Chapter 2
Investigations of Tides from the Antiquity to Laplace

Vincent Deparis, Hilaire Legros, and Jean Souchay

Abstract Tidal phenomena along the coasts were known since the prehistoric era, but a long journey of investigations through the centuries was necessary from the Greco-Roman Antiquity to the modern era to unravel in a quasi-definitive way many secrets of the ebb and flow. These investigations occupied the great scholars from Aristotle to Galileo, Newton, Euler, d'Alembert, Laplace, and the list could go on. We will review the historical steps which contributed to an increasing understanding of the tides.

2.1 Introduction

In the Western world, the first questionings about the ebb and flow date back to the 4th century B.C., when learned people of Greece began to acquire a precise knowledge of motions in the sea thanks to travels mainly driven by conquests. They raised basic questions such as: 'What causes this wide, periodic, breathing-like motion?' 'Why is it so small in the Mediterranean unlike in large oceans?' The phenomenon was disconcerting, for it is extremely regular in time and irregular in space. From that time to Newton and Laplace, explanations of the tidal phenomena were numerous, sometimes contradictory, often ingenious. They constitute an adventure of the human thought, which we will analyse in this chapter. In particular we will try to illustrate: when the origin of the tidal phenomena was discovered; how the mathematical and physical tools to describe the tides were developed; how

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the scientists succeeded in solving a problem which at the beginning had utterly bewildered them.

The concept of tide has two facets. First, we can consider the *tidal force*, which arises any time an extended body A is subject to the gravitation of another body B. The tidal force exists even in case A is a rigid body and not capable of deformation. Second, we can consider the *effects* of the tidal force exerted by B *on deformable parts* of A, for example ocean tides due to the combined gravitational tidal torque exerted by the Moon and the Sun on the bulk of the oceans. Another example is the terrestrial tides, due to the elasticity of the Earth's crust, ordinarily not perceptible but technically observable since the end of the 19th century. In our study, we will focus on the concept of tidal force rather than that of tidal deformation, even if it was via observations of the latter that the scientists have finally understood the real cause of the former.

We have divided our study chronologically. In the first section we discuss observations of tides from the Antiquity to the beginning of the 17th century. We show that learned people of the Antiquity and of the Middle Ages already had a good inkling of the nature and the behavior of ocean tides, and that hypotheses concerning their origin proliferated. In the 17th century several theories dominated the debates, which had little in common with one another. We describe in particular the theories by three great scientists: Kepler, Galileo, and Descartes. One section will be devoted entirely to Newton. We give details on his explanation of tides in his *Philosophiae Naturalis Principia Mathematica* (*Principa* for short), published in 1687. This work, based on a succession of geometric considerations, evaluates among other things the amplitude of the tidal force, and is regarded as the starting point of the true explanation of tides. Another section deals with the works of Daniel Bernoulli, Euler, and d'Alembert: in a relatively short span of time around 1740, these scientists improved the calculations of tidal effects exploiting then new, very efficient tools of calculus.

We conclude our history with the monumental work of Laplace, which was elaborated over a period of more than half a century. Laplace's work, supported by the tool of spherical harmonics, is the foundation of the modern theory of tides. A more complete study of the tides from the antiquity to modern times was done by D.E. Cartwright [2].

2.2 Study of Tides in the Antiquity[1]

The first precise recorded observations of tides go back to the Antiquity, outside the Mediterranean where the ebb and flow phenomena are negligible and cannot be easily detected. Greek mathematicians listened to travelers, often involved in military conquests, to form their description of oceanic tides. Among them, Nearchus,[2]

[1] We make an important use of the work of P. Duhem [8, 9].

[2] Born in Crete around 360 B.C., he participated in the expedition of Alexander the Great, being in charge of a fleet of 120 vessels, transporting 10 000 people. He was in charge of establishing a

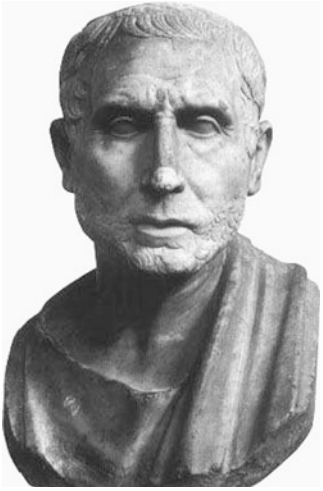

Fig. 2.1 Poseidonios (135–50 B.C.)

around 325 B.C., reported on tides of the Indian Ocean, whereas Pytheas,[3] an adventurer from Marseille, mentioned tides of the Atlantic during his trip from Gades in southern Spain to Brittany, 340–325 B.C. Pytheas noticed the fundamental correlation between ascending tides and the full Moon. The geometer Eratosthenes (276–194 B.C.)[4] carefully studied the flows inside the strait of Sicily. He pointed out that their frequency was nearly half a day, with a positive peak corresponding to the instant when the Moon is in the meridian or anti-meridian direction, and a negative peak when the Moon is close to the horizon. Around 150 B.C. Seleucos, a native of the Red Sea, pushed further the analysis of tides. He noticed that their amplitudes get all the greater as the declination of the Moon is larger.

Poseidonios[5] (Fig. 2.1) (135–50 B.C.), as a member of the Stoic school, was an adept of the Aristotelian conception of the universe, an imperfect sublunar world and a perfect supralunar one. For him the behavior of tides, governed by unexplained powers, confirmed the importance of the Moon in human destiny. This argument prevailed until the Middle Ages. In addition to the observation of the half-diurnal frequency of tides, Poseidonios recognized that their amplitudes (associated with what we call nowadays the tidal coefficient) are strongly linked with the phases of the Moon, being maximal during the syzygies (new or full Moon), minimal during the quadratures (when the Earth-Moon and Earth-Sun directions are perpendicular).

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new maritime route between the Indus and the Persian Gulf. His achievements were described by Strabo in *Geography* (vol. XV).

[3]One of the oldest scientific explorers. His accounts and astronomical observations were used later by Eratosthenes and Hipparchus.

[4]Astronomer, geographer, philosopher, and mathematician, well known for his measurements of the Earth radius by studying shadows produced by the mid-day Sun at Cyrene and Alexandria.

[5]Geographer and historian, he was keen on measurements (meridian length, height of the atmosphere, distance to celestial bodies). He wrote treatises in physics and meteorology.

Fig. 2.2 Strabo (58 B.C.–25 A.C.)

The geographer Strabo (58 B.C.–25 A.C.) (Fig. 2.2), who compiled the scientific knowledge of his times, emphasized the results obtained by Poseidonios. He mentioned that his predecessor recognized that oceanic tides undergo three kinds of motion, each related to an astronomical cycle: diurnal, monthly, and yearly. He also showed how Poseidonios understood that each time the elevation of the Moon reaches about 30°, the sea begins to rise progressively to reach a peak when the Moon crosses the meridian plane. Moreover Strabo reports that Poseidonios observed annual variations with peaks of amplitude around the equinoxes.

Supplementing these observations Pliny the Elder[6] (23–79 A.C.) made a remarkably precise discovery: he revealed a time lag between the instant when the Moon crosses the meridian/anti meridian and the instant when the tide reaches its maximum.

As the above enumeration shows, the Ancients knew the main characteristics of tides with remarkable accuracy and perspicacity. Nevertheless, a physical explanation remained to be found. Seleucos accepted the idea that the Earth rotates around its axis. He explained that this rotation creates a whirlwind which is modified by the presence of the Moon. The resulting effect is an activation of the oceanic motion. From his side, Poseidonios explained that the Moon had a larger influence than the Sun: the Sun, as a powerful fire, destroys all the vapor it creates at the surface of the ocean; the Moon, an attenuated fire, cannot vaporize the fluid masses and thus favors the ebb and flow. The Sun has no direct effect on the tides, but an indirect one as it lights the Moon, which in its turn acts on the oceans.

As far as we can gather from the surviving testimonies, Arab scholars did not add substantial knowledge or theory dealing with the tides. But as in the other fields of astronomy and mathematics, they played a key role in the transmission of scientific knowledge from the Greeks to the Western countries. The astronomer and astrologer

[6]Roman author, naturalist, and philosopher, he wrote *Naturalis Historia*, an encyclopedia of much of the knowledge of his time, the largest single work to have survived from the Roman empire to the present day, encompassing botany, zoology, astronomy, geology, and mineralogy.

Abu Maishar al-Bakhli (787–896), more often called Albumasar (787–886),[7] mentioned the three kinds of cycles accompanying the tides (semi-diurnal, fortnightly, and semi-annual) as well as the leading effect of the Moon as pointed out by Poseidonios. But in contrast with his Greek predecessor, he did not believe that moonlight was the cause of tides. Indeed two facts were difficult to explain according to the theory of Poseidonios: the existence of a peak of amplitude during a diurnal cycle when the Moon is located in the anti-meridian direction, and a maximum of the peak during a monthly cycle when the Moon is in conjunction with the Sun. Both cases correspond to a total absence of Moon light, which contradicts the theoretical foundations above.

Albumasar suggests an alternative explanation. As an astrologer ready to ascribe supernatural powers to celestial bodies, he says the Moon possesses a 'virtue' having the power of driving oceanic motions. The sea itself does not have the capacity to be disturbed under the influence of the moonlight, enhanced by the solar light. The cause of tides should be extrinsic to the sea and, after sieving various alternative explanations, he reaches the conclusion that the Moon is responsible of the uprising of oceanic masses, thanks to its own virtue. He supposes that the Sun too possesses a similar, though attenuated, virtue. Finally, Albumasar explains (correctly) that the lack of significant tidal phenomena in some basins comes from the their configuration, and not from a limitation of the lunar effect.

2.3 Variety of Theories in the Middle Ages

The medieval knowledge about tides came essentially from the writings of Pliny the Elder, until the Albumasar was translated into Latin in 1140. But it is worth mentioning the contribution of the Venerable Bede (about 672–735)[8] one whose interesting features is that it addresses the unexplored tides along the coasts of Great Britain. Bede made very valuable and accurate observations, remarking for instance that the maximum of the tide does not occur at the same time in various harbors along the coast, even when these harbors are located along the same meridian. This constitutes the first recognition of what is nowadays called the 'harbor establishment law'. It proves that Bede's perception of ebb and flow was particularly sharp, at an epoch when science generally stagnated. In contrast with these realistic observations, some original theories were proposed by other scholars. Paul Diacre (720–748), studying the maelströms in the North Sea, remarked that the direction of a whirlpool changed when the tide is reversed: therefore he attributed the tides to some abysses swallowing, then regurgitating, the oceanic masses.

[7]Persian astrologer, astronomer, and Islamic philosopher, he wrote a number of practical manuals on astrology that profoundly influenced the Muslim intellectual history.

[8]English monk at the Northumbrian monastery of St. Peter at Monkwearmouth. Well known as author and scholar, and for *The Ecclesiastical History of the English People*. In *The Reckoning of Time*, he deals with ancient and medieval views of cosmos, including explanations of astronomical phenomena.

In the 4th century, the philosopher and philologist Macrobe (about 370–430) imagined that the ocean had four arms crossed by big currents and that the tides originate from the conflict between these currents. This theory was popularized seven centuries later by the philosopher, mathematician, and naturalist Abelard de Bath (1080–1160), known for his interest in the Arab culture. None of these explanations involves the Moon or the Sun. A little later, we come back to a more convincing explanation of the nature of tides thanks to a professor of theology, Guillaume d'Auvergne (1190–1249), who reinstated the determining influence of the Moon at the center of the discussion. His explanation looks very close to the true one: willing to introduce some astrological principle involving the influence of the Moon, he proposed that the sea gets elevated toward the Moon which acts like a conductor, a disconcerting analogy with magnetism: when the Moon is ascending, it attracts the fluid as a magnet attracts iron when lifted. One of the interesting points in Guillaume d'Auvergne's conception of tides is the foreboding of gravitational attraction, another in the rejection of swallowing or regurgitating of fluid masses: the oceanic mass remains constant and the elevation is due to an agitation created by the Moon. Nevertheless, Guillaume d'Auvergne shows an ignorance of the semi-monthly cycle: he believes that a maximum of the tide occurs each month during the full Moon, attributing the lunar action to its lighting; he does not mention the symmetric case of the new Moon, when the satellite presents its dark face toward us.

Albert the Great (about 1200–1280) proposed similar explanations. For him, the Moon is doubly responsible for the tides. First it is a body of humid nature and so has the ability to attract the oceanic fluid as a magnet attracts iron. Second its brightness creates a heat which leads to the formation of a bulge—some kind of bubbling. He added that the water could be attracted only because of the salinity of the sea. St. Thomas Aquinas (1224–1274) still clung to the idea that the Moon possesses some virtue which gives it the capacity to stimulate motion inside fluids. Thus, the 12th and 13th centuries saw many theories dealing with the formation of tides, broadly based on two postulates. The first says that the Moon has some virtue; the second says that the Moon acts through its light. Either way, such theories face severe inconsistencies. For instance, how can the Moon cause the second semi-diurnal tide when it is located in the anti meridian direction, at a position where its influence should be minimum in terms of power, brightness, or heat? According to some audacious theorists, such as Robert Grosseteste (1175–1253),[9] the power of the Moon, when it is below the horizon, is maintained through the reflection of its light on the celestial sphere. This theory, though highly hypothetical, was supported by many contemporary physicists, such as Roger Bacon (1214–1294). Also, inconsistencies with astrological principles arose: the idea that moonlight acts on oceans by a kind of bubbling is against the principle that the Moon is a humid body cooling down and condensing any vapor. Physicists of the Paris school such as Jean Buridan (1202–1363) hesitate to select one out of these many theories.

In summary, the main difficulties in the theory of tides during the Middle Ages were as follows.

[9]English scholar, bishop of Lincoln. He showed deep interest in geometry and optics.

- Precise observations of tidal mechanisms by Greek and Arab scholars had been forgotten, replaced by obscurantism or wrong beliefs. For instance some theories defend the idea of a monthly cycle for the maximum amplitude instead of the real semi-monthly one.
- The fallacious idea that tides are due to alternating swallowing and regurgitating of water competed the correct idea that oceanic masses remain constant and that a bulge is produced at some place, balanced by a mass deficiency at another place.
- The problem of the geographical variations of tides is unsolved. In particular the following questions do not find any answer. Why are the tides so strong at some coastal locations and nearly non-existent at others? Why is the diurnal inequality (difference of maximum amplitudes between successive semi-diurnal tides) clearly present at some locations like the Red Sea, while it does not appear at others like the Atlantic?
- Does the Moon have an influence? If yes, how to characterize this influence? In terms of its light? Or of some virtue? And how to explain the maximum of the tide when the Moon is lying on the anti-meridian, at its maximum angular distance below the horizon?
- What is the exact role of the Sun? Does it directly raise the water mass by heating? Or does it act indirectly by reflection off the Moon?
- How can we explain the various periods linked to the tides? If the interpretation of the semi-diurnal cycle can be found, what is the cause of the fortnightly and the semi-annual cycles?

2.4 Tides in the Renaissance and the 17th Century

In the Renaissance and especially in the 16th century, the developments of the theory of tides come largely from physicians and astrologers. Their main aim was to establish a link between celestial bodies and phenomena occurring on the Earth. For that aim they made a clear choice between the various explanations prevailing at the end of the Middle Ages and enumerated at the end of the last section. For them the water mass remains constant; tides are obviously caused by the action of the Moon and less predominantly by the Sun; these two bodies do not produce their action through their light but through a specific virtue, which is comparable to the attraction between a magnet and iron.

2.4.1 Renaissance

In the beginning of the 16th century, a physician of Sienna, Lucius Bellantius, explains that the rays with which the Moon attracts the oceanic masses are not light rays, as can be proved during the conjunctions (new Moon), when the Moon shows us its dark face. For him the Moon acts through virtual rays, in the same way as a magnet attracts iron. Another physician, Frederik Grisogono (1472–1538)[10] insists on the modulating influence of the Sun, which in some cases enhances, in others attenuates, the action of the Moon. His intuition is amazingly close to reality. The total tide can be divided into two components, one due to the Moon, the other due to the Sun. They both produce a swelling of the oceanic volume, maximum at the point of the oceanic surface closest to that body, and also at the antipodal point. Grisogono supposes that each of the Moon and the Sun distorts the sea to form an ellipsoid of revolution, whose major axis is oriented toward it. This helps to explain how twice a month, in the full and the new Moon (syzygies) when the two major axes coincide, the amplitude of the tide is maximum. Such ideas from physicians and astrologers spread rapidly, the majority of them supporting the 'magnetic model' of attraction. Jules César Scalinger (1484–1558) claimed that just as iron is moved by a magnet without any physical contact, so the sea can be moved by the presence of a 'noble body' such as the Moon. The English scholar and physician William Gilbert (1544–1603), who undertook pioneering studies in electrostatics and magnetism, discovered that the Earth acts like a giant magnet. He also adhered to the idea that the Moon does not act through its light but through forces analogue to the magnetic one.

2.4.2 Kepler's Views

Kepler (1571–1630) agreed with Scalinger's conception and with the magnetic analogy. Of gravitational phenomena he had a remarkable visions, which opened the path to Newton. First, though himself an occasional astrologer, he was a fierce opponent to the astrological principle according to which the Moon attracts the sea by their common humid nature. He defended the concept of a mutual gravity depending on the sizes of the bodies involved. He explicitly claims that if the Earth ceased its attraction of the oceanic masses, the latter would instantaneously rise toward the Moon. For him gravity is a mutual disposition to join between bodies sensitive to each other. For instance he presumes that if two stones were placed at a little mutual distance apart and far from any other body, they should undergo a mutual attraction, leading to a junction in some intermediary location. Notice that we cannot qualify this attraction as 'universal', because for Kepler the two bodies concerned must be of such nature as to favor attraction. The concept of gravity-driven tides were appreciated among some of Kepler's contemporaries, and by the beginning of the 17th century his ideas had spread quite a lot, even if they encountered opponents such as the mathematician, philosopher, theologian, and astronomer Pierre Gassendi (1592–1655) who rejected the idea that the Sun could have any action on tides, still arguing that the action of the Moon comes from its humid nature. Another strong opponent

[10]Also mathematician, physicist, astronomer, born at Zadan in Croatia and educated at Padova. In addition to *Commentaries on Euclid's Elements*, he developed an important theory of tides, published in Venice in 1528.

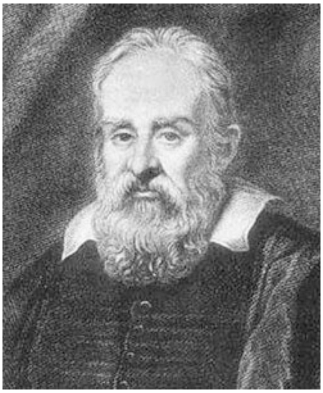


Fig. 2.3 Galileo Galilei (1564–1642)

to Kepler's views was Galileo Galilei (Fig. 2.3) (1564–1642) who expressed his astonishment that such a 'free and subtle spirit' (Kepler's) could defend the idea of any power of the Moon on the water, thus betraying attachment to some occult and childish principles.

2.4.3 Galileo: An Original Concept

Galileo's opposition to Kepler's explanations was motivated by the fact that he himself developed a theory based on new principles of mechanics, very different from all those already described. The fourth day of his *Dialogues Concerning the Two Chief World Systems* [11] (published 1632) is devoted to the problem of tides, and gives a full account of his approach starting from the combination of the Earth's rotation around its axis and its orbital motion around the Sun. This theory could have been imported from the work of Celio Calcagnini[11] (1479–1541) published posthumously at Basel in 1544. Galileo's explanations relies on the analogy between the tides and the motion of water inside a vessel. When the vessel is accelerated or decelerated, the inertia inclines the surface of water toward the back or front side of the vessel. For him, the tidal motions of the oceans follow the same laws, governed by a varying acceleration of the water coming from the combination of the Earth's diurnal rotation plus its annual revolution around the Sun. According to this theory, if only one of these motions existed and not the other, the ocean would be in equilibrium.

These motions, when combined, produce the same kind of displacements as that of water in a vessel. For a given point at the circumference of the Earth, the two velocities due to the rotation and the revolution sometimes add together, sometimes subtract from each other (Fig. 2.4). Therefore the water masses are displaced alternately along the oriental and occidental coasts, causing a diurnal tide. Thus for

[11] Italian humanist and scientist from Ferrara, in his time a reputed astronomer.

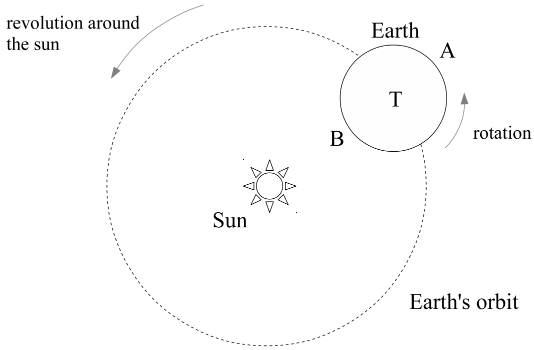


Fig. 2.4 Theory of tides according to Galileo: the ecliptic plane is supposed to coincide with the equator. Rotational and orbital motions are added in **A**, substracted in **B**

Galileo the origin of tides must be found exclusively from the combination of various terrestrial motions, and has no link at all with the influence of the Moon or of the Sun. In retrospect this theory does not look realistic. Nevertheless, the reality of the effect suggested by Galileo deserves some attention.

For him the two motions of rotation and revolution sometimes are added, sometimes substracted. Thus the points on the surface acquire a non-uniform velocity, implying an activation of water motion. Note that Galileo's idea can be associated with the concept which enabled the scholars of the Antiquity to explain the non-uniform motion of the Sun, the Moon, and the planets in the sky through a combination of motions with the help of a deferent and epicycles.

Souffrin [25] analysed this effect, illustrated in Fig. 2.5: the acceleration of a given point M at the surface of the Earth can be divided into two components: the first, γ_1, is the centripetal acceleration of the center of the Earth with respect to the Sun, due to the orbital motion, with angular velocity Ω; the second, γ_2, is the centripetal acceleration due to the rotational motion of the Earth around its center of

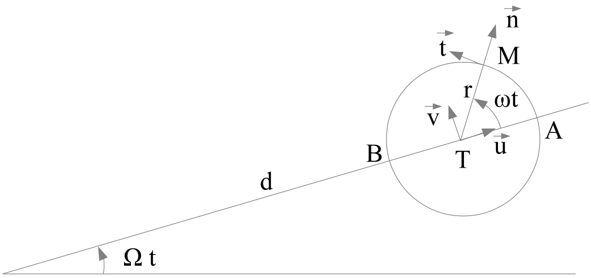


Fig. 2.5 Theory of tides according to Galileo: M is a point on the Earth surface, T the Earth center, S the Sun, r the Earth radius, d the radius of the Earth orbit, supposed circular. Ω is the angular velocity of the orbital motion, $\Omega + \omega$ the angular velocity of the rotational motion

mass, with angular velocity $\Omega + \omega$. Thus the total acceleration of M can be written as[12]

$$\gamma_M = \gamma_1 + \gamma_2 = -\Omega^2 d\mathbf{u} - (\omega + \Omega)^2 r\mathbf{n} \tag{2.1}$$

Since $\mathbf{u} = \cos\omega t\mathbf{n} - \sin\omega t\mathbf{t}$, the decomposition of γ_M along $\mathbf{n}$, $\mathbf{t}$ gives

$$\gamma_M = \gamma_n + \gamma_t = -\big(\Omega^2 d\cos\omega t + (\omega + \Omega)^2 r\big)\mathbf{n} + \Omega^2 d\sin\omega t\mathbf{t} \tag{2.2}$$

Thus the acceleration γ_M for a given point of the Earth has a normal component γ_n and a tangential component γ_t. The normal component has no significant effect, for it acts in the same direction as gravity and is negligible in comparison. The tangential component, though of very small size also, acts perpendicularly to gravity and can have a visible effect. This tangential component $\Omega^2 d\sin\omega t$ comes solely from the orbital motion. Because of the diurnal rotation, it is alternately directed eastward or westward.

Galileo's mistake comes from a misunderstanding of the orbital motion of the Earth. As Newton will show a little less than a century later, the Earth is kept in its orbit by the gravitational attraction of the Sun which acts on all terrestrial matters, the liquid part as well as the solid part. To a first approximation the water as well as the basins are subject to equal gravitational attractions $\Omega^2 d$ of the orbital motion. But the equality is exact only at the center of the Earth and not elsewhere, which constitutes the basis of the explanation of tides by Newton. The water and the basins, attracted in the same manner by the Sun, 'fall' toward it together: the relative motion between the water and the basins does not exist.

Of course Galileo did not know Newton's law of gravitation. No more than Kepler and other contemporaries was he able to understand the orbital motions of the planets. For him the revolution of the Earth is given naturally: it exists without any cause, guided by an imaginary physical principle. Consequently Galileo relied on purely kinematical principles and never adopted a dynamical one. Nothing and nobody could induce him to doubt his solution from the combination of the two Earth motions. Despite his misunderstanding, his will to develop a mechanical theory of tides was fundamentally new and his contribution was essential.

2.5 Descartes and His Theory of Vortices

In his *Principles of Philosophy* published in 1644 [7], Descartes (Fig. 2.6) (1596–1650) proposes an alternative theory of tides, relatively independent of the predecessors. He is convinced that everything in the universe is governed solely by the laws of motion, and that vacuum does not exist: as soon as vacuum arises, it gets filled with subtle matter organized in a system of vortices. This principle is applied to the solar system. The Sun occupies the center of the main vortex, and its proper

[12] Our method of proof, based on accelerations and vectors, is not that of Galileo who works with velocities only. But it is a faithful translation of his idea.

Fig. 2.6 René Descartes (1596–1650)

rotation (discovered at the beginning of the 17th century) is transmitted to the vortex itself, which transports the planets on their orbit. Each planet is at the center of its own vortex. Their proper rotations lead to the rotation of these secondary vortices which transport the satellites in their revolutions (Fig. 2.7).

Thus, the Moon is transported by the Earth's vortex. Starting from this statement, Descartes built up an intricate theory where the Moon, despite being transported by the Earth's vortex, does not move at the same velocity. This creates an obstacle and perturbs the symmetric flow of subtle matter, causing a displacement of the Earth's center with respect to the Earth's vortex. Because of the presence of the Moon, the

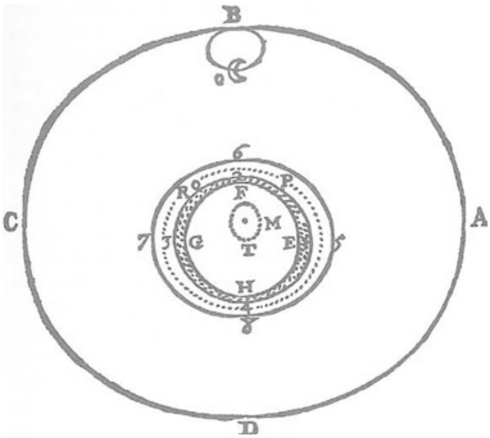


Fig. 2.7 The System of the World according to Descartes (from *Le Monde ou Traité de la lumière*, ed. Adam et Tannery, Paris, 1974). *ABCD* is the vortex of subtle material generated by the proper rotation of the Earth *EFGH*, with center T. *1, 2, 3, 4* represent the sea, and *5, 6, 7, 8* the atmosphere. Because of the presence of the Moon in B, the center M of the vortex does not coincide with T. The tides result from a differential pressure exerted by the vortex matter on the sea

Fig. 2.8 Issac Newton (1642–1724)

vortex material surrounding the Earth cannot flow freely. Consequently it exerts a differential pressure at the surface of the oceans, giving rise to tides.

In addition Descartes explains the existence of two tides per day from the drift between the center of the Earth and the center of the main vortex. Astonishingly, according to his views, the low tides arise when the Moon is located in the meridian and anti-meridian directions, which is opposite what happens in reality. Moreover, Descartes attributes the geographical variations of the high tides along a given coast to the fact that the Earth is not entirely covered by oceans. For him time delays are caused by various factors, such as the form of the coasts, the varying depths of the oceans, the influx from the rivers, as well as the action of winds. These intuitions were valid, for we know today that all these elements have to be taken into account in order to construct accurate tide tables. For Descartes, the semi-monthly period as well as the alternation of large and low tidal amplitudes linked with it come from the non-circularity of the Earth's vortex. All these considerations let us think that his theory of tides, though unrealistic, attests to a remarkable view of mind. In any case it relies on statements which have never been explained or verified, as the existence of vortices, the drift between the center of the Earth and the center of the main vortices, etc. Despite these negative aspects Descartes's explanations became very popular during his life, in particular among his French disciples. In his *Geographia Generalis*, the German geographer Bernard Varenius (1622–1650) [26] adopts Descartes's theory as the preferred one among various others, and this will helped its popularization.

2.6 Newton and the Gravitational Attraction: A Giant Step

In 1687, Isaac Newton (Fig. 2.8) (1642–1727) published his *Philosophiae Naturalis Principia Mathematica* (*Principia* for short) [23], which revolutionized our perception of the Universe. In *Principia*, Newton set out the law of gravitation and the three fundamental laws of motion: the principle of inertia, the principle of the rate of change of momentum, and the law of action-reaction, showing that the behavior of celestial bodies is deducible from these laws. One of the tremendous results of his theory was an explanation of the tidal phenomena. For Newton, the oceanic tides are explained by mechanical principles, as Galileo wanted them to be. Moreover they are a consequence of an attraction at a distance, following Kepler's intuition.

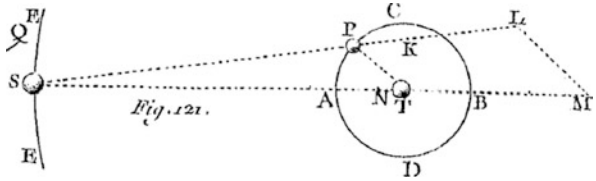


Fig. 2.9 The perturbing forces on the lunar motion (Fig. 121 of *Principia*, vol. I). *S* is the Sun, *T* the Earth, *P* the Moon with orbit *CABD*. The attraction of the Sun on the Earth is represented by the segment NS and the attraction of the Sun on the Moon by the segment LS, which can be divided into two parts: LM and MS

How did Newton reach the explanation of the tides, after explaining the orbital motion of the Earth around the Sun and that of the Moon around the Earth? In fact, his success came from a deep investigation of the orbital motion of the Moon, taking into consideration the departure of its trajectory from an exact ellipse, which was due to the gravitational perturbations of the Sun. In particular he observed the drift of the lunar nodes with respect to the ecliptic, with a 18.6 y period. Extrapolating these solar perturbations he guessed that they should also influence the oceanic masses.

2.6.1 The Solar Perturbation on the Orbital Motion of the Moon

As mentioned above, the tide was analysed by Newton when he was determining the perturbation by the Sun on the Moon orbiting around the Earth. We refer to Proposition 66 of *Principia*, vol. I. In order to solve the problem, Newton used the law of *parallelogram of forces*, well known since the 16th century when it was popularized by the Dutch engineer, mathematician, and philosopher Simon Stevin (1548–1620) and clearly set out by the mathematician Pierre Varignon (1654–1722) in his treatise *New Mechanics* published posthumously (1725). Newton's way of thinking is clearly shown in Fig. 121 of the Principia (see Fig. 2.9). It represents the Sun S, the Moon P, the Earth T, with their mutual distances SP, ST, PT. Newton represents the attraction by the Sun on the Earth (respectively on the Moon) by SN (respectively SL). The attraction SL itself is decomposed into two components: one, SM in the Sun to Earth direction, another one ML in the direction parallel to the Earth-Moon segment. One of the subtleties of Newton's proof is that he depicts the points N and T as coincident, although they have different statuses: T represents the position of the Earth and has a physical meaning, whereas N is used to measure the attraction SN exerted by the Sun, and has a mechanical meaning. From the law of gravitation and by calling F_{SP} (respectively F_{ST}) the forces exerted by the Sun on the Moon (respectively on the Sun)we have, using modern notations:

$$\mathrm{SL} = F_{\mathrm{SP}} = \frac{GM_S}{\mathrm{SP}^2}, \qquad \mathrm{SN} = F_{\mathrm{ST}} = \frac{GM_S}{\mathrm{ST}^2} \tag{2.3}$$

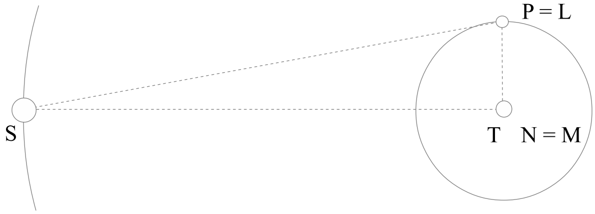


Fig. 2.10 The tidal force when the Moon is in quadrature. Because it is very far from S, P is considered as coinciding with L, and the attraction LS by the Sun on the Moon is practically equal to the attraction NS by the Sun on the Earth. The tidal force is represented by LN, equal to PT. It is added to the attraction of the Moon by the Earth

therefore

$$\frac{\mathrm{SL}}{\mathrm{SN}} = \frac{\mathrm{ST}^2}{\mathrm{SP}^2}, \qquad \mathrm{SL} = \frac{\mathrm{ST}^3}{\mathrm{SP}^2} \tag{2.4}$$

Thus the Sun acts on the Earth with the force F_{ST} represented by SN and on the Moon with the force F_{SP} represented by SL. The length and the direction of the two segments are not the same, and these differences characterize the difference of attraction exerted by the Sun. To determine how the Sun perturbs the orbital motion of the Moon around the Earth, Newton searches which part of SL has a real effect. He remarks that if the accelerations SN and SM are equal, they will change nothing in the relative motion of the two bodies P and T, because they will bring the same attraction, both in amplitude and direction. Thus only the component NM plays a role in the difference of attraction and consequently in the perturbation of the orbital motion of the Moon around the Earth. We must also take into account the component ML, in such a way that finally the perturbing forces exerted by the Sun are reduced to the two segments NM and LM. In modern notation, this way of thinking should be equivalent to calculating the difference between the vectors LS and NS. In the proof above, Newton has just revealed the presence of a solar tidal force, i.e. a differential force which is not due to the total gravitational attraction by the Sun on the Moon, but rather to the difference of attractions by the Sun on the Moon and the on Earth. This is a fundamental discovery in the theory of tides.

2.6.1.1 Case of Quadrature

In quadrature (when ST and TP are perpendicular) the sketch is simplified (Fig. 2.10). M coincides with T, and the lengths of SP and ST can be treated as equal, given the large distance ST from the Sun to the Earth. Moreover we have LM = PT, and LM is oriented along the direction from the Moon to the Earth (*Principia*, vol. I, Proposition 66). We conclude that the tidal force F_{tidal} is reduced to its component PT, and that the ratio of its amplitude to the attraction of the Earth by the Sun is given by (*Principia*, Proposition 66, Corollary 14)

$$\frac{F_{tidal}}{F_{\mathrm{ST}}} = \frac{\mathrm{PT}}{\mathrm{ST}}, \qquad F_{tidal} = \frac{F_{\mathrm{ST}} \times \mathrm{PT}}{\mathrm{ST}} = \frac{GM_S \times \mathrm{PT}}{\mathrm{ST}^3} \tag{2.5}$$

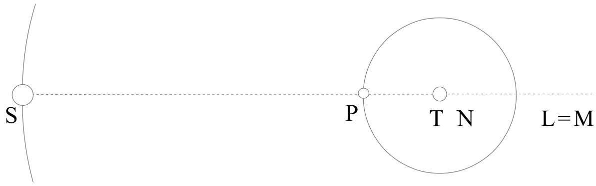


Fig. 2.11 The tidal force when the Moon is in opposition. It is represented by the segment LN, obtained by difference between the attraction LS by the Sun on the Moon and the attraction NS by the Sun on the Earth. Thus the tidal force is in opposition to the attraction of the Moon by the Earth

Thus in this particular case of quadrature we are in presence of a remarkable equivalence between the lengths of the segments representing the forces and the physical distances.

Moreover, applying Kepler's third law to the orbital motion of the Earth, we get

$$\omega_E^2 \mathrm{ST} = \frac{GM_S}{\mathrm{ST}^2}, \qquad \omega_E^2 \mathrm{ST}^3 = GM_S \tag{2.6}$$

where ω_E is the angular velocity, or *mean motion*, of the Earth. In Newton's era, the distance ST from the Earth to the Sun was not known with accuracy. From the last two equations we get

$$F_{tidal} = \omega_E^2 \mathrm{PT} \tag{2.7}$$

Now we can evaluate the ratio of the tidal force to the attraction F_{PT} exerted by the Earth on the Moon:

$$F_{\mathrm{PT}} = \frac{GM_E}{\mathrm{PT}^2} = \omega_M^2 \mathrm{PT} \tag{2.8}$$

where ω_M is the angular velocity, or *mean motion*, of the Moon, and (*Principia*, vol. I, Corollary 17)

$$\frac{F_{tidal}}{F_{\mathrm{PT}}} = \frac{\omega_E^2}{\omega_M^2} \tag{2.9}$$

2.6.1.2 Conjunction and Opposition

When the Moon P is in conjunction with the Sun (Fig. 2.11), L and M coincide. The Moon being closer than the Earth to the Sun, it is subject to a larger gravitational attraction. Newton shows that in this case the perturbing force by the Sun on the Moon is NM = 2 PT. This can be found from the equation

$$\mathrm{SM} = \mathrm{SL} = \frac{\mathrm{ST}^3}{\mathrm{SP}^2} \tag{2.10}$$

with SP = ST − PT. Expanding to the first order we have

$$\mathrm{SM} = \frac{\mathrm{ST}^3}{\mathrm{ST}^2(1-\mathrm{PT}/\mathrm{ST})^2} \approx \mathrm{ST} + 2\,\mathrm{PT} = \mathrm{ST} + \mathrm{NM} \tag{2.11}$$

Therefore, to a first approximation NM = 2 PT. The tidal force NM is twice bigger than in the case of quadrature.

In a similar and symmetric way, when the Moon is in opposition we can easily prove that SM = ST − 2 PT ≈ ST − NM. We still have NM ≈ 2 PT. In conclusion, in syzygies (new Moon and full Moon) the perturbing force exerted by the Sun on the Moon has the same value (2 PT) and is directed in the direction opposite to that of the gravitational attraction exerted by the Earth on the Moon.

Thanks to this ingenious way of geometrical representation of the attraction, Newton could calculate the two components LM and NM of the perturbing force of the Sun on the orbital motion of the Moon, not only in the special cases of syzygies and quadratures but also at any position of the Moon on its orbit. This helped Newton to study in detail the characteristics of the lunar motion, showing in particular that the Moon is accelerated on its orbit from the quadratures towards the syzygies, and that the lunar nodal line[13] undergoes a linear retrogradation.

2.6.2 Ocean Tides

After studying the perturbations exerted by the Sun on the orbital motion of the Moon, Newton in vol. I, Proposition 66, Corollary 19 shows how to apply the same principle to the terrestrial phenomenon of the tides.

2.6.2.1 Analogy Between the Lunar Motion and the Ocean Tides

The fundamental idea consists in substituting for the Moon a set of fluid bodies, then to replace this set by a continuous fluid ring inserted in a canal surrounding the Earth. Under the gravitational attraction of the Sun, the fluid inside the canal undergoes the same kind of gravitational perturbation as the Moon in its orbit, that is to say an acceleration during the syzygies (for the part of fluid oriented in the direction of the Sun and in the opposite direction) and a deceleration during the quadratures (for the part of fluid in the directions perpendicular to the direction of the Sun). This alternating motion gives rise to the tidal phenomena. Thus, thanks to the analogy with the lunar orbital motion, Newton understands that the various parts of the terrestrial globe, located at different distances of the Sun, are subject to different attractions by the Sun, which in their turn cause the oceanic motions.

2.6.2.2 Agreement with the Observed Characteristics of Ocean Tides

In *Principia*, vol. III, Proposition 24, Newton returns to the problem of ocean tides. Here the Moon no longer plays the role of a test body whose irregularities of motion

[13] The intersection of the orbital plane of the Moon with the plane of the ecliptic.

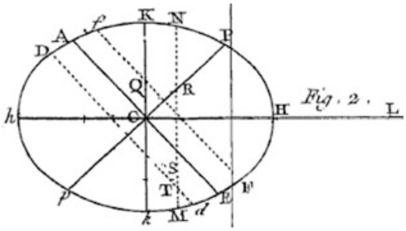


Fig. 2.12 The tidal ellipsoid and the diurnal inequality (Fig. 2 of *Principia*, vol. III). Under the effects of the tidal forces, the oceans change their global shape in an *ellipsoid* whose semi-major axis is directed toward a fictitious body delayed three hours with respect to the real body. *Pp* is the rotation axis, *AE* the equator, *Ff* a parallel. When the body is at a given declination, the two diurnal tides in *F* and *f* do not have the same amplitude: this characterizes the diurnal inequality

reveal the effects of the differential gravitational attraction exerted by the Sun, but as another celestial body providing the same kind of effects on the oceans. His clear objective goes beyond identifying the origin of tides, to explaining the actual tidal phenomena observed along the coasts. Attention is paid to their periodicities, amplitudes, and characteristics as a function of the relative positions of the Moon and the Sun.

Newton asserts that under the action of a celestial body (the Moon or the Sun), the sea at any instant takes the shape of an ellipsoid whose major axis is oriented toward the body. We have mentioned that 16th-century physicians and astrologers had already guessed at this phenomenon intuitively. Newton gives a full justification of the phenomenon, showing that it arises from a symmetry in the tidal force. As the Earth rotates, points on its surface pass alternately through the locations of maximum and minimum elevation of the water.

This explains the succession of low and high tides (Fig. 2.12). Moreover, Newton can explain the monthly periodicity of the tides: during the syzygies, the major axes of the ellipsoids due to the Moon and to the Sun are aligned, leading to the addition of the raising of the sea level, whereas during the quadratures, these axes are perpendicular and the effects at the sea level cancel, leading to an attenuation of the high tides. Newton also remarks that the maximum amplitude of the tides varies according to the distance of the perturbing body, which itself varies because of the ellipticity of its orbit around the Earth. In short, Newton fully explained the various periodicities of the tides, confirming the coherence and the validity of his theoretical assumptions.

By assuming that the seas are distorted into the figure of an ellipsoid, Newton accomplishes a significant step, that of adopting a simple figure of equilibrium, in the same way as he did to express the deformation of the Earth undergoing the effects of its rotation. Following this rotation, the ellipsoid moves in such a way that its major axis always points toward the perturbing body.

2.6.2.3 Remaining Questions

Nevertheless, this basic theoretical model faced several problems when confronted with the observations. First, there are significant time delays between the occurrence of the diurnal tides and the transit of the perturbing body at the meridian or at the anti-meridian. Second, the maxima of tides do not correspond exactly to the syzygies, as predicted by the theory. Third, the amplitudes of tides strongly change depending on the harbors where they are measured, even when the harbors are separated by short distances. Fourth, the diurnal inequality, i.e. the difference of amplitudes between successive high tides, does not show up significantly in observations, whereas theoretical calculations predict them to be large.

In fact the most important factor leading to these discrepancies between theoretical statements and observations lies on the fact that Newton's results are presented in the frame of a static theory: it is necessary to construct a dynamic theory in which the influence of the Coriolis force and the resonance phenomena are taken into account.

Nevertheless, even after recognizing the lack of perfect agreement between his theoretical investigations and the observations, Newton kept the explanations above, mentioning for example that the oceanic motions are delayed by the friction of the bottom of the basins. Notice that this phenomenon of inertia has been fully explored and validated in the 20th century to explain the secular deceleration of the proper rotation of the Earth. Thus in various places of his work, we see that Newton adheres to the idea of the inertia of oceans which necessitates further investigations. For instance he explains the absence of diurnal inequality as well as the presence of time delays between coastal points, by flow effects that conserve perturbing oscillations for some duration, in the same way as water moved in a vessel.

2.6.2.4 Calculation of the Solar Tide

After the periodicities of tides were explained and a hypothesis was made about their time delay, an important challenge remained to be undertaken by Newton: it consisted in starting from the tidal force exerted by the perturbing body (the Moon or the Sun) and deducing the amplitude of the ebb and flow. This could be done for the Sun, thanks to the previous calculations presented previously, of the tidal force exerted by the Sun on the Moon's orbital motion. But it could not be done for the Moon, because its mass was unknown. Thus in a first step, Newton attempted to calculate only the amplitude of the solar tides. Rather than trying to find a final formula, he proceeded by successive numerical approximations gathered in *Principia*, vol. III, Propositions 25 and 36. We saw in Eq. (2.9) that the ratio of the tidal force F_{tidal} exerted by the Sun on the Moon to the force of attraction F_{TP} exerted by the Earth on the Moon in quadrature could be expressed as

$$\frac{F_{tidal}}{F_{\mathrm{PT}}} = \frac{\omega_E^2}{\omega_M^2} = \frac{T_M^2}{T_E^2} = 1/178.725 \tag{2.12}$$

where $T_M = 27.32d$ and $T_E = 365.25d$ are the sidereal periods of revolution of the Moon and of the Earth. Once this result is obtained, it can be used to calculate the solar tide at the surface of the Earth, still in the case of quadrature. In particular it is possible to calculate the ratio of this solar tide to the gravitational acceleration g. Two facts are used for this purpose:

– Since the Moon being 60 times more distant from the center of the Earth than a point on the surface, $g/F_{\mathrm{PT}} = 60^2 = 3600$.
– As the tidal force exerted by the Sun is directly proportional to the distance PT between the point considered and the center of the Earth, this tidal force F'_{tidal} on the surface of the Earth is 60 times smaller than the same tidal force F_{tidal} at the distance of the Moon: $F'_{tidal} = F_{tidal}/60$.

Hence the ratio F'_{tidal}/g is

$$\frac{F'_{tidal}}{g} = \frac{1}{60 \times 3600 \times 178.725} = \frac{1}{38\,604\,600} \tag{2.13}$$

This is the ratio for points on the surface of the Earth in quadrature, i.e. located at 90° with respect to the direction of the Sun. As seen previously, for points in conjunction with the Sun, i.e. for which the Sun is at zenith or at nadir, the ratio is twice bigger. Notice that in quadrature the tidal force is pushing the surface toward the bottom whereas in conjunction it raises the surface toward the attracting body, the Sun. Therefore, the amplitude of the total acting tidal force is 3 times bigger than the amplitude calculated above, that is to say $g/12\,868\,200$.

The next step consists in calculating the elevation of water under the sole action of the Sun. To simplify, Newton considers a fictitious Earth completely covered by oceans, and having the same density. Then he uses the same kind of trick as the one he used to calculate the bulging of the Earth under the centrifugal acceleration due to the rotation: he considers two channels filled with a homogeneous fluid extending radially from the center of the Earth to the surface, one in the direction of the Sun, the other in the direction perpendicular (Fig. 2.14). The first channel is longer, for the tidal force is substracted from the gravity, whereas in the second channel it is added. Assuming proportionality between the bulging of the surface and the perturbing force, he first remarks that the centrifugal force which is 289 times smaller than g at the equator leads to a difference of 27.7 km (in fact 85 472 Paris feet) between the equatorial radius (semi-major axis) and the polar radius (semi-minor axis) of the bulging Earth. By analogy, following the same proportionality, the solar tidal force being 12 868 200 times smaller than g will create a difference of level of 60 cm between a point in quadrature and another point in conjunction with the Sun.

2.6.2.5 Ratio of the Lunar Tide to the Solar Tide and the Mass of the Moon

The mass of the Moon being unknown, Newton cannot calculate directly the amplitude of the lunar tides. In *Principia*, Proposition 37, he considers the inverse problem: knowing the amplitude of the tides as a function of the relative positions of the Moon and the Sun, is it possible to calculate their respective attractions on the oceans and to deduce the mass of the Moon? The basic hypothesis is that the height of the tides caused by each body is proportional to the size of its tidal action. Close to the equinoxes, the two bodies are located on the equatorial plane and during an equinoctial syzygy, the height of the tide is maximal because the actions of the two bodies are maximal and added together. Thus the height of the tide $h_{syz.}$ can be written

$$h_{syz.} = A(M + S) \tag{2.14}$$

where M and S are respectively the actions of the Moon and the Sun, and A is a coefficient of proportionality. About seven days after the syzygy, when the Moon is in equinoctial quadrature, the actions of the Moon and of the Sun must be substracted from each other. Moreover the Moon is no longer on the equator, its declination δ being roughly 23° (if we neglect the inclination of the lunar orbit with the ecliptic). This diminishes the strength of its action, the coefficient of diminution being $\cos^2\delta$. This correction concerns only the semi-diurnal tides, as it will be shown by Laplace. Thus in this case the height of the tide is

$$h_{quad.} = A\left(M\cos^2\delta - S\right) \tag{2.15}$$

From the last two equations, we find

$$\frac{h_{syz.}}{h_{quad.}} = \frac{M/S + 1}{\cos^2\delta(M/S) - 1} = \frac{\mu + 1}{\mu\cos^2\delta - 1} \tag{2.16}$$

where $\mu = M/S$ is the ratio of the action of the Moon to that of the Sun. With this formula it is theoretically possible to calculate μ from the observations of the tides at a given point to the surface of the Earth and in specific configurations (syzygy or quadrature). Equation (2.16) is not exactly that given by Newton, for he took into account the age of the tides, which led him to underestimate the action of the Sun. To apply this formula, he studied the observations made at Bristol harbor, during the days close to the equinoxes, in spring and autumn: he remarked that the tidal range, i.e. the difference of level between the high tide and the low tide, amounted to 45 feet during the syzygies and to 25 feet during the quadratures. From this observational data he concludes that the action of the Moon is 4.4815 times larger than that of the Sun. The value is off by a factor of 2: we know today that the true value is 2.18. The reasons for the discrepancy are first that the quality of the observations of tides is doubtful and second that dynamical effects are not taken into account, the calculations being made in the frame of a static model. Newton did not have at that time the mathematical tools that would have enabled him to tackle them. Laplace, at the end of the 18th century, will undertake Newton's calculations with substantial improvements leading to a ratio, much closer to the true value, of $\mu = 2.35$.

Nevertheless, using his value, Newton could give for the first time an estimate of the mass of the Moon. First proved that the tidal force exerted by the perturbing body (M or S) is proportional to its mass and to the inverse of the cube of its distance to the Earth. Thus by using the same notations as previously the ratio of the tidal force exerted by the Moon to that exerted by the Sun is given by

$$\frac{F_M}{F'_{tidal}} = \frac{M_{Moon}}{\mathrm{PT}^3} \times \frac{\mathrm{ST}^3}{M_{Sun}} = \frac{\rho_{Moon} R^3_{Moon}\mathrm{ST}^3}{\rho_{Sun} R^3_{Sun}\mathrm{PT}^3} \tag{2.17}$$

where ρ_{Moon} and R_{Moon} (respectively ρ_{Sun} and R_{Sun}) stand for the density and the radius of the Moon (respectively of the Sun). Moreover, calling α_{Moon} and α_{Sun} the apparent diameters of the Moon and of the Sun,

$$\alpha_{Moon} = \frac{2R_{Moon}}{\mathrm{PT}}, \qquad \alpha_{Sun} = \frac{2R_{Sun}}{\mathrm{ST}} \tag{2.18}$$

These apparent diameters, varying as an inverse function of the distance (PT or ST), were already known with a very good accuracy in Newton's time, thanks to various astrometric measurements done for a little less than one century, since the first refractor by Galileo around 1610. On average we have $\alpha_{Moon} = 31'16''.5$ and $\alpha_{Sun} = 32'12''$, which immediately give $\alpha_{Sun}/\alpha_{Moon} = 1.0296$. From the equations above we get

$$\frac{F_M}{F'_{tidal}} = \frac{\alpha^3_{Moon}}{\alpha^3_{Sun}} \times \frac{\rho_{Moon}}{\rho_{Sun}}, \qquad \frac{\rho_{Moon}}{\rho_{Sun}} = \frac{\alpha^3_{Sun}}{\alpha^3_{Moon}} \times \frac{F_M}{F'_{tidal}} \tag{2.19}$$

Newton deduced the ratio $F_M/F'_{tidal} = 4.4815$ from the records of tides at Bristol. This led to $\rho_{Moon}/\rho_{Sun} = 4.891$. The next step is to determine ρ_{Sun}/ρ_{Earth}. Indeed, by using Kepler's third law and the value of g, we have

$$GM_{Sun} = \omega^2_{Earth}\mathrm{ST}^3, \qquad g = \frac{GM_{Earth}}{R^2_{Earth}} \tag{2.20}$$

hence

$$\frac{\rho_{Sun}}{\rho_{Earth}} = \frac{\omega^2_{Earth}\mathrm{ST}^3 R^3_{Earth}}{gR^2_{Earth}R^3_{Sun}} = \frac{\omega^2_{Earth}R_{Earth}\mathrm{ST}^3}{gR^3_{Sun}} \tag{2.21}$$

Finally, with $\alpha_{Sun} = 2R_{Sun}/\mathrm{ST}$ we find

$$\frac{\rho_{Sun}}{\rho_{Earth}} = \frac{8\omega^2_{Earth}R_{Earth}}{g\alpha^3_{Sun}} \tag{2.22}$$

In Newton's time all the quantities on the right-hand side of this equation were known with very good relative accuracy. The radius of the Earth had been set by Picard at $R_{Earth} = 6732$ km, while $g = 9.81\ \mathrm{ms}^{-2}$, $\omega_{Earth} = 2\pi/T_{Earth}$ with $T_{Earth} = 365.25d$, the value of α_{Sun} having been given previously. From the juxtaposition of the values found for ρ_{Moon}/ρ_{Sun} and ρ_{Sun}/ρ_{Earth}, a calculation gives $\rho_{Moon}/\rho_{Earth} = 11/9$ (*Principia*, vol. II, Proposition 37, Corollary 3). Thus, for Newton the Moon is slightly denser than the Earth. The ratio of the mass of the Moon to that of the Earth is obviously

$$\frac{M_{Moon}}{M_{Earth}} = \frac{\rho_{Moon}R^3_{Moon}}{\rho_{Earth}R^3_{earth}} \tag{2.23}$$

From determination both of the apparent diameter and of the parallaxes of the Moon, it is possible to provide the ratio R_{Moon}/R_{Earth} which, according to Newton is $1/3.65$. Finally he arrives at the mass ratio

$$\frac{M_{Moon}}{M_{Earth}} = \frac{11}{9} \times \frac{1}{3.65^3} = \frac{1}{39.79} \tag{2.24}$$

This is roughly twice bigger than the true value of 1/81. Newton over-estimated the mass of our satellite by a factor of 2, but recall that we are in presence of the first calculation of the mass of the Moon.

2.6.3 Assessment of Newton's Contribution

From his various calculations detailed in the previous sections, Newton demonstrated one of the most impressive consequences of his law of gravitation, a full explanation of the phenomenon of tides, through the differential gravitational action of the Sun and of the Moon on a particle on the surface of the Earth. These results were rapidly recognized throughout England as a real triumph of his theoretical investigations. This can be seen from the presentation of *Principia* by Edmund Halley (1656–1742) to King James in 1697, during which Halley singled out Newton's work on tides, explaining that his illustrious contemporary solved for the first time the mysterious problem of the ebb and flow.

Nevertheless, as it is well known, the diffusion of Newton's work and of his law of gravitation encountered strong opposition. Among opponents we find Huygens (1629–1695) who, while recognizing the unquestionable advances made by Newton, could not subscribe to his conception of gravity. The main trouble is with action at a distance: Huygens was not convinced that celestial bodies show a natural tendency for mutual attraction. This is confirmed by a letter to Leibnitz in 1690, in which he concedes that he cannot accept the reasons given by Newton on his theory of the ebb and flow as well as on other theories based on the principle of gravitational attraction [12]. Huygens's opinion is widely shared by scholars in France and other countries of the continent. Newton himself was disconcerted by the idea that a matter at rest can act on another matter without mutual contact. In *Principia* (vol. II, book III, scholium) he remarks that he explained celestial phenomena as well as terrestrial ones (tides) thanks to his law of gravitation without being able to assign a cause of this law.

Let us summarize the characteristics of Newton's theory of tides.

- Nowadays this theory is regarded as a by-product of his law of gravitation, but at that time it was taken by his supporters as an emblematic confirmation of his general theory of gravitation, including all the new fundamental tools of physics: law in $1/r^2$, calculus, geometrical combination of forces, etc.
- Newton did not tackle head-on the problem of ocean tides. This problem came gradually to his mind after he studied in detail the inequalities of the lunar orbital motion due to the perturbing gravitational action of the Sun, which in fact is based on the same dynamical principle as that which raises the ocean mass and causes the tides.
- Newton's tricky geometrical reasoning where he represents the attractions by segment lengths judiciously chosen allow him to quantify in a simple way the solar tidal force.

- Thanks to his theory of tides, and by fitting the results of his calculations to observational records of tidal range, Newton could make an estimate of the mass of the Moon relatively to that of the Earth. Moreover, his results explain such well-established characteristics as the presence of two tides per day and the variation of the tidal range according to the lunar phase with extrema during syzygies and quadratures.
- Newton understood perfectly the principle and the origin of tidal forces but his explanations about their consequences are imperfect, essentially because his conception of tides is static and in consequence he did not include the dynamical approach of the oceanic motions. Nevertheless his results mark a turning point and will be fully exploited by his successors like Daniel Bernoulli, Euler, and d'Alembert, and in a quasi-modern form by Laplace at the end of the 18th century.

2.7 Theory of Tides and Analytical Calculations Around 1740

For half a century after Newton, no substantial study on the subject appeared to improve or complete it. But during this same period, mathematics was progressing, notably thanks to the contributions of Leibniz, Jacob and Johann Bernoulli, l'Hôpital, and Varignon. All these mathematicians participated in the birth and the development of calculus. Whereas Newton showed a complex geometrical reasoning, these new tools allowed the development of analytical studies related to mechanics and more specifically to celestial mechanics. At the same time, the metaphysical opposition raised by the principle of Newton's action at a distance was gradually abandoned in the light of the obvious improvements it brought for the resolution of various problems. To illustrate this evolution, we can mention a testimony from Daniel Bernoulli (1700–1782) in 1740, who presents gravitation as an incomprehensible and essential principle that the famous Newton has so well established and that his contemporaries could no longer reject, without harming sublime knowledge and fortunate discoveries of the century. People spoke less of the 'absurdity' of gravitational attraction, and accepted the concept as it was, only preoccupied with investigating its consequences.

At the same time, a lot of systematic observations of tides were being carried out. In the period from 1700 to 1720, Jacques Cassini (1677–1756), a staunch supporter of Descartes's theory of vortices, gathered and discussed tidal observations in French harbors, Le Havre and Dunkerque (1701–1702), Lorient (1711–1712 and 1716–1719) and principally Brest (171–1716) [3, 24]. In the middle of the 18th century, the quasi-totality of scholars recognized the essential correctness of Newton's explanations, but they sometimes emphasized their insufficiencies. Marquise de Châtelet (1706–1749), in her commentaries on *Principia* [14] published posthumously in 1756, explained that people in her time knew that the tides are caused by the inequalities of the action of the Moon and of the Sun on the Earth; she added that Newton had established the mechanism of this cause so well that nobody could express any doubt on its validity. But she also pointed out that the famous scholar (Newton) did not investigate deeply enough the details of the important subject of tides.

In order to encourage scientists to investigate the problem more deeply, the French Académie des Sciences proposed in 1738 the precise elucidation of the tides as a prize to be awarded by the Académie in 1740. Four works received this prize. Three of them were based on the theory of gravitation. They were submitted by Daniel Bernoulli (Fig. 2.13) (1700–1782), Euler (1707–1785), and MacLaurin (1698–1746). The remaining one, by Cavalleri [4], was based on Descartes's theory of vortices, and must surely be, according to Laplace, the very last work dealing with this theory and considered by the Académie. MacLaurin's work entitled *De causa physica fluxus and refluxus maris* [22] is based on proofs of geometrical type. It presents remarkable theorems on the attraction of spheroids, but paradoxally offers few developments on the ocean tides. The two other works, entitled *Traité sur le flux et du reflux de la mer* by Bernoulli [1] and *Inquisto physica in causam fluxus ac refluxus maris* by Euler [10] both represent the real beginning of analytical studies on the subject of tides. These two works fully exploit Newton's calculations but in addition benefit from the drastic improvement accomplished at the beginning of the 18th century in the fields of calculus and of analytical mechanics. Thanks to these advantageous new tools, the two authors did not have to solve the problem of ocean tides by similitude with the problem of the lunar orbital motion perturbed by the tidal action of the Sun, as Newton did. They could directly tackle the resolution of the problem in the frame of terrestrial mechanics.

2.7.1 Prize of the Académie of 1740 for Bernoulli

Daniel Bernoulli's work honored by the prize of the Académie is entirely in the lineage of Newton. He deals with three major problems. The first, the most important according to Bernoulli himself, concerns the elevation of the ocean surface under the attraction of a perturbing body, the Sun. For that purpose he used exactly the same procedure as Newton. But his calculations were much clearer. The second concerns the exact time and amplitude of the high (or low) tides at any point on the surface of the Earth under the combined gravitational action of the two bodies (the Moon and the Sun). The calculations allow him to establish for the first time a tide table. The third concerns the estimation of the mass of the Moon. To this end, he did not follow the same procedure as Newton, based on the height of the tides, but an alternative one based on the interval of time separating successive high (low) tides one day after the other. Marquise de Châtelet, in her *Commentaires des Principes Mathématiques de la Philosophie Naturelle* [14] offers a very clear analysis of Bernoulli's treatise, following his arguments one by one, and often in a more understandable form. One of the fundamental principles of Bernoulli is that the attraction of the Earth by the Sun is rigorously equal to the centrifugal force coming from the revolution of the Earth, if we consider the Earth as a whole. If we consider locally a particle closer

Fig. 2.13 Daniel Bernoulli (1700–1782)

to the Sun than the center of the Earth, the centrifugal acceleration will be the same whereas the attraction will be stronger. This leads to the characterization of the tidal force which can be regarded as the difference between the attraction of the perturbing body (the Sun) and the centrifugal force.

2.7.1.1 Calculation of the Elevation of Water

The most important question tackled by Bernoulli concerns the amplitude of the tides caused by the Sun. For that aim, he starts from the following hypotheses:

- The Earth at rest is spherical, completely covered by the sea, a thin fluid layer.
- Unlike Newton's hypothesis, the Earth is heterogeneous and made of concentric layers, each having its own density. A law of variation of density as a function of depth is given.
- At anytime the figure of equilibrium of the Earth undergoing the action of the perturbing body (here the Sun) is an ellipsoid, whose the major axis is directed toward the perturbing body.

Thus calculating the amplitude of the tides amounts to measuring the difference between the semi-minor and the semi-major axes of the ellipsoid. Following the same reasoning as Newton, Bernoulli imagined two channels, one directed toward the Sun and the other in a perpendicular direction (Fig. 2.14). In the first the tidal force is against the gravity, whereas in the second it increases it. The solution of the problem of the elevation of water is given by the equality of pressure at the bases of the channels. The problem is complicated by the fact that the ellipsoidal deformation alters the gravity of the Earth at any of point of the surface: in order to solve this additional difficulty, Bernoulli initiates subtle analytical calculations giving a model of self-gravity of the Earth. The height difference β between a high tide and the corresponding low tide is equal the difference above between the lengths of the two

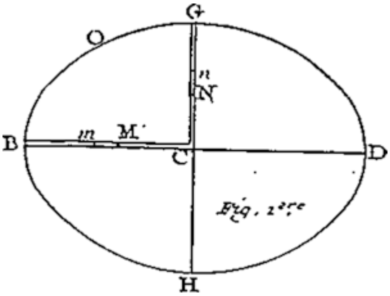


Fig. 2.14 Calculation of the amplitude of the solar tide with the equilibrium of two channels (Fig. 1 of Sect. V of the *Commentaires aux Principes Mathématiques* of la Marquise du Châtelet [14]). The two channels are directed one toward the Sun, the other in a direction perpendicular to the Sun, and join at the center of the Earth

channels. In the case of the simplified hypothesis of a homogeneous Earth with an ocean surrounding it and having the same density, Bernoulli gets

$$\beta = \frac{15}{4}\frac{M_{Sun}}{M_{Earth}}\frac{R_{Earth}^3}{d_S^3}R_{Earth} \tag{2.25}$$

where M_{Sun} and M_{Earth} are the mass of the Sun and of the Earth, d_S is the distance ST from the Earth to the Sun, and R_{Earth} is the Earth radius.

In the first half of the 18th century, d_S was not known with good accuracy. But by using Kepler's third law it is possible to substitute for it the angular velocity ω_E of the Earth around the Sun. Indeed we have the two relationships

$$GM_{Sun} = \omega_{Earth}^2 d_S^3, \qquad GM_{Earth} = \omega_{Moon}^2 d_M^3 \tag{2.26}$$

where ω_{Moon} and d_M are the angular velocity of the Moon around the Earth and the distance between the Moon and the Earth. Substituting these into Eq. (2.25), we get

$$\beta = \frac{15}{4}\frac{\omega_{Earth}^2}{\omega_{Moon}^2}\frac{R_{Earth}^3}{d_{Moon}^3}R_{Earth} \tag{2.27}$$

This formula enables Bernoulli to find the same numerical value as Newton, with $\beta \approx 60$ cm. But this value is obtained with the simplified model of oceans described above (a homogeneous Earth surrounded by an ocean with the same density). Bernoulli is convinced that the value is too small compared with what is expected from the observations of tides. Therefore he expounds various hypotheses about the interior of the Earth, considering for instance the case it is empty, or the case the density of an internal layer is proportional, or inversely proportional, to its radius. In some cases, with a density profile judiciously chosen, he succeeded in obtaining a value of β significantly larger than the value above. This profile corresponds to an increase of density with the depth of the layer, which is quite realistic.

Nevertheless, Bernoulli's calculations are something ambiguous and erroneous: first he believes that only a small part of the oceanic mass is moved by the attraction of the perturbing body, insisting that the Earth as a whole cannot be deformed;

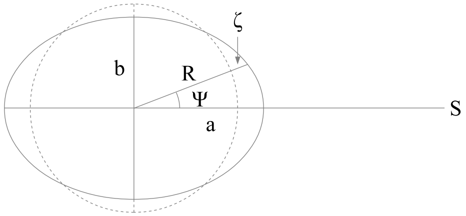


Fig. 2.15 The height of water ζ of a tidal ellipsoid is measured with respect to the initial spherical surface (when the water is at rest), with radius R. Ψ is the geocentric zenithal distance enabling one to define the position of the body S (Moon or Sun)

second, his calculations based on an equilibrium characterized by the ellipsoidal figure with equality of pressures at the bases of the two channels rely on the opposite hypothesis of a global deformation of our planet. There is a contradiction between the method of calculation and the model of the Earth chosen. This was pointed by d'Alembert in his *Reflexions sur la Cause Générale des Vents* in 1746 [5].

Nevertheless, though clumsy and erroneous, Bernoulli's approach deserves great interest. First it acknowledges the discrepancy between Newton's value for the amplitude of tides and the significantly larger amplitudes observed in various harbors. Second it inaugurates to some extent the concept of the internal structure of the Earth, based on a decomposition in layers with a gradual variation of density [6]. This new concept will be fully used a few years later by Clairaut in his study of the figure of the Earth, by Bouguer in 1749 in a study of the variation of amplitude and location-dependent direction of gravity on the surface of the Earth, and by d'Alembert and Euler in their works on the precession of equinoxes.

Bernoulli used geometrical arguments to show that when the perturbing body is at zenith, the elevation of water is twice the size of the depression when it is on the horizon, which is the result found by Newton. Moreover he states that each body (Moon or Sun) acts on the sea independently. In a first step, he makes the approximation that the two celestial bodies move on the celestial equatorial plane, with constant angular velocity. Under their combined action, the height of water with respect to the surface of the sea at rest (Fig. 2.15) is given by

$$\zeta = \beta_S\left(\cos^2\psi_S - \frac{1}{3}\right) + \beta_M\left(\cos^2\psi_M - \frac{1}{3}\right) \tag{2.28}$$

where ψ_S and ψ_M are the zenithal angular distances of the Sun and the Moon, β_S and β_M are the differences between the semi-major and semi-minor axes of the ellipsoid representing the equilibrium tide for the Sun and the Moon (Fig. 2.16). Assuming β_S/β_M known, Bernoulli uses a formula derived from Eq. (2.28) to calculate the exact instant of the high tide during a lunar cycle. For that purpose he corrected his results by taking into account that the relative motions of the Moon and of the Sun are elliptical and inclined with respect to the equator. Finally he could construct the first theoretical tide tables, which proved satisfactory for any harbor where dominant tides have a semi-diurnal frequency. Finally, an important step in Bernoulli's calculations is the ratio of the lunar action to the solar one. We

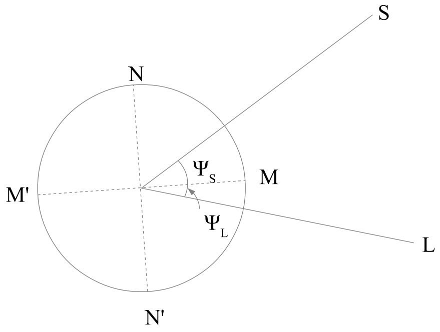


Fig. 2.16 Combined action of the Moon and the Sun. We are in the equatorial plane. S is the Sun, represented by the angle ψ_S. L is the Moon, represented by the angle ψ_L. $MNM'N'$ is the oceanic surface in the case of no deformation. Each body (Moon or Sun) deforms the water surface in an ellipsoid whose semi-major axis is directed toward it. The combination of the two ellipsoids generates a high tide in M and M' and a low tide in N and N'

saw that Newton's value of 4.48 was deduced from a comparison of the heights of tides during syzygies and quadratures. These values obtained in various harbors are not taken in a 'free sea': they show big variations and have consequently a large uncertainty. In contrast Bernoulli proposed an alternative method on the occurrence of tides, which seemed for him much easier to estimate.

Everyday the tides are delayed due to the fact the average time interval between two transits of the Moon at a given meridian is $24^{h}50^{mn}$. But during syzygies, this interval is shorter ($24^{h}35^{mn}$) than during quadratures ($25^{h}25^{mn}$). These differences come from various combinations between lunar and solar tides and enable one to determine the ratio between the action of the Moon and of the Sun. By averaging over various tidal observations, Bernoulli gets a value of 2.5 for this ratio. From this new determination, he could get an updated value of the density and the mass of the Moon, respectively 5/7 (≈ 0.71) and 1/70 of those of the Earth, these two values being much closer to the true values (respectively 0.60 and 1/81) than Newton's ones.

2.7.2 Prize of Académie of 1740 for Euler

A second work honored by the prize of the Académie des Sciences in 1740 was by Euler (Fig. 2.17), who deepened several points of Newton's theory, from which he borrowed the definition of tidal force as the difference between the gravitational force exerted by an external body (the Moon or the Sun) on a point on the surface of the Earth, and this same gravitational force exerted at the center of the Earth. Euler established in a very modern way the analytical expression of the tidal force, which led him to deduce the formula of the radial and tangential components at the point considered. These formulas can be regarded as definitive.

Fig. 2.17 Leonhard Euler (1707–1783)

Euler modeled the Earth as a spherical undeformable globe surrounded by an oceanic layer with limited thickness. Then, exploiting his formula, he defined the figure of equilibrium of the Earth subject to the effect of tides. He showed that at first order this figure is really an ellipsoid, as had been suggested without proof by Newton and Bernoulli. Moreover Euler did not have to rely on the artificial concept of two perpendicular channels joined at their bases. Instead, he found his inspiration in an idea already proposed by Huygens: the ocean surface is at rest on the condition that it is perpendicular to the direction of the vertical, as materialized by the plumbline.

2.7.2.1 Analytical Expressions for the Tidal Force

Euler establishes the expressions of the radial and tangential components of the tidal force in Chap. II, par. 24–27 of a work entitled 'On the lunisolar forces which put the oceans in motion'. Referring to Fig. 2.18, the gravitational force exerted by the Sun at the center C of the Earth and at any point M are given respectively by

$$\frac{GM_{Sun}}{d^2}\mathbf{u}_x, \qquad \frac{GM_{Sun}}{l^2}\mathbf{u}_l \tag{2.29}$$

The tidal force **F** is given by the difference between these two forces. Calling α the angle between SC and SM, the components of **F** are

$$F_x = \frac{GM_{Sun}}{l^2}\cos\alpha - \frac{GM_{Sun}}{d^2}, \qquad F_y = -\frac{GM_{Sun}}{l^2}\sin\alpha \tag{2.30}$$

$$F_x = GM_{Sun}\left(\frac{d-x}{l^3} - \frac{1}{d^2}\right), \qquad F_y = -GM_{Sun}\frac{y}{l^3} \tag{2.31}$$

Then Euler decomposes F_x and F_y into a radial and tangential components

$$F_r = -F_x\cos\psi + F_y\sin\psi \tag{2.32}$$

$$F_t = F_x\sin\psi + F_y\cos\psi \tag{2.33}$$

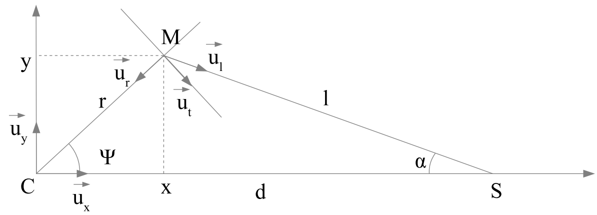


Fig. 2.18 Analytical expression of the tidal forces. S is the Sun, C the center of the Earth, M a point on the Earth for which the tidal force is determined. r, l, d stand respectively for the distances CM, MS, CS

In view of

$$l = \sqrt{(d-x)^2 + y^2}, \qquad \sin\alpha = \frac{y}{l}, \qquad \cos\psi = \frac{x}{r}, \qquad \sin\psi = \frac{y}{r} \tag{2.34}$$

these give

$$F_r = -\frac{GM_{Sun}}{(d^2 - 2dx + r^2)^{3/2}}\left(\frac{(d-x)x}{r} - \frac{y^2}{r}\right) + \frac{GM_{Sun}x}{d^2 r} \tag{2.35}$$

and

$$F_t = \frac{GM_{Sun}}{(d^2 - 2dx + r^2)^{3/2}}\frac{dy}{r} - \frac{GM_{Sun}y}{d^2 r} \tag{2.36}$$

Using the expansion

$$\left(d^2 - 2dx + r^2\right)^{-3/2} = \frac{1}{d^3}\left(1 + \frac{3x}{d} - \frac{3}{2}\frac{x^2+y^2}{d^2} + \frac{15}{2}\frac{x^2}{d^2} + \cdots\right) \tag{2.37}$$

it follows (Chap. II, art. 27, Fig. 15) that

$$F_r = \frac{GM_{Sun}}{d^3\sqrt{x^2+y^2}}\left(y^2 - 2x^2 + \frac{3}{2}\frac{x}{d}\left(3y^2 - 2x^2\right)\right) \tag{2.38}$$

$$F_t = \frac{GM_{Sun}}{d^3\sqrt{x^2+y^2}}\left(3xy + \frac{3}{2}\frac{y}{d}\left(4x^2 - y^2\right)\right) \tag{2.39}$$

These are the formulas established by Euler. To find the modern formula, we can introduce the angle ψ. Then

$$F_r = \frac{GM_{Sun}}{d^2}\left(\frac{r}{d}\left(3\cos^3\psi - 1\right) + \frac{3}{2}\frac{r^2}{d^2}\left(5\cos^3\psi - 3\cos\psi\right)\right) \tag{2.40}$$

$$F_t = \frac{GM_{Sun}}{d^2}\left(\frac{r}{d}(3\cos\psi\sin\psi) + \frac{3}{2}\frac{r^2}{d^2}\sin\psi\left(5\cos^2\psi - 1\right)\right) \tag{2.41}$$

Thus the analytical results obtained by Euler are very clear and give a quasi-definitive form to the expression of the tidal force.

2.7.2.2 Figure of Equilibrium and Tangential Component

A second part of Euler's investigations, particularly interesting, concerns the form of the surface of equilibrium of the oceanic mass under the combined gravitational

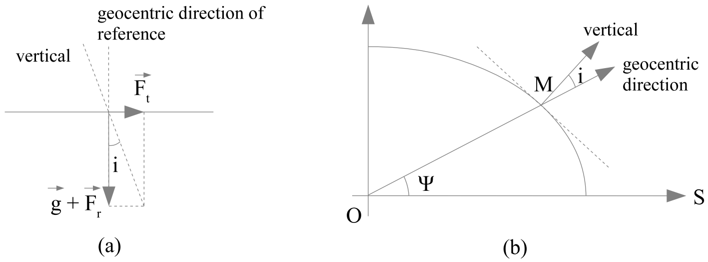


Fig. 2.19 Figure of equilibrium under the action of the tidal force: (**a**) deviation of the vertical under the influence of the tangential component of the tidal force, (**b**) the figure of equilibrium of the fluid layer at each point is perpendicular to the vertical. It is defined by the inclination i of the vertical with respect to the geocentric direction of M

action of the Moon and the Sun. In paragraphs 34 to 38 of Chap. 3, Euler explains that this form depends on the role of the tangential component of the tidal force. Making use of a statement first given by Huygens, Euler considers that the surface of the fluid is in equilibrium when at each point it is perpendicular to the direction of the gravity. The tidal forces modify slightly the direction of the vertical with respect to the geocentric direction of reference (Fig. 2.19): the tangential component F_t of the tidal force is responsible of the deviation i of the vertical line, which at first order can be calculated in a straightforward manner:

$$i \approx \tan i = \frac{F_t}{g + F_r} \approx \frac{F_t}{g} \tag{2.42}$$

By taking into account the constraint that the total mass of water remains constant, Euler gets the general formula of the surface of equilibrium (Chap. II, art. 36)

$$r = R + \frac{GM_{Sun}}{dg}\left[\frac{R^2}{d^2}\left(\frac{3\cos^2\psi - 1}{2}\right) + \frac{R^3}{d^3}\left(\frac{5\cos^3\psi - 3\cos\psi}{2}\right)\right] \tag{2.43}$$

where R is a reference radius and r is the radius form the center to the point considered at the surface of the ellipsoid, from which the angle ψ is measured. This expression by Euler constitutes a determining step toward a modern and accurate theory of the tides. The quantity $\zeta = r - R$ is the height of the equipotential represented by the surface of the ellipsoid. The right-hand side, when multiplied by g, is the tidal potential. The expressions $(3\cos^2\psi - 1)/2$ and $(5\cos^3\psi - 3\cos\psi)/2$ are called the Legendre polynomials of 2nd and 3rd degree respectively. They were introduced by Legendre (1752–1833) around 1780 and appear in Laplace's equations, studied below. By using the equality $g = GM_{Earth}/R^2$, Eq. (2.43) can be rewritten, at the first order in R/d,

$$r = R + \frac{M_{Sun}}{M_{Earth}}\frac{R^3}{d^3}R\frac{3\cos^2\psi - 1}{2} \tag{2.44}$$

that is to say

$$r = b + \beta\cos^2\psi \tag{2.45}$$

Fig. 2.20 Jean Le Rond d'Alembert (1717–1783)

with

$$\beta = \frac{3}{2}\frac{M_{Sun}}{M_{Earth}}\frac{R^3}{d^3}R, \qquad b = R - \frac{\beta}{3} \tag{2.46}$$

This is the equation of an ellipse with semi-major axis b and with difference β between the semi-major and semi-minor axes. From these calculations Euler proves what Newton and Bernoulli supposed without proof: at first order, the figure of equilibrium of the oceans under the action of the Sun or the Moon is an ellipsoid. Moreover the amplitude of β, which can be interpreted as the amplitude of the tides, is 2.5 times smaller than that found by Bernoulli, in the case of a homogeneous model of the Earth. The results by Euler are exact when neglecting the self-gravity of the oceans. The great advantage of Euler's method is that the solution is available for an oceanic surface layer surrounding a solid Earth, whereas in Bernoulli's method, relying on the equality of pressure at the bases of two perpendicular channels, the whole Earth (including the oceans) must be homogeneous and fluid.

2.8 D'Alembert and His 'Reflexions sur la Cause Générale des Vents'

D'Alembert (Fig. 2.20) (1717–1783) did not publish a specific work dealing with oceanic tides but his report entitled *Réflexions sur la Cause Générale des Vents*, submitted in 1746 to the Royal Academy of Sciences of Berlin, and published in 1747 [5], deals with several important points related to the tides. One of the subjects of great interest concerns the characteristics of regular winds in the tropical areas of the Earth. His aim is to study how the tidal forces exerted both by the Moon and the Sun on the atmosphere of the Earth can be regarded as the origin of winds on its surface. He tried to start from the calculation of the atmospheric tides and to infer in some detail the velocity distribution of the winds.

Some remarkable studies are included in the first part of the work. Two of them can be retained as emblematic of a totally new approach to the problem: the first is

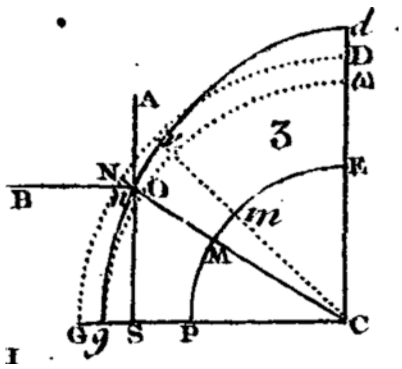


Fig. 2.21 Determination of the figure of equilibrium of the fluid layer by d'Alembert (Fig. 3 of *Réflexions sur la Cause Générale des Vents*)

the determination of the surface of equilibrium of a given fluid surface layer when it is subjected to a given external force. The consequence is the production of an oscillation around a figure of equilibrium. D'Alembert calculates the eigenmodes of this oscillation. The second study is oriented toward the question of self-gravity of a deformed fluid layer. This can be naturally applied to the problem of ocean tides. On this topic, d'Alembert completes the work of his predecessors, making the link between Euler's and Bernoulli's results. In the following we present the main features of these two studies.

2.8.1 Mechanics of Surface Layer of Fluid

D'Alembert does not show any interest on the global deformation of the Earth as Newton and Bernoulli did. In contrast he analyses in detail the deformation of a thin fluid layer surrounding the Earth considered as an undeformable spherical globe. In that sense his method is similar to Euler's.

2.8.1.1 Figure of Equilibrium of the Fluid Layer

Here d'Alembert makes use of the same mathematical formalism both for the tidal force and for the centrifugal force due to the rotation. For both, only the tangential component of the force has an effect on the figure of equilibrium (Fig. 2.21). Moreover these tangential components have the same structure. They can both be expressed by an equation of the type $F_t = \phi \cos\psi \sin\psi$. In the case of the tidal force, ψ stands for the zenithal distance of the perturbing body and ϕ stands for $\phi = 3GM_{Sun}R_{Earth}/d^3$, where M_{Sun} is the mass of the Sun, R_{Earth} is the Earth radius and d is the distance from the body to the Earth. In the case of the centrifugal force ψ is the latitude and $\phi = \omega^2 R_{Earth}$ where ω is the rotational angular velocity of the Earth. D'Alembert found the same results as Euler but in a more straightforward manner, thanks to the use of polar coordinates instead of rectangular ones.

The steps of his argument are similar: after determining the inclination of the gravity under the effect of the tangential component, he finds the difference β between the semi-major and semi-minor axes of the ellipsoid as

$$\beta = \frac{\phi R_{Earth}}{2g} \tag{2.47}$$

In the particular case of tides, this gives

$$\beta = \frac{3}{2}\frac{M_{Sun}}{M_{Earth}}\frac{R_{Earth}^3}{d^3}R_{Earth} \tag{2.48}$$

Then the mass conservation enables him to give the expression of the deformation (Fig. 2.21).

2.8.1.2 Oscillation of the Fluid Layer

D'Alembert's investigations are not restricted to the determination of the figure of equilibrium of the fluid layer, in the frame of a static theory. He wants to deal with a much more difficult problem: what is the law of displacement of the various parts of the fluid? For that purpose he determines the interval of time Δt necessary for a given particle to go from the initial spherical surface to the new ellipsoidal one. In a second step he compares Δt with the interval of time $\Delta\theta$ necessary for the same particle to fall from a height a above the surface when undergoing the acceleration of gravity g. By considering that the period of oscillation of the particle is $T = 4\Delta t$, and with $a = 1/2g\Delta\theta^2$, his calculations lead to the formula $T = 2\pi R_{Earth}/\sqrt{6gh}$, where h is the depth of the fluid layer. With respect to this equation d'Alembert makes the interesting remark: T does not depend on the gravity but only of the parameters R_{Earth} and h. This confirms for the first time the existence of a proper oscillation mode of the fluid, even when forcing is absent. About half a century later Laplace will reinforce this result, inaugurating a long series of similar works.

2.8.2 Self-gravity of the Fluid Surface Layer

A second remarkable study of d'Alembert's concerns the self-gravity of a fluid surface layer. When this layer is deformed, it creates gravitational changes, increasing or decreasing its own initial deformation. D'Alembert's argument relied on MacLaurin's and Daniel Bernoulli's calculations. In art. 49, he shows that to determine the figure of equilibrium of the fluid layer by taking into account its self-gravity, all we have to do is to multiply the perturbing force by a factor

$$\rho = \frac{1}{1 - \frac{3}{5}\frac{\delta}{\Delta}} \tag{2.49}$$

where δ is the density of the fluid layer and Δ is the mean density of the Earth considered as a solid body. Then the self-gravity of the fluid layer is expressed in a rather simple way, by inserting this law in the tidal force. The difference between the semi-major and semi-minor axes of the deformable ellipsoid becomes

$$\beta = \frac{3}{2(1-\frac{3}{5}\frac{\delta}{\Delta})}\frac{M_{Sun}}{M_{Earth}}\frac{R^3}{d^3}R \tag{2.50}$$

D'Alembert suggested that this formula, validated by Laplace half a century later, could enable one to deduce the unknown mean density δ if one takes β from observations of tides, for instance by measuring the difference of height between the low and high tides at a given point of the sea. Nevertheless this kind of measurement is particularly delicate because d'Alembert does not take into account dynamical phenomena which act on the figure of equilibrium. Laplace will make appropriate adjustments in 1790, remarking that the determination of δ/Δ will be more efficient when studying long periodic tidal components, less apt to affect the equilibrium tide.

Here, one of d'Alembert's important conclusions is that the self-gravity of a fluid layer accentuates the amplitude of the tides. But with the real ratio $\delta/\Delta = 1/5.5$, the calculations above lead to a small increase of 13 % of the amplitude of the tides due to the self-gravity. This value remains rather small, in comparison to the amplitudes of resonant phenomena in harbors that exhibit significantly larger effects. Finally, notice that when the auto-gravity of the sea is not taken into account we get Euler's equation (2.48). And when considering the fictitious case in which the sea has the same density as the Earth, we find

$$\beta = \frac{15}{4}\frac{M_{Sun}}{M_{Earth}}\frac{R^3}{d^3}d \tag{2.51}$$

which fits with the expression found by Daniel Bernoulli. Therefore D'Alembert's expressions are completely in accordance with his two contemporaries.

2.9 Laplace's Masterpiece

Pierre Simon de Laplace (Fig. 2.22) (1749–1827) is just 25 years old when he begins his work on tides in 1774. In his introduction, when presenting the theories of his predecessors (Newton, Daniel Bernoulli, Euler, and d'Alembert) he points out their lack of validity when they are confronted with real observed tides. He proposes a complete renewal of the theoretical concepts, emphasizing the necessity to solve in a much more rigorous way what he considers as 'one of the most complex and interesting problems of the whole physical astronomy'. In 1825, after half a century of personal investigations on the subject of tides, he mentions that the motion of fluid covering a planet was a almost entirely new topic when he undertook its treatment in 1774. This terse comment seems excessive if we recall that Newton gave the definition of the tidal force, Euler found its precise formulation, and if we recall the calculations on the static tide by Bernoulli, Euler, and d'Alembert, as well as of the study by this last author of the oscillation and self-gravity of a fluid layer.

Fig. 2.22 Pierre-Simon de Laplace (1749–1829)

However, Laplace's comment rings quite true if we remark that all these predecessors, though broadly explaining the phenomena related to tides, were unable to propose an adequate model describing their effects. Laplace is in fact the first scientist to construct a mathematical model of tides. Moreover for that purpose he invented specific mathematical tools to handle the dynamical equations.

Laplace tackled the problem of tides in four successive memoirs, and gave a synthetic presentation of his calculations in his *Traité de Mécanique Céleste* (book IV, vol. II; book XIII, vol. V) [19, 21]. The first two memoirs, written in 1775 and 1776 and published in 1778 and 1779 [15, 16], are entitled *Recherches sur plusieurs points du Système du Monde*. They are devoted to theoretical aspects, accompanied with general equations and a particular study of the influence of the bathymetry of the sea on the oceanic tides. In the third memoir entitled *Traité du flux et du reflux*, written in 1790 and published in 1797 [18], Laplace proposes a theoretical study of the observations, in particular those recorded by Jacques Cassini at the beginning of the 18th century and gathered in a treatise on tides by Lalande in 1781 [13]. The contents of these memoirs are presented again by Laplace in a very clear and synthetic way in his book IV of *Traité de Mécanique Céleste*, published in 1799 [19]. In particular he makes full use of results acquired in 1782 on the spherical harmonics [17]. The fourth memoir, also entitled *Traité du flux et du reflux de la mer*, written in 1818 and published in 1820 [20], is devoted to observations organized by himself in harbor of Brest. At last in the book XIII of *Traité de Mécanique Céleste*, written in 1824 and published in 1825 [21], Laplace analyse as set of observations carried out at Brest from 1807 to 1822.

2.9.1 Development of Analytical Mechanics

Before taking a look at Laplace's capital contribution to the theory of tides, it is worth making a brief summary of the then recent developments in the fields of analytical mechanics, from which Laplace could build his theoretical investigations.

First of all, we recall that in 1755 Euler published memoirs where he established the general equations of hydrostatics and hydrodynamics, whatever the compressibility of the fluid. By generalizing the ideas of Clairaut, developed in 1743 on the occasion of his researches on the figure of the Earth, Euler introduced the notion of pressure and gave the general condition of equilibrium of a fluid by showing that the pressure counterbalances at each point the effect of the acceleration. He established the general equation of motion of the fluid with respect to an absolute reference frame, introducing the internal force of pressure $-\nabla p$ and the external force $\mathbf{f}$ in such a way that

$$\rho\gamma = -\nabla p + \mathbf{f} \tag{2.52}$$

where ρ is the density and γ the acceleration. Euler also introduced the local equation of the conservation of mass, which characterizes the fact that the variation of mass inside a given volume of fluid is equal to the mass flux through the surface bounding the volume:

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{v} = 0 \tag{2.53}$$

Together with the concept of pressure, Euler introduced also the concept of potential, although the evolution of this concept through his work is rather vague and progressive. As soon as 1736, he defines a function μ whose the differential is exact: $\mathrm{d}\mu = -P\,\mathrm{d}x - Q\,\mathrm{d}y - R\,\mathrm{d}z$, where P, Q and R are the rectangular components of the force per mass unit. In 1743, Clairaut showed the importance of such an expression in the equilibrium of a fluid mass and also proved that it must be an exact differential form. He added that the expression above represents the 'effort' of the gravity. This concept is close to the notion of work. Clairaut also showed that the surface of equilibrium of a fluid is given by setting the integral of $P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ to a constant. The notion of potential, in a latent state, was finalized in 1774 and 1776 by Lagrange who showed that the gravitational attraction derives from a potential Ω and that the components of the force can be obtained by calculating the partial derivatives of this potential. He added that this way of representation of the forces can prove extremely advantageous by its simplicity and its generality. Finally we mention that also in 1774 Lagrange introduced the use of spherical coordinates, which Laplace used extensively later.

2.9.2 The Equations of 1775 and 1776

In 1775, Laplace uses the general equations of the hydrodynamics set up by Euler, to apply them to the Earth, in spherical coordinates. As his predecessors he models the ocean as a uniform fluid layer with variable depth, covering entirely a spherical, solid and undeformable Earth. The fluid is supposed incompressible. Laplace makes an essential hypothesis which simplifies noticeably his calculations: he remarks that the depth of the oceanic layer is small compared with the Earth radius. This implies

$$(6) \qquad y = -\frac{l}{\sin\theta}\,\frac{\partial . u\gamma \sin\theta}{\partial\theta} - l\gamma\,\frac{\partial v}{\partial\varpi},$$

$$(7) \qquad \frac{d^2 u}{dt^2} - 2n\,\frac{dv}{dt}\sin\theta\cos\theta = -g\,\frac{\partial y}{\partial\theta} + B\Delta + \frac{\partial R}{\partial\theta},$$

$$(8) \qquad \frac{d^2 v}{dt^2}\sin^2\theta + 2n\,\frac{du}{dt}\sin\theta\cos\theta = -g\,\frac{\partial y}{\partial\varpi} + C\Delta\sin\theta + \frac{\partial R}{\partial\varpi},$$

R étant égal à $K[\cos\theta\cos v + \sin\theta\sin v\cos(\varphi - nt - \varpi)]^2$.

Fig. 2.23 Laplace equations in 1776. *Equation (6)* characterize the mass conservation. *Equations (7) and (8)* are the dynamic equations. Laplace uses l for the depth of the layer, y for its deformation, n for the angular velocity of the Earth. The Earth radius is chosen to be the unit of length and u and v are horizontal displacements. $B\Delta$ and $C\Delta$ correspond to the self-gravity

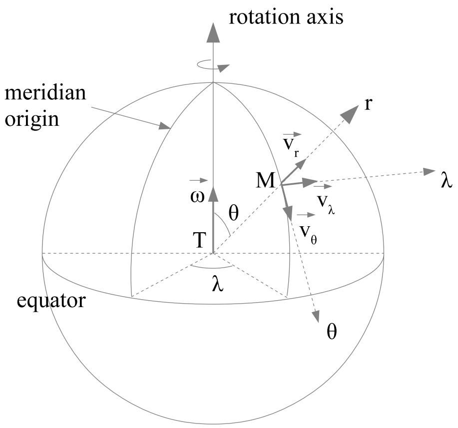


Fig. 2.24 Reference frame and parameters of the Laplace dynamical equations. M is represented by its spherical coordinates: the colatitude θ, the longitude λ, counted positively eastward l. ω is the vector rotation of the Earth

that the real large scale motions of the fluid are quasi-horizontal. In other words, the vertical velocity can be neglected, all the particles belonging to the same vertical line having a priori the same velocity. Then the general problem of tides can be treated by retaining only the tangential components, as a 2-dimensional problem. This fundamental simplification, known as the *long wave approximation*, will be fully used in later geophysical studies. Laplace kept on working on his equations inside his memoirs of 1776 (Fig. 2.23) and 1790, and his *Mécanique Céleste*, vol. IV, until he gave a precise and definitive formulation, which still stands nowadays.

Let θ and λ denote the colatitude and the longitude of a particle of the fluid layer at depth h, v_θ and v_λ the respective North-South and East-West components of the horizontal velocity $\mathbf{v}$ of the particle with respect to the Earth (Fig. 2.24). ζ is the radial deformation of the fluid layer, ω is the angular velocity of the Earth, g is the gravity, V is the tidal potential and Φ is the potential of self-gravity of the fluid layer.

With this notation, the dynamical equation and the equation of mass conservation are written with respect to a reference frame linked to the Earth

$$\frac{\partial \mathbf{v}}{\partial t} + 2\omega \wedge \mathbf{v} = -g\nabla\zeta + \nabla V + \nabla\Phi \tag{2.54}$$

$$\frac{\partial \zeta}{\partial t} + \operatorname{div} h\mathbf{v} = 0 \tag{2.55}$$

The pressure at a given point of the fluid is the sum of two terms

$$p = p_0 + p_\zeta \tag{2.56}$$

where p_0 is the hydrostatic pressure and p_ζ the additional pressure due to the deformation of the oceans. As the vertical accelerations due to the tides are very small in comparison to the gravity, the additional pressure p_ζ results only from the weight of the water column undergoing the deformation

$$p_\zeta = \rho g \zeta \tag{2.57}$$

ρ being the density of the fluid.

The hydrostatic pressure p_0 is counterbalanced by the gravity of the Earth combined to the force due to the rotation in such a way that these terms do not take part in the dynamical equations. In spherical coordinates, the preceding equations become

$$\frac{\partial v_\theta}{\partial t} - 2\omega\cos\theta\, v_\lambda = -\frac{1}{a}\frac{\partial}{\partial\theta}(g\zeta - V - \Phi) \tag{2.58}$$

$$\frac{\partial v_\lambda}{\partial t} + 2\omega\cos\theta\, v_\theta = -\frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda}(g\zeta - V - \Phi) \tag{2.59}$$

$$\frac{\partial \zeta}{\partial t} + \frac{1}{a\sin\theta}\left(\frac{\partial}{\partial\theta}(hv_\theta\sin\theta) + \frac{\partial}{\partial\lambda}(hv_\lambda)\right) = 0 \tag{2.60}$$

2.9.3 Conservation of Mass

The way Laplace takes into account the conservation of mass differs significantly from his predecessors: it does not involve a global conservation of the ocean supposed to cover the whole Earth and whose surface shape changes from a sphere to an ellipsoid. Instead Laplace's calculations express conservation locally. The starting point is the equation given by Euler in 1755

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\mathbf{v} = 0 \tag{2.61}$$

Then Laplace substitutes a variable surface density $\rho(h+\zeta)$ for a constant volume density ρ, h being variable in space but constant in time:

$$\frac{\partial \rho(h+\zeta)}{\partial t} + \operatorname{div} \rho(h+\zeta)\mathbf{v} = \mathbf{0} \tag{2.62}$$

As $\zeta \mathbf{v}$ has a 2nd-order amplitude, and taking into account that ρ is constant, this can be rewritten

$$\frac{\partial \zeta}{\partial t} + \mathrm{div}(h\mathbf{v}) = 0 \tag{2.63}$$

This equation expresses the fact that the mass flux through the walls of a column of water is compensated for by variations in the height of the column.

2.9.4 Complementary Acceleration due to the Rotation of the Earth

One of the most essential improvements offered by the two Eqs. (2.58) and (2.59) is the presence of the components with the factor $2\omega\cos\theta$. They signal the existence of a complementary acceleration in a rotating frame, later called the Coriolis acceleration. They imply a deviation of the motions at the surface of the rotating Earth. Laplace's predecessors considered that the sole effect of the rotation of the Earth was to displace the ellipsoid of the equilibrium tides. In the introduction to his memoir of 1755, Laplace remarked the error of this simplified hypothesis, noticing that the change in the relative position of the Moon and the Sun at the surface of the seas is not the unique effect coming from the rotation of the Earth. This was already pointed out by MacLaurin in his treatise about the ebb and flow, but without any calculation. Laplace completed his remark with the following reasoning: the velocity of a particle of fluid remains the same when staying in the same parallel, its angular velocity increases or decrease according to its distance to the equator, and it drifts in meridian as it moves in parallel. The important fact is that the amplitude of the changes due to this effect is of the same order as the gravitational action of the two perturbing bodies. MacLaurin must not be considered as the only predecessor to mention the effect above. Galileo, in 1632, studied it in the problem of a bullet launched along a meridian, and Hadley, in 1735, when interpreting the deviation of trade winds westward. But Laplace was the first scientist to propose a quantitative analysis far before Coriolis (1792–1843).

2.9.5 A Decisive Innovation: Spherical Harmonics

In 1782, Laplace invented what turned out to be a decisive tool for tackling problems of tidal phenomena: spherical harmonics [17]. They occupy a fundamental place in his work dealing with terrestrial dynamics, notably by leading to a rewriting of his dynamical equations in a more elegant manner. In his memoir of 1790 (art. 2 and 3) he already amends his notation, by introducing the spherical harmonics of order 2 in the expression of the tidal potential and by taking into account in a simple manner the self-gravity of the fluid layer. But the mathematical expressions of the potential as well as of the dynamical equations reach their full maturity in the *Mécanique Céleste* (vols. III and IV) of 1799.

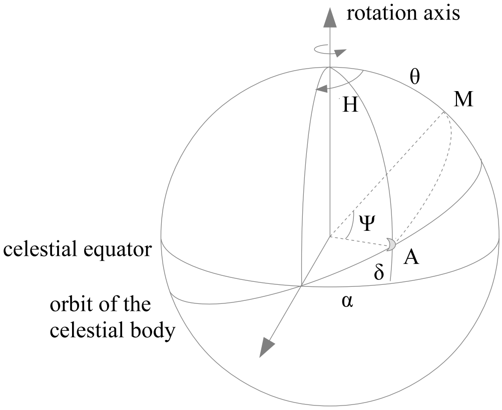


Fig. 2.25 Coordinates of a celestial body in the sky. M is the zenith of a surface point, with colatitude θ and longitude λ (counted positively eastward). A is a point corresponding to the direction of the celestial body (Moon or Sun), with declination δ and right ascension α. H is the hour angle of the intersection of the orbit with the equator. The angle between the 'meridian' of the body and the meridian of the surface point is given by $H - \alpha = \omega t + \lambda - \alpha$

2.9.6 Tidal Potential

In vol. III, art. 23, Laplace establishes the definitive expression for the tidal potential V exerted by a perturbing body (Fig. 2.25):

$$V = \frac{Gm_p}{d}\left(\frac{r^2}{d^2}P_2(\cos\psi) + \frac{r^3}{d^3}P_3(\cos\psi) + \frac{r^4}{d^4}P_4(\cos\psi)\right) \tag{2.64}$$

where d is the distance of the perturbing body from the center of the Earth, m_p its mass, ψ the geocentric angle of zenithal distance, r the radius of the Earth and P_n are the Legendre polynomials defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n[(x^2-1)^n]}{\mathrm{d}x^n} \tag{2.65}$$

Thus the first Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3x^2-1}{2}, \qquad P_3(x) = \frac{5x^3-3x}{2}$$
$$P_4(x) = \frac{35x^4 - 30x^2 + 3}{8} \tag{2.66}$$

This expression of the tidal potential is very close to the expression of the deformation of an oceanic layer as given by Euler in 1740. Thanks to classic relationships in a spherical triangle, Laplace can replace $\cos\psi$ by a function of the colatitude θ and the longitude λ of the point considered on the surface of the Earth, and of the equatorial coordinates of the perturbing body, i.e. its right ascension α and its declination δ:

$$\cos\psi = \cos\theta\sin\delta + \sin\theta\cos\delta\cos(\omega t + \lambda - \alpha) \tag{2.67}$$

By inserting this expression in the tidal potential, Laplace shows that it can be written naturally as a combination of functions called *spherical harmonics* $Y_n(\theta, \lambda)$:

$$V(\theta, \lambda) = V_2 Y_2(\theta, \lambda) + V_3 Y_3(\theta, \lambda) + \cdots = \sum_{n=2}^{\infty} V_n Y_n(\theta, \lambda) \tag{2.68}$$

Here the coefficients V_n themselves can be written as functions of the spherical harmonics $Y_n(\delta, \omega t - \alpha)$

2.9.7 Potential for Self-gravity

Thanks to their various properties, especially of orthogonality, spherical harmonics form a basis in which any surface function can be expanded. Thus Laplace expands the deformation of the fluid layer $\zeta(\theta, \lambda)$ in spherical harmonics

$$\zeta(\theta, \lambda) = \sum_{n=0}^{\infty} \zeta_n Y_n(\theta, \lambda) \tag{2.69}$$

With the help of this expansion Laplace finds a simple and subtle expansion of the potential for self-gravity of the fluid layer (vol. III, art. 11; vol. IV, art. 2)

$$\Phi(\theta, \lambda) = \sum_{n=0}^{\infty} \Phi_n Y_n(\theta, \lambda) \tag{2.70}$$

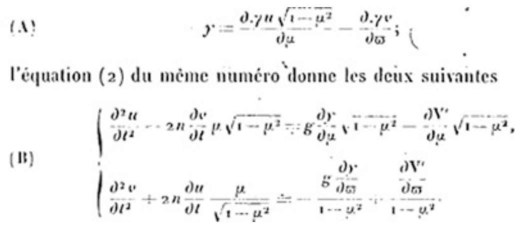
with

$$\Phi_n = \frac{3g\rho_w}{\rho_e} \frac{\zeta_n}{2n+1} \tag{2.71}$$

where ρ_w and ρ_e are the density of water and the mean density of the Earth. A remarkable point of the formula above is that each component of degree n of this potential is expressed as a function only of the corresponding degree of deformation ζ_n. Therefore each spherical harmonics can be treated separately, according to their degree.

2.9.8 Dynamical Equations with Spherical Harmonics

In his *Mécanique Céleste* (book III, art. 3) Laplace rewrites the equations of motion of a fluid particle, which are similar to those given in 1776, by relying on a completely new approach which expands in spherical harmonics all the parameters concerned. This approach is extremely general and valid at each degree of the harmonics, which can be treated independently. They are shown in Fig. 2.26.


(A) $y = \frac{\partial . \gamma u \sqrt{1-\mu^2}}{\partial \mu} - \frac{\partial . \gamma v}{\partial \varpi};$

l'équation (2) du même numéro donne les deux suivantes

(B) $\begin{cases} \frac{\partial^2 u}{\partial t^2} - 2n \frac{\partial v}{\partial t} \mu \sqrt{1-\mu^2} = g \frac{\partial y}{\partial \mu} \sqrt{1-\mu^2} - \frac{\partial V'}{\partial \mu} \sqrt{1-\mu^2}, \\ \frac{\partial^2 v}{\partial t^2} + 2n \frac{\partial u}{\partial t} \frac{\mu}{\sqrt{1-\mu^2}} = - \frac{g \frac{\partial y}{\partial \varpi}}{1-\mu^2} + \frac{\frac{\partial V'}{\partial \varpi}}{1-\mu^2}. \end{cases}$

Fig. 2.26 Laplace equations in 1799. μ corresponds to $\cos\theta$. The potential V' is the combination of the tide potential V and the potential of self-gravity Φ of the fluid layer

2.9.9 Oscillation of the Fluid Layer in Case of a Static Earth

In 1799, after much effort, Laplace succeeded in making progress on the difficult problem already raised by d'Alembert, that of determining the oscillation of the fluid layer in the case of a static Earth. In that specific case, the three Eqs. (2.58), (2.59), (2.60) can combine to give one equation in ζ. By differentiating the equation of mass conservation (2.63) and using the fact that h is constant, Laplace gets

$$\frac{\partial^2 \zeta}{\partial t^2} + \frac{h}{a}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\theta}{\partial t}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\lambda}\left(\frac{\partial v_\lambda}{\partial t}\right)\right] = 0 \tag{2.72}$$

and, by replacing the partial derivatives of the velocities by their expressions from (2.58) and (2.59),

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{h}{a^2}\Delta_t(g\zeta - V - \Phi) \tag{2.73}$$

where Δ_t denotes the tangential Laplacian given by

$$\Delta_t = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2} \tag{2.74}$$

Equation (2.73) was found by Laplace as early as 1776. But it took further 20 years for Laplace to find way of solving this complex equation. Once more thanks to the spherical harmonics, he found a way out, as it shown in *Mécanique Céleste* (vol. IV, art. 2). Each term in the equation is expressed with the help of spherical harmonics, and by use of their remarkable property

$$\Delta_t Y_n = -n(n+1)Y_n \tag{2.75}$$

Equation (2.73) becomes equivalent to a set of equations for different degrees n:

$$\frac{\partial^2 \zeta_n}{\partial t^2} = -n(n+1)\frac{h}{a^2}(g\zeta_n - V_n - \Phi_n) \tag{2.76}$$

with

$$\Phi_n = \frac{3g\rho_w}{\rho_e}\frac{\zeta_n}{2n+1} \tag{2.77}$$

Laplace gets

$$\frac{\partial^2 \zeta_n}{\partial t^2} + n(n+1)\left(1 - \frac{3\rho_w}{(2n+1)\rho_e}\right)\frac{gh}{a^2}\zeta_n = n(n+1)\frac{h}{a^2}V_n \tag{2.78}$$

This represents the equation of an oscillation forced by the tidal potential V_n. For $n = 2$, and by setting the tidal potential to zero, we have

$$\frac{\partial^2 \zeta_2}{\partial t^2} + 6\left(1 - \frac{3\rho_w}{5\rho_e}\right)\frac{gh}{a^2}\zeta_2 = 0 \tag{2.79}$$

This represents an oscillation of the fluid layer, with characteristic period

$$T = \frac{2\pi a}{\sqrt{6gh(1 - \frac{3\rho_w}{5\rho_e})}} \tag{2.80}$$

This corresponds exactly to the eigenmode of oscillation found by d'Alembert. But Laplace generalized the work for all degrees of the spherical harmonics.

2.9.10 Hydrostatic Equilibrium

In his *Mécanique Céleste* (vol. IV, art. 12), Laplace remarks that his equations recover in a simpler manner his predecessors' results on another fundamental topic: the equilibrium tide. A simple hypothesis is adopted that the surface of the sea takes the form induced by the instantaneous forces acting on it, in other words the velocities and their derivatives are ignored. With this hypothesis, the value of the deformation ζ of the fluid layer can be determined immediately for arbitrary depth and density of the sea. Indeed the equation becomes

$$g\nabla\zeta = \nabla V + \nabla\Phi \tag{2.81}$$

or after integration

$$\zeta = \frac{V + \Phi}{g} \tag{2.82}$$

This formula says that equipotentials are equivalent to equipressures. It shows the advantageous of working with potential, which quickly yields the static deformation. Expanding in spherical harmonics and recalling

$$\Phi_n = \frac{3g\rho_w}{\rho_e}\frac{\zeta_n}{2n+1} \tag{2.83}$$

we get, for each harmonic of degree n,

$$\zeta_n = \frac{1}{(1 - \frac{3}{2n+1}\frac{\rho_w}{\rho_e})}\frac{V_n}{g} \tag{2.84}$$

When self-gravity is neglected, this formula is very close to that given by Euler in 1740. Truncating the expression at the second order, we get

$$V = \frac{3Gm_p a^2}{2d^3}\left(\cos^2\psi - \frac{1}{3}\right) \tag{2.85}$$

and

$$\zeta = \frac{3}{2(1 - \frac{3}{5}\frac{\rho_w}{\rho_e})} \frac{m_p a^4}{m_e d^3} \left(\cos^2 \psi - \frac{1}{3} \right) \tag{2.86}$$

which now corresponds to the formula given by d'Alembert. Laplace is interested in the hypothesis of hydrostatic equilibrium in order to show that it conflicts with the observations. He showed that when the Moon and the Sun are in conjunction in the summer solstice, when their declination is maximal, the hypothesis implies that the excess of water at midday high tide over the following low tide should be roughly 8 times bigger than the excess of midnight high tide over the following low tide, whereas the observations show these excesses to be of the same size.

2.9.11 Three Species of Oscillation

An extremely important contribution of Laplace's study concerns the special structure of the tidal potential: he showed that this potential generates three different kinds of oscillations, for which he studied the influence of bathymetry of the oceans. His research on this topic began in the memoirs of 1775 and 1776 (art. 25–28), then in *Mécanique Céleste* (art. 4–10). He shows that when restricting to the degree 2 of spherical harmonics, the leading term, the tidal potential exerted by an external body is

$$V = \frac{3Gm_p a^2}{2d^3} \left(\left(\cos\theta \sin\delta + \sin\theta \cos\delta \cos(\omega t + \lambda - \alpha) \right)^2 - \frac{1}{3} \right) \tag{2.87}$$

where the parameters are either local, like colatitude θ and longitude λ, or related to the ephemerids of the perturbing body, as celestial coordinates α and δ, ωt being the sidereal angle of rotation of the Earth. By expanding (2.87) and combining the terms, we find

$$\begin{aligned} V = & \frac{Gm_p a^2}{d^3} \left(\frac{3\sin^2\delta - 1}{2} \right) \left(\frac{3\cos^2\theta - 1}{2} \right) \\ & + \frac{3Gma^2}{d^3} \sin\theta \cos\theta \sin\delta \cos\delta \cos(\omega t + \lambda - \alpha) \\ & + \frac{3Gma^2}{4d^3} \sin^2\theta \cos^2\delta \cos 2(\omega t + \lambda - \alpha) \end{aligned} \tag{2.88}$$

This modern way of writing the expressions is not exactly the same as Laplace's one, but strictly equivalent. It shows the symmetry between θ, δ, and between λ, $\omega t - \alpha$. The distance d of the external body (Moon or Sun) as well as its equatorial coordinates α and δ vary relatively very slowly with respect to the diurnal variable ωt. This led to the conclusion by Laplace that the three terms of the potential V in Eq. (2.88) give rise to three different species of oscillation. He mentions that the three species mix without interacting and can be studied separately.

2.9.11.1 Oscillations of the First Species

The oscillations of the first species do not depend on the longitude of the surface point, but vary as a function of the orbital parameters of the perturbing body: they have a zonal structure. Among these oscillations figure monthly and fortnightly components for the Moon, annual and semi-annual ones for the Sun. The amplitudes of these oscillations do not depend on the bathymetry of the oceans.

2.9.11.2 Oscillations of the Second Species

These have a quasi-diurnal period (close to $24^{h}50^{mn}$ for the Moon, $24^{h}00^{mn}$ for the Sun) due to the presence of the argument ωt with diurnal frequency. Their amplitudes are modulated by the orbital motion of the body. The amplitude is zero when this body is on the celestial equator and gets all larger as the declination gets high. This occurs during the solstices for the Sun and twice a month for the Moon.

In theory, when the declination is maximal, the oscillations should generate a large difference of amplitude between successive high tides occurring on the same day. This is in fact contrary to what is observed in various harbors of the Atlantic, where these two tides show approximatively the same amplitudes. Laplace discovered that these oscillations depend on the depth of the seas and vanish if the depth is constant. As early as 1775 he expressed his satisfaction in observing his predictions, mentioning that this agreement constituted one of the main accomplishments of his research.

2.9.11.3 Oscillations of the Third Species

They are the most prominent in harbors of the Atlantic. Their period is semi-diurnal (close to $12^{h}25^{mn}$ for the Moon, $12^{h}00^{mn}$ for the Sun) and their amplitudes are also modulated by the relative celestial motion of the celestial body. The amplitude is maximum when the body lies on the celestial equator. Laplace sought the condition in which these oscillations vanish and found that this requires an ocean of infinite depth.

2.10 Methodology, Organization, and Analysis of Observations

The numerous calculations by Laplace on the influence of bathymetry and his research on the necessary conditions for the oscillations of second and third species to vanish reach some limits. The impossibility of explaining the variety of tides by a direct deterministic calculation led him, in his memoir of 1790, then in *Mécanique Céleste* (vols. IV and XIII), to fully exploit observational data and to develop semi-empirical methods.

Thus, if Laplace must be considered as the founder of the dynamical theory of tides, his activities in this field were not restricted to theoretical studies. He was also concerned by more practical aspects and was at the origin of the development of systematic observations of the tides. When in 1790 he sought to determine the local laws of the ebb and flow using a semi-empirical method, the only observations available were those carried out between 1711 and 1716 at Brest harbor and those on Lalande's initiative in 1777. Laplace remarked that they were too vague and incomplete to enable a fruitful analysis. Then he exhorted the scientific community to undertake tidal measurements with 'the same care as astronomical observations'.

In 1803, with the help of Pierre Lévêque and Alexis de Rochon, he participated in a commission in charge of the planning of tidal observations. *Memoir on the observations it is important to carry out on the tides in different harbors of the Republic* was written on this occasion. It establishes an extremely precise protocol of observations, underlying the fact that if earlier the observations guided the theory, now the theory guides the observations. In 1806, following this memoir, a long series of observations was undertaken at Brest. Laplace in *Mécanique Céleste* (vol. XII) analysed the data from 16 years of observations (1807–1822) and in 1843 the Bureau des Longitudes published the observations from 1807 to 1835.

2.10.1 Semi-empirical Methods Based on Partial Flows

Whereas the purely theoretical expressions of the tidal potential established by Laplace reached a quasi-definitive status and incurred few modifications until now, his semi-empirical methods were rather complex and merely gave a starting point for the development in the 19th and 20th centuries. Laplace was aware of why his calculations are not satisfactory: tides are modified by the distribution of continents and oceans, irregularities of the ocean depths, the positions and the slopes of shores, currents, the drag of water. It is true that for these reasons tides have no direct and simple relationship with the tidal potential. But they should obey some laws. Laplace established a principle that should give access to local tides laws and is still used up to the present. It relies on two basic ideas:

- The tidal potential can be decomposed as a series of sinusoidal terms with various periodicities. This decomposition explains the modulation of tidal waves according to the characteristics of the lunar and solar orbital motions (variations of declination, of distance, of longitude). In his decomposition, Laplace introduced a significant number of waves, which are found later in the decompositions used by Lord Kelvin in 1867, Darwin in 1883, and Doodson in 1921.
- Despite numerous perturbations listed above, tides conserve something of their periodicities. In other words, the sea is subject to the same periods as those of the forcing tidal potential: each wave of this potential generates a partial sea flow itself expressed by a sinusoidal function with the same period. The coefficients and phases of the partial flows are modified differently for each harbor and for each wave. The total sea flow at a given point is reconstructed as the sum of the individual partial flows, using the principle of superposition.

2.10.2 Determination of the Amplitudes and Phases of the Partial Flows

The determination of the parameters acting on the water height is possible only from observations. Laplace's method tries to employ a shrewd combination of observations to disentangle the phenomenon being studied. For instance, for the characterization of the semi-diurnal tides, high and low tides were recorded in the vicinity of solstitial and equinoctial syzygies or quadratures. In vol. IV, Laplace uses Cassini's observations from the beginning of the 18th century, refined by observations that he himself organized at Brest between 1807 and 1822. He gathered more than 6000 observations for the purpose. Thus he could determine the fundamental parameters (coefficients and phases) which take part in the diurnal and semi-diurnal oscillations, and find a very good agreement between his semi-empirical formula and observations. In particular he could show that, under the effect of the terrestrial rotation and of various perturbations listed above, the amplitude of the diurnal flow in Brest harbor is reduced by a factor of 1/3 compared with the value predicted by the theoretical equilibrium tide, whereas the semi-diurnal flows is multiplied by a factor of 16. Pushing further the treatment of observations, Laplace sought to put in evidence the flow depending on the lunar potential of degree 3, that is to say involving $1/d_M^4$. His semi-empirical method was powerful indeed.

2.10.3 Determination of the Ratio of Lunar/Solar Tides

Finally, Laplace could attempt a fresh estimate of the ratio μ of the amplitude of the lunar tide to that of the solar tide. We saw that in 1687 Newton had set the value $\mu = 4.5$ by using the height of tides in Bristol harbor, and that later Bernoulli lowered this value to $\mu = 2.5$ by using the precise times of high tides instead of their height. Then in 1749 d'Alembert and Euler lowered the value further to 2.33 thanks to the study of the precession-nutation of the Earth. Finally Laplace found the value 2.35 and concluded that 'the agreement of values found by various means is remarkable'. This ratio also enabled him to calculate the mass of the Moon, for which he found 1/75 of the mass of the Earth, very close to the real fraction of 1/81.

2.10.4 Laplace and Atmospheric Tides

On the margin of his research about the oceanic tides, it is worth mentioning that Laplace was also interested in atmospheric tides. The subject had already been initiated by Daniel Bernoulli and d'Alembert: Laplace explained that the gravitational influence of the Moon and the Sun generates in the atmosphere periodic motions similar to the oceanic ones but extremely weak. The barometer variation he calculated theoretically should be of the order of 0.6 mm of mercury (80 Pa). These variations are too small to explain the strange variations of the barometer, with a 12-hour period and 1.5 mm amplitude observed in the 18th century in tropical areas, in particular by Lamanon in 1785 during La Perouse expedition (1785–1788). Laplace concluded that these variations should be due to thermal forcing. In 1825, using the analysis of 8 years of pressure measurements at the Observatoire de Paris, he tried to show the existence of atmospheric tides with a lunar origin: he found an oscillation of 0.055 mm of mercury, but emphasized that his results are not statistically convincing. The existence of such a tide was demonstrated for the first time in 1842, from observations on the island of St. Helens.

2.11 Conclusion on Laplace's Work

Laplace's work is a landmark in the study of tides. We can condense our discursive text above on his contributions into three bullet points.

- The origin of the tides and the ultimate outcome of Newton's ideas. Instead of the generating force of tides, Laplace used the fruitful tidal generating potential, and pioneered the use of the spherical harmonics.
- Even more fundamental, the establishment of a dynamical theory of tides. By neglecting the vertical velocity in the fluid layer, Laplace establishes the general equation of the dynamics of water in the oceans, which to this day remains the basis of tidal theory. He highlighted the Coriolis force and the fact that each oscillation of the tidal potential generates a partial flow with the same period which, mixed with various local perturbations, gives a great variety of geography-dependent tidal behaviors.
- The organization of an observational network, with a very precise protocol. In parallel he developed a method of analysis of observations.

From these various point of view, Laplace can be considered as the true founder of the modern science of ocean tides.

2.12 Overall Conclusion

Since the first ideas on the influence of the Moon put forward by the Ancients, until the mathematical work of Laplace, the improvements of the theory of tides have been considerable. But why did it take such a long time to solve the problem? Three reasons can be found.

- First, solving the problem necessitated the discovery of the universal law of gravitation, so had to wait Newton.
- Second, tides mingle two causes, a deterministic and precise law of gravitation and perturbations by local environments. To understand the tides, we had to separate the two causes. Newton took the first step by giving the tidal force. Laplace

took the second step, by showing that the sea flows in each harbor have the same periodicities as the tidal potential but with phases and amplitudes depending on the local characteristics of each site.

- Third, mathematical tools had to be invented, not only calculus but also spherical harmonics and the equations of fluid mechanics. By the end Laplace's career, all the tools are ready and provide the theoretical basis for the future.

After Laplace, improvements continued. Several of them are:

– The increase of the observations, in particular thanks to the floating tide gauges invented in 1843 by Rémi Chazallon (1802–1872).
– Understanding that tides result largely from resonances of basins to astronomical excitations: pioneering work by John William Lubbock and William Whewell in 1830–1840, then by Rollin Harris in 1897.
– Refining the harmonic expansion of the tidal potential: 91 terms for George Darwin in 1883, 378 for Doodson in 1921, and 12 935 for Hartmann and Wenzel in 1995.
– Understanding that tides concern only oceanic masses but also the solid part of the Earth: elastic deformations.
– Last but not least, fantastic computing tools that integrate the dynamical equations, replacing the partial integrations done by George Biddell Airy, Lord Kelvin, Henri Poincaré, Carl Gustav Rossby, and others.

In addition to the numerical modeling of tides, the problem today consists in dealing with the tides in a global way to determine the motions of a deformable Earth, partially covered by oceans, containing a fluid core and subject to the action of external bodies.

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