s, x)$  is bounded, there is no problem showing the Feynman-Kac formula. There are various ways of treating the formula: purely analytically as perturbation, in terms of semi-group theory, or using Itô's formula. However, if  $c(s, x)$  is unbounded (or singular), there are several points which must be carefully treated, and the advantage or disadvantage of the three methods mentioned above will come out. This will be discussed later on.

Let us assume the diffusion process  $X_r$  is given as a solution of the stochastic differential equation

$$dX_r = \sigma(r, X_r) \cdot dB_r + \tilde{b}(r, X_r) dr,$$

where  $B_r$  is a  $d$ -dimensional Brownian motion.<sup>22</sup> Then, applying formulae (2.29) and (2.30) of Itô's stochastic calculus to

<sup>21</sup> Cf., e.g. Liptser-Shiryayev (1977), Ikeda-Watanabe (1981, 89)

<sup>22</sup> If  $\sigma(t, x)$  and  $\tilde{b}(t, x)$  are bounded and Lipschitz continuous, then solutions exist

$$Z_r = p(r, X_r), \quad Y_r = e^\alpha, \quad \alpha = \int_s^r c(u, X_u) du, \quad \text{for } r \in [s, t],$$

we have, after a routine manipulation using Itô's stochastic calculus,

$$(2.34) \quad u(s, X_s) - f(t, X_t) \exp\left(\int_s^t c(r, X_r) dr\right) = a \text{ martingale},$$

because of (2.33), where the expectation of the right-hand side vanishes.

Therefore, taking the expectation of both sides of (2.34), we get the *Feynman-Kac formula*

$$(2.35) \quad u(s, x) = P_{(s,x)}\left[\exp\left(\int_s^t c(r, X_r) dr\right) f(t, X_t)\right], \quad \text{for } s \in (a, t).$$

Conversely, if we define a function  $u(s, x)$  by (2.35), it is easy to see that  $u(s, x)$  satisfies an integral equation

$$(2.36) \quad u(s, x) = P_{t-s} f(s, x) + \int_s^t P_{(s,x)}[c(r, X_r) u(r, X_r)] dr, \quad \text{for } s \in (a, t),$$

where  $P_t$  denotes the semi-group of the unperturbed space-time diffusion process  $\{(t, X_t), P_{(s,x)}, (s, x) \in [a, b] \times \mathbf{R}^d\}$ . In fact, expanding Kac's multiplicative functional in the right-hand side of (2.35) as

$$\begin{aligned} \exp\left(\int_s^t c(r, X_r) dr\right) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\int_s^t c(r, X_r) dr\right)^k \\ &= 1 + \int_s^t c(r, X_r) dr \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(\int_r^t c(u, X_u) du\right)^{k-1}; \end{aligned}$$

taking the expectation and applying the Markov property, we have

$$\begin{aligned} P_{(s,x)}\left[\int_s^t c(r, X_r) dr \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(\int_r^t c(\tau, X_\tau) d\tau\right)^{k-1} f(t, X_t)\right] \\ = \int_s^t P_{(s,x)}[c(r, X_r) u(r, X_r)] dr, \end{aligned}$$

and hence the right-hand side of (2.36). Therefore, the  $u(s, x)$  defined at (2.35) satisfies equation (2.33) weakly.

If the function  $c(t, x)$  is not bounded, we shall see that the method of applying Itô's formula explained above is not always best for the Feynman-Kac formula. This point will be discussed in Chapter 6.

Now let us adopt an analytic method: in this case we consider the integral equation (2.36) instead of the diffusion equation (2.33). A solution can be constructed as follows: Define successively

$$u^{(0)}(s, x) = P_{t-s} f(s, x) = P_{(s,x)}[f(t, X_t)],$$

$$u^{(k)}(s, x) = \int_s^t P_{(s,x)}[c(r, X_r)u^{(k-1)}(r, X_r)]dr, \text{ for } k \geq 1.$$

Then it is easy to see that

$$u(s, x) = \sum_{k=0}^{\infty} u^{(k)}(s, x)$$

converges, is bounded, and satisfies equation (2.36). Moreover we can show easily by induction

$$u^{(k)}(s, x) = P_{(s,x)}\left[\frac{1}{k!} \left(\int_s^t c(r, X_r)dr\right)^k f(t, X_t)\right].$$

Therefore, the solution  $u(s, x)$  has the expression of (2.35). The uniqueness of solutions of the integral equation (2.36) is easy to show, if  $c(t, x)$  is bounded. For uniqueness, see Lemma 6.1 and Section 6.5, in which the case of singular  $c(t, x)$  will be treated.

## 2.7. Kac's Semi-Group and its Renormalization

Let  $\{X_t, P_{(s,x)}; (s, x) \in [a, b] \times \mathbf{R}^d\}$  be a basic (unperturbed) space-time diffusion process and let  $c(t, x)$  be a bounded measurable function.<sup>23</sup>

Kac's multiplicative functional is defined by

<sup>23</sup> The boundedness assumption is just for simplicity, and will be removed in later sections

$$(2.37) \quad m_s^t = \exp\left(\int_s^t c(r, X_r) dr\right),$$

which does not satisfy normality condition (2.24).

In terms of Kac's multiplicative functional, we can define a new semi-group  $P_{t-s}^c$  by

$$(2.38) \quad P_{t-s}^c f((s, x)) = P_{(s, x)}[m_s^t f(t, X_t)],$$

which is called *Kac's semi-group*.

The semi-group property of  $\{P_{t-s}^c\}$  follows immediately from the multiplicativity of Kac's functional and the Markov property of the basic unperturbed process  $\{X_t, P_{(s, x)}; (s, x) \in [a, b] \times \mathbf{R}^d\}$ :

$$\begin{aligned} P_{r-s}^c P_{t-r}^c f((s, x)) &= P_{(s, x)}[m_s^r P_{(r, X_r)}[m_r^t f(t, X_t)]] \\ &= P_{(s, x)}[m_s^r m_r^t f(t, X_t)] \\ &= P_{t-s}^c f((s, x)). \end{aligned}$$

for  $a \leq s \leq r \leq t \leq b$ .

If the function  $c(t, x)$  is non-positive, then  $P_{t-s}^c 1((s, x)) \leq 1$ , and hence we can construct a space-time diffusion process *with killing* on an extended probability space such that its semi-group coincides with the semi-group  $\{P_{t-s}^c\}$  defined by (2.38). This is well known.<sup>24</sup> However, if the function  $c(t, x)$  takes both positive and negative values, one cannot construct a diffusion process which has the semi-group  $\{P_{t-s}^c\}$ .

In fact, when we define a probability measure applying the formula (2.9) of finite dimensional distributions, we need a normalized, i.e., transition *probability*. The well-known problem with Kac's semi-group  $\{P_{t-s}^c\}$  is that it is not normalized, i.e., it happens to be  $P_{t-s}^c 1 \geq 1$  ( $P_{t-s}^c 1 \leq 1$  causes no trouble, as remarked above). Therefore, one cannot apply formula (2.9) to the transition function defined through the semi-group  $\{P_{t-s}^c\}$ . Probabilistically the positive part of the function  $c(t, x)$  represents the existence of "creation of particles", which is problematic. We will encounter

<sup>24</sup> Cf., e.g. Dynkin (1965), Blumenthal-Gettoor (1968). However, the killed processes will not be employed in this monograph; instead, we will apply "renormalization"

this problem later on in Chapters 3, 5 and 6.<sup>25</sup>

On the other hand, we can "comfortably" define measures  $P_{(s,x)}^c$  with creation and killing by

$$(2.39) \quad P_{(s,x)}^c[F] = P_{(s,x)}[m_s^b F],$$

where  $F$  is any bounded  $\mathcal{F}_s^b$ -measurable function on  $\Omega$ . However, the system of measures  $\{P_{(s,x)}^c, (s,x) \in [a,b] \times \mathbf{R}^d\}$  does not define a Markov process or a semi-group. Nonetheless, through the **renormalization** of the measure with creation and killing  $P_{(s,x)}^c$  we can get a Markov process as follows.

Let us define

$$(2.40) \quad \xi(s,x) = P_{(s,x)}^c[1] = P_{(s,x)}[m_s^b].$$

Since  $c(t,x)$  is bounded, it is clear that

$$(2.41) \quad 0 < \xi(s,x) < \infty.$$

With the function  $\xi(s,x)$  we define a system of **renormalized measures**  $\{\bar{P}_{(s,x)}\}$  of  $\{P_{(s,x)}^c\}$  by

$$(2.42) \quad \bar{P}_{(s,x)}[F] = \frac{1}{\xi(s,x)} P_{(s,x)}^c[F].$$

Then, the renormalized measures define a new space-time diffusion process  $\{(t, X_t), \bar{P}_{(s,x)} : (s,x) \in [a,b] \times \mathbf{R}^d\}$ , which will be called the **renormalized process**. Its semi-group  $\bar{P}_{t-s}f$  as a space-time process is given by

$$(2.43) \quad \bar{P}_{t-s}f((s,x)) = \frac{1}{\xi(s,x)} P_{(s,x)}[e^{\int_s^t c(r, X_r) dr} f(t, X_t) \xi(t, X_t)],$$

which is the  $\xi$ -*transformation* of Kac's semi-group  $P_{t-s}^c$  defined in (2.38), namely,

$$\bar{P}_{t-s}f((s,x)) = \frac{1}{\xi} P_{t-s}^c(f\xi)(s,x).$$

---

<sup>25</sup> There are various ways to handle "creation of particles", introducing additional structures and interpreting induced semi-groups, cf. Nagasawa (1969), Mitro (1979). See Chapter 12 on branching processes

Formula (2.43) can be shown easily applying the Markov property of the basic unperturbed process:

$$\begin{aligned}\bar{P}_{t-s} f((s, x)) &= \frac{1}{\xi(s, x)} P_{(s, x)}[m_s^t f(t, X_t) m_t^b] \\ &= \frac{1}{\xi(s, x)} P_{(s, x)}[m_s^t f(t, X_t) P_{(t, X_t)}[m_t^b]] \\ &= \frac{1}{\xi(s, x)} P_{(s, x)}[m_s^t f(t, X_t) \xi(t, X_t)],\end{aligned}$$

where  $P_{(t, x)}[m_t^b] = \xi(t, x)$  is substituted, and hence we have (2.43).<sup>26</sup>

Thus we have shown<sup>27</sup>

**Theorem 2.2.** *Let  $\bar{P}_{(s, x)}$  be the renormalization of  $P_{(s, x)}^c$  defined in (2.42) and let  $\{(t, X_t), P_{(s, x)}: (s, x) \in [a, b] \times \mathbf{R}^d\}$  be the renormalized process. Then, its semi-group  $\bar{P}_{t-s}$  is the  $\xi$ -transformation of Kac's semi-group  $P_{t-s}^c$  defined by (2.38), namely,*

$$(2.44) \quad \bar{P}_{t-s} f((s, x)) = \frac{1}{\xi} P_{t-s}^c(f \xi)(s, x).$$

We can formulate Theorem 2.2 as a corollary of Theorem 2.1 applied to the renormalization of Kac's functional  $m_s^t$  defined by

$$(2.45) \quad n_s^t = \frac{1}{\xi(s, X_s)} m_s^t \xi(t, X_t).$$

In fact we have

**Theorem 2.3.** *The renormalized Kac functional  $n_s^t$  defined in (2.45) satisfies normality condition (2.24).*

**Proof.** Because of definition (2.40) and of the multiplicativity of  $m_s^t$ , we have

<sup>26</sup> Therefore, the renormalized process is a conditional space-time diffusion process in terms of the survival condition  $\xi(s, x) = P_{(s, x)}^c[1]$

<sup>27</sup> The case of unbounded or singular  $c(t, x)$  can be handled similarly under an integrability condition, see Chapters 5 and 6



$$\begin{aligned}
P_{(s,x)}[n_s^a] &= \frac{1}{\xi(s,x)} P_{(s,x)}[m_s^a \xi(t, X_t)] = \frac{1}{\xi(s,x)} P_{(s,x)}[m_s^a P_{(t,X_t)}[m_t^a]] \\
&= \frac{1}{\xi(s,x)} P_{(s,x)}[m_s^a m_t^a] = \frac{1}{\xi(s,x)} P_{(s,x)}[m_s^a] \\
&= \frac{1}{\xi(s,x)} P_{(s,x)}^c[1] = 1,
\end{aligned}$$

completing the proof.

Therefore, when one treats Kac's semi-group, it is better to consider the renormalized process  $\{(t, X_t), \bar{P}_{(s,x)}: (s, x) \in [a, b] \times \mathbf{R}^d\}$ , from which one can always recover Kac's semi-group. Namely, one uses the renormalized process (it is a conservative diffusion process !) in computation, and when one needs Kac's semi-group, one applies formula (2.44) the other way round

$$(2.46) \quad P_{t-s}^c(f) = \xi \bar{P}_{t-s}(f \frac{1}{\xi}).$$

The crucial fact is this:  $\bar{P}_{t-s}$  is the semi-group of a *diffusion process* but Kac's one  $P_{t-s}^c$  is not.

*Remark.* The renormalized process will play an important role in Chapter 5, in which we consider the case of creation and killing  $c(t, x)$  with singularity. If  $c(t, x)$  is singular, some additional conditions will be needed to guarantee property (2.41) of the function  $\xi(s, x)$  defined in (2.40).

## 2.8. Time Change

In this and the following sections, we consider time-homogeneous diffusion processes. If we observe a diffusion process  $\{X_t, P_x\}$  with a defective clock, then the movement of the process looks slower or faster even though it stays on the same path. This is the so-called "time change".

Let  $c(x)$  be a positive continuous function and set

$$\tau(t, \omega) = \int_0^t c(X_s(\omega)) ds,$$

and the *time-change function*

$$\tau^{-1}(s, \omega) = \sup \{t: \tau(t, \omega) \leq s\}.$$

We define a new process by

$$(2.47) \quad \begin{aligned} Y_t &= X_{\tau^{-1}(t)}, \quad t < \zeta, \\ &= \Delta, \quad t \geq \zeta, \end{aligned}$$

where  $\zeta(\omega) = \tau(\infty, \omega)$  and  $\Delta$  is an extra point.

**Lemma 2.1.** (Nagasawa-Sato (1963)) *Let  $G_\lambda$  and  $G_\lambda^Y$ ,  $\lambda > 0$ , be the resolvent operators of  $\{X_t, P_x\}$  and  $\{Y_t, \zeta, P_x\}$ , respectively. Then, they satisfy*

$$(2.48) \quad G_\lambda^Y f = G_\lambda(c f) - \lambda G_\lambda\{(c-1)G_\lambda^Y f\},$$

$$(2.49) \quad G_\lambda f = G_\lambda^Y(f c^{-1}) - \lambda G_\lambda^Y\{(c^{-1}-1)G_\lambda f\}.$$
<sup>28</sup>

**Proof.** We set  $f(\Delta) = 0$ . By the definition of resolvent operators of semi-groups

$$(2.50) \quad G_\lambda^Y f(x) = P_x \left[ \int_0^\infty f(Y_t) e^{-\lambda t} dt \right] = P_x \left[ \int_0^\infty f(X_s) e^{-\lambda \tau(s)} d\tau(s) \right].$$

Therefore, we have

$$\begin{aligned} \lambda G_\lambda\{(c-1)G_\lambda^Y f\} &= \lambda P_x \left[ \int_0^\infty \{c(X_t) - 1\} G_\lambda^Y f(X_t) e^{-\lambda t} dt \right] \\ &= \lambda \int_0^\infty dt P_x \left[ e^{-\lambda t} (c(X_t) - 1) P_{X_t} \left[ \int_0^\infty d\tau(r) f(X_r) e^{-\lambda \tau(r)} \right] \right]. \end{aligned}$$

Because of the Markov property and  $\tau(t+r, \omega) = \tau(t, \omega) + \tau(r, \theta_t \omega)$ , where  $\theta_t$  is the shift operator,  $X_r(\theta_t \omega) = X_{t+r}(\omega)$ ,

$$= \lambda \int_0^\infty dt P_x \left[ e^{-\lambda t} (c(X_t) - 1) e^{\lambda \tau(t)} \int_t^\infty d\tau(s) f(X_s) e^{-\lambda \tau(s)} \right]$$

---

<sup>28</sup> For a general form of the formulae, cf. Theorem 2.1 in Nagasawa-Sato (1963)

$$\begin{aligned}
&= P_x \left[ \int_0^\infty d\tau(s) e^{-\lambda \tau(s)} f(X_s) \int_0^s dt (\lambda c(X_t) - \lambda) e^{-\lambda t + \lambda \tau(t)} \right] \\
&= P_x \left[ \int_0^\infty d\tau(s) e^{-\lambda \tau(s)} f(X_s) (e^{-\lambda s + \lambda \tau(s)} - 1) \right] \\
&= P_x \left[ \int_0^\infty d\tau(s) e^{-\lambda s} f(X_s) \right] - P_x \left[ \int_0^\infty d\tau(s) e^{-\lambda \tau(s)} f(X_s) \right] \\
&= G_\lambda(c f) - G_\lambda^Y(f),
\end{aligned}$$

which proves formula (2.48). Formula (2.49) can be shown in the same way.

Then we have

**Theorem 2.4.** *The time changed process  $\{Y_t, \zeta, P_x\}$  is a diffusion process with the generator*

$$(2.51) \quad A^Y = \frac{1}{c} A,$$

where  $A$  and  $A^Y$  denote the generators of the diffusion process  $\{X_t, P_x\}$  and  $\{Y_t, \zeta, P_x\}$  defined respectively through

$$\lambda - A = G_\lambda^{-1},$$

$$\lambda - A^Y = (G_\lambda^Y)^{-1}.$$

**Proof.** Apply  $(\lambda - A)$  to both sides of formula (2.48). Then

$$\begin{aligned}
(\lambda - A) G_\lambda^Y f &= (\lambda - A) G_\lambda \{c f - \lambda(c - 1) G_\lambda^Y f\} \\
&= c f - \lambda(c - 1) G_\lambda^Y f,
\end{aligned}$$

since  $(\lambda - A) = G_\lambda^{-1}$ . Therefore,

$$\lambda c G_\lambda^Y f - A G_\lambda^Y f = c f,$$

from which follows

$$\left(\lambda - \frac{1}{c}A\right) G_\lambda^Y f = f,$$

which implies (2.51).

## 2.9. Dirichlet Problem

Let  $\{X_t, P_x\}$  be a diffusion process on  $\mathbf{R}^d$  determined by an elliptic differential operator  $A$  given in (2.1). Let  $D$  be a compact connected domain in  $\mathbf{R}^d$  with a smooth boundary  $\partial D$ , and let  $T$  be the first hitting time to the boundary  $\partial D$ . Moreover, let  $g(x)$  be a continuous function on the boundary  $\partial D$ . Then

$$(2.52) \quad u(x) = P_x[e^{-\lambda T} g(X_T)]$$

solves the Dirichlet problem

$$(2.53) \quad \begin{aligned} (\lambda - A)u(x) &= 0, & \text{in } D, \\ u(x) &= g(x), & \text{on } \partial D. \end{aligned}$$

This assertion is treated in standard textbooks on Markov processes and potentials under more general problem setting,<sup>29</sup> but we shall need no such generality in this book.

Let  $U = U_\varepsilon$  be the first leaving time from an  $\varepsilon$ -neighbourhood of a point  $x \in D$ . Then

$$(2.54) \quad \begin{aligned} P_x[e^{-\lambda U} u(X_U)] &= P_x[e^{-\lambda U} P_{X_U}[e^{-\lambda T} g(X_T)]] \\ &= P_x[e^{-\lambda(U(\omega) + T(\theta_U\omega))} g(X_{U(\omega) + T(\theta_U\omega)}(\omega))] \\ &= P_x[e^{-\lambda T} g(X_T)] \\ &= u(x), \end{aligned}$$

where  $U(\omega) + T(\theta_U\omega) = T(\omega)$  and the strong Markov property have been applied.

<sup>29</sup> Cf. Dynkin (1965), Blumenthal-Gettoor (1968), Port-Stone (1978), Doob (1984), ...

Let  $f(x)$  be a bounded continuous function on  $\mathbf{R}^d$ . Then

$$\begin{aligned}
 (2.55) \quad \mathbb{P}_x\left[\int_0^U dt e^{-\lambda t} f(X_t)\right] &= \mathbb{P}_x\left[\int_0^\infty dt e^{-\lambda t} f(X_t)\right] - \mathbb{P}_x\left[\int_U^\infty dt e^{-\lambda t} f(X_t)\right] \\
 &= G_\lambda f(x) - \mathbb{P}_x\left[e^{-\lambda U} \mathbb{P}_{X_U}\left[\int_0^\infty dt e^{-\lambda t} f(X_t)\right]\right] \\
 &= G_\lambda f(x) - \mathbb{P}_x\left[e^{-\lambda U} G_\lambda f(X_U)\right].
 \end{aligned}$$

Since  $(\lambda - A)u = f$  holds for  $u(x) = G_\lambda f(x)$ , formula (2.55) yields

$$(2.56) \quad \frac{1}{\mathbb{P}_x[U]} \{\mathbb{P}_x[e^{-\lambda U} u(X_U)] - u(x)\} = \frac{1}{\mathbb{P}_x[U]} \mathbb{P}_x\left[\int_0^U dt e^{-\lambda t} (A - \lambda)u(X_t)\right],$$

for any  $u$  in the domain of the generator  $A$ .

The generator  $A$  is in the sense of the one in Theorem 2.4, which coincides with Dynkin's one in our case, cf. Dynkin (1965). Since various generators are defined for a semi-group depending on purposes, when we speak of "the generator" of a semi-group, we must be aware of its domain of definition. For detail see books mentioned at footnote 29.

Since the function  $u(x)$  defined at (2.52) is  $\lambda$ -harmonic, as is shown in (2.54), if we define " $\lambda$ -harmonic measure" by

$$H_U(x, B) = \mathbb{P}_x[e^{-\lambda U} 1_B(X_U)],$$

then we have

$$u(x) = \int_{\partial U} u(\xi) H_U(x, d\xi),$$

where  $\partial U$  denotes the boundary of the  $\varepsilon$ -neighbourhood, and hence it is differentiable.

Letting  $\varepsilon$  tend to zero, formula (2.56), which is called Dynkin's formula, yields the first equation of the Dirichlet problem (2.53) for the function  $u(x)$  defined in (2.52). Therefore, the second equality being clear, the  $u(x)$  solves the Dirichlet problem.

## 2.10. Feller's One-Dimensional Diffusion Processes

Let us consider a second order differential operator

$$(2.57) \quad A = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \quad a(x) > 0,$$

in an open interval  $(\alpha, \beta)$ , and define

$$(2.58) \quad W(x) = \int_c^x dy \frac{2b(y)}{a(y)},$$

where  $c \in (\alpha, \beta)$  is arbitrary but fixed. Then, with the function  $W(x)$ , the operator  $A$  can be represented in a divergence form

$$(2.59) \quad A = \frac{1}{2}a(x)e^{-W(x)}\frac{d}{dx}\left(e^{W(x)}\frac{d}{dx}\right).$$

As an example let us consider

$$A = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{d-1}{2}\frac{1}{x} - x\right)\frac{d}{dx}, \quad d \geq 2,$$

in  $(0, \infty)$ . Then,  $W(x) = (d-1)\log x - x^2$  and hence

$$A = \frac{1}{2}x^{1-d}e^{x^2}\frac{d}{dx}\left(x^{d-1}e^{-x^2}\frac{d}{dx}\right).$$

This case will be treated in Section 7.9 as an example of diffusion processes of Schrödinger equations with singular potentials.

In general, with a given continuous function  $W(x)$ , we define Feller's *canonical scale*  $S(x)$  by

$$(2.60) \quad S(x) = \int_c^x dy e^{-W(y)},$$

where  $c \in (\alpha, \beta)$  is arbitrary but fixed, and Feller's *speed measure*  $M$ , with a positive continuous function  $a(x)$ , by

$$(2.61) \quad \frac{dM}{dx} = \frac{1}{a(x)}e^{W(x)}.$$

In terms of the canonical scale  $S$  and the speed measure  $M$ , we define Feller's *canonical operator*  $\mathcal{A}$  by

$$(2.62) \quad \mathcal{A} = \frac{1}{2} \frac{d}{dM} \frac{d^+}{dS}$$

on a subset

$$\mathcal{D}(\mathcal{A}) = \{f : f \in C([\alpha, \beta]), d^+f \ll dS, \\ d(\frac{d^+f}{dS}) \ll dM, \text{ and } \frac{d}{dM} (\frac{d^+f}{dS}) \in C([\alpha, \beta]),\}^{30}$$

where  $d^+f/dS$  denotes the Radon-Nikodym derivative of the signed measure induced by  $f$  with respect to the measure induced by  $S$ . If  $f$  is differentiable, then  $d^+f/dS = df/dS$ .

The diffusion process determined by Feller's canonical operator  $\mathcal{A}$  with an appropriate boundary condition is called *Feller's one-dimensional diffusion process*.<sup>31</sup> Feller's diffusion process was constructed by Feller with the help of Hille-Yosida's semi-group theory; and by Itô-McKean (1965) using the transformation theory of diffusion processes; their method will be explained in the following.

First we construct a diffusion process determined by

$$(2.63) \quad \frac{1}{2} \frac{d}{dS} \frac{d^+}{dS}$$

Assume that  $S(x)$  is defined on  $\mathcal{D}_S = [\alpha, \beta]$ , and let  $\mathcal{R}_S = \text{the range of } S$ .

Let  $\{B_t, P_x\}$  be a one-dimensional Brownian motion, and set

$$(2.64) \quad \zeta(\omega) = \inf \{t : B_t(\omega) \notin \mathcal{R}_S\}.$$

Then we define a diffusion process on the transformed state space  $\mathcal{D}_S$  by

$$(2.65) \quad Y_t = \begin{cases} S^{-1}(B_t), & \text{for } t < \zeta, \\ \Delta, & \text{for } t \geq \zeta, \end{cases}$$

---

<sup>30</sup> The interval may be half-open or open

<sup>31</sup> Interesting phenomena of Feller's diffusion process are discussed in Brox (1986) when  $W(x)$  is a Brownian path. Cf. also Tanaka (1987), Kawazu-Tamura-Tanaka (1992)

$$Q_x = P_{S(x)}, \quad \text{for } x \in \mathcal{D}_S,$$

where  $\Delta$  denotes an extra point.

**Lemma 2.2.** *The diffusion process  $\{Y_t, t < \zeta, Q_x, x \in \mathcal{D}_S\}$  defined in (2.65) is determined by the second order differential operator*

$$\frac{1}{2} \frac{d}{dS} \frac{d^+}{dS}.$$

**Proof.** The (strong) Markov property of the transformed process is easy to show and left as an exercise. For  $f \in C^2(\mathcal{D}_S)$ ,  $f(\Delta) = 0$ ,

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{1}{h} \{Q_x[f(Y_h)] - Q_x[f(Y_0)]\} \\ &= \lim_{h \downarrow 0} \frac{1}{h} \{P_{S(x)}[f(S^{-1}(B_h))] - P_{S(x)}[f(S^{-1}(B_0))]\} \\ &= \frac{1}{2} \frac{d}{dy} \frac{d}{dy} f(S^{-1}(y)), \quad \text{where set } y = S(x), \\ &= \frac{1}{2} \frac{d}{dS(x)} \frac{d}{dS(x)} f(x), \end{aligned}$$

which completes the proof.

Let us consider a simple example:

$$\mathcal{A} = \frac{1}{2} x^2 \frac{d}{dx} \left( x^2 \frac{d}{dx} \right),$$

with

$$S(x) = -\frac{1}{x} + 1,$$

where  $\mathcal{D}_S = (0, \infty)$  and  $\mathcal{R}_S = (-\infty, 1)$ . Then

$$T_0(Y) = \inf \{t: Y_t = 0\} = \inf \{t: B_t = -\infty\} = T_{-\infty}(B) = \infty,$$

$$T_\infty(Y) = \inf \{t: Y_t = \infty\} = \inf \{t: B_t = 1\} = T_1(B) < \infty.$$

Therefore, the origin  $\{0\}$  is an inaccessible point of the diffusion process  $Y_t$ , while  $\{\infty\}$  is accessible. Since  $S^{-1}(y) = (1 - y)^{-1}$ ,



$$Y_t = \frac{1}{1 - B_t} \quad \text{for } t < \zeta,$$

$$Q_x = P_{1 - \frac{1}{x}} \quad \text{for } x \in \mathcal{D}_S = (0, \infty).$$

Now returning to the starting point, we consider the diffusion process determined by Feller's canonical operator

$$\begin{aligned} \mathcal{A} &= \frac{1}{2} \frac{d}{dM} \frac{d^+}{dS} \\ &= \frac{1}{2} a(x) e^{-2W(x)} \frac{d}{dS} \frac{d^+}{dS}, \end{aligned}$$

the expression of which suggests an application of time change. Define Kac's additive functional

$$(2.66) \quad \tau(t) = \int_0^t dr \frac{e^{2W(Y_r)}}{a(Y_r)},$$

with which we apply "time-change" to the diffusion process  $\{Y_t, t < \zeta, Q_x, x \in \mathcal{D}_S\}$  in Lemma 2.2.

Then we get

**Theorem 2.5.** (Itô-McKean (1965)) *Feller's canonical diffusion process  $\{Z_t, t < \tau(\zeta), Q_x, x \in \mathcal{D}_S\}$ <sup>32</sup> determined by*

$$\begin{aligned} \mathcal{A} &= \frac{1}{2} \frac{d}{dM} \frac{d^+}{dS} \\ &= \frac{1}{2} a(x) e^{-W(x)} \frac{d}{dx} \left( e^{W(x)} \frac{d^+}{dx} \right) \end{aligned}$$

is given through time change of the diffusion process  $Y_t$  in Lemma 2.2, namely,

$$(2.67) \quad \begin{aligned} Z_t &= Y(\tau^{-1}(t)) = S^{-1}(B(\tau^{-1}(t))), \\ Q_x &= P_{S(x)}, \quad \text{for } x \in \mathcal{D}_S, \end{aligned}$$

---

<sup>32</sup>  $\zeta$  is defined at (2.59)

where  $\tau$  is defined at (2.66) and, with  $c \in (\alpha, \beta)$ ,

$$S(x) = \int_c^x dy e^{-W(y)}.$$

## 2.11. Feller's Test<sup>33</sup>

Let  $\{X_t, t < \tau(\zeta), Q_x, x \in \mathcal{D}_S = (\alpha, \beta)\}$ <sup>34</sup> be Feller's canonical diffusion process. We assume that the process is regular in  $(\alpha, \beta)$ , namely,

$$P_x[T_y < \infty] > 0, \text{ for } \forall x, y \in (\alpha, \beta),$$

where  $T_y$  is the first hitting time

$$(2.68) \quad T_y = \inf \{t > 0: X_t = y\}.$$

A classification of boundary points was given by Feller (1957) in terms of the canonical scale  $S$  and the speed measure  $M$ . Let us formulate it for the left boundary point  $\{\alpha\}$ . Denote

$$(2.69) \quad S(\alpha, x] = S(x) - \lim_{y \downarrow \alpha} S(y),$$

$$M(\alpha, x] = M((\alpha, x]).$$

Then, the boundary point  $\{\alpha\}$  is classified as follows (*Feller's Test*):

$\{\alpha\}$ is	<b>Regular</b>	<b>Exit</b>	<b>Entrance</b>	<b>Natural</b>
if				
$S(\alpha, x]$	$< \infty$	$< \infty$	$= \infty$	$= \infty$
and				
$M(\alpha, x]$	$< \infty$	$= \infty$	$< \infty$	$= \infty$

<sup>33</sup> At the first reading this section may be skipped until Section 7.9

<sup>34</sup> We denote Feller's canonical diffusion process again by  $X_t$  instead of  $Z_t$

**Theorem 2.6.** (Feller (1957)) *If the left boundary point  $\{\alpha\}$  is*

- (i) *regular, then  $\{\alpha\}$  is accessible from  $(\alpha, \beta)$ , and  $(\alpha, \beta)$  is accessible from  $\{\alpha\}$ ,*
- (ii) *exit, then  $\{\alpha\}$  is accessible from  $(\alpha, \beta)$ , but  $(\alpha, \beta)$  is inaccessible from  $\{\alpha\}$ ,*
- (iii) *entrance, then  $\{\alpha\}$  is inaccessible from  $(\alpha, \beta)$ , but  $(\alpha, \beta)$  is accessible from  $\{\alpha\}$ ,*
- (iv) *natural, then  $\{\alpha\}$  is inaccessible from  $(\alpha, \beta)$ , and  $(\alpha, \beta)$  is inaccessible from  $\{\alpha\}$ .*

**Proof.** Let us define for  $\lambda > 0$  and  $y \in (\alpha, \beta)$

$$(2.70) \quad P_\alpha[e^{-\lambda T_y}] = \lim_{x \downarrow \alpha} P_x[e^{-\lambda T_y}],$$

$$P_y[e^{-\lambda T_\alpha}] = \lim_{x \downarrow \alpha} P_y[e^{-\lambda T_x}].$$

It is clear that  $\{\alpha\}$  is accessible from  $y \in (\alpha, \beta)$  if  $P_y[e^{-\lambda T_\alpha}] > 0$ , while inaccessible if  $P_y[e^{-\lambda T_\alpha}] = 0$ ; and  $(\alpha, \beta)$  is accessible from  $\{\alpha\}$ , if  $P_\alpha[e^{-\lambda T_y}] > 0$ , but inaccessible if  $P_\alpha[e^{-\lambda T_y}] = 0$ . Therefore, our proof is reduced to the evaluation of

$$(2.71) \quad u^y(x) = P_x[e^{-\lambda T_y}] \quad \text{or} \quad w^y(x) = \frac{1}{P_y[e^{-\lambda T_x}]}.$$

Let  $y \in (\alpha, \beta)$  be fixed. We have shown in Section 2.9 that

$$(2.72) \quad u(x) = P_x[e^{-\lambda T_y} g(X_{T_y})]$$

solves the Dirichlet problem

$$(2.73) \quad (\lambda - \mathcal{A})u = 0, \quad \text{in } (\alpha, \beta) \setminus \{y\},$$

$$u(y) = g(y),$$

in one-dimension. Therefore,  $u(x) = u^y(x) = P_x[e^{-\lambda T_y}]$ , for a fixed  $y$ , satisfies (2.73) with  $g(y) = 1$ , namely it is  $\lambda$ -harmonic in  $(\alpha, \beta) \setminus \{y\}$ .

Now let us define

$$(2.74) \quad \sigma_S [c, y] = \int_c^y S[c, z]M(dz),$$

$$\sigma_M [c, y] = \int_c^y M[c, z]dS(z),$$

and

$$(2.75) \quad \sigma_S(\alpha, y) = \lim_{c \downarrow \alpha} \sigma_S[c, y],$$

$$\sigma_M(\alpha, y) = \lim_{c \downarrow \alpha} \sigma_M[c, y].$$

Then, it is routine to check that the boundary point  $\{\alpha\}$  can be classified also in terms of  $\sigma_S(\alpha, y)$  and  $\sigma_M(\alpha, y)$  as follows (*Feller's Test*):

$\{\alpha\}$ is	<b>Regular</b>	<b>Exit</b>	<b>Entrance</b>	<b>Natural</b>
if				
$\sigma_S(\alpha, y]$	$< \infty$	$< \infty$	$= \infty$	$= \infty$
and				
$\sigma_M(\alpha, y]$	$< \infty$	$= \infty$	$< \infty$	$= \infty$

**Lemma 2.3.**

$$(2.76) \quad P_y[e^{-\lambda T_\alpha}] > 0, \text{ for } y \in (\alpha, \beta) \Leftrightarrow \sigma_S(\alpha, \cdot] < \infty,$$

$$(2.77) \quad P_y[e^{-\lambda T_\alpha}] = 0, \text{ for } y \in (\alpha, \beta) \Leftrightarrow \sigma_S(\alpha, \cdot] = \infty;$$

*namely the left boundary point  $\{\alpha\}$  is accessible, if it is regular or exit, while inaccessible if it is entrance or natural.*

**Proof.** Assume  $\sigma_S(\alpha, \cdot] < \infty$ . Then  $S(\alpha, \cdot] < \infty$ . Let  $w(x)$  be a non-negative decreasing solution of

$$(2.78) \quad (\lambda - \mathcal{A})w = 0, \text{ in } (\alpha, y), \text{ with } w(y) = 1,$$

and hence

$$\frac{1}{2} \frac{d}{dM} \frac{dw}{dS} = \lambda w, \text{ in } (\alpha, y).$$

Integrating twice with respect to  $dM$  and then  $dS$ , we have

$$\frac{dw}{dS}(y) S[x, y] - \{w(y) - w(x)\} = 2\lambda \int_x^y dS(\xi) \int_{\xi}^y w(z) dM(z),$$

which yields, through the substitution  $w(y) = 1$ ,

$$(2.79) \quad w(x) = 1 + \left(-\frac{dw}{dS}(y)\right) S[x, y] + 2\lambda \int_x^y dS(\xi) \int_{\xi}^y w(z) dM(z).$$

Since  $w(x) \geq w(z)$  for  $x \geq z$  and

$$(2.80) \quad \int_x^y dS(\xi) \int_{\xi}^y dM(z) = \int_x^y dS(\xi) M[\xi, y] = \int_x^y S[x, \xi] dM(\xi) = \sigma_S[x, y]$$

by partial integration, we have

$$w(x) \leq 1 + \left(-\frac{dw}{dS}(y)\right) S[x, y] + 2\lambda w(x) \sigma_S[x, y].$$

Because of the assumption  $\sigma_S(\alpha, \cdot] < \infty$ , there exists  $y_0$  such that

$$2\lambda \sigma_S[x, y_0] < \frac{1}{2},$$

and hence

$$\frac{1}{2} w(x) \leq 1 + \left(-\frac{dw}{dS}(y_0)\right) S[x, y_0],$$

which yields

$$\lim_{x \downarrow \alpha} w(x) \leq 2 \left\{ 1 + \left(-\frac{dw}{dS}(y_0)\right) S(\alpha, y_0] \right\} < \infty.$$

Let  $\alpha < c < x < y$ . Then, by the strong Markov property, we have

$$(2.81) \quad \mathbb{P}_y[e^{-\lambda T_c}] = \mathbb{P}_y[e^{-\lambda T_x}] \mathbb{P}_x[e^{-\lambda T_c}].$$

Therefore, for a fixed  $y$ ,

$$(2.82) \quad w(x) = w^y(x) = \frac{1}{\mathbb{P}_y[e^{-\lambda T_x}]} = \frac{\mathbb{P}_x[e^{-\lambda T_c}]}{\mathbb{P}_y[e^{-\lambda T_c}]} = \text{const } u(x),$$

where  $u(x) = \mathbb{P}_x[e^{-\lambda T_c}]$  for a fixed  $c$ , and hence the  $w(x)$  is monotone decreasing and satisfies  $w(y) = 1$  and

$$(\lambda - \mathcal{A})w = 0, \quad \text{in } (\alpha, y),$$

since we can let  $c \downarrow \alpha$ , it consequently satisfies (2.78). Therefore, we have

$$\lim_{x \downarrow \alpha} w^y(x) = \frac{1}{\mathbb{P}_y[e^{-\lambda T_\alpha}]} < \infty,$$

and hence

$$\mathbb{P}_y[e^{-\lambda T_\alpha}] > 0.$$

Conversely, assume  $\sigma_S(\alpha, \cdot] = \infty$ . Since  $w(z) \geq w(y) = 1$  for  $z \leq y$ , formula (2.79) yields

$$w(x) \geq 1 + \left(-\frac{dw}{dS}(y)\right) S[x, y] + 2\lambda \int_x^y dS(\xi) \int_\xi^y dM(z),$$

and hence because of (2.80) we have

$$\lim_{x \downarrow \alpha} w(x) \geq 1 + \left(-\frac{dw}{dS}(y)\right) S(\alpha, y] + 2\lambda \sigma_S(\alpha, y] = \infty.$$

Since  $w(x) = w^y(x) = \frac{1}{\mathbb{P}_y[e^{-\lambda T_x}]}$  is a non-negative decreasing solution of (2.78), we have

$$\lim_{x \downarrow \alpha} w^y(x) = \lim_{x \downarrow \alpha} \frac{1}{\mathbb{P}_y[e^{-\lambda T_x}]} = \infty,$$

and hence

$$P_y[e^{-\lambda T_\alpha}] = 0,$$

which completes the proof of the lemma.

**Lemma 2.4.** (i) *If the left boundary point  $\{\alpha\}$  is regular or entrance, then*

$$P_\alpha[e^{-\lambda T_x}] > 0, \quad \text{for } x \in (\alpha, \beta);$$

*namely  $(\alpha, \beta)$  is accessible from  $\{\alpha\}$ .*

(ii) *If  $\{\alpha\}$  is exit or natural, then*

$$P_\alpha[e^{-\lambda T_x}] = 0, \quad \text{for } x \in (\alpha, \beta);$$

*namely  $(\alpha, \beta)$  is inaccessible from  $\{\alpha\}$ .*

**Proof.** Assume that the left boundary point  $\{\alpha\}$  is regular or entrance. Let  $u(x)$  be an increasing non-negative solution of the Dirichlet problem

$$(2.83) \quad (\lambda - \mathcal{A})u = 0 \quad \text{in } (\alpha, y), \quad \text{with } u(y) = 1,$$

and hence

$$\frac{1}{2} \frac{d}{dM} \frac{du}{dS} = \lambda u,$$

from which we have

$$(2.84) \quad \frac{du}{dS}(x) - \frac{du}{dS}(c) = 2\lambda \int_c^x u(z)M(dz).$$

Since  $u(x)$  is increasing,  $du/dS \geq 0$ . On the other hand

$$\frac{d}{dx} \left( \frac{du}{dS} \right) = 2m \mathcal{A} u = 2m \lambda u > 0,$$

where  $m = e^{W/a}$ , and hence  $du/dS$  is strictly increasing in the interval  $(\alpha, y)$ . Consequently, we have

$$(2.85) \quad 1 \geq u(y) - u(\alpha) = \int_\alpha^y dS \frac{du}{dS} > \frac{du}{dS}(\alpha) S(\alpha, y),$$

which yields

$$(2.86) \quad \frac{du}{dS}(\alpha) S(\alpha, y] \leq \gamma \{u(y) - u(\alpha)\}, \quad \text{with } \gamma < 1.$$

Therefore, (2.84) together with (2.86), implies

$$u(y) - u(\alpha) = 2\lambda \int_{\alpha}^y dS(\xi) \int_{\alpha}^{\xi} u(z) M(dz) + \gamma \{u(y) - u(\alpha)\}.$$

Since  $u(z) \leq u(y)$  for  $z \leq y$ , we have

$$\begin{aligned} (1 - \gamma) \{u(y) - u(\alpha)\} \\ \leq 2\lambda u(y) \int_{\alpha}^y dS(\xi) \int_{\alpha}^{\xi} M(dz) = 2\lambda u(y) \sigma_M(\alpha, y] < \infty, \end{aligned}$$

where  $\sigma_M(\alpha, y] < \infty$ , because of the assumption that  $\{\alpha\}$  is regular or entrance. Since  $\sigma_M(\alpha, y] \downarrow 0$  (as  $y \downarrow \alpha$ ), there exist  $y_0 \in (\alpha, y)$  such that  $2\lambda \sigma_M(\alpha, y_0] < 1 - \gamma$ , and hence we have

$$u(y_0) - u(\alpha) < u(y_0),$$

namely,

$$u(\alpha) > 0.$$

Therefore, since the function

$$u(x) = u^y(x) = P_x[e^{-\lambda T_y}]$$

is an increasing non-negative solution of the Dirichlet problem (2.83), we have

$$u^y(\alpha) = P_{\alpha}[e^{-\lambda T_y}] > 0,$$

which proves the first assertion of the lemma.

Let us prove the second assertion. Assume  $\sigma_M(\alpha, \cdot] = \infty$ . Integrating both sides of (2.84) with respect to  $dS$ , we have



$$u(x) - u(c) - \frac{du}{dS}(c)S[c, x] = 2\lambda \int_c^x dS(\xi) \int_c^\xi u(z) M(dz).$$

Since  $u(c) \leq u(z) \leq u(y) = 1$ ,

$$1 \geq u(x) - u(c) \geq \frac{du}{dS}(c)S[c, x] + 2\lambda u(c) \int_c^x dS(\xi) M[c, \xi],$$

and hence

$$1 \geq 2\lambda \lim_{c \downarrow \alpha} u(c) \sigma_M(\alpha, x),$$

which implies  $\lim_{c \downarrow \alpha} u(c) = 0$ , since  $\sigma_M(\alpha, x) = \infty$ . Applying this to  $u(x) = u^y(x) = P_\alpha[e^{-\lambda T_y}]$ , we have

$$\lim_{c \downarrow \alpha} u(c) = P_\alpha[e^{-\lambda T_y}] = 0,$$

which proves the second assertion of the lemma.

Lemma 2.3 and Lemma 2.4 complete the proof of Theorem 2.6.

As an example let us consider

$$(2.87) \quad A = \frac{1}{2} \frac{d^2}{dx^2} + \varepsilon \frac{1}{x} \frac{d}{dx},$$

in  $(0, \infty)$ , which includes the case of the radial part of the  $d$ -dimensional Brownian motion, i.e.,  $d$ -Bessel process with  $\varepsilon = (d - 1)/2$ . Then,

$$(2.88) \quad W(x) = 2\varepsilon \log x,$$

and hence

$$(2.89) \quad M(c, x) = \int_c^x dy e^{W(y)} = \int_c^x dy y^{2\varepsilon} \\ = \begin{cases} \frac{1}{1 + 2\varepsilon} (x^{1+2\varepsilon} - c^{1+2\varepsilon}), & \text{for } \varepsilon \neq -1/2, \\ \log x - \log c, & \text{for } \varepsilon = -1/2. \end{cases}$$

Therefore

$$(2.90) \quad M(0, x] \begin{cases} < \infty, & \text{if } \varepsilon > -1/2, \\ = \infty, & \text{if } \varepsilon \leq -1/2. \end{cases}$$

On the other hand

$$(2.91) \quad S[c, x] = \int_c^x dy e^{-W(y)} = \int_c^x dy y^{-2\varepsilon} \\ = \begin{cases} \frac{1}{1-2\varepsilon} (x^{1-2\varepsilon} - c^{1-2\varepsilon}), & \text{for } \varepsilon \neq 1/2, \\ \log x - \log c, & \text{for } \varepsilon = 1/2, \end{cases}$$

and hence

$$(2.92) \quad S(0, x] \begin{cases} < \infty, & \text{if } \varepsilon < 1/2, \\ = \infty, & \text{if } \varepsilon \geq 1/2. \end{cases}$$

Consequently, the origin  $\{0\}$  is

$$(2.93) \quad \begin{aligned} & \text{"exit",} && \text{if } \varepsilon \leq -1/2, \\ & \text{"regular",} && \text{if } -1/2 < \varepsilon < 1/2, \\ & \text{"entrance",} && \text{if } 1/2 \leq \varepsilon. \end{aligned}$$

Another example which will be considered in Chapter 7 is

$$(2.94) \quad A = \frac{1}{2} \frac{d^2}{dx^2} + \left( \varepsilon \frac{1}{x} - x \right) \frac{d}{dx}, \quad \text{in } (0, \infty).$$

In this case

$$(2.95) \quad W(x) = 2\varepsilon \log x - x^2.$$

Since the term  $-x^2$  vanishes near the origin, it does not contribute the divergence or convergence of the integrals  $M(0, x]$  and  $S(0, x]$ , and hence (2.93) also holds for the diffusion process determined by (2.94).



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